Myopia, Farsightedness, and Stability in the Housing Matching Model

Author
C. van Ravenswaaij

Student number
2114854

Supervisors
P.J.J. Herings
R.L.P. Hendrickx

Master Thesis
Econometrics and Mathematical Economics
Tilburg University
The Netherlands
March 13, 2025

Contents

1 Introduction

Matching theory has many applications, such as the marriage and the college admission problems of Gale and Shapley (1962), the kidney exchange problem, and the housing matching model of Shapley and Scarf (1974). In most of the literature, agents are either assumed to be myopic or fully farsighted. Furthermore, many different definitions for a matching to be stable can be found in the literature.

In this thesis, we study multiple stability concepts of the housing matching model of Shapley and Scarf (1974), which differ by the degree of farsightedness of the agents. Throughout the thesis, we assume that all agents have the same degree of foresight. Like in most of the literature, we first assume that agents are either myopic or fully farsighted. Furthermore, we look at the case in which agents have a limited degree of farsightedness, which has not been studied often but leads to some interesting results. Within the degree of farsightedness, we also vary between weak dominance and strict dominance. An overview of the stability concepts that we study under different degrees of farsightedness and with respect to strict or weak dominance can be found in Table 1 at the end of the introduction.

Shapley and Scarf (1974) introduced the housing matching model, which can be represented by a tuple that consists of the finite set of agents and the preferences of the agents over the indivisible items of the agents. Each agent has an indivisible good, for example a house, and has preferences over the set of indivisible goods. It is assumed that each agent has no use for more than one item. The goal is to redistribute the indivisible items among the agents with respect to their preferences.

Demuynck, Herings, Saulle, and Seel (2019a) defined a social environment as a tuple that consists of the finite set of agents, the state space, an effectivity correspondence, and the preferences of the agents over the states. The housing matching model of Shapley and Scarf (1974) is a special case of a social environment.

In this thesis, we assume that each agent has strict preferences over the indivisible items, and we refer to the housing matching model of Shapley and Scarf (1974) as the housing matching model. Moreover, we define a social environment corresponding to a housing matching model. As in Shapley and Scarf (1974), the goal is to redistribute the indivisible items among the agents with respect to their preferences.

As in Chwe (1994), Demuynck, Herings, Saulle, and Seel (2019b), Kawasaki (2010), Kawasaki (2019) and Klaus, Klijn, and Walzl (2010), we assume that during a sequence of trades each agent remains in possession of his own item until the end outcome of this sequence of trades is reached. If this end outcome is stable in the sense that no coalition of agents decides to deviate from it, then and only then the agents trade their items. Hence, the agents only trade at the end of a sequence of trades, if this end outcome is stable, and not during the sequence of trades. Thus, at each time during the sequence of trades, the outcomes are considered to be realizations of what happens if a coalition of agents decides to trade, and a realization can be interrupted by a coalition of agents which proposes another outcome.

1.1 Myopia

The most common stability concept for cooperative games is the core, the set of outcomes which are not dominated. The core can be seen as a myopic concept, in the sense that agents or coalitions of agents only look one step ahead, i.e. they do not anticipate that another coalition of agents might deviate after their own deviation. For each housing matching model with not necessarily strict preferences over the houses, Shapley and Scarf (1974) showed with the help of the top trading cycle algorithm of Gale that the core is non-empty. For each housing matching model with strict preferences over the houses, Roth and Postlewaite (1977) proved that the strong core consists of the unique top trading cycle allocation.
Another myopic concept is the von Neumann-Morgenstern stable set, which was introduced by von Neumann and Morgenstern (1944). A set is a von Neumann-Morgenstern (vNM) stable set if it satisfies the following two conditions: each outcome inside the set is not dominated by another outcome inside the set, this is known as internal stability, and each outcome outside the set is dominated by an outcome inside the set, this is known as external stability. A von Neumann-Morgenstern stable set may fail to exist and if it exists, it cannot be empty, but does not have to be unique.
Therefore, Demuynck et al. (2019a) introduced the myopic stable set of a social environment. For a finite state space, a set is a myopic stable set if it satisfies deterrence of external deviations, iterated external stability and minimality. A set satisfies deterrence of external deviations if all states that dominate a state inside the set are part of the set. In other words, no coalition of agents can benefit from deviating from a state inside the set to a set outside the set. A set satisfies iterated external stability if from any state A outside the set there is a finite sequence of dominations leading to some outcome A' inside the set. Note that each deviating coalition prefers the next outcome in the sequence to the current one. A set satisfies minimality if there is no proper subset of the set that satisfies deterrence of external deviations and iterated external stability. Demuynck et al. (2019a) showed that a myopic stable set always exists and that it is unique for a finite state space. For each housing matching model with strict preferences, Demuynck et al. (2019b) showed that the weak dominance myopic stable set is equal to the strong core.

In this thesis, we study the core, the von Neumann-Morgenstern stable set and the myopic stable set of the social environment corresponding to a housing matching model under the variation of strict dominance and weak dominance. With the proof in Shapley and Scarf (1974), we show that the top trading cycle allocation, which is A^*, is an element of the core. Also, with the help of the proof in Roth and Postlewaite (1977) we show that the strong core of our social environment corresponding to a housing matching model is equal to $\{A^*\}$. We show that each vNM stable set contains the core, each weak dominance vNM stable set contains A^* and that there are housing matching models for which a vNM stable set does not exist. Moreover, we show that the myopic stable set contains the core, and we rewrite the proof given in Demuynck et al. (2019b), in the context of our social environment corresponding to a housing matching model, to show that the weak dominance myopic stable set is equal to the strong core.

1.2 Full Farsightedness

In most of the literature, agents are either assumed to be myopic, meaning that agents can only look one step ahead, or fully farsighted. Full farsightedness represents that each agent or each coalition of agents anticipates that another coalition of agents might react on their deviation without any limit. This means that each coalition of agents can see each possible chain of deviations without any restriction on the length of the chain. Chwe (1994) defined indirect dominance, the farsighted notion of dominance, as the following: an outcome A' indirectly dominates another outcome A if there is a finite sequence of outcomes starting from A and ending at A', such that each deviating coalition that moves from outcome A^{k-1} to the consecutive outcome A^k prefers the end outcome A' to the outcome that they are moving from, which is A^{k-1}.

A farsighted solution concept is the farsighted core, which consists of all the outcomes that are not indirectly dominated. In the context of a general game, Chwe (1994) introduced the farsighted vNM stable set and the largest consistent set as two other farsighted solution concepts. A farsighted vNM stable set is a set that satisfies internal stability and external stability with respect to indirect dominance. Like in the myopic case, a farsighted vNM stable set may fail to exist, and if it exists, it cannot be empty, but may not be unique.

Therefore, Chwe (1994) introduced the largest consistent set, the consistent set which contains all consistent sets. A set is a consistent set if and only if it consists of all the outcomes that satisfy deterrence of deviations. Deterrence of deviations means that each deviation from any outcome A inside the set to an arbitrary outcome A' is deterred by the credible threat of ending in another outcome A'' inside the set. By threat, we mean that A'' compared to A is worse or equally well for at least one agent in the deviating coalition. This threat is credible if either $A'' = A'$ or A'' indirectly dominates A'. A consistent set requires deterrence of internal deviations and deterrence of external deviations. A consistent set does not have to be unique and $\emptyset$ is a consistent set. Chwe (1994) showed that the largest consistent set always exists and that it is unique. The idea behind the largest consistent set is the following: if an outcome is not in the largest consistent set, then this outcome cannot be stable, and if an outcome is in the largest consistent set, then it is possible that this outcome is stable. Hence, the largest consistent set rules out with confidence. Chwe (1994) showed that under some conditions the largest consistent set is nonempty. In Herings, Mauleon, and Vannetelbosch (2004), it is shown that in certain social environments the largest consistent set might rule out too much.

Another farsighted solution concept is a pairwise farsightedly stable set, which in the context of networks was introduced in Herings, Mauleon, and Vannetelbosch (2009). In Herings, Mauleon, and Vannetelbosch (2010), this concept was applied to the context of coalition formation games and they introduced a farsightedly stable set. In the literature, a farsightedly stable set is called a DEM farsighted stable set, see for example Kimya (2023). Herings et al. (2010) showed that a DEM farsighted stable set always exists and cannot be empty, but does not have to be unique. A set is a DEM farsighted stable set if it satisfies deterrence of external deviations, external stability and minimality. A set satisfies deterrence of external deviations if each deviation from any outcome inside the set to an outcome outside the set is deterred by the possibility of ending worse off or equally well off. A set satisfies external stability if each outcome outside the set is indirectly dominated by an outcome within the set. Minimality means that there does not exist a proper subset of the set that satisfies the other two conditions.

In this thesis, we study the farsighted core, the farsighted vNM stable set, the largest consistent set and the DEM farsighted stable set with respect to three different definitions of indirect dominance.
We use indirect dominance to denote that each deviating coalition that moves from an outcome A^{k-1} to the consecutive outcome A^k strictly prefers the end outcome A' to the outcome from which they are moving, which is A^{k-1}. With respect to indirect dominance, we prove for all housing matching models that the set $\{A^*\}$ is the farsighted core, the unique farsighted vNM stable set and the unique DEM farsighted stable set. Moreover, we show that the largest consistent set always contains A^*, but that it can contain more than A^*.
As in Mauleon and Vannetelbosch (2004), we define indirect weak dominance as indirect dominance, such that each deviating coalition that moves from an outcome A^{k-1} to the consecutive outcome A^k weakly prefers the end outcome A' to the outcome from which they are moving, which is A^{k-1}. For all housing matching models, we prove that with respect to indirect weak dominance, the farsighted core is either $\emptyset$ or $\{A^*\}$, and that the largest consistent set contains A^*.
Moreover, for all housing matching models such that the farsighted core is equal to $\{A^*\}$, we show that with respect to indirect weak dominance, $\{A^*\}$ is the unique farsighted vNM stable set and the unique DEM farsighted stable set.
For all housing matching models such that the farsighted core is $\emptyset$, we show that with respect to indirect weak dominance, $\{A^*\}$ is a farsighted vNM stable set and a DEM farsighted stable set. Furthermore, we show that $\{A^*\}$ is not necessarily the unique farsighted vNM stable set and that there is at least one other DEM farsighted stable set.
Kawasaki (2010) introduced the concept of indirect antisymmetric weak dominance, which compared to indirect weak dominance has one additional restriction. This restriction is that each agent in the deviating coalition, which is indifferent between the end outcome and the outcome that he is moving from gets the same item in the next outcome in the sequence. In other words, this agent is only be part of the deviating coalition if the deviation does not change which item he gets.
We rewrite the proof given in Kawasaki (2010), in the context of our social environment corresponding to a housing matching model, to show for all housing matching models that the farsighted core with respect to indirect antisymmetric weak dominance is equal to $\{A^*\}$. Moreover, for all housing matching models we notice that with respect to indirect antisymmetric weak dominance, $\{A^*\}$ is the unique farsighted vNM stable set and the unique DEM farsighted stable set. Furthermore, for all housing matching models, we prove that the largest consistent set with respect to indirect antisymmetric weak dominance is equal to $\{A^*\}$.

1.3 Horizon-K Farsightedness

The interesting case is the case in which agents have a limited degree of farsightedness, this is the intermediate case between myopia and full farsightedness. We denote the degree of farsightedness by K, which represents the number of steps agents can look ahead. For this intermediate case, two models have been developed: horizon-K farsightedness by

Herings, Mauleon, and Vannetelbosch (2019) and level-K farsightedness by Herings and Khan (2022).

In the context of networks, Herings et al. (2019) introduced the concept of a horizon-K farsighted set to study the influence of the degree of farsightedness on the stability of networks. A set is a horizon-K farsighted set if the set satisfies horizon-K deterrence of external deviations, horizon-K external stability and minimality.
A set satisfies horizon-K deterrence of external deviations if each deviation from any outcome A inside the set to an outcome A' outside the set is deterred by the credible threat of ending in another outcome A'', which compared to A is not strictly preferred by all agents in the deviating coalition. With a credible threat, we mean that A'' is such that either A'' can be reached from A' by a sequence of outcomes of a length smaller than or equal to $K-2$ and A'' belongs to the set or A'' can be reached from A' by a sequence of a length equal to $K-1$ and there does not exist a sequence of a length smaller than $K-1$ starting at A' and ending at A''.
A set satisfies horizon-K external stability if from each outcome outside the set there is a finite sequence, which consists of sequences of outcomes of length smaller than or equal to K, leading to an outcome inside the set. Minimality means that there does not exist a proper subset of the set that satisfies the above two conditions. Herings et al. (2019) showed that a horizon-K farsighted set always exists and that the horizon-1 farsighted set is unique.

In the context of networks, Herings and Khan (2022) introduced the concept of a level-K stable set and the concept of heterogeneity in the degree of foresight. In comparison to the horizon-K farsightedness in Herings et al. (2019), level-K farsightedness in Herings and Khan (2022) is defined in an inductive way. In order to define when a deviation from an outcome to another outcome is a level-K deviation, one need to know what the level-$(K-k)$ deviations are for $k \in \{1, \ldots, K-1\}$.
A set is a level-K stable set if it satisfies deterrence of external deviations, iterated external stability and minimality, as defined in Herings and Khan (2022). Herings and Khan (2022) proved that there always exists a unique level-K stable set.

In this thesis, we study the horizon-K farsighted core, the horizon-K von Neumann-Morgenstern stale set and the horizon-K farsighted set of Herings et al. (2019), which we call the horizon-K farsighted stable set, of the social environment corresponding to a housing matching model.
Horizon-K farsightedness is the intermediate case between myopia and full farsightedness. Horizon-1 farsightedness is equivalent to myopia in the sense that agents can only look one step ahead. We notice that the horizon-1 farsighted core is the core and that horizon-1 farsighted vNM stable sets are vNM stable sets. Moreover, we prove that the myopic stable set is the unique horizon-1 farsighted stable set.
Horizon-∞ farsightedness and full farsightedness are related in the sense that agents can look infinitely many steps ahead. We show that the horizon-∞ farsighted core is the farsighted core, that the unique horizon-∞ vNM stable set is the farsighted vNM stable set and that $\{A^*\}$ is the unique horizon-∞ farsighted stable set.

The more interesting case is the case in which agents are neither myopic nor fully farsighted, i.e. agents can look K steps ahead with $1 < K < \infty$. For all housing matching models and for all $K \geq 2$, we show that the horizon-K farsighted core is equal to $\{A^*\}$

and that $\{A^*\}$ is the unique horizon-K vNM stable set.
The results of the horizon-K farsighted stable set depend on whether agents can look two steps ahead or at least three steps ahead. For the case that agents can look two steps ahead, we prove that the core is a subset of each horizon-2 farsighted stable set.
For the case that agents can look at least three steps ahead, i.e. $K \geq 3$, we prove for all housing matching models that $\{A^*\}$ is the unique horizon-K farsighted stable set.

1.4 Section overview

The thesis is organized as follows. In Section 2, we give a formal definition of the housing matching model and the social environment corresponding to this housing matching model. In Section 3, we determine the core, the vNM stable set and the myopic stable set, as defined in Demuynck et al. (2019a), for our social environment corresponding to a housing matching model.
Under the assumption that all agents are fully farsighted, we study the farsighted core, the farsighted vNM stable set, the largest consistent set and the DEM farsighted stable set with respect to indirect dominance in Section 4, with respect to indirect weak dominance in Section 5 and with respect to indirect antisymmetric weak dominance in Section 6.
In Section 7, we determine the core and the von Neumann-Morgenstern set under the assumption that agents can only look K steps ahead and we determine the horizon-K farsighted stable set as defined in Herings et al. (2019). Finally in Section 8, we give an overview of all the results that hold for all housing matching models and we conclude.

		Degree of farsightedness		
		Myopia	Fully farsighted	Horizon-K farsighted
dominance	strict	core: 3.1 vNM stable set: 3.3 myopic stable set: 3.4	core: 4.2 vNM stable set: 4.3 DEM farsighted stable set: 4.5 largest consistent set: 4.4	core: 7.1 vNM stable set: 7.2 horizon-K farsighted stable set: 7.3
	weak	core: 3.2 vNM stable set: 3.3 myopic stable set: 3.4	core: 5.2 and 6.2 vNM stable set: 5.3 and 6.2 DEM farsighted stable set: 5.5 and 6.2 largest consistent set: 5.4 and 6.3	We do not study horizon-K farsightedness with respect to weak dominance

Table 1: Overview of the stability concepts that are studied under different degrees of farsightedness and with respect to strict or weak dominance. The numbers refer to the subsections in which the mentioned stability concept is studied.

2 Housing Matching Model

Let $n \geq 1$ and let $N = \{1, \ldots, n\}$ be the finite set of all agents. Each agent $i \in N$ has an item i, which is an indivisible good, and has strict preferences over all the items. These strict preferences are described by an $n \times n$ matrix P with rows representing the agents and columns representing the items. Let $P_{ij} \in \mathbb{R}$ denote the entry of matrix P in the ith row and the jth column. The entries of P have the following meaning: $P_{ij} > P_{ik}$ means that agent i strictly prefers item j to item k. Hence, only the ordering of the entries of P matters, i.e. the preferences are purely ordinal. Note that we do not take expectations over the preferences. We use non-positive numbers as entries of P to avoid any confusion with the agents in N, i.e. $P_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i, j \in N$.

The **housing matching model** is defined as the tuple (N, P) with N and P as above. In this section, we construct the social environment, defined as in Demuynck et al. (2019a), corresponding to the housing matching model (N, P). A social environment is a tuple consisting of a finite set of agents, a state space, an effectivity correspondence and preferences of the agents over the state space. An effectivity correspondence defines for each pair (A, A') of states the set of coalitions that can change state A into state A'.

2.1 Preferences over the state space

The goal is to redistribute the items among the agents with respect to their preferences, such that each agent gets one item. This can be described by an $n \times n$ matrix A, called an allocation, with rows representing the agents and columns representing the items. Let $A_{ij} \in \{0, 1\}$ denote the entry of matrix A in the ith row and the jth column and let it be defined as

$$A_{ij} = \begin{cases} 1 & \text{if agent } i \text{ gets item } j, \\ 0 & \text{if agent } i \text{ does not get item } j. \end{cases}$$

Note that the matrix A is a permutation matrix, which is a zero-one matrix with all row-sums and all column-sums equal to 1. Let A_i denote the ith row of A. Let the **state space** X be the set of all permutation matrices.

As in Demuynck et al. (2019b), the preferences of the agents over the set X, denoted by $(\succsim_i)_{i \in N}$, are induced by their preferences P over the indivisible items.

Definition 2.1 (Preferences $(\succsim_i)_{i \in N}$ over the set X)**.**
Let $i \in N$ and let $A, A' \in X$ be two different permutation matrices. Let $j, k \in N$ be, such that $A_{ij} = 1$ and $A'_{ik} = 1$. Then we define **the preferences of the agents** $(\succsim_i)_{i \in N}$ **over the state space** as follows:

(1) $A \succ_i A'$ if and only if $P_{ij} > P_{ik}$,

(2) $A \sim_i A$ if and only if $j = k$.

Let 2^N denote the set of all subsets of N. An element of 2^N is called a coalition. The set of coalitions that can change a permutation matrix A into a permutation matrix A' is defined by the effectivity correspondence. In order to define it, we need to use another description of permutation matrices.

2.2 Cycle decomposition

In Dummit and Foote (2004), the concept of a cycle was introduced. We give a formal definition of a cycle in the context of our housing matching model (N, P). In this thesis, we use that $\mathbb{N}$ is the set of all positive integers, i.e. $\mathbb{N} = \{1, 2, 3, \dots\}$.

Definition 2.2 (Cycle c and $S(c)$)**.**
Let $\ell \in \mathbb{N}$ with $\ell \leq n$ and let $c_k \in N$ for $k \in \{1, \dots, \ell\}$ be distinct agents with $c_1 < c_k$ for all $k \in \{2, \dots, \ell\}$. Then define a **cycle** $c = (c_1 c_2 \cdots c_\ell)$ as the permutation that sends the item of agent c_k to agent c_{k+1} for $1 \leq k \leq \ell - 1$ and that sends the item of agent c_ℓ to agent c_1, and define **the set of agents involved in cycle** c, $S(c) \in 2^N \setminus \{\emptyset\}$, as

$$S(c) = \{i \in N \mid \exists\, k \in \{1, \dots, \ell\} \text{ such that } i = c_k\}.$$

We say that the cycle $c = (c_1 c_2 \cdots c_\ell)$ contains agent $i \in N$ if and only if $i \in S(c)$. Let $|S(c)| = \ell$ denote the length of cycle c.

Two cycles c and d, which are not necessarily of the same length, are called **disjoint** if $S(c) \cap S(d) = \emptyset$. Hence, two cycles are disjoint if they have no agents in common.

In order to understand which permutation a cycle represents and when two cycles are disjoint, we look at an example.

Example 2.3. The cycle (123) represents that agent 1 gets item 3, agent 2 gets item 1 and that agent 3 gets item 2. Note that the cycles (123) and (245) are not disjoint, because $2 \in S(123)$ and $2 \in S(245)$. The cycles (123) and (45) are disjoint, because they have no agents in common. △

Let Σ_n denote the set of all bijections from N to N, i.e. the set of all permutations of N. Note that $|\Sigma_n| = n!$. In Chapter 4.1 of Dummit and Foote (2004) it is shown that each permutation $\sigma \in \Sigma_n$ can be written as a unique finite product of disjoint cycles, this is called the **cycle decomposition** of σ. In our housing matching model this unique product of cycles represents the trades that are being executed. Note that disjoint cycles commute in the sense that (1)(23) and (23)(1) both represent that agent 1 gets his own item and that agents 2 and 3 trade with each other. Thus, each permutation has a unique cycle decomposition up to rearranging its cycles. Suppose that the cycle decomposition of σ is as follows:

$$\sigma = (c_1 c_2 \cdots c_{\ell^1})(c_{\ell^1+1} c_{\ell^1+2} \cdots c_{\ell^2}) \cdots (c_{\ell^{m-1}+1} c_{\ell^{m-1}+2} \cdots c_{\ell^m}),$$

then the agents in N are partitioned into m cycles and for $j \in N$ we have that

$$\sigma(j) = \begin{cases} c_{\ell^{s-1}+1} & \text{if } j = c_{\ell^s} \text{ for some } s \in \{1, \dots, m\}, \\ c_{r+1} & \text{if } j = c_r \text{ with } r \neq \ell^s \text{ for all } s \in \{1, \dots, m\}, \end{cases}$$

with $\ell^0 = 0$. Note that in our housing matching model (N, P) we have that $\sigma(j)$ is the agent that gets the item of agent j.

In Dummit and Foote (2004), it is proved that each permutation has a unique cycle decomposition. Our state space X is the set of all permutation matrices. Hence, in the following lemma, we show that each $A \in X$ has a unique cycle decomposition.

Lemma 2.4. *Each permutation matrix can be uniquely described by a finite product of disjoint cycles, in which a cycle of one agent means that the agent gets his own item and a cycle of multiple agents means that each agent in the cycle gets the item belonging to the previous agent in the cycle.*

Proof. Let the map $\mathscr{A} : \Sigma_n \to X$ be defined as

$$\mathscr{A}(\sigma)_{ij} = \delta_{i\sigma(j)} = \begin{cases} 1 & \text{if } \sigma(j) = i, \\ 0 & \text{if } \sigma(j) \neq i, \end{cases}$$

with $\sigma(j) = i$ meaning that agent i gets item j. We show that this map is a bijection. First, we show that this map is injective. Let $\sigma, \theta \in \Sigma_n$ be two bijections. Suppose that $\mathscr{A}(\sigma) = \mathscr{A}(\theta)$, then we have that $\delta_{i\sigma(j)} = \mathscr{A}(\sigma)_{ij} = \mathscr{A}(\theta)_{ij} = \delta_{i\theta(j)}$ for all $i, j \in N$. Hence, we get that $\theta = \sigma$.

Now, we show that the map is surjective. Let $A \in X$. Define $\sigma \in \Sigma_n$ as the permutation with $\sigma(j) = i$ if $A_{ij} = 1$, then we have that $A = \mathscr{A}(\sigma)$. Hence, the map is bijective. Thus, each permutation matrix has a unique cycle decomposition. □

For $n = 3$, the cycle decomposition of the six permutation matrices is shown in Table 2.

Permutation matrices	Cycle decomposition
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(1)(2)(3)
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(123)
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(132)
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(1)(23)
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	(2)(13)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(3)(12)

Table 2: Illustration of the cycle decomposition of permutation matrices.

For a housing matching model (N, P), we can conclude from Table 2, that the permutation matrix which allocates every agent his own item, has a cycle decomposition consisting of n cycles. Hence, it seems reasonable to define a set that contains all the cycles in the cycle decomposition of a permutation matrix.

Definition 2.5. $[C(A)$, $\mathcal{S}(A)$, c_A and $c_A^i]$

Let $k \geq 1$, let $A \in X$ and let $c^1, \ldots, c^k$ be the disjoint cycles, such that $\prod_{\ell=1}^{k} c^\ell$ is the cycle decomposition of A. Then define the following:

(1) $C(A)$ as the set of cycles that are in the cycle decomposition of A, i.e.

$$C(A) = \{c^1, \ldots, c^k\},$$

(2) $\mathcal{S}(A)$ as the set of coalitions that form a cycle in A, i.e.

$$\mathcal{S}(A) = \{S(c^1), \ldots, S(c^k)\} = \bigcup_{c \in C(A)} \{S(c)\},$$

(3) c_A as an arbitrary cycle in the cycle decomposition of A, i.e. $c_A \in C(A)$,

(4) c_A^i as the cycle in the cycle decomposition of A that contains agent i, i.e. $c_A^i \in C(A)$ such that $i \in S(c_A^i)$.

The following example explains what $C(A)$, $\mathcal{S}(A)$, c_A and c_A^i mean.

Example 2.6. From Table 2, we know that the cycle decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is (1)(23). Hence, allocation A is the product of two disjoint cycles (1) and (23), $C(A) = \{(1), (23)\}$, $\mathcal{S}(A) = \{\{1\}, \{2, 3\}\}$, $c_A^1 = (1)$ and $c_A^2 = c_A^3 = (23)$. △

For $n = 3$, an illustration of the meaning of the definitions in Definition 2.5 can be found in Table 3.

Permutation matrices	Cycle decomposition	$C(A)$	$\mathcal{S}(A)$	c_A^i
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(1)(2)(3)	$\{(1), (2), (3)\}$	$\{\{1\}, \{2\}, \{3\}\}$	$c_A^1 = (1),\ c_A^2 = (2),\ c_A^3 = (3)$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(123)	$\{(123)\}$	$\{\{1, 2, 3\}\}$	$c_A^i = (123)\ \forall\, i \in N$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(132)	$\{(132)\}$	$\{\{1, 2, 3\}\}$	$c_A^i = (132)\ \forall\, i \in N$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(1)(23)	$\{(1), (23)\}$	$\{\{1\}, \{2, 3\}\}$	$c_A^1 = (1),\ c_A^i = (23)\ \forall\, i \in \{2, 3\}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	(2)(13)	$\{(2), (13)\}$	$\{\{2\}, \{1, 3\}\}$	$c_A^2 = (2),\ c_A^i = (13)\ \forall\, i \in \{1, 3\}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(3)(12)	$\{(3), (12)\}$	$\{\{3\}, \{1, 2\}\}$	$c_A^3 = (3),\ c_A^i = (12)\ \forall\, i \in \{1, 2\}$

Table 3: Illustration of the meaning of the definitions in Definition 2.5 for $n = 3$.

2.3 Effectivity correspondence

There are multiple ways to define an effectivity correspondence, which defines for each pair (A, A') of permutation matrices the set of coalitions that can change a permutation matrix A into a permutation matrix A'. In general an effectivity correspondence is a correspondence $E : X \times X \to 2^N$ with $E(A, A')$ as the set of coalitions that can change state A into state A'.

As in Chwe (1994), Demuynck et al. (2019b), Kawasaki (2010), Kawasaki (2019) and Klaus et al. (2010), we assume that each agent remains in possession of his own item until a stable state is reached, then and only then, the agents trade their items. With a stable state, we mean that no coalition of agents decides to change it. Thus, states are considered to be realizations of what happens if a coalition of agents decides to trade, and a realization can be interrupted by a coalition of agents which proposes another outcome.

In Roth and Postlewaite (1977), a coalition S is effective for the allocation A', if the agents in S reallocate their initial items among themselves according to A'. This is the same as an effectivity correspondence.
Hence, Roth and Postlewaite (1977) defined the effectivity correspondence as follows: $S \in E(A, A')$ if and only if for every $i \in S$ there exists $j \in S$ such that $A'_{ij} = 1$. In other words, a coalition S can change A into A', if and only if, according to A', the items of agents in S are reallocated among the agents in S.

Example 2.7. Suppose that the effectivity correspondence is defined as in Roth and Postlewaite (1977). Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then $E(A, A') = \{\{1\}, \{2, 3\}, \{1, 2, 3\}\}$. We have that $\{1\} \in E(A, A')$, this means that agent 1 can claim his own item, and at the same time, he determines that agents 2 and 3 trade with each other. △

In Example 2.7, it is unnatural that agent 1 determines what the other agents do if he claims his own item, because agent 2 might rather have his own item than the item of agent 3. Hence, we need restrictions in the definition of the effectivity correspondence on what happens to agents outside the deviating coalition S.

In Kawasaki (2010), a coalition S can change A into A', if and only if allocation A' satisfies the following three conditions:

(1) the initial items of the agents in S are reallocated among S itself,

(2) the agents in $N \setminus S$, that are not affected by the deviating coalition, get the same item as in A,

(3) the agents in $N \setminus S$, that are affected by the deviating coalition, get their own item.

This principle agrees with how people behave in real life settings, therefore we give Kawasaki (2010)'s definition of an effectivity correspondence in the context of our housing matching model (N, P).

Definition 2.8 (Effectivity correspondence E).
Define the **effectivity correspondence** as the correspondence $E : X \times X \to 2^N$ such that $\forall\ S \in 2^N \setminus \{\emptyset\}$ and $\forall\ A, A' \in X$, we have that $S \in E(A, A')$ if and only if the following three conditions are satisfied:

(1) $\forall\, i \in S$ there exists $j \in S$, such that $A'_{ij} = 1$,

(2) $\forall\, i \in N \setminus S$ such that $S(c^i_A) \cap S = \emptyset$, it holds that: if $A_{ij} = 1$, then $A'_{ij} = 1$,

(3) $\forall\, i \in N \setminus S$ such that $S(c^i_A) \cap S \neq \emptyset$, it holds that: $A'_{ii} = 1$.

We say that a coalition S can move or deviate from A to A' when $S \in E(A, A')$. Note that each $S \in E(A, A')$ can be written as $S = \bigcup_{i \in S} S(c^i_{A'})$ and that $N \in E(A, A')$ for all $A, A' \in X$. Thus, it holds that $E(A, A') \neq \emptyset$ for all $A, A' \in X$. In condition (2) of Definition 2.8 we assume coalitional sovereignty defined in Ray and Vohra (2015) and Herings, Mauleon, and Vannetelbosch (2017) as that a deviating coalition S cannot enforce what unaffected agents outside S do.

Example 2.9. Let $n = 3$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then with Table 2 and Definition 2.8, we get that $A = (1)(2)(3)$, $A' = (1)(23)$ and that

$$E(A, A') = \{\{2, 3\}, \{1, 2, 3\}\}.$$

Note that $\{1\} \notin E(A, A')$. In other words, agent 1 cannot determine that agents 2 and 3 have to trade with each other when he claims his own item. Moreover, we have that $C(A') = \{(1), (23)\}$, that $S = \{2, 3\} \in E(A, A')$ is equal to $S(23)$, and that $N \in E(A, A')$ can be written as $N = S(1) \cup S(23)$.

With Table 2 and Definition 2.8, we also get that

$$E(A', A) = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Note that $\{1\} \notin E(A', A)$. In other words, agent 1 cannot determine that agents 2 and 3 do not trade with each other anymore. Also, note that for example we have that $\{2\} = S(2)$, $\{1, 2\} = S(1) \cup S(2)$ and that $N = S(1) \cup S(2) \cup S(3)$. △

Since we described the state space, the preferences of the agents over the state space and the effectivity correspondence, we can now, as in Demuynck et al. (2019a), give a formal definition of the social environment corresponding to a housing matching model (N, P).

Definition 2.10 (Social environment).
Let (N, P) be a housing matching model. The **social environment corresponding to** (N, P) is the tuple

$$\mathcal{E}(N, P) = (N, X, E, (\succsim_i)_{i \in N})$$

consisting of the finite set of agents N, the state space X, which is the set of all permutation matrices, the effectivity correspondence E on X, as in Definition 2.8, and the preferences $(\succsim_i)_{i \in N}$ over X as in Definition 2.1.

For the housing matching model of Shapley and Scarf (1974), Demuynck et al. (2019b) showed that the myopic stable set of Demuynck et al. (2019a), which we study in Section 3, has a nice property under the assumption that the effectivity correspondence satisfies two conditions. These two conditions are given as follows:

(1) $\forall\, S \in 2^N \setminus \{\emptyset\}$ and $\forall\, A, A' \in X$, if $S \in E(A, A')$ then for all $i \in S$, there exists $j \in S$ such that $A'_{ij} = 1$,

(2) $\forall\, S \in 2^N \setminus \{\emptyset\}$, $\forall\, A \in X$ and for all bijections $\phi : S \to S$, there exists $A' \in X$ such that $S \in E(A, A')$ and $\forall\, i \in S$ it holds that $A'_{i\phi(i)} = 1$.

In other words, their second condition means that every redistribution of the items within a coalition can be achieved.

In the following lemma, we show that our effectivity correspondence satisfies the above two conditions of Demuynck et al. (2019b). Note that our effectivity correspondence is not the same as the effectivity correspondence of Demuynck et al. (2019b), because we have restrictions on how the items of agents outside the deviating coalition S are reallocated.

Lemma 2.11. *The effectivity correspondence, defined as in Definition 2.8, satisfies the above two conditions of Demuynck et al. (2019b).*

Proof. Note that condition (1) of Definition 2.8 implies that our effectivity correspondence satisfies the first condition of Demuynck et al. (2019b). We show that our effectivity correspondence also satisfies their second condition.

Take $S \in 2^N \setminus \{\emptyset\}$, $A \in X$ and a bijection $\phi : S \to S$. Then let $A' \in X$ be the permutation matrix that satisfies the following three conditions:

(1) $\forall\, i \in S$ it holds that $A'_{i\phi(i)} = 1$,

(2) $\forall\, i \in N \setminus S$ such that $S(c^i_A) \cap S = \emptyset$, it holds that: if $A_{ij} = 1$, then $A'_{ij} = 1$,

(3) $\forall\, i \in N \setminus S$ such that $S(c^i_A) \cap S \neq \emptyset$, it holds that: $A'_{ii} = 1$.

Then, with Definition 2.8, we have that $S \in E(A, A')$. Thus, our effectivity correspondence satisfies their second condition. □

3 Myopia

The most common stability concept in the literature on cooperative games is the core, the set of permutation matrices such that no coalition wants to deviate from it. In the literature, the core is seen as a myopic concept, in the sense that coalitions of agents do not anticipate that their deviations can lead to further deviations by other coalitions of agents.

Under the assumption that all agents are myopic, we study the core, the von Neumann-Morgenstern stable set, introduced in von Neumann and Morgenstern (1944) and the myopic stable set, which was introduced in Demuynck et al. (2019a), of the social environment corresponding to the housing matching model (N, P).

3.1 Core

The literature distinguishes two types of dominance. We give these definitions in the context of our social environment corresponding to the housing matching model (N, P).

Definition 3.1 (Strict dominance and weak dominance)**.**
Let (N, P) be a housing matching model and let $A, A' \in X$ be two different permutation matrices. Then define strict dominance and weak dominance as the following:

(1) the permutation matrix A' **strictly dominates** A in $\mathcal{E}(N, P)$ if there exists a coalition $S \in E(A, A')$ such that $A' \succ_i A$ for all $i \in S$,

(2) the permutation matrix A' **weakly dominates** A in $\mathcal{E}(N, P)$ if there exists a coalition $S \in E(A, A')$ such that $A' \succsim_i A$ for all $i \in S$ and $A' \succ_j A$ for at least one $j \in S$.

Let 2^X denote the set of all subsets of X. In Demuynck et al. (2019a), the following definitions are given. The **dominance correspondence** is defined as the correspondence $f : X \to 2^X$ such that $f(A)$ denotes the subset of X that contains A and all the permutation matrices A' that strictly dominate A. In other words,

$$f(A) = \{A\} \cup \{A' \in X \mid A' \text{ strictly dominates } A \text{ in } \mathcal{E}(N, P)\}.$$

Define the correspondence $f^2 : X \to 2^X$ such that $f^2(A)$ is the set of permutation matrices that can be reached from A by at most two consecutive strict dominations, i.e.

$$f^2(A) = \{A'' \in X \mid \exists\, A' \in X \text{ such that } A' \in f(A) \text{ and } A'' \in f(A')\}.$$

For $k \geq 1$, define the correspondence $f^k : X \to 2^X$ such that $f^k(A)$ is the set of permutation matrices that can be reached from A by at most k consecutive strict dominations and define the correspondence $f^{\mathbb{N}} : X \to 2^X$ such that $f^{\mathbb{N}}(A)$ is the set of permutation matrices that can be reached from A by a finite number of strict dominations:

$$f^{\mathbb{N}}(A) = \bigcup_{k \in \mathbb{N}} f^k(A).$$

Note that the correspondences $f^k : X \to 2^X$ and $f^{\mathbb{N}} : X \to 2^X$ are myopic in the sense that coalitions only deviate when they see an immediate gain by doing so. We can also do the same for weak dominance. As in Demuynck et al. (2019a), let the **weak dominance correspondence** be denoted by $\tilde{f}$. Now, we are ready to define the core. The definition that we use was introduced in Demuynck et al. (2019a) and it is written down in the context of our social environment corresponding to the housing matching model (N, P).

Definition 3.2 (Core and strong core).
Let (N, P) be a housing matching model. Then define the core and the strong core as the following:

(1) the **core** CO of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not strictly dominated:
$$CO = \{A \in X \mid f(A) = \{A\}\},$$

(2) the **strong core** SCO of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not weakly dominated:
$$SCO = \{A \in X \mid \tilde{f}(A) = \{A\}\}.$$

Note that strict dominance implies weak dominance, hence the strong core is a subset of the core. With the help of the top trading cycle algorithm of Gale, which is described in Shapley and Scarf (1974), we show that the core is nonempty. In Shapley and Scarf (1974), the definition of a top trading cycle is given. We give their definition in the context of our housing matching model (N, P) and our definition of a cycle. In this thesis, we denote a strict inclusion by $\subsetneq$ and we denote a weak inclusion by $\subseteq$.

Definition 3.3 (Top trading cycle).
Let (N, P) be a housing matching model, let $c = (i_1 \cdots i_\ell)$ be a cycle and let $i_0 = i_\ell$. Then c is a **top trading cycle** for $S \subseteq N$ if $\emptyset \subsetneq S(c) \subseteq S$ and if for each $k \in \{1, \ldots, \ell\}$ it holds that $P_{i_k i_{k-1}} > P_{i_k j}$ for all $j \in S \setminus \{i_{k-1}\}$.

Recall that $S(c)$ in Definition 3.3 is the coalition consisting of the agents belonging to cycle c, i.e. $S(c) = \{i_1, \ldots, i_\ell\}$. Since the set of agents is finite, we get for each nonempty coalition $S \subseteq N$ that there exists at least one top trading cycle. From Definition 3.3 and Definition 2.2, we get that in a top trading cycle for S each agent gets his most preferred item in S.

Notation 3.4 (Top trading cycle).
We denote a top trading cycle c by tc to indicate the difference between a cycle and a top trading cycle derived from the top trading cycle algorithm.

Shapley and Scarf (1974) also constructed an allocation, which is derived from the top trading cycle algorithm of Gale. We give a formal formulation of the top trading cycle algorithm and we construct an $n \times n$ permutation matrix from this algorithm.

Algorithm 1 Top trading cycle algorithm

Let A^* be an $n \times n$ matrix with all entries equal to zero.

Input : A housing matching model (N, P).

Step 1 : Let $tc^1 = (i_1^1 \cdots i_{\ell^1}^1)$ with $i_0^1 = i_{\ell^1}^1$ be a top trading cycle for N and let $A^*_{i_k^1 i_{k-1}^1} = 1$ for $k \in \{1, \ldots, \ell^1\}$.

Step 2 : If $N \setminus S(tc^1) \neq \emptyset$, let $tc^2 = (i_1^2 \cdots i_{\ell^2}^2)$ with $i_0^2 = i_{\ell^2}^2$ be a top trading cycle for $N \setminus S(tc^1)$ and let $A^*_{i_k^2 i_{k-1}^2} = 1$ for $k \in \{1, \ldots, \ell^2\}$.

For $\tau \geq 3$, continue with step τ until there are no agents left.

Step τ : If $N \setminus \left(\bigcup_{1 \leq r \leq \tau-1} S(tc^r) \right) \neq \emptyset$, let $tc^\tau = (i_1^\tau \cdots i_{\ell^\tau}^\tau)$ with $i_0^\tau = i_{\ell^\tau}^\tau$ be a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau-1} S(tc^r) \right)$ and let $A^*_{i_k^\tau i_{k-1}^\tau} = 1$ for $k \in \{1, \ldots, \ell^\tau\}$.

Output: The result that the set of agents N is partitioned into T disjoint coalitions $S(tc^\tau)$, i.e. $N = \bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau-1} S(tc^r) \right)$ and a permutation matrix A^* such that for all $\tau \in \{1, \ldots, T\}$, we have that $A^*_{i_k^\tau i_{k-1}^\tau} = 1$ for all $k \in \{1, \ldots, \ell^\tau\}$.

Note that the coalitions $S(tc^1), \ldots, S(tc^T)$ are disjoint, i.e. they have no agents in common. Hence, even when there exist multiple top trading cycles for a coalition $S \in 2^N \setminus \{\emptyset\}$, we get with the fact that preferences over the items are strict, that there is a unique permutation matrix that follows from the top trading cycle algorithm. We denote this unique top trading cycle permutation matrix by A^*. Note that the permutation matrix A^* can be viewed as iteratively carrying out the indicated trades within the top trading cycles.

Remark 3.5. Note that with Definition 2.2 and Lemma 2.4, the cycle decomposition of A^* is the product of the top trading cycles $tc^\tau = (i_1^\tau i_2^\tau \cdots i_{\ell^\tau - 1}^\tau i_{\ell^\tau}^\tau)$ for $\tau \in \{1, \ldots, T\}$. In other words, $C(A^*) = \{tc^1, \ldots, tc^T\}$ and $A^* = \prod_{\tau=1}^{T} tc^\tau$.

To see the above, we look at an example.

Example 3.6. Let $n = 3$ and let the preference matrix be

$$P = \begin{pmatrix} 0 & -1 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

Then both (1) and (23) are top trading cycles for N. Suppose that $tc^1 = (1)$. Then we still have that (23) is a top trading cycle for $N \setminus \{1\} = \{2, 3\}$. Thus, we get that $A^* = (1)(23)$. Now, suppose that $tc^1 = (23)$. Note that (1) is also a top trading cycle for $N \setminus \{2, 3\} = \{1\}$. Hence, again, we get that $A^* = (1)(23)$. △

From Example 3.6, we can conclude that when there exist multiple top trading cycles for a coalition S, the order in which they are selected in the top trading algorithm does not matter in the sense that the top trading permutation matrix remains the same.

Example 3.7. Let $n = 3$, $N = \{1, 2, 3\}$ and let the preference matrix be

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

The preference matrix P represents for each agent the preferences over the items. For example, the first choice of agent 1 is item 2, his second choice is item 1 and his third choice is item 3.

The unique top trading cycle for N is $tc^1 = (23)$ and for $N \setminus S^1 = \{1\}$, the top trading cycle is $tc^2 = (1)$. Note that there are no agents left. Thus, agent 1 gets his own item and agents 2 and 3 trade with each other. This is represented in the permutation matrix derived from the top trading cycle algorithm:

$$A^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that agent 2 and agent 3 get their top choice. Hence, only the coalition $\{1\}$ could strictly improve upon A^*. Since agent 1 already gets his own item in A^*, we have that $f(A^*) = \{A^*\}$. With Definition 3.2(1), we get that $A^* \in CO$.

We show that each permutation matrix not equal to A^* is strictly dominated. For this, we use the notation of permutation matrices as cycles given in Table 2. With Definition 2.8, Definition 3.1(1) and Table 4, we get that each permutation matrix not equal to A^* is strictly dominated.

Sequence	$E(A, A')$	Preference
$(1)(2)(3) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2, 3\} \in E((1)(2)(3), (1)(23))$	$(1)(23) \succ_i (1)(2)(3)\ \forall\, i \in S$
$(2)(13) \xrightarrow[S=\{1,3\}]{} (1)(2)(3)$	$\{1, 3\} \in E((2)(13), (1)(2)(3))$	$(1)(2)(3) \succ_i (2)(13)\ \forall\, i \in S$
$(3)(12) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2, 3\} \in E((3)(12), (1)(23))$	$(1)(23) \succ_i (3)(12)\ \forall\, i \in S$
$(123) \xrightarrow[S=\{1\}]{} (1)(2)(3)$	$\{1\} \in E((123), (1)(2)(3))$	$(1)(2)(3) \succ_1 (123)$
$(132) \xrightarrow[S=\{3\}]{} (1)(2)(3)$	$\{3\} \in E((132), (1)(2)(3))$	$(1)(2)(3) \succ_3 (132)$

Table 4: Illustration that each permutation matrix not equal to A^* is strictly dominated.

Hence, with Definition 3.2(1), we get that $CO = \{A^*\}$. △

In Example 3.7, we showed that the core of a specific housing matching model contains the permutation matrix A^*. This result can be generalized to all housing matching models (N, P). The proof of the following theorem is the proof in Shapley and Scarf (1974), but written down in the context of our social environment corresponding to the housing matching model (N, P).

Theorem 3.8. *For all housing matching models (N, P), the core of $\mathcal{E}(N, P)$ is nonempty.*

Proof. Let (N, P) be a housing matching model. We show that the permutation matrix A^*, which is derived from the top trading cycle algorithm, is an element of the core. From the top trading cycle algorithm, we get that N is partitioned into T disjoint coalitions $S(tc^\tau)$, i.e. $N = \bigcup_{1 \le \tau \le T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \le r \le \tau - 1} S(tc^r) \right)$ and that the cycle decomposition of A^* is the product of all top trading cycles.

Take an arbitrary coalition $S \in 2^N \setminus \{\emptyset\}$, and let $s \in \{1, \ldots, T\}$ be the first index such that $S \cap S(tc^s) \neq \emptyset$, so we have that $S \subseteq N \setminus \left(\bigcup_{1 \le r \le s-1} S(tc^r) \right)$. Recall that $tc^s = (i_1^s \cdots i_{\ell^s}^s)$ with $i_0^s = i_{\ell^s}^s$.

Take $i = i_k^s \in S \cap S(tc^s)$, then we have that $A^*_{i i_{k-1}^s} = 1$. According to Definition 3.3, agent i's most preferred item in $N \setminus \left(\bigcup_{1 \le r \le s-1} S(tc^r) \right)$ is item i_{k-1}^s. Thus, with Definition 2.1, condition (1) of Definition 2.8 and Definition 3.1(1), the coalition S cannot strictly improve upon A^*. In other words, we have that $f(A^*) = \{A^*\}$. Thus, we can conclude that $A^* \in CO$. $\square$

Recall that in Theorem 3.8 we showed that $A^* \in CO$ and that in Example 3.7 the core was equal to $\{A^*\}$. With this example, we could draw the wrong conclusion that the core of each housing matching model is equal to $\{A^*\}$.

Example 3.9. Let $n = 3$, $N = \{1, 2, 3\}$ and let the preference matrix P be given as

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Then we have that $A^* = (1)(23)$, because the top trading cycle for N is $tc^1 = (23)$ and the top trading cycle for $N \setminus S(tc^1) = \{1\}$ is $tc^2 = (1)$. We already know that $A^* \in CO$, and with Table 5, we also know that $(1)(2)(3)$, $(2)(13)$ and $(3)(12)$ are strictly dominated.

Sequence	$E(A, A')$	Preference
$(1)(2)(3) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((1)(2)(3), (1)(23))$	$(1)(23) \succ_i (1)(2)(3)\ \forall i \in S$
$(2)(13) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((2)(13), (1)(23))$	$(1)(23) \succ_i (2)(13)\ \forall i \in S$
$(3)(12) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((3)(12), (1)(23))$	$(1)(23) \succ_i (3)(12)\ \forall i \in S$

Table 5: Illustration that three permutation matrices are strictly dominated.

The permutation matrix $A' = (132)$ represents that agent 1 gets item 2, that agent 2 gets item 3 and that agent 3 gets item 1. Note that agents 1 and 2 get their top choice and that agent 3 gets an item that he prefers to his own item. Hence, there does not exist a coalition S such that it can strictly improve upon A', i.e. $A' \in CO$.

In the permutation matrix $A'' = (123)$ agent 3 gets his top choice, and agents 1 and 2 both get an item that they prefer to their own item. Hence, only the coalition $\{1, 2\}$ could strictly improve upon A'', but according to A'', agent 2 already gets item 1. Hence, we have that $A'' \in CO$. Thus, we get $CO = \{A^*, A', A''\}$. △

From Example 3.9, we can draw the conclusion that in general, the core is not equal to $\{A^*\}$, because the core can contain permutation matrices that are not equal to A^*. We already know that the strong core is a subset of the core. In the following subsection, we show that the strong core is not equal to the core for each housing matching model.

3.2 Strong core

In Example 3.9, we showed for a specific housing matching model that the core contains more than A^*. In the following example, we determine the strong core for this specific housing matching model.

Example 3.10 (Example 3.9 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 3.9:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$, and that $CO = \{(1)(23), (132), (123)\}$. We determine the strong core. Note that $A' = (132)$ and $A'' = (123)$ are both weakly dominated by A^*, for coalition $S = \{2, 3\}$. Recall that strict dominance implies weak dominance. Thus, we have that $SCO = \{A^*\}$. Hence, we have found two permutation matrices, A' and A'', that are in the core, but are not in the strong core. △

From Example 3.10, we can conclude that the strong core is not equal to the core for each housing matching model. Now, we determine whether there is a relation between the strong core and A^*. In Roth and Postlewaite (1977), it is shown that when the preferences over the items are strict and if the effectivity correspondence is defined by condition (1) of Definition 2.8, the strong core consists of one unique element. With the help of the paper of Shapley and Scarf (1974), this unique element is A^*, which is the permutation matrix derived from the top trading cycle algorithm.

Since our definition of an effectivity correspondence satisfies condition (1) of Definition 2.8 and has some additional restrictions, it can happen that the proofs in Roth and Postlewaite (1977) are not valid anymore. Hence, we still need to prove that the strong core is equal to $\{A^*\}$ for our effectivity correspondence.

In particular, it is shown in Roth and Postlewaite (1977), that if preferences are strict and the effectivity correspondence is only defined by condition (1) in Definition 2.8, the top trading cycle allocation weakly dominates all other allocations in each housing matching model. In the following example, we show that this is not the case with our effectivity correspondence.

Example 3.11. Let $n = 3$, $N = \{1, 2, 3\}$ and let the preference matrix be given as

$$P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Then we have that $A^* = (1)(23)$. For $A = (2)(13)$, we have that $E(A, A^*) = \{\{2, 3\}, N\}$ and that $A \succ_3 A^*$. Hence, there does not exist a coalition $S \in E(A, A^*)$ such that $A^* \succsim_i A$ for all $i \in S$ and $A^* \succ_j A$ for at least one $j \in S$. With Definition 3.1(2), we get that A^* does not weakly dominate A. △

From Example 3.11, we get for each housing matching model (N, P) that with respect to our effectivity correspondence, the allocation A^* does not necessarily weakly dominate all other permutation matrices in $\mathcal{E}(N, P)$. Thus, we need to prove for each housing matching model (N, P) that each permutation matrix not equal to A^* is weakly dominated by some $A' \in X$, which is not necessarily A^*, in $\mathcal{E}(N, P)$.

Theorem 3.12. *For all housing matching models (N, P), we have that each permutation matrix $A \in X \setminus \{A^*\}$ is weakly dominated in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $A \in X \setminus \{A^*\}$. From the top trading cycle algorithm, we get that N is partitioned into T disjoint coalitions $S(tc^\tau)$, i.e. $N = \bigcup_{1 \le \tau \le T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \le r \le \tau - 1} S(tc^r) \right)$ and that the cycle decomposition of A^* is the product of all top trading cycles. Recall that for each $\tau \in \{1, \ldots, T\}$, we have that $S(tc^\tau) = \{i_1^\tau, \ldots, i_{\ell^\tau}^\tau\}$, $tc^\tau = (i_1^\tau \cdots i_{\ell^\tau}^\tau)$ and $i_0^\tau = i_{\ell^\tau}^\tau$.

Let $s \in \{1, \ldots, T\}$ be the first index such that $tc^s \notin C(A)$. In other words, tc^s is the first top trading cycle that is not in the cycle decomposition of A. Define

$$\phi : \bigcup_{1 \le r \le s} S(tc^r) \to \bigcup_{1 \le r \le s} S(tc^r)$$

as

$$\phi(i_k^r) = i_{k-1}^r \quad \forall\, k \in \{1, \ldots, \ell^r\} \text{ and } \forall\, r \in \{1, \ldots, s\}.$$

Note that by construction, ϕ is a bijection and that $\phi(i) = j$ means that agent i gets item j. Then from Lemma 2.11, we know that there exists $A' \in X$ such that $\bigcup_{1 \le r \le s} S(tc^r) \in E(A, A')$, and such that $A'_{i\phi(i)} = 1$ for all $i \in \bigcup_{1 \le r \le s} S(tc^r)$. Thus, we have that $tc^r \in C(A')$ for all $r \in \{1, \ldots, s\}$. This means that, $tc^1, \ldots, tc^s$ are all in the cycle decomposition of A'. Note that $tc^r \in C(A)$ for all $r \in \{1, \ldots, s-1\}$. In other words, $tc^1, \ldots, tc^{s-1}$ are all in the cycle decomposition of A. Thus, we have that $A' \sim_i A$ for all $i \in \bigcup_{1 \le r \le s-1} S(tc^r)$.

Since $tc^s \in C(A')$, $tc^s \notin C(A)$ and according to A', each agent in $S(tc^s)$ gets his most preferred item of the remaining in $N \setminus \left(\bigcup_{1 \le r \le s-1} S(tc^r) \right)$, we get that $A' \succsim_i A$ for all $i \in S(tc^s)$ and $A' \succ_j A$ for at least one $j \in S(tc^s)$. With Definition 3.1(2), we get that A' weakly dominates A in $\mathcal{E}(N, P)$. □

For each market model with indivisible goods with strict preferences over the goods, Roth and Postlewaite (1977) showed that the top trading cycle allocation is not weakly dominated when the effectivity correspondence is defined as: $S \in E(A, A')$ if and only if $\forall\, i \in S$ there exists $j \in S$ such that $A'_{ij} = 1$.

Kawasaki (2010) showed that if the same effectivity correspondence, as in Definition 2.8, is used, and when the agents can anticipate that coalitions might react to their deviations, the top trading cycle allocation is not indirectly antisymmetrically weakly dominated. In Section 6, we look at this dominance relation.

Because we assume that agents cannot anticipate that coalitions might react to their deviations, we prove that A^* is not weakly dominated by a combination of the proofs in Roth and Postlewaite (1977) and Kawasaki (2010).

Theorem 3.13. *For all housing matching models (N, P), it holds that the permutation matrix A^* is not weakly dominated in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $A' \in X \setminus \{A^*\}$. Suppose that A' weakly dominates A^* in $\mathcal{E}(N, P)$. Then according to Definition 3.1(2), there exists $S \in E(A^*, A')$ such that $A' \succsim_i A^*$ for all $i \in S$ and $A' \succ_j A^*$ for at least one $j \in S$.

Note that A^* is the top trading cycle permutation matrix. Hence, we have that N is partitioned into T disjoint coalitions $S(tc^\tau)$, i.e. $N = \bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau - 1} S(tc^r) \right)$ and the cycle decomposition of A^* is the product of the top trading cycles, i.e. $A^* = \prod_{\tau=1}^{T} tc^\tau$. We show that $A' = A^*$ by induction on $\tau \in \{1, \ldots, T\}$. Let $\tau \in \{1, \ldots, T\}$, then define the statement

$$P(\tau): \ \forall\, i \in \bigcup_{1 \leq r \leq \tau} S(tc^r) \text{ it holds that } A^*_i = A'_i.$$

First, we show that the statement is true for $\tau = 1$. In other words, we need to show that $A^*_i = A'_i$ for all $i \in S(tc^1)$. We have two cases: $S(tc^1) \cap S = \emptyset$ and $S(tc^1) \cap S \neq \emptyset$. If $S(tc^1) \cap S = \emptyset$, then by condition (2) in Definition 2.8, we get that $A^*_i = A'_i$ for all $i \in S(tc^1)$.
Now suppose that $S(tc^1) \cap S \neq \emptyset$. Let $i \in S(tc^1) \cap S$. We know that $i \in S(tc^1)$ implies that according to A^* agent i gets his most preferred item. Hence, in particular, it holds that $A^* \succsim_i A'$. Since $i \in S$, we also have that $A' \succsim_i A^*$. Hence, we get that $A^* \sim_i A'$. With Definition 2.1, we get that $A^*_{ij} = A'_{ij} = 1$. Note that we must have that $j \in S(tc^1) \cap S$, because of the top trading cycle algorithm and condition (1) in Definition 2.8. Thus, we get that $A^*_{jk} = A'_{jk} = 1$ and again, we must have that $k \in S(tc^1) \cap S$. If we repeat this, we get that $S(tc^1) \subseteq S$. Thus, for all $i \in S(tc^1)$ we have that $A^*_i = A'_i$. Hence, the statement is true for $\tau = 1$.

Assume that the statement $P(\tau)$ holds for some $\tau \in \{1, \ldots, T\}$. In other words,

$$\forall\, i \in \bigcup_{1 \leq r \leq \tau} S(tc^r) \text{ it holds that } A^*_i = A'_i.$$

We need to show that $P(\tau + 1)$ holds. In other words, we need to show that

$$\forall\, i \in \bigcup_{1 \leq r \leq \tau+1} S(tc^r) \text{ it holds that } A_i^* = A_i'.$$

Using the induction hypothesis, we need to show that $A_i^* = A_i'$ for all $i \in S(tc^{\tau+1})$. We have two cases: $S(tc^{\tau+1}) \cap S = \emptyset$ and $S(tc^{\tau+1}) \cap S \neq \emptyset$. In the former case, we get by condition (2) in Definition 2.8, that $A_i^* = A_i'$ for all $i \in S(tc^{\tau+1})$. Now, suppose that $S(tc^{\tau+1}) \cap S \neq \emptyset$. Using the same reasoning as in the base case, we have that $S(tc^{\tau+1}) \subseteq S$. Hence, we get that $A' \succsim_i A^*$ for all $i \in S(tc^{\tau+1})$. Since the statement $P(\tau)$ is true, we have that according to A' each agent in $S(tc^{\tau+1})$ can only receive items from agents in $N \setminus \left(\bigcup_{1 \leq r \leq \tau} S(tc^r) \right)$. Hence, with the fact that according to A^* each agent gets his most preferred item of $N \setminus \left(\bigcup_{1 \leq r \leq \tau} S(tc^r) \right)$, we get that $A^* \succsim_i A'$ for all $i \in S(tc^{\tau+1})$. Hence, we have that $A_i^* = A_i'$ for all $i \in S(tc^{\tau+1})$. This shows that the statement $P(\tau + 1)$ is true. Hence, with induction we showed that $A_i^* = A_i'$ for all $i \in \bigcup_{1 \leq \tau \leq T} S(tc^\tau) = N$. Thus, we have that $A^* = A'$. This contradicts the fact that $A' \in X \setminus \{A^*\}$. Hence, we can conclude that A^* is not weakly dominated in $\mathcal{E}(N, P)$. □

With Theorem 3.12 and Theorem 3.13, we get the following corollary.

Corollary 3.14. *For all housing matching models* (N, P)*, it holds that* $SCO = \{A^*\}$*.*

3.3 von Neumann-Morgenstern stable set

In von Neumann and Morgenstern (1944), the von Neumann-Morgenstern stable set was introduced as a solution concept for a state space with a dominance relation over this space. A von Neumann-Morgenstern stable set may fail to exist and if it exists, it cannot be empty, but does not have to be unique.

Definition 3.15 (von Neumann-Morgenstern stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **von Neumann-Morgenstern (vNM) stable set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) **internal stability**: $\forall\, A \in \mathcal{A}$ we have that $f(A) \cap \mathcal{A} = \{A\}$,

(2) **external stability**: $\forall\, A \notin \mathcal{A}$ it holds that $f(A) \cap \mathcal{A} \neq \emptyset$.

The internal stability condition says that each permutation matrix inside $\mathcal{A}$ is not strictly dominated by another permutation matrix inside $\mathcal{A}$. External stability means that each permutation matrix outside $\mathcal{A}$ is strictly dominated by a permutation matrix inside $\mathcal{A}$. If we replace f with $\tilde{f}$, we get the **weak dominance vNM stable set**.

Proposition 3.16. *Let* (N, P) *be a housing matching model for which a von Neumann-Morgenstern stable set exists. Then each vNM stable set must contain the core.*

Proof. Let (N, P) be a housing matching model for which a vNM stable set exists and let $\mathcal{A} \subseteq X$ be a vNM stable set with $CO \not\subseteq \mathcal{A}$, then there exists $A \in CO \setminus \mathcal{A}$. Hence, with Definition 3.2(1), we get that $f(A) = \{A\}$. Thus, we have that $f(A) \cap \mathcal{A} = \emptyset$. This contradicts the fact that $\mathcal{A}$ satisfies external stability. Hence, $\mathcal{A}$ is not a vNM stable set. Thus, we can conclude that each vNM stable set contains the core. $\square$

Note that if the core is a vNM stable set, then it is the unique vNM stable set. In Theorem 3.8, we showed that the core contains A^*, therefore we have the following result.

Corollary 3.17. *Let (N, P) be a housing matching model for which a von Neumann-Morgenstern stable set exists. Then each vNM stable set contains A^* and consists of permutation matrices that are not strictly dominated by A^*.*

We have a similar result for the weak dominance vNM stable set.

Proposition 3.18. *Let (N, P) be a housing matching model for which a weak dominance vNM stable set exists. Then each weak dominance vNM stable set must contain A^* and consists of permutation matrices that are not weakly dominated by A^*.*

Proof. Let (N, P) be a housing matching model for which a weak dominance vNM stable set exists and let $\mathcal{A} \subseteq X$ be a weak dominance vNM stable set with $A^* \notin \mathcal{A}$. From Corollary 3.14, we know that $SCO = \{A^*\}$, thus with Definition 3.2(2), we get that $\tilde{f}(A^*) = \{A^*\}$. Hence, we have that $\tilde{f}(A^*) \cap \mathcal{A} = \emptyset$. This contradicts the fact that $\mathcal{A}$ satisfies external stability. Thus, we can conclude that each weak dominance vNM stable set contains A^*. Therefore, with internal stability, we get that each weak dominance vNM stable set consists of permutation matrices that are not weakly dominated by A^*. $\square$

From Proposition 3.16, one could wonder whether the core is a vNM stable set for each housing matching model (N, P). We show in the following example that in general the core is not a vNM stable set.

Example 3.19 (Example 3.7 continued)**.**
Let $n = 3$ and let the preference matrix be given as in Example 3.7:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$, and that $CO = \{A^*\}$. Note that $(123) \in X \setminus \{A^*\}$ and that (123) is not strictly dominated by A^*, since $(123) \sim_3 (1)(23)$ and $E((123), (1)(23)) = \{\{2, 3\}, N\}$. Thus, we have that $A^* \notin f((123))$, hence $\{A^*\}$ does not satisfy external stability. We can conclude that the core is not a vNM stable set.

Now, we show that $\mathcal{A} = \{(1)(23), (123), (132)\}$ is a vNM stable set. We already know that $f((1)(23)) = \{(1)(23)\}$. From Table 4, we can conclude that $f((123)) = \{(123), (1)(2)(3)\}$ and that $f((132)) = \{(132), (1)(2)(3)\}$. Hence, for all $A \in \mathcal{A}$ we have that $f(A) \cap \mathcal{A} = \{A\}$. Thus, $\mathcal{A}$ satisfies internal stability. We show that $\mathcal{A}$ also satisfies external stability. From Table 4, we already know that $A^* \in f((1)(2)(3))$ and that $A^* \in f((3)(12))$. Note that $(1)(23) \succ_i (2)(13)$ for all $i \in \{2, 3\}$ and that $\{2, 3\} \in E((2)(13), (1)(23))$. Thus, we have that $(1)(23) \in f((2)(13))$. Hence, for all $A \notin \mathcal{A}$, we have that $f(A) \cap \mathcal{A} \neq \emptyset$. Thus, the set $\mathcal{A} = \{(1)(23), (123), (132)\}$ is a vNM stable set.

Now, we show that $SCO = \{A^*\}$ is the unique weak dominance vNM stable set. Recall that $\tilde{f}(A^*) = \{A^*\}$, thus $\{A^*\}$ satisfies internal stability. From Table 6, we know that each $A \in X \setminus \{A^*\}$ is weakly dominated by A^*.

Sequence	$E(A, A^*)$	Preference
$(1)(2)(3) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((1)(2)(3), (1)(23))$	$(1)(23) \succ_i (1)(2)(3)\ \forall\, i \in S$
$(2)(13) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((2)(13), (1)(23))$	$(1)(23) \succ_i (2)(13)\ \forall\, i \in S$
$(3)(12) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((3)(12), (1)(23))$	$(1)(23) \succ_i (3)(12)\ \forall\, i \in S$
$(123) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((123), (1)(23))$	$(1)(23) \succ_2 (123)$ and $(1)(23) \sim_3 (123)$
$(132) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((132), (1)(23))$	$(1)(23) \sim_2 (132)$ and $(1)(23) \succ_3 (132)$

Table 6: Illustration that each $A \in X \setminus \{A^*\}$ is weakly dominated by A^*.

Thus, the set $\{A^*\}$ satisfies external stability. This shows that $\{A^*\}$ is a weak dominance vNM stable set. From Proposition 3.18, it follows that $\{A^*\}$ is the unique weak dominance vNM stable set. △

From Proposition 3.18 and the fact that $\tilde{f}(A^*) = \{A^*\}$, we can conclude that $\{A^*\}$ is the unique weak dominance vNM stable set for each housing matching model (N, P), such that A^* weakly dominates all other permutation matrices. In the following example, we show that $\{A^*\}$ is not a weak dominance vNM stable set for each housing matching model.

Example 3.20 (Example 3.11 continued)**.**
Let $n = 3$, $N = \{1, 2, 3\}$ and let the preference matrix be given as in Example 3.11:

$$P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$. First, we determine which permutation matrices are weakly dominated by A^*. Note that $(1)(23) \succ_i (1)(2)(3)$ for all $i \in \{2, 3\}$, $(123) \succ_2 (1)(23)$, $(132) \succ_3 (1)(23)$, $(2)(13) \succ_3 (1)(23)$ and $(3)(12) \succ_2 (1)(23)$. Thus, A^* only weakly dominates $(1)(2)(3)$. Hence, we get that $\{A^*\}$ does not satisfy weak dominance external stability.

Now, we determine the weak dominance vNM stable sets for this housing matching model. Let $\mathcal{A} \subseteq X$ be a weak dominance vNM stable set. From Proposition 3.18, we can conclude that $A^* \in \mathcal{A}$ and that $(1)(2)(3) \notin \mathcal{A}$. In Table 7, all weak dominations for this housing matching model can be found.

Sequence	$E(A,A')$	Preference
$(1)(2)(3) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((1)(2)(3),(1)(23))$	$(1)(23) \succ_i (1)(2)(3)\ \forall\, i \in S$
$(2)(13) \xrightarrow[S=\{1\}]{} (1)(2)(3)$	$\{1\} \in E((2)(13),(1)(2)(3))$	$(1)(2)(3) \succ_1 (2)(13)$
$(2)(13) \xrightarrow[S=N]{} (132)$	$N \in E((2)(13),(132))$	$(132) \succ_i (2)(13)\ \forall\, i \in \{1,2\}$ and $(132) \sim_3 (2)(13)$
$(2)(13) \xrightarrow[S=\{1,2\}]{} (3)(12)$	$\{1,2\} \in E((2)(13),(3)(12))$	$(3)(12) \succ_i (2)(13)\ \forall\, i \in S$
$(3)(12) \xrightarrow[S=\{1\}]{} (1)(2)(3)$	$\{1\} \in E((3)(12),(1)(2)(3))$	$(1)(2)(3) \succ_1 (3)(12)$
$(123) \xrightarrow[S=\{1\}]{} (1)(2)(3)$	$\{1\} \in E((123),(1)(2)(3))$	$(1)(2)(3) \succ_1 (123)$
$(123) \xrightarrow[S=\{1,3\}]{} (2)(13)$	$\{1,3\} \in E((123),(2)(13))$	$(2)(13) \sim_1 (123)$ and $(2)(13) \succ_3 (123)$
$(123) \xrightarrow[S=\{1,2\}]{} (3)(12)$	$\{1,2\} \in E((123),(3)(12))$	$(3)(12) \succ_1 (123)$ and $(3)(12) \sim_2 (123)$
$(132) \xrightarrow[S=\{1\}]{} (1)(2)(3)$	$\{1\} \in E((132),(1)(2)(3))$	$(1)(2)(3) \succ_1 (132)$
$(132) \xrightarrow[S=\{1,2\}]{} (3)(12)$	$\{1,2\} \in E((132),(3)(12))$	$(3)(12) \sim_1 (132)$ and $(3)(12) \succ_2 (132)$

Table 7: Illustration of all weak dominations.

Hence, we can conclude that

$$\begin{aligned}
\tilde{f}((3)(12)) &= \{(1)(2)(3),(3)(12)\},\\
\tilde{f}((2)(13)) &= \{(2)(13),(1)(2)(3),(132),(3)(12)\},\\
\tilde{f}((132)) &= \{(132),(1)(2)(3),(3)(12)\},\\
\tilde{f}((123)) &= \{(123),(1)(2)(3),(2)(13),(3)(12)\}.
\end{aligned}$$

Suppose that $(3)(12) \notin \mathcal{A}$, then we get with $(1)(2)(3) \notin \mathcal{A}$ that $\mathcal{A}$ does not satisfy weak dominance external stability. Thus, we can conclude that $(3)(12) \in \mathcal{A}$. In order for $\mathcal{A}$ to satisfy weak dominance internal stability we have that $\{(2)(13),(123),(132)\} \cap \mathcal{A} = \emptyset$. Hence, we get that $\mathcal{A} = \{(1)(23),(3)(12)\}$ is the unique weak dominance vNM stable set. △

In the following example, we show that there exists a housing matching model (N,P) for which a strict von Neumann-Morgenstern stable set does not exist.

Example 3.21.
Let $n = 3, N = \{1,2,3\}$ and let the preference matrix be given as

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

We show that there does not exist a vNM stable set. Suppose to the contrary that $\mathcal{A} \subseteq X$ is a vNM stable set. We have that $A^* = (132)$ and from Table 8, we get that $CO = \{A^*\}$. We can conclude from Proposition 3.16 that $A^* \in \mathcal{A}$.

Sequence	$E(A, A')$	Preference
$(1)(2)(3) \xrightarrow[S=N]{} (132)$	$N \in E((1)(2)(3), (132))$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in S$
$(1)(23) \xrightarrow[S=\{1,3\}]{} (2)(13)$	$\{1,3\} \in E((1)(23), (2)(13))$	$(2)(13) \succ_i (1)(23)\ \forall\, i \in S$
$(2)(13) \xrightarrow[S=\{1,2\}]{} (3)(12)$	$\{1,2\} \in E((2)(13), (3)(12))$	$(3)(12) \succ_i (2)(13)\ \forall\, i \in S$
$(3)(12) \xrightarrow[S=\{2,3\}]{} (1)(23)$	$\{2,3\} \in E((3)(12), (1)(23))$	$(1)(23) \succ_i (3)(12)\ \forall\, i \in S$
$(123) \xrightarrow[S=N]{} (132)$	$N \in E((123), (132))$	$(132) \succ_i (123)\ \forall\, i \in S$

Table 8: Illustration that each permutation matrix not equal to A^* is strictly dominated.

Now, we determine $f((1)(23))$. According to $(1)(23)$ agent 2 gets his most preferred item, hence he is not part of any deviating coalition. Also, since according to $(1)(23)$ agents 1 and 3 both get an item they prefer more than their own item, the only deviating coalition is $\{1,3\}$. Hence, it holds that $f((1)(23)) = \{(1)(23), (2)(13)\}$. Similarly, one can show that $f((2)(13)) = \{(2)(13), (3)(12)\}$ and $f((3)(12)) = \{(3)(12), (1)(23)\}$. Note that $f((1)(2)(3)) = X$ and that $f((123)) = \{(123), (132)\}$. Thus, with $A^* \in \mathcal{A}$ and the internal stability condition we get that $(1)(2)(3) \notin \mathcal{A}$ and $(123) \notin \mathcal{A}$. This can be seen in Figure 1. Hence, we have that $\{(1)(2)(3), (123)\} \cap \mathcal{A} = \emptyset$.

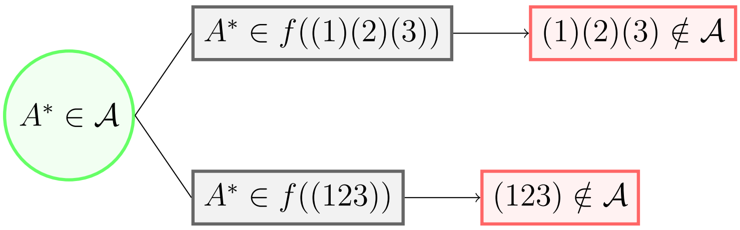

Figure 1: Illustration to show that $\{(1)(2)(3), (123)\} \cap \mathcal{A} = \emptyset$.

First, note that we can conclude that $\{A^*\}$ is not a vNM stable set, because we have that $A^* \notin f((1)(23))$.

Suppose that $(1)(23) \in \mathcal{A}$, then the internal stability condition gives us that $(3)(12) \notin \mathcal{A}$ and that $(2)(13) \notin \mathcal{A}$. This can be seen in Figure 2.

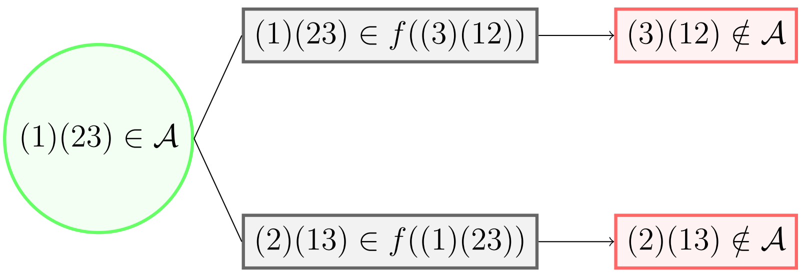

Figure 2: Illustration to show that if $(1)(23) \in \mathcal{A}$, then $\{(2)(13), (3)(12)\} \cap \mathcal{A} = \emptyset$.

Hence, with $\{(1)(2)(3), (123)\} \cap \mathcal{A} = \emptyset$, we get that $\mathcal{A} = \{(132), (1)(23)\}$. Note that

$$f((2)(13)) \cap \mathcal{A} = \{(2)(13), (3)(12)\} \cap \{(132), (1)(23)\} = \emptyset.$$

Hence, $\mathcal{A} = \{(132), (1)(23)\}$ does not satisfy the external stability condition. Thus, we can conclude that $(1)(23) \notin \mathcal{A}$.

Now suppose that $(2)(13) \in \mathcal{A}$, then from the internal stability condition we get that $(3)(12) \notin \mathcal{A}$. This is shown in Figure 3.

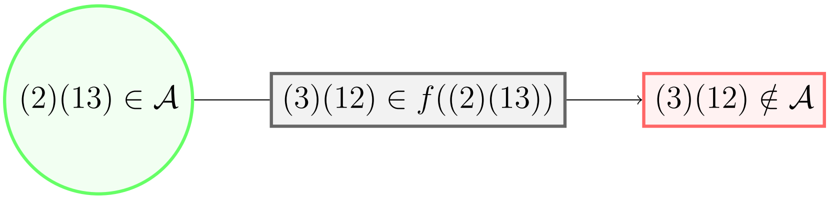

Figure 3: Illustration to show that if $(2)(13) \in \mathcal{A}$, then $(3)(12) \notin \mathcal{A}$.

Hence, with $\{(1)(2)(3), (123), (1)(23)\} \cap \mathcal{A} = \emptyset$, we have that $\mathcal{A} = \{(132), (2)(13)\}$. Note that

$$f((3)(12)) \cap \mathcal{A} = \{(3)(12), (1)(23)\} \cap \{(132), (2)(13)\} = \emptyset.$$

Thus, we have that $\mathcal{A} = \{(132), (2)(13)\}$ does not satisfy external stability. We can conclude that $(2)(13) \notin \mathcal{A}$.

Suppose that $(3)(12) \in \mathcal{A}$, then with $\{(1)(2)(3), (123), (1)(23), (2)(13)\} \cap \mathcal{A} = \emptyset$, we have that $\mathcal{A} = \{(132), (3)(12)\}$. Note that

$$f((1)(23)) \cap \mathcal{A} = \{(1)(23), (2)(13)\} \cap \{(132), (3)(12)\} = \emptyset.$$

Hence, $\mathcal{A} = \{(132), (3)(12)\}$ does not satisfy external stability. Thus, it holds that $(3)(12) \notin \mathcal{A}$.

Hence, we have that $\mathcal{A} \cap \{(1)(2)(3), (123), (1)(23), (2)(13), (3)(12)\} = \emptyset$. With the fact that $\mathcal{A} \subseteq X$, we get that $\mathcal{A} = \{A^*\}$. Recall that $\{A^*\}$ is not a vNM stable set, hence there does not exist a vNM stable set. $\triangle$

From Example 3.21, we can conclude that there are housing matching models for which a von Neumann-Morgenstern stable set does not exist. Therefore, we look at another stability concept which exists for all housing matching models.

3.4 Myopic stable set

In Demuynck et al. (2019a), the definition of a myopic stable set is given. Note that the set of permutation matrices X is finite and that with Lemma 2.11, our effectivity correspondence satisfies the conditions given in Demuynck et al. (2019b). Therefore, we give the definition of a myopic stable set as given in Demuynck et al. (2019a) for a finite state space.

Definition 3.22 (Myopic Stable Set (MSS))**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **myopic stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three properties:

(1) **deterrence of external deviations**: $\forall\, A \in \mathcal{A}$ it holds that $f(A) \subseteq \mathcal{A}$,

(2) **iterated external stability**: $\forall\, A \notin \mathcal{A}$ it holds that $f^{\mathbb{N}}(A) \cap \mathcal{A} \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ that satisfies (1) and (2).

Deterrence of external deviations says that no coalition of myopic agents can strictly benefit from changing $A \in \mathcal{A}$ into a state outside $\mathcal{A}$. Iterated external stability means that from any state outside $\mathcal{A}$ a state inside $\mathcal{A}$ is reached by a finite number of strict dominations. Note that each deviating coalition prefers the next outcome in the sequence to the current one. Iterated external stability implies that each MSS is nonempty.

Note that the set X satisfies conditions (1) and (2) in Definition 3.22, hence we need the minimality condition. If we replace f with $\tilde{f}$, we get the **weak dominance myopic stable set**. Since X is finite, we know from Demuynck et al. (2019a), that there is a unique myopic stable set.

One might wonder what kind of relation there is between the core and the myopic stable set.

Lemma 3.23. *For all housing matching models (N, P), the myopic stable set $\mathcal{A}$ of $\mathcal{E}(N, P)$ must contain the core, i.e. $CO \subseteq \mathcal{A}$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$ be a myopic stable set with $CO \nsubseteq \mathcal{A}$. Hence, there exists an allocation $A \in CO$ such that $A \notin \mathcal{A}$. From Definition 3.2(1), we know that $f(A) = \{A\}$. Thus, it holds that

$$\begin{aligned} f^2(A) &= \{A'' \in X \mid \exists\, A' \in X \text{ such that } A' \in f(A) \text{ and } A'' \in f(A')\} \\ &= \{A'' \in X \mid A'' \in f(A)\} = \{A\}. \end{aligned}$$

Hence, we get that $f^k(A) = \{A\}$ for all $k \in \mathbb{N}$. This gives us $f^{\mathbb{N}}(A) = \{A\}$. Thus, for $A \notin \mathcal{A}$, we have that $f^{\mathbb{N}}(A) \cap \mathcal{A} = \emptyset$. This contradicts condition (2) of Definition 3.22. Hence, we can conclude that each myopic stable set must contain the core. □

With Theorem 3.8, we can conclude that the myopic stable set contains A^* for each housing matching model. From Lemma 3.23, one could wonder whether the core is the myopic stable set for each housing matching model (N, P).
In Example 3.21, we showed for a specific housing matching model that a vNM stable set does not exist. In the following example, we determine the myopic stable set for this specific housing matching model and we show that it is not equal to the core. This example was introduced in Demuynck et al. (2019b) and is written down in the context of our social environment corresponding to the housing matching model (N, P).

Example 3.24 (Example 3.21 continued)**.**
Let $n = 3, N = \{1, 2, 3\}$ and let the preference matrix be given as in Example 3.21:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

Recall that $A^* = (132)$ and that we have $CO = \{A^*\}$. Let $\mathcal{A} \subseteq X$ be the myopic stable set. From Lemma 3.23, we know that $A^* \in \mathcal{A}$.

From Table 8 in Example 3.21, we can see that we have the following sequence:

$$(1)(23) \xrightarrow[S=\{1,3\}]{} (2)(13) \xrightarrow[S=\{1,2\}]{} (3)(12) \xrightarrow[S=\{2,3\}]{} (1)(23).$$

Recall that $f((1)(23)) = \{(1)(23), (2)(13)\}$, $f((2)(13)) = \{(2)(13), (3)(12)\}$ and $f((3)(12)) = \{(3)(12), (1)(23)\}$. Hence, we have that

$$f^{\mathbb{N}}((1)(23)) = f^{\mathbb{N}}((2)(13)) = f^{\mathbb{N}}((3)(12)) = \{(1)(23), (2)(13), (3)(12)\}.$$

Suppose that $\{(1)(23), (2)(13), (3)(12)\} \cap \mathcal{A} = \emptyset$, then condition (2) in Definition 3.22 is not satisfied. Hence, in order for $\mathcal{A}$ to be a myopic stable set, we must have that $\{(1)(23), (2)(13), (3)(12)\} \cap \mathcal{A} \neq \emptyset$. Suppose without loss of generality that $(1)(23) \in \mathcal{A}$, then with $f((1)(23)) = \{(1)(23), (2)(13)\}$ and condition (1) in Definition 3.22, we get that $(2)(13) \in \mathcal{A}$. Again, with $f((2)(13)) = \{(2)(13), (3)(12)\}$ and condition (1) in Definition 3.22, we get that $(3)(12) \in \mathcal{A}$. Hence, we have that $\{(1)(23), (2)(13), (3)(12)\} \subseteq \mathcal{A}$.

Hence, we get that $\mathcal{A}' = \{(1)(23), (2)(13), (3)(12), (132)\} \subseteq \mathcal{A}$. Note that $\mathcal{A}'$ satisfies deterrence of external deviations and minimality. Also since $(132) \in f((1)(2)(3))$ and $(132) \in f((123))$, we have that $\mathcal{A}'$ satisfies iterated external stability. Hence, the set $\mathcal{A} = \{(1)(23), (2)(13), (3)(12), (132)\}$ is the myopic stable set. △

From Example 3.24, we get that the myopic stable set is not equal to the core. One could wonder what happens when we have weak dominance rather than strict dominance.

In Demuynck et al. (2019b), it is shown that the weak dominance myopic stable set is equal to the strong core, if the effectivity correspondence satisfies their two conditions. From Lemma 2.11, we know that our effectivity correspondence satisfies their conditions. Hence, the weak dominance myopic stable set of $\mathcal{E}(N, P)$ is the strong core of $\mathcal{E}(N, P)$. To make the thesis self-contained, a proof is given below.

Theorem 3.25. *For all housing matching models (N, P), the weak dominance MSS of $\mathcal{E}(N, P)$ is equal to the strong core of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. Recall that the strong core of $\mathcal{E}(N, P)$ is equal to $\{A^*\}$. By Definition 3.2(2), we get that $\tilde{f}(A^*) = \{A^*\}$, hence the strong core satisfies deterrence of external deviations. Moreover, because the strong core has exactly one element, it automatically satisfies minimality. We show that the strong core satisfies iterated external stability with the help of the top trading cycle algorithm. In other words, we need to show that for all $A \notin \{A^*\}$ it holds that $\tilde{f}^{\mathbb{N}}(A) \cap \{A^*\} \neq \emptyset$.

From the top trading cycle algorithm, we get that N is partitioned into T disjoint coalitions, i.e. $N = \bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau - 1} S(tc^r) \right)$. Let A be a permutation matrix with $A \neq A^*$. Let $A^0 = A$ and let $\ell = 1$. Recall that A_i denotes the ith row of A. By using the steps listed below, we now construct a sequence of permutation matrices $(A^\ell)_{\ell=1}^T$ with $A^\ell \in \tilde{f}^\ell(A)$ and $A^T = A^*$.

(1) If $\ell = T+1$, then stop.

(2) If $\forall\, i \in S(tc^\ell)$, we have that $A_i^{\ell-1} = A_i^*$, then let $A^\ell = A^{\ell-1}$. Increase ℓ by one and go back to the first step.

(3) If $\exists\, i \in S(tc^\ell)$ such that $A_i^{\ell-1} \neq A_i^*$, then define coalition S^ℓ as $S^\ell = \bigcup_{1\leq\tau\leq\ell} S(tc^\tau)$ and the bijection $\phi : S^\ell \to S^\ell$ as $\phi(i) = j$ if $A_{ij}^* = 1$. From Lemma 2.11, we get that there exists $A^\ell \in X$ such that $S^\ell \in E(A^{\ell-1}, A^\ell)$ and such that $\forall\, i \in S^\ell$ we have that $A_i^\ell = A_i^*$. Increase ℓ by one and go back to the first step.

Note that by construction we have that $A^T = A^*$. We show that $\forall\, \ell \in \{1,\dots,T\}$ it holds that $A^\ell \in \tilde{f}(A^{\ell-1})$, because then we have that $A^\ell \in \tilde{f}^\ell(A)$. Take $\ell \in \{1,\dots,T\}$. We have two cases: $\forall\, i \in S(tc^\ell)$ it holds that $A_i^{\ell-1} = A_i^*$ and $\exists\, i \in S(tc^\ell)$ such that $A_i^{\ell-1} \neq A_i^*$.

Note that if $\forall\, i \in S(tc^\ell)$ it holds that $A_i^{\ell-1} = A_i^*$, then we have that $A^\ell = A^{\ell-1}$. Thus, it holds that $A^\ell \in \tilde{f}(A^{\ell-1})$.

If $\exists\, i \in S(tc^\ell)$ such that $A_i^{\ell-1} \neq A_i^*$, then A^ℓ is constructed such that $A_j^\ell = A_j^*$ for all $j \in S^\ell$ and such that $S^\ell \in E(A^{\ell-1}, A^\ell)$. Note that $\{tc^1,\dots,tc^{\ell-1}\} \subseteq C(A^{\ell-1})$ and that $\{tc^1,\dots,tc^\ell\} \subseteq C(A^\ell)$. From the top trading cycle algorithm, we know for agent $i \in S(tc^\ell)$, that the best item available in $N \setminus S^{\ell-1}$ is the item corresponding to A_i^*. Thus, it holds that $A^\ell \sim_i A^* \succ_i A^{\ell-1}$. For all other $j \in S^\ell$, we have that $A^\ell \sim_j A^* \succsim_j A^{\ell-1}$. Hence, by Definition 3.1(2), we get that $A^\ell \in \tilde{f}(A^{\ell-1})$.

Hence, we can conclude that $A^\ell \in \tilde{f}(A^{\ell-1})$ for all $\ell \in \{1,\dots,T\}$. In particular, we get that $A^* \in \tilde{f}^T(A)$. This proves that the strong core satisfies iterated external stability. The weak dominance MSS of $\mathcal{E}(N,P)$ is thus equal to the strong core of $\mathcal{E}(N,P)$. $\square$

4 Full Farsightedness with Strict Dominance

In Section 3, we studied stability concepts under the assumption that all agents are myopic, in the sense that they only look one step ahead. In Definition 3.1(1), we defined strict dominance with this assumption.

In this section, we assume that all agents are fully farsighted. This means that each coalition of agents can anticipate the deviations of other coalitions without any limit. In other words, each coalition of agents can see each possible chain of deviations without any restriction on the length of the chain. Like in Definition 3.1(1), we can define when a permutation matrix A is dominated by A' under the assumption that all agents are fully farsighted. In the literature, this is known as indirect dominance.
In Section 3, we studied the core and the vNM stable set under the assumption that all agents are myopic. In this section, we study these stability concepts under the assumption that agents are fully farsighted. Moreover, we study the largest consistent set, which was introduced by Chwe (1994), and the DEM farsighted stable set of Herings et al. (2010).

4.1 Indirect dominance

In Chwe (1994), the definition of indirect dominance, the farsighted notion of dominance, is given in the context of a general game with strict preferences. Note that the social environment corresponding to the housing matching model (N, P), $\mathcal{E}(N, P)$, is a general game with not necessarily strict preferences over the set X. Hence, we give this definition in the context of $\mathcal{E}(N, P)$.

Definition 4.1 (Indirect dominance)**.**
Let (N, P) be a housing matching model and let $A, A' \in X$ be two different permutation matrices. The permutation matrix A' **indirectly dominates** A in $\mathcal{E}(N, P)$, denoted by $A' \gg A$, if there is a sequence of permutation matrices $A^0, \dots, A^m \in X$ with $A^0 = A$ and $A^m = A'$ and there are coalitions $S^1, \dots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \dots, m\}$ the following two conditions hold:

(1) $S^k \in E(A^{k-1}, A^k)$,

(2) $A' \succ_i A^{k-1}$ for all $i \in S^k$.

Note that a coalition S^k can move from A^{k-1} to A^k with A^k not necessarily strictly preferred to A^{k-1} for all $i \in S^k$. Furthermore, a coalition S^k can move from A^{k-1} to A^k with $A^{k-1} \succ_i A^k$ for all $i \in S^k$. In other words, a coalition can move to a state that is less preferred than the current state, but because the agents are farsighted they anticipate that at the end they get a more preferred item. Indirect dominance has a nice property.

Lemma 4.2. *For all housing matching models (N, P), the following holds. Let $A, A', A'' \in X$ be different permutation matrices and let $S \in E(A, A')$. If $A'' \gg A'$ and $A'' \succ_i A$ for all $i \in S$, then we have that $A'' \gg A$.*

Proof. Let (N, P) be a housing matching model, let $A, A', A'' \in X$ be different permutation matrices and let $S \in E(A, A')$. Suppose that $A'' \gg A'$ and that $A'' \succ_i A$ for all $i \in S$. We need to show that $A'' \gg A$. Thus, we need to show that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A$ and $A^m = A''$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \ldots, m\}$ we have that $S^k \in E(A^{k-1}, A^k)$ and that $A'' \succ_i A^{k-1}$ for all $i \in S^k$.

Take $A^0 = A$, $A^1 = A'$ and $S^1 = S \in E(A, A')$, then we know that $A'' \succ_i A$ for all $i \in S^1$. From $A'' \gg A'$, we know that there is a sequence of permutation matrices $B^0, \ldots, B^{m'} \in X$ with $B^0 = A'$ and $B^{m'} = A''$ and there are coalitions $R^1, \ldots, R^{m'} \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k' \in \{1, \ldots, m'\}$ we have that $R^{k'} \in E(B^{k'-1}, B^{k'})$ and $A'' \succ_j B^{k'-1}$ for all $j \in R^{k'}$. For $k \in \{2, \ldots, m'+1\}$, let $S^k = R^{k-1}$ and let $A^k = B^{k-1}$, then we have that $S^k \in E(A^{k-1}, A^k)$ and that $A'' \succ_i A^{k-1}$ for all $i \in S^k$. This shows that $A'' \gg A$. $\square$

4.2 Farsighted core

In Diamantoudi and Xue (2003), the definition of the farsighted core is given in the context of partitions instead of permutations and in the context of a hedonic game, which is a tuple consisting of a finite set of players and a preference relation over the coalitions that contain that agent. We give this definition in the context of our social environment corresponding to the housing matching model (N, P).

Definition 4.3 (Farsighted core)**.**
Let (N, P) be a housing matching model. The **farsighted core** FCO of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not indirectly dominated:

$$FCO = \{A \in X \mid \nexists\, A' \in X \text{ such that } A' \gg A\}.$$

Note that with $m = 1$ strict dominance implies indirect dominance, hence we get that $FCO \subseteq CO$. We can also describe the farsighted core in terms of the dominance correspondence. Define the **indirect dominance correspondence** as the correspondence $f_{\gg} : X \to 2^X$ such that

$$f_{\gg}(A) = \{A\} \cup \{A' \in X \mid A' \gg A\}.$$

Remark 4.4. Let (N, P) be a housing matching model. The farsighted core of $\mathcal{E}(N, P)$ can also be described as

$$FCO = \{A \in X \mid f_{\gg}(A) = \{A\}\}.$$

Note that the correspondence $f^{\mathbb{N}} : X \to 2^X$ from Section 3 and the correspondence $f_{\gg} : X \to 2^X$ are not the same. To see this, note that $A' \in f^{\mathbb{N}}(A)$ if there exists a finite sequence of strict dominations from A to A' and that $A' \in f_{\gg}(A)$ if there exists a finite farsighted sequence from A to A'. With a sequence of strict dominations, we mean that each coalition S^k only deviates from A^{k-1} to A^k if A^k is strictly preferred to A^{k-1} and with a farsighted sequence, we mean that coalition S^k only deviates from A^{k-1} to A^k if the end outcome A' is strictly preferred to A^{k-1}.

Example 4.5 (Example 3.19 continued)**.**
Let $N = \{1, 2, 3\}$ and let the preference matrix P be as in Example 3.19:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$. We already know that $CO = \{A^*\}$ and that $FCO \subseteq CO$. Hence, we have that either $FCO = \emptyset$ or $FCO = \{A^*\}$. Suppose that A^* is indirectly dominated in $\mathcal{E}(N, P)$. In other words, there exists $A' \in X \setminus \{A^*\}$ such that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A^*$ and $A^m = A'$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\ k \in \{1, \ldots, m\}$ we have that $S^k \in E(A^{k-1}, A^k)$ and $A' \succ_i A^{k-1}$ for all $i \in S^k$.

Note that according to A^* agents 2 and 3 get their top choice and that agent 1 gets his own item. Thus, we have that $A^* \succsim_i A$ for all $i \in \{2, 3\}$ and for all $A \in X \setminus \{A^*\}$. Hence, we get that $A^* \succsim_i A'$ for all $i \in \{2, 3\}$. For the nonempty coalition $S^1 \in E(A^*, A^1)$, it must hold that $A' \succ_i A^*$ for all $i \in S^1$. Thus, we have that $S^1 \cap \{2, 3\} = \emptyset$. Hence, we have that $S^1 = \{1\}$. Note that agent 1 can only move from A^* to A^* itself. Hence, we get that $A^* = A^1$. Proceeding in this manner we get that $A^* = A^1 = A^2 = \cdots = A^m = A'$. This contradicts the fact that $A' \in X \setminus \{A^*\}$. Thus, we can conclude that A^* is not indirectly dominated, i.e. $FCO = \{A^*\}$. △

The result in Example 4.5 that $A^* \in FCO$ can be generalized to all housing matching models.

Theorem 4.6. *For all housing matching models (N, P), we have that the permutation matrix A^* is not indirectly dominated in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $A' \in X \setminus \{A^*\}$. We show that A' does not indirectly dominate A^* in $\mathcal{E}(N, P)$ by a proof by contradiction. Suppose that there exists a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A^*$ and $A^m = A'$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that for all $k \in \{1, \ldots, m\}$ it holds that $S^k \in E(A^{k-1}, A^k)$ and that $A' \succ_i A^{k-1}$ for all $i \in S^k$.

From the top trading cycle algorithm, we know that N is partitioned into T disjoint coalitions $S(tc^\tau)$, $N = \bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau - 1} S(tc^r) \right)$ and that $A^* = \prod_{\tau=1}^{T} tc^\tau$.

By induction on the parameter τ, we show that $S^k \cap S(tc^\tau) = \emptyset$ for all $k \in \{1, \ldots, m\}$ and for all $\tau \in \{1, \ldots, T\}$. Let $\tau \in \{1, \ldots, T\}$, then define the statement

$$P(\tau) : S^k \cap \left(\bigcup_{1 \leq r \leq \tau} S(tc^r) \right) = \emptyset \text{ for all } k \in \{1, \ldots, m\}.$$

First, we show that the statement is true for $\tau = 1$. According to A^* agents in $S(tc^1)$ get their top choice, hence $A^* \succsim_i A'$ for all $i \in S(tc^1)$. Thus, with the fact that for coalition

$S^1 \in E(A^*, A^1)$ it must hold that $A' \succ_j A^*$ for all $j \in S^1$, we get that $S^1 \cap S(tc^1) = \emptyset$. This implies with Definition 2.8 that $tc^1 \in C(A^1)$, i.e. $A^*_i = A^1_i$ for all $i \in S(tc^1)$. Hence, we get that $A^1 \succsim_i A'$ for all $i \in S(tc^1)$. Thus, with the fact that for coalition $S^2 \in E(A^1, A^2)$ it must hold that $A' \succ_j A^1$ for all $j \in S^2$, we get that $S^2 \cap S(tc^1) = \emptyset$. This implies with Definition 2.8 that $tc^1 \in C(A^2)$, i.e. $A^*_i = A^2_i$ for all $i \in S(tc^1)$. Hence, we get that $A^2 \succsim_i A'$ for all $i \in S(tc^1)$. Thus, we have that $S^3 \cap S(tc^1) = \emptyset$. If we repeat this, we get for all $k \in \{1, \ldots, m\}$ that $tc^1 \in C(A^k)$ and that $S^k \cap S(tc^1) = \emptyset$. This shows that the statement $P(\tau)$ holds for $\tau = 1$.

Assume that the statement $P(\tau)$ holds for some $\tau \in \{1, \ldots, T\}$. In other words, we have that

$$S^k \cap \left(\bigcup_{1 \leq r \leq \tau} S(tc^r) \right) = \emptyset \text{ for all } k \in \{1, \ldots, m\}.$$

We need to show that statement $P(\tau + 1)$ holds. Hence, we need to show that

$$S^k \cap \left(\bigcup_{1 \leq r \leq \tau+1} S(tc^r) \right) = \emptyset \text{ for all } k \in \{1, \ldots, m\}.$$

Hence, we only need to show that $S^k \cap S(tc^{\tau+1}) = \emptyset$ for all $k \in \{1, \ldots, m\}$. From the induction hypothesis and Definition 2.8, we get that $tc^r \in C(A^k)$ for all $k \in \{1, \ldots, m\}$ and for all $r \in \{1, \ldots, \tau\}$. Since $\{tc^1, \ldots, tc^\tau\} \subseteq C(A')$ and since according to A^* each agent in $S(tc^{\tau+1})$ gets their top choice of the remaining items belonging to agents in $N \setminus \left(\bigcup_{1 \leq r \leq \tau} S(tc^r) \right)$, we get that $A^* \succsim_i A'$ for all $i \in S(tc^{\tau+1})$. Thus, with the fact that for coalition $S^1 \in E(A^*, A^1)$ we have that $A' \succ_j A^*$ for all $j \in S^1$, we get that $S^1 \cap S(tc^{\tau+1}) = \emptyset$. Hence, with Definition 2.8 we can conclude that $tc^{\tau+1} \in C(A^1)$, i.e. $A^*_i = A^1_i$ for all $i \in S(tc^{\tau+1})$. Thus, we get that $A^1 \succsim_i A'$ for all $i \in S(tc^{\tau+1})$. Hence, with the fact that for coalition $S^2 \in E(A^1, A^2)$ it must hold that $A' \succ_j A^1$ for all $j \in S^2$, we get that $S^2 \cap S(tc^{\tau+1}) = \emptyset$. Again, with Definition 2.8, we can conclude that $tc^{\tau+1} \in C(A^2)$. If we repeat this, we can conclude that $S^k \cap S(tc^{\tau+1}) = \emptyset$ for all $k \in \{1, \ldots, m\}$. This shows that the statement $P(\tau + 1)$ is true.

Hence, with induction we showed that

$$\emptyset = S^k \cap \left(\bigcup_{1 \leq \tau \leq T} S(tc^\tau) \right) = S^k \cap N \text{ for all } k \in \{1, \ldots, m\}.$$

Thus, we get that $S^k = \emptyset$ for all $k \in \{1, \ldots, m\}$. This contradicts that $S^k \in 2^N \setminus \{\emptyset\}$. We can conclude that A' does not indirectly dominate A^* in $\mathcal{E}(N, P)$. Hence, the permutation matrix A^* is not indirectly dominated in $\mathcal{E}(N, P)$. □

Corollary 4.7. *With Definition 4.3 and Theorem 4.6, we get for all housing matching models (N, P) that $A^* \in FCO$.*

One can also wonder what happens when the core contains more than one permutation matrix.

Example 4.8 (Example 3.10 continued)**.**
Let $N = \{1, 2, 3\}$ and let the preference matrix P be as in Example 3.10:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

We know that $FCO \subseteq CO$ and that $CO = \{A^*, A', A''\}$ with $A^* = (1)(23)$, $A' = (132)$ and $A'' = (123)$. With the following sequences

$$\begin{aligned} A' &\xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=\{2,3\}]{} A^* \\ A'' &\xrightarrow[S^1=\{2\}]{} (1)(2)(3) \xrightarrow[S^2=\{2,3\}]{} A^* \end{aligned}$$

we get respectively that A^* indirectly dominates A' and that A^* indirectly dominates A''. From Corollary 4.7, we know that $A^* \in FCO$. To see this, note that according to A^* agents 2 and 3 get their top choice, hence $\{2, 3\} \cap S^1 = \emptyset$. Suppose that $S^1 = \{1\}$, then the only allocation that S^1 can move to from A^* is A^* itself. Hence, A^* is not indirectly dominated. Thus, we have that $FCO = \{A^*\}$. △

In Example 4.8, we have that A^* indirectly dominates the permutation matrices in $CO \setminus \{A^*\}$. This result can be generalized to all housing matching models and to all permutation matrices that are not equal to A^*.

In Diamantoudi and Xue (2003), it is shown for a hedonic game with strict preferences that each state in the core indirectly dominates any other state. The pair $(N, (\succsim_i)_{i \in N})$ is a hedonic game and does not necessarily have strict preferences over X. Therefore, we rewrite the theorem and the proof given in Diamantoudi and Xue (2003) in our context that allows indifference between two permutation matrices. Our result is that A^* indirectly dominates all other permutation matrices.

To prove it, we use the cycle decomposition of the permutation matrices as in Lemma 2.4. The permutation matrix A^* has a nice property that we also need in the proof.

Lemma 4.9. *For all housing matching models (N, P), the following holds. For all $A \in X \setminus \{A^*\}$, we have for every $c_A \in C(A) \setminus C(A^*)$, that there exists an agent $i \in S(c_A)$ such that $A^* \succ_i A$.*

Proof. Let (N, P) be a housing matching model, let $A \in X \setminus \{A^*\}$ and let $c_A \in C(A) \setminus C(A^*)$. Suppose that $c_A = (c_1 \cdots c_\ell)$ with $c_0 = c_\ell$. Define $\phi : S(c_A) \to S(c_A)$ as the bijection

$$\phi(c_k) = c_{k-1} \quad \forall\, k \in \{1, \ldots, \ell\}.$$

From Lemma 2.11, we know that there exists $A' \in X$, such that $S(c_A) \in E(A^*, A')$ and $A'_{j\phi(j)} = 1$ for all $j \in S(c_A)$. Hence, we get that $c_A \in C(A')$. Thus, we have that $A' \sim_j A$ for all $j \in S(c_A)$. Recall from Theorem 3.13, that the allocation A^* is not weakly dominated. Thus, in particular, there exists $i \in S(c_A)$ such that $A^* \succ_i A'$ or we have that $A^* \succsim_j A'$ for all $j \in S(c_A)$. Note that $c_A \notin C(A^*)$, thus we cannot have that $A^* \sim_j A'$ for all $j \in S(c_A)$. We can conclude that there exists $i \in S(c_A)$ such that $A^* \succ_i A'$. Thus, with the fact that $A' \sim_j A$ for all $j \in S(c_A)$, we get that there exists $i \in S(c_A)$ such that $A^* \succ_i A$. □

Before we give a proof that A^* indirectly dominates all other permutation matrices in each housing matching model, we look at Example 4.8 to show that there is a construction to prove this by using Lemma 4.9.

Example 4.10 (Example 4.8 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 4.8. We know that $A^* = (1)(23)$, $X \setminus \{A^*\} = \{(1)(2)(3), (2)(13), (3)(12), (123), (132)\}$ and from Table 3, that $C(A^*) = \{(1), (23)\}$. We show that A^* indirectly dominates all $A \in X \setminus \{A^*\}$.

First, we show that A^* indirectly dominates $A = (1)(2)(3)$. Note that $C(A) \setminus C(A^*) = \{(2), (3)\}$. Hence, according to Lemma 4.9 agents 2 and 3 strictly prefer A^* to A. We also have that $\{2, 3\} \in E(A, A^*)$ and that $C(A^*) \setminus C(A) = \{(23)\}$. Thus, we get that A^* strictly dominates A with the following sequence

$$(1)(2)(3) \xrightarrow[S^1=\{2,3\}]{} (1)(23).$$

Since strict dominance implies indirect dominance, we get that $A^* \gg (1)(2)(3)$.

Secondly, we show that A^* indirectly dominates $A = (2)(13)$. Note that $C(A) \setminus C(A^*) = \{(2), (13)\}$. Hence, according to Lemma 4.9 there must exist an agent in $S(13) = \{1, 3\}$ such that he strictly prefers A^* to A. Note that $A \succ_1 A^*$ and that $A^* \succ_3 A$. Hence, let $S^1 = \{3\}$ and let $A^1 = (1)(2)(3)$, then we have that $S^1 \in E(A, A^1)$ and that $C(A^1) \setminus C(A^*) = \{(2), (3)\}$. From the above, we already know that $A^* \succ_i A^1$ for all $i \in S^2 = \{2, 3\}$ with $S^2 \in E(A^1, A^*)$ and $C(A^*) \setminus C(A^1) = \{(23)\}$. Thus, with the following sequence

$$(2)(13) \xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=\{2,3\}]{} A^*,$$

we get that $A^* \gg (2)(13)$. The proof that A^* indirectly dominates $(3)(12)$ is similar.

Thirdly, we show that A^* indirectly dominates $A = (123)$. Note that $C(A) \setminus C(A^*) = \{(123)\}$. Hence, according to Lemma 4.9 there must exist an agent in $S(123) = N$ such that he strictly prefers A^* to A. Note that $A \succ_1 A^*$, $A^* \succ_2 A$ and that $A^* \sim_3 A$. Hence, let $S^1 = \{2\}$ and let $A^1 = (1)(2)(3)$, then we have that $S^1 \in E(A, A^1)$ and that $C(A^1) \setminus C(A^*) = \{(2), (3)\}$. From the above, we already know that $A^* \succ_i A^1$ for all $i \in S^2 = \{2, 3\}$ with $S^2 \in E(A^1, A^*)$ and $C(A^*) \setminus C(A^1) = \{(23)\}$. Hence, we get the following sequence

$$(123) \xrightarrow[S^1=\{2\}]{} (1)(2)(3) \xrightarrow[S^2=\{2,3\}]{} A^*.$$

Thus, we have that $A^* \gg (123)$. The proof that A^* indirectly dominates $A = (132)$ is similar.

We can conclude that $A^* \gg A$ for all $A \in X \setminus \{A^*\}$. △

Note that in Example 4.10, all the cycles in $C(A) \setminus C(A^*)$ of a length greater than 1 are decomposed, one at a time, until we have reached allocation A^ℓ, such that either $A^\ell = A^*$ or $C(A^\ell) \setminus C(A^*)$ only contains cycles of length 1, and then the cycles in $C(A^*) \setminus C(A^\ell)$ are formed, one at a time. This construction is used to prove that A^* indirectly dominates all other permutation matrices.

Theorem 4.11. *For all housing matching models (N, P), it holds that the permutation matrix A^* indirectly dominates any $A \in X \setminus \{A^*\}$ in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $A \in X \setminus \{A^*\}$. We need to show that $A^* \gg A$. In other words, we need to construct a sequence of permutation matrices starting with A and ending at A^* where each agent in coalition $S^k \in E(A^{k-1}, A^k)$ strictly prefers A^* to A^{k-1}. Take $A^0 = A$.

The first part of the sequence is decomposing the cycles in $C(A) \setminus C(A^*)$ of a length greater than 1 into cycles consisting of one agent. From Lemma 4.9, we know that for all $c_A \in C(A) \setminus C(A^*)$ there exists an agent $i \in S(c_A)$ such that $A^* \succ_i A$.

Let $c_A \in C(A) \setminus C(A^*)$ be such that $|S(c_A)| > 1$, let $i \in S(c_A)$ be an agent such that $A^* \succ_i A$ and let $S^1 = \{i\}$. Define $\phi : S^1 \to S^1$ as the bijection $\phi(i) = i$. From Lemma 2.11, we know that there exists $A^1 \in X$ such that $S^1 \in E(A, A^1)$ and such that $(i) \in C(A^1)$.

Again, from Lemma 4.9, we know that for all $c_{A^1} \in C(A^1) \setminus C(A^*)$, there exists $j \in S(c_{A^1})$ such that $A^* \succ_j A^1$. Let $c_{A^1} \in C(A^1) \setminus C(A^*)$ be such that $|S(c_{A^1})| > 1$, let $j \in S(c_{A^1})$ be an agent such that $A^* \succ_j A^1$ and let $S^2 = \{j\}$. Let $\phi : S^2 \to S^2$ be defined as the bijection $\phi(j) = j$. From Lemma 2.11, we know that there exists $A^2 \in X$ such that $S^2 \in E(A^1, A^2)$ and such that $\{(i), (j)\} \subseteq C(A^2)$.

We can repeat above reasoning until we have reached the permutation matrix A^ℓ, such that either $C(A^\ell) \setminus C(A^*) = \emptyset$ or for all $c_{A^\ell} \in C(A^\ell) \setminus C(A^*)$ we have that $|S(c_{A^\ell})| = 1$.

Thus, so far, we have the sequence of the first part

$$A \underset{S^1}{\longrightarrow} A^1 \underset{S^2}{\longrightarrow} A^2 \underset{S^3}{\longrightarrow} \dots \underset{S^\ell}{\longrightarrow} A^\ell$$

with for all $k \in \{1, \dots, \ell\}$ the following: $S^k \in E(A^{k-1}, A^k)$, $|S^k| = 1$ and $A^* \succ_i A^{k-1}$ for all $i \in S^k$. Note that $C(A^\ell) \setminus C(A^*) = \emptyset$ if and only if $A^\ell = A^*$. Hence, if it holds that $C(A^\ell) \setminus C(A^*) = \emptyset$, then we already showed, with the above sequence, that $A^* \gg A$.

Suppose that $A^* \neq A^\ell$, then we continue with the following step. The second part of the sequence is to construct A^* using the decomposition in the previous part. Note that $C(A^*) \setminus C(A^\ell) = \{c^1, \dots, c^h\}$ is the set of disjoint cycles that are in the cycle decomposition of A^* and not in the cycle decomposition of A^ℓ. Also note that $C(A^*) \setminus C(A^\ell) \neq \emptyset$, since $A^* \neq A^\ell$.

We show that each cycle in $C(A^*) \setminus C(A^\ell)$ has a length greater than 1. Note that for all $A \in X \setminus \{A^*\}$, we have that

$$N = \bigcup_{c_A \in C(A)} S(c_A) = \left(\bigcup_{c_A \in C(A^*) \cap C(A)} S(c_A) \right) \bigcup \left(\bigcup_{c_A \in C(A) \setminus C(A^*)} S(c_A) \right)$$

and that

$$N = \bigcup_{c_{A^*} \in C(A^*)} S(c_{A^*}) = \left(\bigcup_{c_{A^*} \in C(A^*) \cap C(A)} S(c_{A^*}) \right) \bigcup \left(\bigcup_{c_{A^*} \in C(A^*) \setminus C(A)} S(c_{A^*}) \right).$$

Hence, for all $A \in X$ we have that

$$\bigcup_{c_A \in C(A) \setminus C(A^*)} S(c_A) = \bigcup_{c_{A^*} \in C(A^*) \setminus C(A)} S(c_{A^*}).$$

Since each $c_{A^\ell} \in C(A^\ell) \setminus C(A^*)$ consists of one agent, we get with the above that each $c_{A^*} \in C(A^*) \setminus C(A^\ell)$ consists of multiple agents.

First, we construct the cycle c^1. Let $S^{\ell+1} = S(c^1)$ and suppose that $c^1 = (c^1_1 \cdots c^1_{|S^{\ell+1}|})$ with $c^1_0 = c^1_{|S^{\ell+1}|}$. Define $\phi : S^{\ell+1} \to S^{\ell+1}$ as the bijection

$$\phi(c^1_s) = c^1_{s-1} \quad \forall\, s \in \{1, \ldots, |S^{\ell+1}|\}.$$

From Lemma 2.11, we know that there exists $A^{\ell+1} \in X$, such that $S^{\ell+1} \in E(A^\ell, A^{\ell+1})$ and $A^{\ell+1}_{i\phi(i)} = 1$ for all $i \in S^{\ell+1}$. Thus, we have that $c^1 \in C(A^{\ell+1})$. We show that $A^* \succ_i A^\ell$ for all $i \in S^{\ell+1}$. According to A^*, we have for all $s \in \{1, \ldots, |S^{\ell+1}|\}$ that each agent $c^1_s \in S^{\ell+1}$ gets item $c^1_{s-1} \in S^{\ell+1}$. Moreover, according to A^ℓ, each agent in $S^{\ell+1}$ gets his own item. Note that c^1 consists of multiple agents, hence by the top trading cycle algorithm we can conclude that $A^* \succ_i A^\ell$ for all $i \in S^{\ell+1}$.

Now, we construct the cycle c^2. Let $S^{\ell+2} = S(c^2)$ and suppose that $c^2 = (c^2_1 \cdots c^2_{|S^{\ell+2}|})$ with $c^2_0 = c^2_{|S^{\ell+2}|}$. Define $\phi : S^{\ell+2} \to S^{\ell+2}$ as the bijection

$$\phi(c^2_s) = c^2_{s-1} \quad \forall\, s \in \{1, \ldots, |S^{\ell+2}|\}.$$

Again, from Lemma 2.11, we know that there exists $A^{\ell+2} \in X$, such that $S^{\ell+2} \in E(A^{\ell+1}, A^{\ell+2})$ and $\{c^1, c^2\} \subseteq C(A^{\ell+2})$. Again, from the top trading cycle algorithm, we get that $A^* \succ_i A^{\ell+1}$ for all $i \in S^{\ell+2}$.

We repeat this until all the cycles $c^1, \ldots, c^h$ are formed. Hence, the sequence of the first and the second part is equal to

$$A \underset{S^1}{\longrightarrow} A^1 \underset{S^2}{\longrightarrow} \ldots \underset{S^\ell}{\longrightarrow} A^\ell \underset{S^{\ell+1}}{\longrightarrow} A^{\ell+1} \underset{S^{\ell+2}}{\longrightarrow} A^{\ell+2} \underset{S^{\ell+3}}{\longrightarrow} \ldots \underset{S^{\ell+h-1}}{\longrightarrow} A^{\ell+h-1} \underset{S^{\ell+h}}{\longrightarrow} A^{\ell+h} = A^*$$

with for all $k \in \{1, \ldots, \ell+h\}$ it holds that $S^k \in E(A^{k-1}, A^k)$ and that $A^* \succ_i A^{k-1}$ for all $i \in S^k$. In particular, we have that $|S^k| = 1$ for all $k \in \{1, \ldots, \ell\}$ and $S^k = S(c^{k-\ell})$ for all $k \in \{\ell+1, \ldots, \ell+h\}$. Thus, we can conclude that $A^* \gg A$.

Hence, the permutation matrix A^* indirectly dominates any $A \in X \setminus \{A^*\}$ in $\mathcal{E}(N, P)$ □

From Theorem 4.6 and Theorem 4.11, we get the following corollary.

Corollary 4.12. *For all housing matching models (N, P), we have that $FCO = \{A^*\}$.*

4.3 Farsighted von Neumann-Morgenstern stable set

In Chwe (1994), the farsighted von Neumann-Morgenstern stable set was introduced in the context of a general game and in Mauleon, Vannetelbosch, and Vergote (2011) the definition is given in the context of the marriage problem of Gale and Shapley (1962). Like in the myopic case, a farsighted von Neumann-Morgenstern stable set may fail to exist and if it exists, it cannot be empty, but does not have to be unique.

In the context of the housing matching model of Shapley and Scarf (1974), Klaus et al. (2010) studied the farsighted vNM stable set with the same effectivity correspondence as in 2.8, but with a preference relation such that no agent is indifferent between his own item and an item of another agent. We give the definition in the context of our social environment corresponding to the housing matching model (N, P).

Definition 4.13 (Farsighted von Neumann-Morgenstern stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **farsighted von Neumann-Morgenstern (vNM) stable set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) **internal stability**: $\forall\, A \in \mathcal{A}$ we have that $f_{\gg}(A) \cap \mathcal{A} = \{A\}$,

(2) **external stability**: $\forall\, A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$.

Internal stability says that no permutation matrix belonging to $\mathcal{A}$ is indirectly dominated by another permutation matrix belonging to $\mathcal{A}$. External stability means that each permutation matrix not belonging to $\mathcal{A}$ is indirectly dominated by a permutation matrix inside $\mathcal{A}$.

Note that the internal stability condition is automatically satisfied for a set $\mathcal{A}$ with $|\mathcal{A}| = 1$. Hence, if we want to know whether a set $\mathcal{A}$ with $|\mathcal{A}| = 1$ is a farsighted vNM stable set, we only need to check whether it satisfies the external stability condition.

In Mauleon et al. (2011), it is shown for the marriage problem of Gale and Shapley (1962) with strict preferences, that each singleton consisting of a matching in the core is a farsighted vNM stable set. In our context with not necessarily strict preferences over the set of all permutation matrices, we have a stronger result.

Theorem 4.14. *For all housing matching models* (N, P)*, the set* $\mathcal{A} = \{A^*\}$ *is a farsighted vNM stable set of* $\mathcal{E}(N, P)$*.*

Proof. Let (N, P) be a housing matching model. Note that $|\mathcal{A}| = 1$, hence we only need to show that $\mathcal{A} = \{A^*\}$ satisfies the external stability condition. From Theorem 4.11, we get that A^* indirectly dominates all $A \in X \setminus \{A^*\}$, i.e. $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$. Thus, the set $\mathcal{A} = \{A^*\}$ is a farsighted vNM stable set. □

In Theorem 4.14, we showed that $\{A^*\}$ is a farsighted vNM stable set. For each $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$, we get with Theorem 4.11, that A^* indirectly dominates any permutation matrix $A \in \mathcal{A} \setminus \{A^*\}$. Hence, the internal stability condition is not satisfied. Thus, each farsighted vNM stable set cannot be of the form $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$. Therefore, the remaining question is, can a set that does not contain A^* be a farsighted vNM stable set? In the following example, we show that $\{A^*\}$ is the unique farsighted vNM stable set for a specific housing matching model.

Example 4.15. Let $n = 3$, $N = \{1, 2, 3\}$ and let the preference matrix be given as:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & -1 & 0 \end{pmatrix}.$$

Then we have that $A^* = (3)(12)$. Let $\mathcal{A}$ be a farsighted vNM stable set. In Table 9, the construction of Theorem 4.11 is used to show that A^* indirectly dominates any $A \in X \setminus \{A^*\}$. Hence, with condition (1) in Definition 4.13, we have that either $\mathcal{A} = \{A^*\}$ or $A^* \notin \mathcal{A}$.

Sequence	$E(A, A^1)$ and $E(A^1, A)$	Preference
$(1)(2)(3) \xrightarrow[S^1=\{1,2\}]{} (3)(12)$	$\{1,2\} \in E((1)(2)(3), (3)(12))$	$(3)(12) \succ_i (1)(2)(3)\ \forall\, i \in S^1$
$(1)(23) \xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=\{1,2\}]{} (3)(12)$	$\{3\} \in E((1)(23), (1)(2)(3))$	$(3)(12) \succ_3 (1)(23)$
	$\{1,2\} \in E((1)(2)(3), (3)(12))$	$(3)(12) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(2)(13) \xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=\{1,2\}]{} (3)(12)$	$\{3\} \in E((2)(13), (1)(2)(3))$	$(3)(12) \succ_3 (3)(12)$
	$\{1,2\} \in E((1)(2)(3), (3)(12))$	$(3)(12) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(123) \xrightarrow[S^1=\{1\}]{} (1)(2)(3) \xrightarrow[S^2=\{1,2\}]{} (3)(12)$	$\{1\} \in E((123), (1)(2)(3))$	$(3)(12) \succ_1 (123)$
	$\{1,2\} \in E((1)(2)(3), (3)(12))$	$(3)(12) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(132) \xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=\{1,2\}]{} (3)(12)$	$\{3\} \in E((132), (1)(2)(3))$	$(3)(12) \succ_3 (132)$
	$\{1,2\} \in E((1)(2)(3), (3)(12))$	$(3)(12) \succ_i (1)(2)(3)\ \forall\, i \in S^2$

Table 9: Illustration that each $A \in X \setminus \{A^*\}$ is indirectly dominated by A^*.

Suppose that $A^* \notin \mathcal{A}$. Since $\mathcal{A}$ is a farsighted vNM stable set, we must have that there exists $A' \in \mathcal{A}$ such that $A' \gg A^*$. With Theorem 4.6, we know that A^* is not indirectly dominated, thus we have that $\mathcal{A} = \{A^*\}$. △

The remaining question is, can the result in Example 4.15 be generalized to all housing matching models? For the marriage problem of Gale and Shapley (1962) with strict preferences, Mauleon et al. (2011) proved that each farsighted vNM stable set must be a singleton consisting of a matching in the core. In our context with not necessarily strict preferences over the set of all permutation matrices, there is a stronger result.

Theorem 4.16. *For all housing matching models (N, P), it holds that if $\mathcal{A} \subseteq X$ is a farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A} = \{A^*\}$.*

Proof. Let (N, P) be a housing matching model. We prove that if $\mathcal{A} \neq \{A^*\}$, then we have that $\mathcal{A}$ is not a farsighted vNM stable set. Suppose that $\mathcal{A} \subseteq X$ with $\mathcal{A} \neq \{A^*\}$. We have two cases: $A^* \notin \mathcal{A}$ and $A^* \in \mathcal{A}$ with $|\mathcal{A}| > 1$. For the latter case, we already discussed that it cannot be a farsighted vNM stable set. Thus, each farsighted vNM stable set either is equal to $\{A^*\}$ or it does not contain A^*.

Suppose that $A^* \notin \mathcal{A}$, then from Theorem 4.6 we know that A^* is not indirectly dominated. Hence, for each set $\mathcal{A} \subseteq X$ with $A^* \notin \mathcal{A}$ the external stability condition is not satisfied. Thus, each $\mathcal{A} \subseteq X$ that does not contain A^* is not a farsighted vNM stable set. Hence, if $\mathcal{A} \subseteq X$ is a farsighted vNM stable set, then we have that $\mathcal{A} = \{A^*\}$. □

From Theorem 4.14 and Theorem 4.16, we get the following corollary.

Corollary 4.17. *For all housing matching models (N, P), the unique farsighted vNM stable set of $\mathcal{E}(N, P)$ is $\{A^*\}$.*

4.4 Largest consistent set

In Chwe (1994), a consistent set and the largest consistent set are defined for a general game with strict preferences. We give these definitions in the context of our social environment corresponding to a housing matching model with not necessarily strict preferences over X, and we use some small adjustments in order to make the definitions more understandable. For the definition of a consistent set, we need another definition.

Definition 4.18 (($\mathcal{A}, \gg$)-deterrence of deviations)**.**
Let (N, P) be a housing matching model. Let $\mathcal{A} \subseteq X$ and let $A \in X$. We say that A satisfies **($\mathcal{A}, \gg$)-deterrence of deviations** if for all $A' \in X$ and for all $S \in E(A, A')$, there exists $A'' \in \mathcal{A}$ such that the following two conditions hold:

(1) either $A'' = A'$ or $A'' \gg A'$,

(2) there exists $i \in S$ with $A \succsim_i A''$.

That A satisfies ($\mathcal{A}, \gg$)-deterrence of deviations means that each deviation from permutation matrix A to an arbitrary permutation matrix A' is deterred by the possibility of ending up at a permutation matrix A'' inside $\mathcal{A}$, which is not strictly preferred to A by at least one agent in the deviating coalition.

Definition 4.19 (Consistent set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ is a **consistent set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) if $A \in \mathcal{A}$, then A satisfies ($\mathcal{A}, \gg$)-deterrence of deviations,

(2) for all $A \notin \mathcal{A}$ we have that A does not satisfy ($\mathcal{A}, \gg$)-deterrence of deviations.

In the following remark, we give an intuition of ($\mathcal{A}, \gg$)-deterrence of external deviations.

Remark 4.20. Let (N, P) be a housing matching model, let $\mathcal{A} \subseteq X$ and let $A \in X$. Suppose that A does not satisfy ($\mathcal{A}, \gg$)-deterrence of deviations. Then we have that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that for all $A'' \in \mathcal{A}$, we have that at least one of the following conditions hold:

(1) $A'' \neq A'$ and $A'' \not\gg A'$,

(2) $A'' \succ_i A$ for all $i \in S$.

In particular, we have that for all $A'' \in \mathcal{A}$ with $A'' \gg A'$, it must hold that $A'' \succ_i A$ for all $i \in S$. Hence, from Lemma 4.2, we get that $A'' \gg A$. Also note that for $A'' \in \mathcal{A}$ with $A'' = A'$, it must hold that $A' \succ_i A$ for all $i \in S$. Hence, with Definition 3.1(1), we can conclude that A' strictly dominates A.

As in Chwe (1994), define the **consistent correspondence** as the correspondence $g_{\gg} : 2^X \to 2^X$ such that

$$g_{\gg}(\mathcal{A}) = \{A \in X \mid A \text{ satisfies } (\mathcal{A}, \gg)\text{-deterrence of deviations}\}.$$

Remark 4.21. From Definition 4.19 and the definition of the consistent correspondence, we get that $\mathcal{A} \subseteq X$ is consistent, if and only if $\mathcal{A}$ is a fixed point of $g_{\gg}$, i.e. $\mathcal{A} = g_{\gg}(\mathcal{A})$,

Note that for $A \in \mathcal{A}$ and $A' = A$, we have for $A'' = A$ that $A'' = A'$ and $A \sim_i A''$ for all $i \in N$. Hence, in order to check whether $A \in \mathcal{A}$ satisfies ($\mathcal{A}, \gg$)-deterrence of deviations, we need to check it for all $A' \in X \setminus \{A\}$.

Lemma 4.22. *For all housing matching models (N, P), we have that $\emptyset$ is a consistent set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. Note that the statement $A \in \emptyset$ is false for all $A \in X$. Also note that the statement $A \in X$ satisfies $(\emptyset, \gg)$-deterrence of deviations means that for all $A' \in X$ and for all $S \in E(A, A')$, there exists $A'' \in \emptyset$ such that either $A'' = A'$ or $A'' \gg A'$, and such that $\exists\ i \in S$ with $A \succsim_i A''$. Hence, the statement $A \in X$ satisfies $(\emptyset, \gg)$-deterrence of deviations is also false for all $A \in X$. With Chartrand, Polimeni, and Zhang (2018) we can conclude that the statements imply each other. Thus, $\emptyset$ is a consistent set of $\mathcal{E}(N, P)$. □

In Chwe (1994), it is shown that there exists a unique consistent set, called the largest consistent set Y, that contains all consistent sets, i.e. if Y' is consistent, then $Y' \subseteq Y$. In our context the definition of the largest consistent set given in Chwe (1994) is:

$$Y = \bigcup_{\mathcal{A} \subseteq g_{\gg}(\mathcal{A})} \mathcal{A}.$$

We give a slightly different definition, but we show that our definition of the largest consistent set and the definition in Chwe (1994) are equivalent.

Definition 4.23 (Largest consistent set)**.**
The **largest consistent set** $\mathbb{A}_{\gg}$ of $\mathcal{E}(N, P)$ is the union of all consistent sets:

$$\mathbb{A}_{\gg} = \bigcup_{\mathcal{A} = g_{\gg}(\mathcal{A})} \mathcal{A}$$

equivalently

$$\mathbb{A}_{\gg} = \{A \in X \mid \exists\ \mathcal{A} \subseteq X \text{ with } \mathcal{A} = g_{\gg}(\mathcal{A}) \text{ and } A \in \mathcal{A}\}.$$

In the following lemma, we show that the largest consistent set, as defined in Definition 4.23, is the definition of Chwe (1994).

Lemma 4.24. *For all housing matching models (N, P), the largest consistent set $\mathbb{A}_{\gg}$ is the largest consistent set of Chwe (1994).*

Proof. Let (N, P) be a housing matching model. Note that

$$\mathbb{A}_{\gg} = \bigcup_{\mathcal{A} = g_{\gg}(\mathcal{A})} \mathcal{A} \subseteq \bigcup_{\mathcal{A} \subseteq g_{\gg}(\mathcal{A})} \mathcal{A} = Y.$$

We show that Y is a consistent set of $\mathcal{E}(N, P)$ by rewriting the proof given in Chwe (1994). Note that $g_{\gg} : 2^X \to 2^X$ is isotonic, which means that if $\mathcal{A}' \subseteq \mathcal{A}$, then $g_{\gg}(\mathcal{A}') \subseteq g_{\gg}(\mathcal{A})$. Thus, for all $\mathcal{A} \subseteq X$ such that $\mathcal{A} \subseteq g_{\gg}(\mathcal{A})$, we have that $g_{\gg}(\mathcal{A}) \subseteq g_{\gg}(Y)$. Hence, we get that

$$Y = \bigcup_{\mathcal{A} \subseteq g_{\gg}(\mathcal{A})} \mathcal{A} \subseteq \bigcup_{\mathcal{A} \subseteq g_{\gg}(\mathcal{A})} g_{\gg}(\mathcal{A}) \subseteq g_{\gg}(Y).$$

Since $g_{\gg} : 2^X \to 2^X$ is isotonic, we have that $g_{\gg}(Y) \subseteq g_{\gg}(g_{\gg}(Y))$. With the definition of Y we can conclude that $g_{\gg}(Y) \subseteq Y$. Thus, we have that $Y = g_{\gg}(Y)$, i.e. Y is a consistent set of $\mathcal{E}(N, P)$. Hence, we get that $Y \subseteq \mathbb{A}_{\gg}$. This gives us that $Y = \mathbb{A}_{\gg}$. In other words, the largest consistent set $\mathbb{A}_{\gg}$ is the largest consistent set of Chwe (1994). □

From the proof of Lemma 4.24, we get the following corollary.

Corollary 4.25. *For all housing matching models* (N, P)*, the largest consistent set* $\mathbb{A}_{\gg}$ *of* $\mathcal{E}(N, P)$ *is a consistent set.*

Note that the state space X is finite and that $A \sim_i A$ for all $i \in N$ and for all $A \in X$, hence with a corollary in Chwe (1994) we can conclude that the largest consistent set $\mathbb{A}_{\gg}$ is nonempty and satisfies external stability, which means that $\forall\ A \notin \mathbb{A}_{\gg}$ it holds that $f_{\gg}(A) \cap \mathbb{A}_{\gg} \neq \emptyset$.

In the following proposition, we show that each nonempty consistent set must contain the top trading cycle permutation matrix.

Proposition 4.26. *For all housing matching models* (N, P)*, we have that each nonempty set* $\mathcal{A} \subseteq X \setminus \{A^*\}$ *is not a consistent set of* $\mathcal{E}(N, P)$*.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X \setminus \{A^*\}$ be a nonempty subset. In order to prove that $\mathcal{A}$ is not a consistent set, we show that each $A \in \mathcal{A}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Let $A \in \mathcal{A}$, then we need to show that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that for all $A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\gg A'$, or we have that $A'' \succ_i A$ for all $i \in S$.

We show that there exists $A' \in X \setminus \mathcal{A}$, such that for all $A'' \in \mathcal{A}$ we have that $A'' \not\gg A'$. Take $A' = A^* \in X \setminus \mathcal{A}$, then by Theorem 4.6, we know that A^* is not indirectly dominated. Hence, in particular, for all $A'' \in \mathcal{A}$, we have that $A'' \not\gg A^*$. This shows that A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Thus, we can conclude that $\mathcal{A}$ is not a consistent set of $\mathcal{E}(N, P)$. □

From Proposition 4.26, we know that each nonempty consistent set $\mathcal{A}$ must contain A^*. Without any further restrictions on the set $\mathcal{A}$ we can show that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Lemma 4.27. *For all housing matching models* (N, P)*, it holds for each* $\mathcal{A} \subseteq X$ *with* $A^* \in \mathcal{A}$*, that* A^* *satisfies* $(\mathcal{A}, \gg)$*-deterrence of deviations.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$. We need to show that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. For all $A' \in X$ and for all $S \in E(A^*, A')$, let $A'' = A^* \in \mathcal{A}$. We show that it holds that either $A^* = A'$ or $A^* \gg A'$, and that $\exists\ i \in S$ with $A^* \succsim_i A^*$. Note that $A^* \sim_i A^*$ for all $i \in N$. Hence, for all $A' \in X \setminus \{A^*\}$ we need to show that $A^* \gg A'$. This follows from Theorem 4.11. Thus, A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. □

With the help of the following corollary we can show that $\mathcal{A} = \{A^*\}$ is a consistent set.

Corollary 4.28. *For all housing matching models* (N, P)*, it follows from Lemma 4.9 that for each* $A \in X \setminus \{A^*\}$ *there exists* $i \in N$ *such that* $A^* \succ_i A$*.*

Theorem 4.29. *For all housing matching models* (N, P)*, the set* $\mathcal{A} = \{A^*\}$ *is a consistent set of* $\mathcal{E}(N, P)$*.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} = \{A^*\}$. The first condition in Definition 4.19 is satisfied, because from Lemma 4.27 it follows that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Thus, for all permutation matrices $A \in X \setminus \{A^*\}$, we need to show that A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Hence, for all $A \in X \setminus \{A^*\}$, we need to show that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $A^* \neq A'$ and $A^* \not\gg A'$, or such that $\forall\, i \in S$ it holds that $A^* \succ_i A$. Since A^* indirectly dominates all other permutation matrices, we need to show for all $A \in X \setminus \{A^*\}$, that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $\forall\, i \in S$ it holds that $A^* \succ_i A$.

Let $A \in X \setminus \{A^*\}$. Then from Corollary 4.28, we know that there exists $i \in N$ such that $A^* \succ_i A$. Take $S = \{i\}$ and define $\phi : S \to S$ as $\phi(i) = i$, then ϕ is a bijection. From Lemma 2.11, we know that E satisfies condition (2) of Demuynck et al. (2019b), hence there exists $A' \in X$ such that $S = \{i\} \in E(A, A')$ and $A'_{ii} = 1$. Hence, A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Thus, all permutation matrices not equal to A^*, do not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations.

Thus, we can conclude that $\mathcal{A} = \{A^*\}$ is a consistent set. □

From Proposition 4.26 and Lemma 4.27, we know for each nonempty consistent set $\mathcal{A}$ that $\{A^*\} \subseteq \mathcal{A}$. From Theorem 4.29, we know that $\{A^*\}$ is a consistent set, hence we get the following corollary.

Corollary 4.30. *For all housing matching models (N, P), we have that $A^* \in \mathbb{A}_{\gg}$.*

In the following example, we look at whether $\{A^*\}$ can be the largest consistent set.

Example 4.31 (Example 4.5 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 4.5:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

Let $\mathcal{A} \subseteq X$ be a nonempty consistent set. From Proposition 4.26, we know that $A^* = (1)(23) \in \mathcal{A}$. We show that $\mathcal{A}$ must be equal to $\{A^*\}$. In other words, we show that $\{A^*\}$ is the only nonempty consistent set. Since $\mathcal{A}$ is an arbitrary nonempty consistent set, it is sufficient to show that each $A \in X \setminus \{A^*\}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Hence, we need to show that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $\forall\, A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\gg A'$, or we have that $A'' \succ_i A$ for all $i \in S$.

Let $A \in \{(1)(2)(3), (2)(13), (3)(12)\}$. Suppose that $A' = A^*$, then we have that $E(A, A^*) = \{\{2, 3\}, N\}$. For each $A'' \in \mathcal{A} \setminus \{A^*\}$, we have that $A'' \not\gg A^*$. Note that each $A \in \{(1)(2)(3), (2)(13), (3)(12)\}$ is strictly dominated by A^* for $S = \{2, 3\}$. Thus, for $A'' = A^*$ we have that $A^* \succ_i A$ for all $i \in \{2, 3\}$. We can conclude that A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Since $\mathcal{A}$ is consistent, we get that $\{(1)(2)(3), (2)(13), (3)(12)\} \cap \mathcal{A} = \emptyset$. Thus, we have that $\mathcal{A} \subseteq \{(1)(23), (123), (132)\}$.

Let $A = (123)$. Suppose that $A' = (1)(2)(3)$, then we have that $E(A, A') = 2^N \setminus \{\emptyset\}$. Let $S = \{1\} \in E(A, A')$. For $A'' \in \{A^*, (132)\}$, we have that $A'' \succ_1 (123)$ and for

$A'' = (123)$, we have that $(123) \not\gg (1)(2)(3)$. Thus, we can conclude that (123) does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Since $\mathcal{A}$ is consistent, we get that $(123) \notin \mathcal{A}$.

Let $A = (132)$. Suppose that $A' = (1)(2)(3)$, then we have that $E(A, A') = 2^N \setminus \{\emptyset\}$. Let $S = \{3\} \in E(A, A')$. For $A'' \in \{A^*, (123)\}$, we have that $A'' \succ_3 (132)$ and for $A'' = (132)$, we have that $(132) \not\gg (1)(2)(3)$. Thus, we can conclude that (132) does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Hence, we have that $(132) \notin \mathcal{A}$, since $\mathcal{A}$ is consistent.

We can conclude that $\mathcal{A} = \{A^*\}$ is the unique nonempty consistent set. Thus, we also have that $\mathbb{A}_{\gg} = \{A^*\}$. △

The result in Example 4.31 that a permutation matrix that is strictly dominated by A^* cannot be in any nonempty consistent set can be generalized to all housing matching models.

Proposition 4.32. *For all housing matching models (N, P), we have that the following holds. Let $\mathcal{A} \subseteq X$ be a nonempty consistent set of $\mathcal{E}(N, P)$. If A^* strictly dominates $A \in X \setminus \{A^*\}$, then we have that $A \notin \mathcal{A}$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$ be a nonempty consistent set of $\mathcal{E}(N, P)$. We give a proof by contradiction. Suppose that there exists $A \in \mathcal{A} \setminus \{A^*\}$ such that A^* strictly dominates A.

We show that $A \in \mathcal{A}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations, which gives a contradiction with the fact that $\mathcal{A}$ is a consistent set of $\mathcal{E}(N, P)$. Hence, we need to show that $\exists\ A' \in X$ and $\exists\ S \in E(A, A')$, such that $\forall\ A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\gg A'$, or we have that $A'' \succ_i A$ for all $i \in S$. From Proposition 4.26, we know that each nonempty consistent set must contain A^*, hence we have that $A^* \in \mathcal{A}$.

Let $A' = A^*$. Note that from Theorem 4.6, we know that A^* is not indirectly dominated. Hence, for all $A'' \in \mathcal{A} \setminus \{A^*\}$, we have that $A'' \neq A^*$ and $A'' \not\gg A^*$. Now, let $A'' = A^*$. Recall that A is strictly dominated by A^*. Thus, according to Definition 3.1(1) there exists $S \in E(A, A^*)$ such that $A^* \succ_i A$ for all $i \in S$.

We can conclude that $A \in \mathcal{A}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Thus, we showed that if A^* strictly dominates $A \in X \setminus \{A^*\}$, then we have that $A \notin \mathcal{A}$ for any nonempty consistent set $\mathcal{A}$ of $\mathcal{E}(N, P)$. □

In the following example, we show that a set $\mathcal{A}$ with $|\mathcal{A}| > 1$ and $A^* \in \mathcal{A}$ can be a consistent set. This shows that the largest consistent set can contain permutation matrices that are not equal to A^*.

Example 4.33 (Example 3.24 continued)**.**
Let the preference matrix like in Example 3.24, be given as

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

Recall that $A^* = (132)$. Let $\mathcal{A} = \{(132), (1)(23), (2)(13), (3)(12)\} \subseteq X$. We show that $\mathcal{A}$ is a consistent set of $\mathcal{E}(N, P)$.

First, we show that each permutation matrix outside $\mathcal{A}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Note that $X \setminus \mathcal{A} = \{(1)(2)(3), (123)\}$. We know from Theorem 4.6 that $A^* = (132)$ is not indirectly dominated. Note that for all $A \in X \setminus \mathcal{A}$ we have that $E(A, A^*) = \{N\}$ and that $\nexists\, i \in N$ such that $A \succsim_i A^*$. Hence, $A \in X \setminus \mathcal{A}$ implies that A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations.

Secondly, we show that $A \in \mathcal{A}$ implies that A satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. In other words, for all $A \in \mathcal{A}$, we need to show that $\forall\, A' \in X$ and $\forall\, S \in E(A, A')$, there exists $A'' \in \mathcal{A}$, such that either $A'' = A'$ or $A'' \gg A'$, and such that $\exists\, i \in S$ with $A \succsim_i A''$. For $A^* = (132) \in \mathcal{A}$, we know from Lemma 4.27 that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

With the fact that strict dominance implies indirect dominance, we know from Example 3.21 that $(132) \gg (123)$, $(2)(13) \gg (1)(23)$, $(3)(12) \gg (2)(13)$ and $(1)(23) \gg (3)(12)$. Now, we show that $(1)(23) \in \mathcal{A}$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Let $A' = (1)(2)(3)$, then with the sequence

$$(1)(2)(3) \xrightarrow[S=\{2,3\}]{} (1)(23)$$

we get that $(1)(23) \in \mathcal{A}$ strictly dominates $(1)(2)(3)$. Thus, we have that $(1)(23) \gg (1)(2)(3)$. Note that $E((1)(23), (1)(2)(3)) = 2^N \setminus \{\emptyset, \{1\}\}$ and that $(1)(23) \sim_i (1)(23)$ for all $i \in N$. Hence, the deviation from $(1)(23)$ to $(1)(2)(3)$ is deterred by the possibility of ending up at $(1)(23)$. For $A' = (132) \in \mathcal{A}$, we have that $E((1)(23), (132)) = \{N\}$ and that $(1)(23) \sim_2 (132)$. Now, let $A' = (123)$, then we have that $E((1)(23), (123)) = \{N\}$, that $(132) \gg (123)$ with $(132) \in \mathcal{A}$ and $(1)(23) \sim_2 (132)$. Thus, the deviation from $(1)(23)$ to (123) is deterred for agent 2 by the possibility of ending up at (132).

Suppose that $A' = (2)(13)$, then we have that $(3)(12) \in \mathcal{A}$ with $(3)(12) \gg (2)(13)$, $E((1)(23), (2)(13)) = \{\{1, 3\}, N\}$ and that $(1)(23) \succ_3 (3)(12)$. Hence, the deviation from $(1)(23)$ to $(2)(13)$ is deterred for agent 3 by the possibility of ending up at $(3)(12)$. For $A' = (3)(12)$, it holds that $(1)(23) \in \mathcal{A}$ with $(1)(23) \gg (3)(12)$, $E((1)(23), (3)(12)) = \{\{1, 2\}, N\}$ and $(1)(23) \sim_i (1)(23)$ for all $i \in N$.

We showed for all $A' \in X$ and for all $S \in E((1)(23), A')$ that there exists $A'' \in \mathcal{A}$, such that either $A'' = A'$ or $A'' \gg A'$, and such that there exists $i \in S$ with $(1)(23) \succsim_i A''$. Thus, we get that $(1)(23)$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. An overview of the above can be found in Table 10.

$A' \in X$	$S \in E(A, A')$	$A'' \in \mathcal{A}$	$A'' = A'$ or $A'' \gg A'$	$\exists\, i \in S$ with $A \succsim_i A''$
$(1)(2)(3)$	$E((1)(23), (1)(2)(3)) = 2^N \setminus \{\emptyset, \{1\}\}$ $\forall\, S \in \{\{2\}, \{1, 2\}, \{2, 3\}, N\}$ $\forall\, S \in \{\{3\}, \{1, 3\}, \{2, 3\}, N\}$	$(1)(23)$	$A'' \gg A'$	 $(1)(23) \sim_2 (1)(23)$ $(1)(23) \sim_3 (1)(23)$
(132)	$E((1)(23), (132)) = \{N\}$	(132)	$A'' = A'$	$(1)(23) \sim_2 (132)$
(123)	$E((1)(23), (123)) = \{N\}$	(132)	$A'' \gg A'$	$(1)(23) \sim_2 (132)$
$(2)(13)$	$E((1)(23), (2)(13)) = \{\{1, 3\}, N\}$	$(3)(12)$	$A'' \gg A'$	$(1)(23) \succ_3 (3)(12)$
$(3)(12)$	$E((1)(23), (3)(12)) = \{\{1, 2\}, N\}$	$(1)(23)$	$A'' \gg A'$	$(1)(23) \sim_2 (1)(23)$

Table 10: Illustration that $A = (1)(23)$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Similarly, we can show that (2)(13) $\in \mathcal{A}$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. An overview of this can be found in Table 11. Note that with the sequence

$$(1)(2)(3) \xrightarrow[S=\{1,3\}]{} (2)(13)$$

we get that (2)(13) strictly dominates (1)(2)(3). Thus, it holds that $(2)(13) \gg (1)(2)(3)$.

$A' \in X$	$S \in E(A, A')$	$A'' \in \mathcal{A}$	$A'' = A'$ or $A'' \gg A'$	$\exists\, i \in S$ with $A \succsim_i A''$
(1)(2)(3)	$E((2)(13), (1)(2)(3)) = 2^N \setminus \{\emptyset, \{2\}\}$ $\forall\, S \in \{\{1\}, \{1,2\}, \{1,3\}, N\}$ $\forall\, S \in \{\{3\}, \{1,3\}, \{2,3\}, N\}$	(2)(13)	$A'' \gg A'$	 $(2)(13) \sim_1 (2)(13)$ $(2)(13) \sim_3 (2)(13)$
(132)	$E((2)(13), (132)) = \{N\}$	(132)	$A'' = A'$	$(2)(13) \sim_3 (132)$
(123)	$E((2)(13), (123)) = \{N\}$	(132)	$A'' \gg A'$	$(2)(13) \sim_3 (132)$
(1)(23)	$E((2)(13), (1)(23)) = \{\{2,3\}, N\}$	(2)(13)	$A'' \gg A'$	$(2)(13) \sim_3 (2)(13)$
(3)(12)	$E((2)(13), (3)(12)) = \{\{1,2\}, N\}$	(1)(23)	$A'' \gg A'$	$(2)(13) \succ_1 (1)(23)$

Table 11: Illustration that $A = (2)(13)$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Now consider (3)(12) $\in \mathcal{A}$, then with the sequence

$$(1)(2)(3) \xrightarrow[S=\{1,2\}]{} (3)(12),$$

we get that (3)(12) strictly dominates (1)(2)(3). Thus, with Table 12, we have that (3)(12) satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

$A' \in X$	$S \in E(A, A')$	$A'' \in \mathcal{A}$	$A'' = A'$ or $A'' \gg A'$	$\exists\, i \in S$ with $A \succsim_i A''$
(1)(2)(3)	$E((3)(12), (1)(2)(3)) = 2^N \setminus \{\emptyset, \{3\}\}$ $\forall\, S \in \{\{1\}, \{1,2\}, \{1,3\}, N\}$ $\forall\, S \in \{\{2\}, \{1,2\}, \{2,3\}, N\}$	(3)(12)	$A'' \gg A'$	 $(3)(12) \sim_1 (3)(12)$ $(3)(12) \sim_2 (3)(12)$
(132)	$E((3)(12), (132)) = \{N\}$	(132)	$A'' = A'$	$(3)(12) \sim_1 (132)$
(123)	$E((3)(12), (123)) = \{N\}$	(132)	$A'' \gg A'$	$(3)(12) \sim_1 (132)$
(1)(23)	$E((3)(12), (1)(23)) = \{\{2,3\}, N\}$	(2)(13)	$A'' \gg A'$	$(3)(12) \succ_2 (2)(13)$
(2)(13)	$E((3)(12), (2)(13)) = \{\{1,3\}, N\}$	(3)(12)	$A'' \gg A'$	$(3)(12) \sim_1 (3)(12)$

Table 12: Illustration that $A = (3)(12)$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Hence, we get that $A \in \mathcal{A}$ implies that A satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. We have showed that $\mathcal{A}$ satisfies the two conditions in Definition 4.19, thus

$$\mathcal{A} = \{(132), (1)(23), (2)(13), (3)(12)\}$$

is a consistent set of $\mathcal{E}(N, P)$. △

From Example 4.31 and Example 4.33, we can conclude that for some housing matching models $\{A^*\}$ is the largest consistent set, but it is not the largest consistent set for each housing matching model (N, P).

4.5 DEM farsighted stable set

In the context of networks, Herings et al. (2009) introduced the concept of a pairwise farsightedly stable set. In Herings et al. (2010), this concept was applied to the context of coalition formation games and a farsightedly stable set was introduced. We give their definition in the context of our social environment corresponding to the housing matching model (N, P) and Definition 4.1. As in Kimya (2023), we give it the name DEM farsighted stable set.
Recall that $f_{\gg}$ is the indirect dominance correspondence, i.e.

$$f_{\gg}(A) = \{A\} \cup \{A' \in X \mid A' \gg A\}.$$

Definition 4.34 (DEM farsighted stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **DEM farsighted stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three properties:

(1) **deterrence of external deviations**: $\forall\ A \in \mathcal{A}$, $\forall\ A' \notin \mathcal{A}$ and $\forall\ S \in E(A, A')$, there exists $A'' \in \mathcal{A}$ such that $A'' \gg A'$ and $\exists\ i \in S$ with $A \succsim_i A''$,

(2) **external stability**: $\forall\ A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ that satisfies (1) and (2).

Deterrence of external deviations says that each deviation from a permutation matrix A inside $\mathcal{A}$ to a permutation matrix A' outside $\mathcal{A}$ is deterred by the possibility of ending up at another permutation matrix A'' inside $\mathcal{A}$, which is not strictly preferred by at least one agent in the deviating coalition.

Note that deterrence of external deviations looks a lot like $(\mathcal{A}, \gg)$-deterrence of deviations: $\forall\ A' \in X$ and $\forall\ S \in E(A, A')$, there exists $A'' \in \mathcal{A}$, such that either $A'' = A'$ or $A'' \gg A'$, and such that $\exists\ i \in S$ with $A \succsim_i A''$. Hence, that $A \in \mathcal{A}$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations says that each deviation from a permutation matrix A to an arbitrary permutation matrix $A' \in X$, not necessarily outside $\mathcal{A}$, is deterred by the possibility of ending up at another permutation matrix A'' inside $\mathcal{A}$, which is not strictly preferred by at least one agent in the deviating coalition. Thus, $A \in \mathcal{A}$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations implies that $A \in \mathcal{A}$ satisfies deterrence of external deviations.

External stability means that each permutation matrix outside $\mathcal{A}$ is indirectly dominated by a permutation matrix inside $\mathcal{A}$. Note that external stability implies that for all housing matching models (N, P) we have that $\emptyset$ is not a DEM farsighted stable set of $\mathcal{E}(N, P)$.

Hence, the DEM farsighted stable requires deterrence of external deviations, the external stability condition and minimality and the largest consistent set requires deterrence of external deviations, deterrence of internal deviations and maximality.

Theorem 4.35. *For all housing matching models (N, P), the set $\mathcal{A} = \{A^*\}$ is the unique DEM farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. First, we show that $\mathcal{A} = \{A^*\}$ is a DEM farsighted stable set of $\mathcal{E}(N, P)$. The first condition in Definition 4.34 becomes $\forall\ A' \in X \setminus \{A^*\}$ it holds that $A^* \gg A'$. This follows from Theorem 4.11, hence $\mathcal{A}$

satisfies deterrence of external deviations. From Theorem 4.11, we get that $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$. Hence, $\mathcal{A}$ satisfies external stability. The set $\mathcal{A}$ automatically satisfies the minimality condition, since $|\mathcal{A}| = 1$. Thus, $\mathcal{A}$ is a DEM farsighted stable set of $\mathcal{E}(N, P)$.

Secondly, we show that each set $\mathcal{A}' \subseteq X$ with $A^* \in \mathcal{A}'$ and $|\mathcal{A}'| > 1$ is not a DEM farsighted stable set of $\mathcal{E}(N, P)$. Let $\mathcal{A}' \subseteq X$ with $A^* \in \mathcal{A}'$ and $|\mathcal{A}'| > 1$, then from the above $\mathcal{A} = \{A^*\} \subsetneq \mathcal{A}'$ is a proper subset that satisfies (1) and (2) in Definition 4.34. Hence, $\mathcal{A}'$ does not satisfy the minimality condition. Thus, $\mathcal{A}'$ is not a DEM farsighted stable set of $\mathcal{E}(N, P)$.

Thirdly, we show that each set $\mathcal{A}'' \subseteq X \setminus \{A^*\}$ is not a DEM farsighted stable set of $\mathcal{E}(N, P)$. Let $\mathcal{A}'' \subseteq X \setminus \{A^*\}$ and let $A \in \mathcal{A}''$. Note that $A^* \notin \mathcal{A}''$. From Theorem 4.6, we know that $f_{\gg}(A^*) = \{A^*\}$. Hence, we have that $f_{\gg}(A^*) \cap \mathcal{A}'' = \emptyset$. This shows that $\mathcal{A}''$ does not satisfy the external stability condition. Thus, $\mathcal{A}''$ is not a DEM farsighted stable set of $\mathcal{E}(N, P)$.

We can conclude that $\mathcal{A} = \{A^*\}$ is the unique DEM farsighted stable set of $\mathcal{E}(N, P)$. $\square$

5 Full Farsightedness with Weak Dominance

In Section 4, we studied the core, the vNM stable set, the largest consistent set and the DEM farsighted stable set under the assumption that all agents are fully farsighted. In this section, we also assume that all agents are fully farsighted and we study these stability concepts with respect to indirect weak dominance instead of indirect dominance.

5.1 Indirect weak dominance

In Mauleon and Vannetelbosch (2004), the definition of indirect weak dominance is given in the context of a coalition formation game. We give this definition in the context of our social environment corresponding to the housing matching model (N, P).

Definition 5.1 (Indirect weak dominance)**.**
Let (N, P) be a housing matching model. Let $A, A' \in X$ be two different permutation matrices. The permutation matrix A' **indirectly weakly dominates** A in $\mathcal{E}(N, P)$, denoted by $A' \gg A$, if there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A$ and $A^m = A'$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \ldots, m\}$ the following two conditions hold:

(1) $S^k \in E(A^{k-1}, A^k)$,

(2) $A' \succsim_i A^{k-1}$ for all $i \in S^k$ and $A' \succ_j A^{k-1}$ for at least one $j \in S^k$.

Note that with $m = 1$ weak dominance implies indirect weak dominance. Also note that indirect dominance implies indirect weak dominance and that from Theorem 4.11, we know that A^* indirectly dominates all other permutation matrices for all housing matching models. Thus, we get the following corollary.

Corollary 5.2. *For all housing matching models* (N, P)*, each* $A \in X \setminus \{A^*\}$ *is indirectly weakly dominated by* A^* *in* $\mathcal{E}(N, P)$*.*

In Lemma 4.2, we showed that indirect dominance has a nice property. Indirect weak dominance has a similar property.

Lemma 5.3. *For all housing matching models* (N, P)*, the following holds. Let* $A, A', A'' \in X$ *be different permutation matrices and let* $S \in E(A, A')$*. If* $A'' \gg A'$*,* $A'' \succsim_i A$ *for all* $i \in S$ *and* $A'' \succ_j A$ *for at least one* $j \in S$*, then we have that* $A'' \gg A$*.*

Proof. Let (N, P) be a housing matching model, let $A, A', A'' \in X$ be different permutation matrices and let $S \in E(A, A')$. Suppose that $A'' \gg A'$, $A'' \succsim_i A$ for all $i \in S$ and that $A'' \succ_j A$ for at least one $j \in S$. We need to show that $A'' \gg A$. Thus, we need to show that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A$ and $A^m = A''$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \ldots, m\}$, we have that $S^k \in E(A^{k-1}, A^k)$, $A'' \succsim_i A^{k-1}$ for all $i \in S^k$ and $\exists\, j \in S^k$ with $A'' \succ_j A^{k-1}$.

Take $A^0 = A$, $A^1 = A'$ and $S^1 = S \in E(A, A')$, then we know that $A'' \succsim_i A$ for all $i \in S^1$ and that $\exists\, j \in S^1$ with $A'' \succ_j A$. From $A'' \gg A'$, we know that there is a sequence of permutation matrices $B^0, \ldots, B^{m'} \in X$ with $B^0 = A'$ and $B^{m'} = A''$ and there are coalitions $R^1, \ldots, R^{m'} \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k' \in \{1, \ldots, m'\}$, we have that

$R^{k'} \in E(B^{k'-1}, B^{k'})$, $A'' \succsim_i B^{k'-1}$ for all $i \in R^{k'}$ and $\exists\ j \in R^{k'}$ with $A'' \succ_j B^{k'-1}$. For $k \in \{2, \ldots, m'+1\}$, let $S^k = R^{k-1}$ and let $A^k = B^{k-1}$, then we have that $S^k \in E(A^{k-1}, A^k)$, $A'' \succsim_i A^{k-1}$ for all $i \in S^k$ and that $\exists\ j \in S^k$ with $A'' \succ_j A^{k-1}$. This shows that $A'' \ggeq A$. □

5.2 Strong farsighted core

In Section 4, we defined the farsighted core as the set of permutation matrices that are not indirectly dominated. In this section, we look at the farsighted core with respect to indirect weak dominance. Define the **indirect weak dominance correspondence** as the correspondence $f_{\ggeq} : X \to 2^X$ such that

$$f_{\ggeq}(A) = \{A\} \cup \{A' \in X \mid A' \ggeq A\}.$$

Definition 5.4 (Strong farsighted core)**.**
Let (N, P) be a housing matching model. The **strong farsighted core** $SFCO$ of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not indirectly weakly dominated:

$$SFCO = \{A \in X \mid f_{\ggeq}(A) = \{A\}\}.$$

Recall that weak dominance implies indirect weak dominance and that indirect dominance implies indirect weak dominance, hence we have that $SFCO \subseteq SCO$ and $SFCO \subseteq FCO$. Recall from Corollary 4.12, that the farsighted core of each housing matching model is equal to $\{A^*\}$, hence we get the following corollary.

Corollary 5.5. *For all housing matching models* (N, P)*, we have that the strong farsighted core of* $\mathcal{E}(N, P)$ *is either* $\emptyset$ *or* $\{A^*\}$*.*

In the following example, we show, for a specific housing matching model, that the strong farsighted core is equal to $\{A^*\}$.

Example 5.6. Let $n = 3, N = \{1, 2, 3\}$ and let the preference matrix be given as

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & -2 \\ -1 & 0 & -2 \end{pmatrix}.$$

Then we have that $A^* = (123)$. Note that according to A^* each agent $i \in N$ gets his top choice, thus we have that $A^* \succsim_i A'$ for all $i \in N$ and for all $A' \in X \setminus \{A^*\}$. Hence, with Definition 5.1, we can conclude that A^* is not indirectly weakly dominated. From Corollary 5.2, we know that each permutation matrix in $X \setminus \{A^*\}$ is indirectly weakly dominated by A^*, hence we get that $SFCO = \{A^*\}$. △

From Example 5.6, we could draw the wrong conclusion that for all housing matching models (N, P), we have that the strong farsighted core of $\mathcal{E}(N, P)$ is equal to $\{A^*\}$. In the following example, we show that A^* can be indirectly weakly dominated.

Example 5.7 (Example 4.10 continued).
Let $n = 3$ and let the preference matrix P be as in Example 4.10:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$. Take $A' = (132)$, then for the following sequence:

$$(1)(23) \xrightarrow[S^1=\{1,2\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (132),$$

we have that $S^1 = \{1,2\} \in E((1)(23),(1)(2)(3))$, $(132) \succ_1 (1)(23)$, $(132) \sim_2 (1)(23)$, $S^2 = N \in E((1)(2)(3),(132))$ and $(132) \succ_i (1)(2)(3)$ for all $i \in N$. Hence, $A' = (132)$ indirectly weakly dominates A^*. The intuition behind this, is the following. Agents 1 and 2 can deviate from (1)(23) to (1)(2)(3), knowing that allocation (1)(2)(3) gives each agent his worst choice and thus knowing that all agents want to deviate from it. A possible deviation from (1)(2)(3) is allocation (132).

From Corollary 5.2, we know that each permutation matrix in $X \setminus \{A^*\}$ is indirectly weakly dominated by A^*. Hence, the strong farsighted core of $\mathcal{E}(N,P)$ is empty, i.e. $SFCO = \emptyset$. △

In the following example, we show that if there exists an agent which according to A^* does not receive his top choice, the strong farsighted core does not have to be $\emptyset$.

Example 5.8 (Example 4.31 continued).
Let $n = 3$ and let the preference matrix be as in Example 4.31:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ -1 & -2 & 0 \\ -2 & 0 & -1 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$. We show that $SFCO = \{A^*\}$. From Corollary 5.2, we know that each permutation matrix in $X \setminus \{A^*\}$ is indirectly weakly dominated by A^*. Hence, we need to show that A^* is not indirectly weakly dominated. Suppose to the contrary that A^* is indirectly weakly dominated in $\mathcal{E}(N,P)$. In other words, there exists $A' \in X \setminus \{A^*\}$, such that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A$ and $A^m = A'$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1,\ldots,m\}$, we have that $S^k \in E(A^{k-1}, A^k)$, $A' \succsim_i A^{k-1}$ for all $i \in S^k$ and $\exists\, j \in S^k$ with $A' \succ_j A^{k-1}$. Note that $2^N \setminus \{\emptyset\} = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},N\}$.

Note that according to A^* agents 2 and 3 get their top choice, thus we have that $A^* \succsim_i A$ for all $i \in \{2,3\}$ and for all $A \in X \setminus \{A^*\}$. With the fact that for the nonempty coalition $S^1 \in E(A^*, A^1)$ it must hold that there exists $j \in S^1$ with $A' \succ_j A^*$, we can conclude that $S^1 \notin \{\{2\},\{3\},\{2,3\}\}$. Hence, we have that $S^1 \in \{\{1\},\{1,2\},\{1,3\},N\}$.

Suppose that $S^1 = N$, then it must hold that $A' \succsim_i A^*$ for all $i \in N$. We also have that $A^* \succsim_i A'$ for all $i \in \{2,3\}$. Hence, we get that $A' \sim_i A^*$ for all $i \in \{2,3\}$, i.e. we have that $(23) \in C(A')$. Thus, according to A' agent 1 gets his own item, i.e. $A' = (1)(23) = A^*$. This gives a contradiction with $A' \in X \setminus \{A^*\}$, thus we have that $S^1 \neq N$.

Now, suppose that $S = \{1, 3\}$, then it must hold that $A' \succsim_3 A^*$. Recall that $A^* \succsim_3 A'$, hence we have that $A' \sim_3 A^*$. This means that according to A' agent 3 gets item 2. With the fact that for the nonempty coalition hold that there exists $j \in S^1$ with $A' \succ_j A^*$, we get that it must hold that $A' \succ_1 A^*$. According to A^* agent 1 receives item 1. Note that $P_{11} > P_{13}$ and that $P_{12} > P_{11}$. Thus, according to A' agent 1 should get item 2. This gives a contradiction with the fact that agent 3 should also get item 2, hence we have that $S^1 \neq \{1, 3\}$. Thus, we can conclude that $S^1 \in \{\{1\}, \{1, 2\}\}$.

Suppose that $S^1 = \{1, 2\}$, then it must hold that $A' \succsim_2 A^*$. Recall that $A^* \succsim_2 A'$, hence we have that $A' \sim_2 A^*$. This means that according to A' agent 2 gets item 3. With the fact that there exists $j \in S^1$ with $A' \succ_j A^*$, we get that $A' \succ_1 A^*$. Thus, with the above we get that according to A' agent 1 should get item 2. Hence, we have that $A' = (132)$. In order for this to happen, we must have that there exists $k \in \{1, \ldots, m\}$ such that $3 \in S^k$. Note that S^1 can only move from (1)(23) to (1)(23), (1)(2)(3) and (3)(12). Note that $(1)(23) \succ_3 (132)$, $(1)(2)(3) \succ_3 (132)$ and that $(3)(12) \succ_3 (132)$. Hence, we get that $3 \notin S^2$. Coalition $S \in \{\{1\}, \{2\}, \{1, 2\}\}$ can only move from (1)(2)(3) to (1)(2)(3), or (3)(12). Coalition $S \in \{\{1\}, \{2\}, \{1, 2\}\}$ can only move from (3)(12) to (3)(12), or (1)(2)(3). Hence, we get that $3 \notin S^k$ for all $k \in \{1, \ldots, m\}$. Thus, we can conclude that $S^1 \neq \{1, 2\}$.

Thus, we must have that $S^1 = \{1\}$. Note that agent 1 can only move from (1)(23) to (1)(23) itself. Thus, with the above we get that there does not exist $S^1 \in 2^N \setminus \{\emptyset\}$, such that $A' \succsim_i A^*$ for all $i \in S^1$ and $\exists\ j \in S^1$ with $A' \succ_j A^*$. This shows that A^* is not indirectly weakly dominated. Hence, we have that $SFCO = \{A^*\}$. △

From Example 5.7, we can conclude that the strong farsighted core can be empty. Therefore, we look at another solution concept.

5.3 Weak farsighted von Neumann-Morgenstern stable set

In Section 4, we defined the farsighted vNM stable set. In this section, we look at the farsighted vNM stable set with respect to $\gg$, which we call the weak farsighted vNM stable set.

Definition 5.9 (Weak farsighted von Neumann-Morgenstern stable set).
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **weak farsighted von Neumann-Morgenstern (vNM) stable set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) **internal stability**: $\forall\ A \in \mathcal{A}$ we have that $f_{\gg}(A) \cap \mathcal{A} = \{A\}$,

(2) **external stability**: $\forall\ A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$.

In the following theorem, we show that for all housing matching models (N, P) there exists at least one weak farsighted vNM stable set.

Theorem 5.10. *For all housing matching models (N, P), the set $\mathcal{A} = \{A^*\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. Note that $|\mathcal{A}| = 1$, hence $\mathcal{A}$ automatically satisfies internal stability. From Corollary 5.2, we know that A^* indirectly weakly dominates all other permutation matrices. Hence, we have that $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$. This shows that $\mathcal{A}$ satisfies external stability. Thus, the set $\mathcal{A} = \{A^*\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. □

In Theorem 5.10, we showed that $\{A^*\}$ is a weak farsighted vNM stable set for all housing matching models (N, P). From Corollary 5.2, we know that $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$, hence each set $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$ does not satisfy internal stability. Thus, for all housing matching models (N, P), we have that each weak farsighted vNM stable set of $\mathcal{E}(N, P)$ either is equal to $\{A^*\}$ or it does not contain A^*.

Theorem 5.11. *For all housing matching models (N, P) such that $SFCO = \{A^*\}$, we have that $\mathcal{A} = \{A^*\}$ is the unique weak farsighted vNM stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model such that $SFCO = \{A^*\}$. In Theorem 5.10, we already showed that $\mathcal{A} = \{A^*\}$ is a weak farsighted vNM stable set. Recall that each set $\mathcal{A}' \subseteq X$ with $A^* \in \mathcal{A}$ and $A^* \in \mathcal{A}$ does not satisfy internal stability. Thus, we only need to show that each set $\mathcal{A}'' \subseteq X$ with $A^* \notin \mathcal{A}'$ is not a weak farsighted vNM stable set.

Let $\mathcal{A}'' \subseteq X$ with $A^* \notin \mathcal{A}'$. Note that $SFCO = \{A^*\}$, thus we have that A^* is not indirectly weakly dominated. Hence, it holds that

$$f_{\gg}(A^*) \cap \mathcal{A}'' = \{A^*\} \cap \mathcal{A}'' = \emptyset.$$

Thus, we can conclude that $\mathcal{A}''$ does not satisfy external stability. This shows that the set $\mathcal{A} = \{A^*\}$ is the unique weak farsighted vNM stable set of $\mathcal{E}(N, P)$. □

In the following example, we show for the specific housing matching model in Example 5.7, that there exists a weak farsighted vNM stable set that does not contain A^*.

Example 5.12 (Example 5.7 continued)**.**
Let $n = 3$ and let the preference matrix P be as in Example 5.7:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$ and that $SFCO = \emptyset$. We determine all weak farsighted vNM stable sets of $\mathcal{E}(N, P)$. From Theorem 5.10, we know that $\mathcal{A} = \{A^*\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. First, we show that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both weak farsighted vNM stable sets of $\mathcal{E}(N, P)$ by using Table 13, then we show that $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$ are the only weak farsighted vNM stable sets of $\mathcal{E}(N, P)$.

$A \in X$	Preferences compared to (123)	Preferences compared to (132)
(1)(2)(3)	$(123) \succ_i (1)(2)(3)\ \forall\, i \in N$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in N$
(2)(13)	$(123) \sim_1 (2)(13)$ $(123) \succ_2 (2)(13)$ $(123) \succ_3 (2)(13)$	$(132) \succ_1 (2)(13)$ $(132) \succ_2 (2)(13)$ $(132) \sim_3 (2)(13)$
(3)(12)	$(3)(12) \succ_1 (123)$ $(123) \sim_2 (3)(12)$ $(123) \succ_3 (3)(12)$	$(132) \sim_1 (3)(12)$ $(132) \succ_2 (3)(12)$ $(132) \succ_3 (3)(12)$
(1)(23)	$(123) \succ_1 (1)(23)$ $(1)(23) \succ_2 (123)$ $(123) \sim_3 (1)(23)$	$(132) \succ_1 (1)(23)$ $(132) \sim_2 (1)(23)$ $(1)(23) \succ_3 (132)$
(123)	$(123) \sim_i (123)\ \forall\, i \in N$	$(132) \succ_1 (123)$ $(132) \succ_2 (123)$ $(123) \succ_3 (132)$

Table 13: Illustration of the comparison of the preferences of each $A \in X$ to $A' \in \{(123), (132)\}$.

First, we show that $\mathcal{A}' = \{(132)\}$ is a weak farsighted vNM stable set. Since $|\mathcal{A}'| = 1$, we only need to show that $\mathcal{A}'$ satisfies external stability. In other words, we need to show that $(132) \in f_{\gg}(A)$ for all $A \in X \setminus \{(132)\}$. For all $A \in \{(1)(2)(3), (2)(13), (3)(12)\}$, we can conclude from Table 13 and $N \in E(A, (132))$, that $(132) \ggcurly A$. From Example 5.7, we know that $(132) \ggcurly (1)(23)$. Hence, we only need to show that $(132) \ggcurly (123)$. From Table 13, we know that $(132) \succ_i (123)$ for all $i \in \{1, 2\}$. Hence, agents 1 and 2 can deviate from (123) to (1)(2)(3) knowing that agent 3 is willing to trade such that allocation (132) is constructed. Hence, we have that $(132) \ggcurly (123)$. This shows that $\mathcal{A}' = \{(132)\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. An illustration of the sequences can be found in Table 14.

Sequence	$E(A, A^1)$ and $E(A^1, A')$	Preference
$(1)(2)(3) \xrightarrow[S^1=N]{} (132)$	$N \in E((1)(2)(3), (132))$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in S^1$
$(2)(13) \xrightarrow[S^1=N]{} (132)$	$N \in E((2)(13), (132))$	$(132) \succ_i (2)(13)\ \forall\, i \in \{1, 2\}$ and $(132) \sim_3 (2)(13)$
$(3)(12) \xrightarrow[S^1=N]{} (132)$	$N \in E((3)(12), (132))$	$(132) \sim_1 (3)(12)$ and $(132) \succ_i (3)(12)\ \forall\, i \in \{2, 3\}$
$(1)(23) \xrightarrow[S^1=\{1,2\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (132)$	$\{1, 2\} \in E((1)(23), (1)(2)(3))$	$(132) \succ_1 (1)(23)$ and $(132) \sim_2 (1)(23)$
	$N \in E((1)(2)(3), (132))$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(123) \xrightarrow[S^1=\{1,2\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (132)$	$\{1, 2\} \in E((123), (1)(2)(3))$	$(132) \succ_i (123)\ \forall\, i \in S^1$
	$N \in E((1)(2)(3), (132))$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in S^2$

Table 14: Illustration that each $A \in X \setminus \{(132)\}$ is indirectly weakly dominated by $A' = (132)$.

Now we show that $\mathcal{A}'' = \{(123)\}$ is a weak farsighted vNM stable set. Hence, we need to show that $(123) \in f_{\gg}(A)$ for all $A \in X \setminus \{(123)\}$. Let $A = (1)(2)(3)$, then we have that $N \in E((1)(2)(3), (123))$ and that $(123) \succ_i (1)(2)(3)$ for all $i \in N$. Thus, we have that (123) strictly dominates (1)(2)(3). This gives us that $(123) \gg (1)(2)(3)$. Now, let $A = (2)(13)$, then all agents can deviate from (2)(13) to (123) knowing that all agents weakly prefer (123) to (2)(13). From Table 13, we know that $(123) \sim_1 (2)(13)$ and that $(123) \succ_i (2)(13)$ for all $i \in \{2, 3\}$. With the fact that $N \in E((2)(13), (123))$, we get that (123) weakly dominates (2)(13). This gives us that $(123) \gg (2)(13)$.

Consider $A = (3)(12)$, then agents 2 and 3 can deviate from (3)(12) to (1)(2)(3) knowing that all agents want to deviate from (1)(2)(3) to (123). From Table 13, we know that $(123) \sim_2 (3)(12)$ and that $(123) \succ_3 (3)(12)$. Thus, with $\{2, 3\} \in E((3)(12), (1)(2)(3))$, we have that $(123) \gg (3)(12)$. Now, we show that (123) indirectly weakly dominates $A^* = (1)(23)$. From Table 13, we know that $(123) \succ_1 (1)(23)$, $(1)(23) \succ_2 (123)$ and that $(123) \sim_3 (1)(23)$. Hence, agents 1 and 3 can deviate from (1)(23) to (1)(2)(3) knowing that there is a possibility that all agents deviate from (1)(2)(3) to (123). Hence, we have that $(123) \gg (1)(23)$. Consider $A = (132)$, then agent 3 can deviate from (132) to (1)(2)(3) knowing that there is a possibility that all agents deviate from (1)(2)(3) to (123). Note that $\{3\} \in E((132), (1)(2)(3))$ and that $(123) \succ_3 (132)$. Hence, we have that $(123) \gg (132)$. An illustration of the above can be found in Table 15. Hence, we get that $\mathcal{A}''$ satisfies external stability. Thus, we have that $\mathcal{A}'' = \{(123)\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$.

Sequence	$E(A, A^1)$ and $E(A^1, A')$	Preference
$(1)(2)(3) \xrightarrow[S^1=N]{} (123)$	$N \in E((1)(2)(3), (123))$	$(123) \succ_i (1)(2)(3)\ \forall\, i \in S^1$
$(2)(13) \xrightarrow[S^1=N]{} (123)$	$N \in E((2)(13), (123))$	$(123) \sim_1 (2)(13)$ and $(123) \succ_i (2)(13)\ \forall\, i \in \{2, 3\}$
$(3)(12) \xrightarrow[S^1=\{2,3\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (123)$	$\{2, 3\} \in E((3)(12), (1)(2)(3))$	$(123) \sim_2 (3)(12)$ and $(123) \succ_3 (3)(12)$
	$N \in E((1)(2)(3), (123))$	$(123) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(1)(23) \xrightarrow[S^1=\{1,3\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (123)$	$\{1, 3\} \in E((1)(23), (1)(2)(3))$	$(123) \succ_1 (1)(23)$ and $(123) \sim_3 (1)(23)$
	$N \in E((1)(2)(3), (123))$	$(123) \succ_i (1)(2)(3)\ \forall\, i \in S^2$
$(132) \xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (123)$	$\{3\} \in E((132), (1)(2)(3))$	$(123) \succ_3 (132)$
	$N \in E((1)(2)(3), (123))$	$(123) \succ_i (1)(2)(3)\ \forall\, i \in S^2$

Table 15: Illustration that each $A \in X \setminus \{(123)\}$ is indirectly weakly dominated by $A' = (123)$.

Now we show that $\mathcal{A} = \{A^*\}$, $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are the only weak farsighted vNM stable sets of $\mathcal{E}(N, P)$. We know that $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$, that $(132) \in f_{\gg}(A')$ for all $A' \in X \setminus \{(132)\}$ and that $(123) \in f_{\gg}(A'')$ for all $A'' \in X \setminus \{(123)\}$. Hence, with the internal stability condition we can conclude that each set $\mathcal{A}''' \subseteq X$ with $|\mathcal{A}'''| > 1$ and $\{A^*, (132), (123)\} \cap \mathcal{A}''' \neq \emptyset$ is not a weak farsighted vNM stable set.

Note that $f_{\gg}((1)(2)(3)) = X$, hence with the internal stability condition we get that each set that contains (1)(2)(3) and at least one other permutation matrix is not a weak farsighted vNM stable set. Thus, each set $\mathcal{A}''' \subseteq X$ with $|\mathcal{A}'''| > 1$ and

$$\{A^*, (132), (123), (1)(2)(3)\} \cap \mathcal{A}''' \neq \emptyset$$

is not a weak farsighted vNM stable set. Note that $(123) \succ_i (1)(2)(3)$ for all $i \in N$, thus we have that $(1)(2)(3) \not\gg (123)$. Hence, the set $\{(1)(2)(3)\}$ is not a weak farsighted vNM stable set. Thus, we only need to show that the sets $\{(2)(13)\}$, $\{(3)(12)\}$ and $\{(2)(13), (3)(12)\}$ are not weak farsighted vNM stable sets. Note that $(3)(12) \succ_i (2)(13)$ for all $i \in \{1, 2\}$ and that $\{1, 2\} \in E((2)(13), (3)(12))$. Hence, with the following sequence

$$(2)(13) \xrightarrow[S=\{1,2\}]{} (3)(12),$$

we get that $(3)(12)$ weakly dominates $(2)(13)$. Thus, with internal stability, we get that $\{(2)(13), (3)(12)\}$ is not a weak farsighted vNM stable set. For all $A \in \{(2)(13), (3)(12)\}$, we have that $A \succ_1 (1)(23)$ and that $(1)(23) \succ_i A$ for all $i \in \{2, 3\}$. Thus, we have that $(2)(13) \not\gg (1)(23)$ and that $(3)(12) \not\gg (1)(23)$. This shows that the sets $\{(2)(13)\}$ and $\{(3)(12)\}$ are not weak farsighted vNM stable sets. Hence, the sets $\mathcal{A} = \{A^*\}$, $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are the only weak farsighted vNM stable sets of $\mathcal{E}(N, P)$. △

For the housing matching model (N, P) in Example 5.12, we know that there exist three weak farsighted vNM stable sets of $\mathcal{E}(N, P)$. Hence, there are housing matching models for which there exist multiple weak farsighted vNM stable sets. Therefore, we look at another solution concept.

5.4 Largest consistent set

In the case with indirect weak dominance, the definition of a consistent set differs from the one with indirect dominance. To denote the difference, we give the consistent set with respect to $\gg$ the name consistent $\gg$ set. In order to define a consistent $\gg$ set, we need to have a new definition of deterrence of deviations.

Definition 5.13 ($(\mathcal{A}, \gg)$-deterrence of deviations)**.**
Let (N, P) be a housing matching model, let $\mathcal{A} \subseteq X$ and let $A \in X$. We say that A satisfies **$(\mathcal{A}, \gg)$-deterrence of deviations** if for all $A' \in X$ and for all $S \in E(A, A')$, there exists $A'' \in \mathcal{A}$ such that the following two conditions hold:

(1) either $A'' = A'$ or $A'' \gg A'$,

(2) there exists $i \in S$ with $A \succ_i A''$ or for all $j \in S$ we have that $A \succsim_j A''$.

That $A \in X$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations means that each deviation from A to an arbitrary permutation matrix $A' \in X$ is deterred by the possibility of ending up at a permutation matrix A'' inside $\mathcal{A}$, such that A is strictly preferred for at least one agent in the deviating coalition or A is weakly preferred by all agents in the deviating coalition.

Definition 5.14 (Consistent set with respect to $\gtrsim\!\!\gg$)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ is a **consistent $\underline{\gg}$ set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) if $A \in \mathcal{A}$, then A satisfies $(\mathcal{A}, \underline{\gg})$-deterrence of deviations,

(2) for all $A \notin \mathcal{A}$ we have that A does not satisfy $(\mathcal{A}, \underline{\gg})$-deterrence of deviations.

In the following remark, we give an intuition of $(\mathcal{A}, \underline{\gg})$-deterrence of deviations.

Remark 5.15. Let (N, P) be a housing matching model, let $\mathcal{A} \subseteq X$ and let $A \in X$. Suppose that A does not satisfy $(\mathcal{A}, \underline{\gg})$-deterrence of deviations. Then we have that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that for all $A'' \in \mathcal{A}$, we have that at least one of the following conditions hold:

(1) $A'' \neq A'$ and $A'' \not\underline{\gg} A'$,

(2) $A'' \succsim_i A$ for all $i \in S$ and $\exists\, j \in S$ with $A'' \succ_j A$.

In particular, we have that for all $A'' \in \mathcal{A}$ with $A'' \underline{\gg} A'$, it must hold that $A'' \succsim_i A$ for all $i \in S$ and $\exists\, j \in S$ with $A'' \succ_j A$. Hence, from Lemma 5.3 we get that $A'' \underline{\gg} A$. Also note that for $A'' \in \mathcal{A}$ with $A'' = A'$, it must hold that $A' \succsim_i A$ for all $i \in S$ and that $\exists\, j \in S$ with $A' \succ_j A$. Hence, with Definition 3.1(2), we can conclude that A' weakly dominates A.

Define the **consistent $\underline{\gg}$ correspondence** $g_{\underline{\gg}} : 2^X \to 2^X$ as the correspondence such that

$$g_{\underline{\gg}}(\mathcal{A}) = \{A \in X \mid A \text{ satisfies } (\mathcal{A}, \underline{\gg})\text{-deterrence of deviations}\}.$$

Then we have that $\mathcal{A}$ is a consistent $\underline{\gg}$ set if and only if $\mathcal{A} = g_{\underline{\gg}}(\mathcal{A})$. For all housing matching models (N, P), note that $\emptyset$ is a consistent $\underline{\gg}$ set of $\mathcal{E}(N, P)$.

Definition 5.16 (Largest consistent set with respect to $\underline{\gg}$)**.**
Let (N, P) be a housing matching model. The **largest consistent $\underline{\gg}$ set** of $\mathcal{E}(N, P)$, denoted by $\mathbb{A}_{\underline{\gg}}$, is the union of all consistent $\underline{\gg}$ sets:

$$\mathbb{A}_{\underline{\gg}} = \bigcup_{\mathcal{A} = g_{\underline{\gg}}(\mathcal{A})} \mathcal{A}.$$

Remark 5.17. If we replace $\gg$ by $\underline{\gg}$ the proof of Lemma 4.24 remains valid, thus we have that the definition of $\mathbb{A}_{\underline{\gg}}$ and the definition given in Chwe (1994) with respect to $\underline{\gg}$ are equivalent. Thus, we have that

$$\mathbb{A}_{\underline{\gg}} = \bigcup_{\mathcal{A} = g_{\underline{\gg}}(\mathcal{A})} \mathcal{A} = \bigcup_{\mathcal{A} \subseteq g_{\underline{\gg}}(\mathcal{A})} \mathcal{A}.$$

In Section 4, we showed that $\{A^*\}$ is a consistent set. In the following theorem, we show that $\{A^*\}$ is a consistent $\underline{\gg}$ set.

Theorem 5.18. *For all housing matching models (N, P), the set $\mathcal{A} = \{A^*\}$ is a consistent $\underline{\gg}$ set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} = \{A^*\}$. First, we show that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations. In other words, we need to show that for all $A' \in X$ and for all $S \in E(A^*, A')$, we have that either $A^* = A'$ or $A^* \gg A'$, and that $\exists\, i \in S$ with $A^* \succ_i A^*$ or for all $j \in S$ we have that $A^* \succsim_j A^*$. Note that $A^* \sim_j A^*$ for all $j \in N$. Hence, we need to show that for all $A' \in X$ and for all $S \in E(A^*, A')$, either $A^* = A'$ or $A^* \gg A'$. From Corollary 5.2, we know that $A^* \gg A'$ for all $A' \in X \setminus \{A^*\}$. This shows that A^* satisfies $(\mathcal{A}, \gg)$-deterrence of deviations.

Now we show that each $A \in X \setminus \{A^*\}$ does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations. Let $A \in X \setminus \{A^*\}$, then we need to show that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $A^* \neq A'$ and $A^* \not\gg A'$, or such that $\forall\, i \in S$ we have that $A^* \succsim_i A$ and $\exists\, j \in S$ with $A^* \succ_j A$. Since A^* indirectly weakly dominates all other permutation matrices, we need to show that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that $\forall\, i \in S$ we have that $A^* \succsim_i A$ and $\exists\, j \in S$ with $A^* \succ_j A$.

From Theorem 3.12, we know that there exists $A' \in X$ such that $A \in X \setminus \{A^*\}$ is weakly dominated by A' in $\mathcal{E}(N, P)$ for coalition $S = \bigcup_{1 \leq r \leq s} S(tc^r) \in E(A, A')$, with $s \in \{1, \dots, T\}$ the first index such that $tc^s \notin C(A)$ and with $tc^1, \dots, tc^s$ in the cycle decomposition of A'. In particular, we know from Theorem 3.12, that $A' \sim_i A$ for all $i \in \bigcup_{1 \leq r \leq s} S(tc^r)$ and that $A' \succ_j A$ for at least one $j \in S(tc^s)$. Recall that $C(A^*) = \{tc^1, \dots, tc^T\}$ and that $\{tc^1, \dots, tc^s\} \in C(A')$, thus we have that $A^* \sim_k A'$ for all $k \in S$. Hence, with the above we get that $\forall\, i \in S$ it holds that $A^* \succsim_i A$ and $\exists\, j \in S$ with $A^* \succ_j A$. Hence, we showed that A does not satisfy $(\mathcal{A}, \gg)$-deterrence of deviations.

Thus, we can conclude that $\mathcal{A} = \{A^*\}$ is a consistent $\gg$ set of $\mathcal{E}(N, P)$. $\square$

From Theorem 5.18, we get the following result.

Corollary 5.19. *For all housing matching models (N, P), we have that $A^* \in \mathbb{A}_{\gg}$.*

In the following example, we show that there is a housing matching model, for which there exists a consistent $\gg$ set that does not contain A^*.

Example 5.20 (Example 5.12 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 5.12:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

We show that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both consistent $\gg$ sets of $\mathcal{E}(N, P)$.

First, we show that $\mathcal{A}' = \{(132)\}$ is a consistent $\gg$ set. Hence, we need to show that (132) satisfies $(\mathcal{A}', \gg)$-deterrence of deviations and that each $A \in X \setminus \{(132)\}$ does not satisfy $(\mathcal{A}', \gg)$-deterrence of deviations. From Example 5.12, we know that each $A' \in X \setminus \{(132)\}$ is indirectly weakly dominated by (132). Hence, for each $A' \in X \setminus \{(132)\}$, let $A'' = (132) \in \mathcal{A}'$, then we have that $A'' \gg A'$ and that $A'' \sim_j (132)$ for all $j \in N$. Thus, we can conclude that (132) satisfies $(\mathcal{A}', \gg)$-deterrence of deviations.

Now, we show that each $A \in X \setminus \{(132)\}$ does not satisfy $(\mathcal{A}', \gg)$-deterrence of deviations. Hence, we need to show that for each $A \in X \setminus \{(132)\}$, $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $(132) \neq A'$ and $(132) \not\gg A'$, or such that $\forall\, i \in S$ we have that $(132) \succsim_i A$ and $\exists\, j \in S$ with $(132) \succ_j A$. Since (132) indirectly weakly dominates all other permutation matrices, we need to show for each $A \in X \setminus \{(132)\}$, that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that $\forall\, i \in S$ we have that $(132) \succsim_i A$ and $\exists\, j \in S$ with $(132) \succ_j A$. From Table 13 in Example 5.12, we get Table 16, which shows that each $A \in X \setminus \{(132)\}$ does not satisfy $(\mathcal{A}', \gg)$-deterrence of deviations. Thus, we have that $\mathcal{A}'$ is a consistent $\gg$ set of $\mathcal{E}(N, P)$.

$A \in X \setminus \{(132)\}$	$A' \in X$	$S \in E(A, A')$	$\forall\, i \in S$ $(132) \succsim_i A$ and $\exists\, j \in S$ with $(132) \succ_j A$
(1)(2)(3)	(132)	$N \in E((1)(2)(3), (132))$	$(132) \succ_i (1)(2)(3)\ \forall\, i \in N$
(2)(13)	(132)	$N \in E((2)(13), (132))$	$(132) \succ_i (2)(13)\ \forall\, i \in \{1, 2\}$ and $(132) \sim_3 (2)(13)$
(3)(12)	(132)	$N \in E((3)(12), (132))$	$(132) \sim_1 (3)(12)$ and $(132) \succ_i (3)(12)\ \forall\, i \in \{2, 3\}$
(1)(23)	(3)(12)	$\{1, 2\} \in E((1)(23), (3)(12))$	$(132) \succ_1 (1)(23)$ and $(132) \sim_2 (1)(23)$
(123)	(3)(12)	$\{1, 2\} \in E((123), (3)(12))$	$(132) \succ_i (123)\ \forall\, i \in \{1, 2\}$

Table 16: Illustration that each $A \in X \setminus \{(132)\}$ does not satisfy $(\mathcal{A}', \gg)$-deterrence of deviations.

Now we show that $\mathcal{A}'' = \{(123)\}$ is a consistent $\gg$ set. First, we show that (123) satisfies $(\mathcal{A}'', \gg)$-deterrence of deviations. Recall from Example 5.12, that each $A' \in X \setminus \{(123)\}$ is indirectly weakly dominated by (123). Hence, for each $A' \in X \setminus \{(123)\}$, let $A'' = (123) \in \mathcal{A}'$, then we have that $A'' \gg A'$ and that $A'' \sim_j (123)$ for all $j \in N$. Now, we show that each $A \in X \setminus \{(123)\}$ does not satisfy $(\mathcal{A}'', \gg)$-deterrence of deviations. Since $(123) \gg A'$ for all $A' \in X \setminus \{(123)\}$, we need to show for each $A \in X \setminus \{(123)\}$, that there exists $A' \in X$ and that there exists $S \in E(A, A')$, such that $\forall\, i \in S$ we have that $(123) \succsim_i A$ and $\exists\, j \in S$ with $(123) \succ_j A$. Note that with Table 13 in Example 5.12, we get Table 17, which shows that each $A \in X \setminus \{(123)\}$ does not satisfy $(\mathcal{A}'', \gg)$-deterrence of deviations. Thus, we have that $\mathcal{A}'' = \{(123)\}$ is a consistent $\gg$ set of $\mathcal{E}(N, P)$.

$A \in X \setminus \{(123)\}$	$A' \in X$	$S \in E(A, A')$	$\forall\, i \in S$ $(123) \succsim_i A$ and $\exists\, j \in S$ with $(123) \succ_j A$
(1)(2)(3)	(123)	$N \in E((1)(2)(3), (123))$	$(123) \succ_i (1)(2)(3)\ \forall\, i \in N$
(2)(13)	(123)	$N \in E((2)(13), (123))$	$(123) \sim_1 (2)(13)$ and $(123) \succ_i (2)(13)\ \forall\, i \in \{2, 3\}$
(3)(12)	(1)(23)	$\{2, 3\} \in E((3)(12), (1)(23))$	$(123) \sim_2 (3)(12)$ $(123) \succ_3 (3)(12)$
(1)(23)	(2)(13)	$\{1, 3\} \in E((1)(23), (2)(13))$	$(123) \succ_1 (1)(23)$ and $(123) \sim_3 (1)(23)$
(132)	(1)(2)(3)	$\{3\} \in E((132), (1)(2)(3))$	$(123) \succ_3 (132)$

Table 17: Illustration that each $A \in X \setminus \{(123)\}$ does not satisfy $(\mathcal{A}'', \gg)$-deterrence of deviations.

Thus, we can conclude that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both consistent $\gg$ sets of $\mathcal{E}(N, P)$. $\triangle$

For the housing matching model in Example 5.20, we showed that there exist at least two consistent $\gg$ sets that do not contain A^*. In the following example, we determine the largest consistent set for this housing matching model.

Example 5.21 (Example 5.20 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 5.20:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

We show that $\mathbb{A}_{\gg} = \{(1)(23), (132), (123)\}$. Recall that $\mathbb{A}_{\gg}$ is the union of all consistent $\gg$ sets, i.e.

$$\mathbb{A}_{\gg} = \bigcup_{\mathcal{A}=g_{\gg}(\mathcal{A})} \mathcal{A}.$$

From Theorem 5.18, we know that $\{A^*\}$ is a consistent $\gg$ set of $\mathcal{E}(N, P)$. From Example 5.20, we know that $\{(132)\}$ and that $\{(123)\}$ are both consistent $\gg$ sets of $\mathcal{E}(N, P)$. Thus, we have that $\{(1)(23), (132), (123)\} \subseteq \mathbb{A}_{\gg}$.

Now, we show that for each $A \in \{(1)(2)(3), (2)(13), (3)(12)\}$ it holds that $A \notin \mathbb{A}_{\gg}$. Note that $g_{\gg} : 2^X \to 2^X$ is isotonic, which means that if $\mathcal{A}' \subseteq \mathcal{A}$, then it holds that $g_{\gg}(\mathcal{A}') \subseteq g_{\gg}(\mathcal{A})$. Thus, in particular for all sets $\mathcal{A} \subseteq X$ that are consistent $\gg$ sets of $\mathcal{E}(N, P)$, it holds that $\mathcal{A} = g_{\gg}(\mathcal{A}) \subseteq g_{\gg}(X)$. Thus, we have that $\{(1)(23), (132), (123)\} \subseteq g_{\gg}(X)$. We show that

$$g_{\gg}(X) \cap \{(1)(2)(3), (2)(13), (3)(12)\} = \emptyset.$$

Note that

$$g_{\gg}(X) = \{A \in X \mid A \text{ satisfies } (X, \gg)\text{-deterrence of deviations}\}.$$

Thus, we show that each $A \in \{(1)(2)(3), (2)(13), (3)(12)\}$ does not satisfy $(X, \gg)$-deterrence of deviations. Let $A = (1)(2)(3)$, $A' = (1)(23)$ and $S = \{2, 3\} \in E(A, A')$, then for each $A'' \in \{(1)(23), (132), (123)\}$ we have that $A'' \succ_i A$ for all $i \in S$ and for each $A'' \in \{(1)(2)(3), (2)(13), (3)(12)\}$ we have that $A'' \neq A'$ and $A'' \not\gg A'$. Thus, $(1)(2)(3)$ does not satisfy $(X, \gg)$-deterrence of deviations.

Let $A = (2)(13)$, $A' = (1)(23)$ and $S = \{2, 3\} \in E(A, A')$, then for $A'' \in \{(1)(23), (123)\}$ we have that $A'' \succ_i A$ for all $i \in S$, for $A'' = (132)$ we have that $A'' \succ_2 A$ and $A'' \sim_3 A$ and for each $A'' \in \{(1)(2)(3), (2)(13), (3)(12)\}$ we have that $A'' \neq A'$ and $A'' \not\gg A'$. Hence, $(2)(13)$ does not satisfy $(X, \gg)$-deterrence of deviations.

Let $A = (3)(12)$, $A' = (1)(23)$ and $S = \{2, 3\} \in E(A, A')$, then for $A'' \in \{(1)(23), (132)\}$ we have that $A'' \succ_i A$ for all $i \in S$, for $A'' = (123)$ we have that $A'' \sim_2 A$ and $A'' \succ_3 A$ and for each $A'' \in \{(1)(2)(3), (2)(13), (3)(12)\}$ we have that $A'' \neq A'$ and $A'' \not\gg A'$. Hence, $(3)(12)$ does not satisfy $(X, \gg)$-deterrence of deviations. Thus, we have that

$$g_{\gg}(X) = \{(1)(23), (132), (123)\}.$$

Thus, we can conclude that for all $\mathcal{A} \subseteq X$, which are consistent $\gg$ sets, we must have that $\mathcal{A} \subseteq \{(1)(23), (132), (123)\}$. This shows that $\{(1)(2)(3), (2)(13), (3)(12)\} \cap \mathbb{A}_{\gg} = \emptyset$. Thus, we have that $\mathbb{A}_{\gg} = \{(1)(23), (132), (123)\}$. △

For the housing matching model (N, P) as in Example 5.21, we showed in Example 5.12 that $\{(1)(23)\}$, $\{(132)\}$, $\{(123)\}$ are the only weak farsighted vNM stable sets and in Example 5.21, we showed that $\mathbb{A}_{\gg} = \{(1)(23), (132), (123)\}$. Thus, for this specific housing matching model, we have that each weak farsighted vNM stable set is a subset of the largest consistent $\gg$ set. This result can be generalized to all housing matching models. For a general game with strict preferences, Chwe (1994) proved this with respect to indirect dominance. We give this proof in our context and with respect to indirect weak dominance instead of indirect dominance.

Proposition 5.22. *For all housing matching models (N, P), the following holds. If $\mathcal{A}' \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A}' \subseteq \mathbb{A}_{\gg}$.*

Proof. Let (N, P) be a housing matching model. Suppose that $\mathcal{A}' \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. From Remark 5.17, we know that

$$\mathbb{A}_{\gg} = \bigcup_{\mathcal{A} = g_{\gg}(\mathcal{A})} \mathcal{A} = \bigcup_{\mathcal{A} \subseteq g_{\gg}(\mathcal{A})} \mathcal{A}.$$

We need to show that $\mathcal{A}' \subseteq \mathbb{A}_{\gg}$. Hence, with the above it is sufficient to show that $\mathcal{A}' \subseteq g_{\gg}(\mathcal{A}')$. Suppose to the contrary that $A \in \mathcal{A}' \setminus g_{\gg}(\mathcal{A}')$. Since $A \notin g_{\gg}(\mathcal{A}')$, we have that there exists $A' \in X$ and $S \in E(A, A')$, such that for all $A'' \in \mathcal{A}'$, we have that $A'' \neq A'$ and $A'' \not\gg A'$, or we have that $A'' \succsim_i A$ for all $i \in S$ and there exists $j \in S$ with $A'' \succ_j A$. Hence, in particular for all $A'' \in \mathcal{A}'$ with either $A'' = A'$ or $A'' \gg A'$, it must hold that $A'' \succsim_i A$ for all $i \in S$ and there exists $j \in S$ with $A'' \succ_j A$.

We have two cases $A' \in \mathcal{A}'$ and $A' \notin \mathcal{A}'$. First, suppose that $A' \in \mathcal{A}'$, then with the above we get that $A' \succsim_i A$ for all $i \in S$ and there exists $j \in S$ with $A' \succ_j A$. With Definition 3.1(2), we get that A' weakly dominates A for coalition $S \in E(A, A')$. Hence, we get that $A' \gg A$. Thus, we have that $A' \in f_{\gg}(A) \cap \mathcal{A}'$. This gives a contradiction with the internal stability of $\mathcal{A}'$.

Now, suppose that $A' \notin \mathcal{A}'$, then from the external stability of $\mathcal{A}'$ we get that there exists $A'' \in \mathcal{A}'$ such that $A'' \gg A$. Hence, with the above we get that $A'' \succsim_i A$ for all $i \in S$ and there exists $j \in S$ with $A'' \succ_j A$. From Lemma 5.3, we get that $A'' \gg A$. Thus, we have that $A'' \in f_{\gg}(A) \cap \mathcal{A}'$. This gives a contradiction with the internal stability of $\mathcal{A}'$.

Thus, we can conclude that $\mathcal{A}' \subseteq g_{\gg}(\mathcal{A}')$. Hence, we can conclude that if $\mathcal{A}' \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A}' \subseteq \mathbb{A}_{\gg}$. □

5.5 Weak DEM farsighted stable set

In Section 4, we showed for all housing matching models (N, P) that $\mathcal{A} = \{A^*\}$ is the unique DEM farsighted stable set of $\mathcal{E}(N, P)$. In this section, we show that there exists a housing matching model for which there are multiple DEM farsighted stable sets with respect to $\gg$.

In the context of coalition formation games, Herings et al. (2010) defined a farsightedly stable set. We give it the name weak DEM farsighted stable set.

Definition 5.23 (Weak DEM farsighted stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ of permutation matrices is a **weak DEM farsighted stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three properties:

(1) **deterrence of external deviations**: $\forall\, A \in \mathcal{A}$, $\forall\, A' \notin \mathcal{A}$ and $\forall\, S \in E(A, A')$, there exists $A'' \in \mathcal{A} \cap f_{\gtrsim\!\!\!>}(A')$ such that we have that $\exists\, i \in S$ with $A \succ_i A''$ or that $A \succsim_j A''$ for all $j \in S$,

(2) **external stability**: $\forall\, A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ that satisfies (1) and (2).

Note that external stability implies that for all housing matching models (N, P) we have that $\emptyset$ is not a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. Hence, each set $\mathcal{A} \subseteq X$ with $|\mathcal{A}| = 1$ satisfies minimality. Like in Section 4, there is a relation between $(\mathcal{A}, \gg)$-deterrence of deviations and deterrence of external deviations.

Remark 5.24. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$. Suppose that $A \in \mathcal{A}$ satisfies $(\mathcal{A}, \gg)$-deterrence of deviations, then we have that $A \in \mathcal{A}$ satisfies deterrence of external deviations.

Since from Corollary 5.2, we know that A^* indirectly weakly dominates all other permutation matrices, we have that $\{A^*\}$ is a weak DEM farsighted stable set for all housing matching models.

Theorem 5.25. *For all housing matching models (N, P), the set $\mathcal{A} = \{A^*\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. We show that $\mathcal{A} = \{A^*\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. Since $|\mathcal{A}| = 1$, we have that the set $\mathcal{A}$ satisfies minimality. First, we show that $\mathcal{A}$ satisfies deterrence of external deviations. In other words, we need to show that $\forall\, A' \in X \setminus \{A^*\}$ and $\forall\, S \in E(A^*, A')$, it holds that $A^* \gg A'$ and that $\exists\, i \in S$ with $A^* \succ_i A^*$ or that $A^* \succsim_j A^*$ for all $j \in S$. Note that $A^* \sim_j A^*$ for all $j \in N$. Thus, we need to show that $\forall\, A' \in X \setminus \{A^*\}$ it holds that $A^* \gg A'$. This follows from Corollary 5.2, hence $\mathcal{A}$ satisfies deterrence of external deviations. From Corollary 5.2, we also know that $A^* \in f_{\gg}(A)$ for all $A \in X \setminus \{A^*\}$. Hence, $\mathcal{A}$ satisfies external stability. Thus, the set $\mathcal{A} = \{A^*\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. □

From Theorem 5.25 and the minimality condition, we get the following result.

Corollary 5.26. *For all housing matching models (N, P), we have that each set $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$ is not a weak DEM farsighted stable set of $\mathcal{E}(N, P)$.*

For all housing matching models for which A^* is not indirectly weakly dominated, we have that there exists a unique weak DEM farsighted stable set.

Theorem 5.27. *For all housing matching models (N, P) such that $SFCO = \{A^*\}$, we have that $\mathcal{A} = \{A^*\}$ is the unique weak DEM farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model such that $SFCO = \{A^*\}$. In Theorem 5.25, we already showed that $\mathcal{A} = \{A^*\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. From Corollary 5.26, we know that each set $\mathcal{A}' \subseteq X$ with $A^* \in \mathcal{A}$ and $A^* \in \mathcal{A}$ is not a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. Hence, it is sufficient to show that each set $\mathcal{A}'' \subseteq X$ with $A^* \notin \mathcal{A}'$ is not a weak DEM farsighted stable set.

Let $\mathcal{A}'' \subseteq X$ with $A^* \notin \mathcal{A}'$. Note that $SFCO = \{A^*\}$, i.e. A^* is not indirectly weakly dominated. Thus, we have that

$$f_{\gg}(A^*) \cap \mathcal{A}'' = \{A^*\} \cap \mathcal{A}'' = \emptyset.$$

Hence, we can conclude that $\mathcal{A}''$ does not satisfy external stability. This shows that the set $\mathcal{A} = \{A^*\}$ is the unique weak DEM farsighted stable set of $\mathcal{E}(N, P)$. □

In the following example, we show that there exists a housing matching model for which there are multiple weak DEM farsighted stable sets.

Example 5.28 (Example 5.21 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 5.21:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$ and that $SFCO = \emptyset$. We show that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both weak DEM farsighted stable sets of $\mathcal{E}(N, P)$. Since $|\mathcal{A}'| = |\mathcal{A}''| = 1$, we have that both sets satisfy minimality. In Example 5.12, we showed that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both weak farsighted vNM stable sets of $\mathcal{E}(N, P)$. Thus, we have that $(132) \in f_{\gg}(A')$ for all $A' \in X \setminus \{(132)\}$ and that $(123) \in f_{\gg}(A'')$ for all $A'' \in X \setminus \{(123)\}$. Hence, both sets satisfy external stability.

In Example 5.20, we showed that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both consistent $\gg$ sets. Thus, we have that (132) satisfies $(\mathcal{A}', \gg)$-deterrence of deviations and that (123) satisfies $(\mathcal{A}'', \gg)$-deterrence of deviations. Hence, with Remark 5.24, we get that $\mathcal{A}'$ and $\mathcal{A}''$ satisfy deterrence of external deviations. This show that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both weak DEM farsighted stable sets of $\mathcal{E}(N, P)$. △

The result in Example 5.28 that each weak farsighted vNM stable set is a weak DEM farsighted stable set, can be generalized to all housing matching models. In Herings et al. (2010), this is shown in the context of coalition formation games. We give this proof in the context of our social environment corresponding to the housing matching model (N, P).

Proposition 5.29. *For all housing matching models (N, P), the following holds. If $\mathcal{A} \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. Suppose that $\mathcal{A} \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A}$ satisfies internal stability, $f_{\gg}(A) \cap \mathcal{A} = \{A\}$ for all $A \in \mathcal{A}$, and that $\mathcal{A}$ satisfies external stability, $\forall\ A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$. Hence, condition (2) in Definition 5.23 is automatically satisfied.

Suppose that $\mathcal{A}$ does not satisfy deterrence of deviations. Thus, we have that there exist $A \in \mathcal{A}$, $A' \notin \mathcal{A}$ and $S \in E(A, A')$, such that $\forall\ A'' \in \mathcal{A} \cap f_{\gg}(A')$ it holds that $A'' \succsim_i A$ for all $i \in S$ and $A'' \succ_j A$ for at least one $j \in S$. Since $A' \notin \mathcal{A}$ and $\mathcal{A}$ is a weak farsighted vNM stable set, we know that there exists $A'' \in \mathcal{A}$ such that $A'' \gg A'$. Hence, from the above we get that $A'' \succsim_i A$ for all $i \in S$ and $A'' \succ_j A$ for at least one $j \in S$. From Lemma 5.3, we can conclude that $A'' \gg A$, i.e. $A'' \in f_{\gg}(A)$. This violates the internal stability of $\mathcal{A}$. Thus, the set $\mathcal{A}$ satisfies deterrence of deviations.

Suppose that $\mathcal{A}$ does not satisfy minimality. Hence, there exists a proper subset $\mathcal{A}' \subsetneq \mathcal{A}$ that satisfies conditions (1) and (2) in Definition 5.23. Let $A \in \mathcal{A} \setminus \mathcal{A}'$, then since $\mathcal{A}$ satisfies internal stability we have that

$$f_{\gg}(A) \cap \mathcal{A}' \subseteq f_{\gg}(A) \cap \mathcal{A} = \{A\}.$$

Since $A \in \mathcal{A} \setminus \mathcal{A}'$, we get that $f_{\gg}(A) \cap \mathcal{A}' = \emptyset$. This violates the external stability of $\mathcal{A}'$. Thus, $\mathcal{A}$ satisfies minimality.

Thus, $\mathcal{A}$ satisfies deterrence of external deviations, external stability and minimality. Hence, we have that $\mathcal{A}$ is a weak DEM farsighted stable set.

Thus, if $\mathcal{A} \subseteq X$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$, then we have that $\mathcal{A}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. □

Note that a weak DEM farsighted stable set is not necessarily a weak farsighted vNM stable set, since a weak DEM farsighted stable set does not require the internal stability condition as in Definition 5.9. Since the internal stability condition is automatically satisfied for a set that consists of one element, we have the following result, which is proved in Herings et al. (2010).

Proposition 5.30. *For all housing matching models (N, P), the following holds. The set $\mathcal{A} = \{A\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$ if and only if $\mathcal{A} = \{A\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. Suppose that $\mathcal{A} = \{A\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$, then we have that condition (2) in Definition 5.9 is automatically satisfied. Hence, since $|\mathcal{A}| = 1$, we have that $\mathcal{A}$ is a weak farsighted vNM stable set.

Suppose that $\mathcal{A} = \{A\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. Then condition (2) and (3) in Definition 5.23 are automatically satisfied. Since $\mathcal{A} = \{A\}$ satisfies external stability, we have that $A \in f_{\gg}(A')$ for all $A' \in X \setminus \{A\}$. Hence, $\mathcal{A}$ also satisfies deterrence of external deviations, since for all $A' \notin \mathcal{A}$ and $S \in E(A, A')$ we can take $A \in \mathcal{A} \cap f_{\gg}(A)$ with $A \sim_j A$ for all $j \in N$. This shows that $\mathcal{A} = \{A\}$ is a weak DEM farsighted stable set.

Thus, the set $\mathcal{A} = \{A\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$ if and only if $\mathcal{A} = \{A\}$ is a weak farsighted vNM stable set of $\mathcal{E}(N, P)$. □

In Example 5.28, we showed for a specific housing matching model with $SFCO = \emptyset$, that there are multiple weak DEM farsighted stable sets. This result can be generalized to all housing matching models (N, P) such that $SFCO = \emptyset$.

In the context of coalition formation games, Herings et al. (2010) proved that a set is the unique weak DEM farsighted stable set if and only if the following two conditions are satisfied: it consists of all the outcomes that are not indirectly weakly dominated and each outcome outside the set is indirectly weakly dominated by an outcome inside the set. Hence, if there does not exist an outcome which is not indirectly weakly dominated, then there are multiple DEM farsighted stable sets. We use a part of their proof and we rewrite it in the context of our social environment corresponding to a housing matching model.

Theorem 5.31. *For all housing matching models (N, P) such that $SFCO = \emptyset$, we have that there are multiple weak DEM farsighted stable sets of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model such that $SFCO = \emptyset$. In Theorem 5.25, we already showed that $\mathcal{A} = \{A^*\}$ is a weak DEM farsighted stable set of $\mathcal{E}(N, P)$. Hence, we only need to show that there exists at least one other weak DEM farsighted stable set of $\mathcal{E}(N, P)$.

Since $SFCO = \emptyset$, we know that there exists $A' \in X \setminus \{A^*\}$ such that $A' \ggg A^*$. Define $\mathcal{A}' \subseteq X$ as

$$\mathcal{A}' = \{A'\} \cup \{A \in X \mid A' \notin f_{\ggg}(A)\}.$$

Note that $A^* \notin \mathcal{A}'$, since $A' \ggg A^*$. Thus, in particular, we have that $\mathcal{A}' \neq \{A^*\}$. We show that $\mathcal{A}'$ satisfies deterrence of external deviations and external stability. Note that by construction of $\mathcal{A}'$ we have for any $A \notin \mathcal{A}'$ that $A' \in f_{\ggg}(A)$. Thus, with $A' \in \mathcal{A}'$, we get that $\mathcal{A}'$ satisfies external stability.

Now, we show that $\mathcal{A}'$ satisfies deterrence of external deviations. Hence, we need to show that $\forall\ A \in \mathcal{A}'$, $\forall\ B \notin \mathcal{A}$ and $\forall\ S \in E(A, B)$, there exists $C \in \mathcal{A} \cap f_{\ggg}(B)$ such that we have that $\exists\ i \in S$ with $A \succ_i C$ or that $A \succsim_j C$ for all $j \in S$.

Suppose that $A = A' \in \mathcal{A}'$. Then for all $B \notin \mathcal{A}$ we know that $A' \in \mathcal{A} \cap f_{\ggg}(B)$. Hence, by taking $C = A'$ each deviation from A' to any $B \notin \mathcal{A}'$ is deterred. Now, suppose that $A \in \mathcal{A}' \setminus \{A'\}$, then by construction of $\mathcal{A}'$ we have that $A' \notin f_{\ggg}(A)$. Suppose to the contrary that there exist $B \notin \mathcal{A}$ and $S \in E(A, B)$, such that for all $C \in \mathcal{A} \cap f_{\ggg}(B)$ we have that $C \succsim_i A$ for all $i \in S$ and $C \succ_j A$ for at least one $j \in S$. From Lemma 5.3, we can conclude that $\mathcal{A} \cap f_{\ggg}(B) \subseteq f_{\ggg}(A)$. Since $B \notin \mathcal{A}'$, we have by construction of $\mathcal{A}'$ that $A' \in f_{\ggg}(B)$. Hence, we get that $A' \in \mathcal{A} \cap f_{\ggg}(B) \subseteq f_{\ggg}(A)$, this contradicts the fact that $A' \notin f_{\ggg}(A)$. Thus, we can conclude that each deviation from $A \in \mathcal{A}' \setminus \{A'\}$ to any $B \notin \mathcal{A}'$ is deterred. Hence, $\mathcal{A}'$ satisfies deterrence of external deviations.

We have two cases: $\mathcal{A}'$ satisfies minimality and $\mathcal{A}'$ does not satisfy minimality. In the former case we get with the above that $\mathcal{A}'$ is a weak DEM farsighted stable set. Hence, with $\mathcal{A}' \neq \{A^*\}$, we showed that there are multiple weak DEM farsighted stable sets. In the latter case, we know from Herings et al. (2010) with the fact that the cardinality of $\mathcal{A}'$ is finite, that there is a proper subset $\mathcal{A}''' \subsetneq \mathcal{A}'$ that satisfies conditions (1), (2) and (3) in Definition 5.23. Hence, with $A^* \notin \mathcal{A}'$, we get that there are multiple weak DEM farsighted stable sets.

We can conclude for all housing matching models (N, P) such that $SFCO = \emptyset$, that there are multiple weak DEM farsighted stable sets of $\mathcal{E}(N, P)$. □

6 Full Farsightedness with Antisymmetric Weak Dominance

In Section 5, we had the following results for the housing matching model (N, P) as in Example 5.28: the strong farsighted core is empty, there does not exist a unique weak farsighted vNM stable set, the largest consistent set contains a lot of permutation matrices and there are multiple weak DEM farsighted stable sets. Therefore, in this section, we study these stability concepts with respect to a slightly different definition of indirect weak dominance, which compared to indirect weak dominance has one additional restriction.

6.1 Indirect antisymmetric weak dominance

In Kawasaki (2010), indirect antisymmetric weak dominance is defined in the context of the housing matching model of Shapley and Scarf (1974) with the same effectivity correspondence as in Definition 2.8. We give this definition in the context of our social environment corresponding to the housing matching model (N, P) and we denote it by $\gg_a$.

Definition 6.1 (Indirect antisymmetric weak dominance).
Let (N, P) be a housing matching model. Let $A, A' \in X$ be two different permutation matrices. The permutation matrix A' **indirectly antisymmetrically weakly dominates** A in $\mathcal{E}(N, P)$, denoted by $A' \gg_a A$, if there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A$ and $A^m = A'$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \ldots, m\}$ the following three conditions hold:

(1) $S^k \in E(A^{k-1}, A^k)$,

(2) $A' \succsim_i A^{k-1}$ for all $i \in S^k$ and $A' \succ_j A^{k-1}$ for at least one $j \in S^k$,

(3) if $A' \sim_i A^{k-1}$ for some $i \in S^k$, then we have that $A_i^{k-1} = A_i^k$.

Note that the difference between indirect antisymmetric weak dominance and indirect weak dominance is that indirect antisymmetric weak dominance has one additional restriction, namely condition (3) in Definition 6.1. Hence, indirect antisymmetric weak dominance implies indirect weak dominance. Condition (3) says that if agent $i \in S^k$ is indifferent between A^{k-1} and A', i.e. he gets the same item according to A^{k-1} as to A', then he never wants another item. Hence, he only deviates from A^{k-1} to A^k if he gets the same item according to A^k as according to A^{k-1}. In the following example, we show the difference between $\gg$ and $\gg_a$.

Example 6.2 (Example 5.28 continued).
Let $n = 3$ and let the preference matrix P be as in Example 5.28:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$ and that in Example 5.7, we showed that $(132) \gg (1)(23)$ with the following sequence:

$$(1)(23) \xrightarrow[S^1=\{1,2\}]{} (1)(2)(3) \xrightarrow[S^2=N]{} (132).$$

We show that $(132) \not\gg_a (1)(23)$. Suppose to the contrary that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = (1)(23)$ and $A^m = (132)$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\, k \in \{1, \ldots, m\}$ the three conditions in Definition 6.1 hold.

For the nonempty coalition S^1, it must hold that $(132) \succsim_i (1)(23)$ for all $i \in S^1$ and that there exists $j \in S^1$ such that $(132) \succ_j (1)(23)$. Note that $(1)(23) \succ_3 (132)$, hence we have that $3 \notin S^1$. Thus, we have that $S^1 \in \{\{1\}, \{2\}, \{1,2\}\}$. Note that $(132) \succ_1 (1)(23)$ and that $(132) \sim_2 (1)(23)$, hence we have that $S^1 \in \{\{1\}, \{1,2\}\}$. Suppose that $S^1 = \{1,2\}$, then with condition (3) in Definition 6.1, we get that $A^1_2 = A^0_2$. Hence, according to A^1 agent 2 should get item 3. In order for this to happen, we must have that $3 \in S^1$. But we already showed that $3 \notin S^1$. Thus, we have that $S^1 = \{1\}$, but agent 1 can only move from (1)(23) to (1)(23) itself. Hence, we can conclude that $(132) \not\gg_a (1)(23)$.

With $(123) \succ_1 (1)(23)$, $(1)(23) \succ_2 (123)$ and $(123) \sim_3 (1)(23)$, one can show, in a similar way, that $(123) \not\gg_a (1)(23)$.

Recall that $(1)(2)(3) \not\gg (1)(23)$, $(2)(13) \not\gg (1)(23)$ and that $(3)(12) \not\gg (1)(23)$. Hence, with the fact that indirect antisymmetric weak dominance implies indirect weak dominance, we get that $(1)(2)(3) \not\gg_a (1)(23)$, $(2)(13) \not\gg_a (1)(23)$ and that $(3)(12) \not\gg_a (1)(23)$. Thus, we have that A^* is not indirectly antisymmetrically weakly dominated in $\mathcal{E}(N, P)$. △

Remark 6.3. Note that for a sequence of length 1, $m = 1$, $A \underset{S}{\rightarrow} A'$ condition (3) in Definition 6.1 becomes: if $A' \sim_i A$ for some $i \in S$, then we have that $A_i = A'_i$. From Definition 2.1, this is automatically satisfied, hence with $m = 1$ weak dominance implies indirect antisymmetric weak dominance. Also note that if $A' \gg A$, then conditions (1), (2) and (3) in Definition 6.1 are automatically satisfied. Hence, we have that $A' \gg A$ implies $A' \gg_a A$. In other words, indirect dominance implies indirect antisymmetric weak dominance.

In the following example, we show the difference between $\gg$ and $\gg_a$.

Example 6.4 (Example 3.20 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 3.20:

$$P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

We show that there exist two different permutation matrices $A', A \in X$, such that $A' \gg_a A$ and $A' \not\gg A$. Let $A' = (3)(12)$ and let $A = (132)$, then we have that $\{1,2\} \in E(A, A')$, $(3)(12) \sim_1 (132)$ and that $(3)(12) \succ_2 (132)$. Hence, with Definition 3.1(2), we get that (3)(12) weakly dominates (132), thus it holds that $(3)(12) \gg_a (132)$.

Suppose that $(3)(12) \gg (132)$. With Definition 4.1, we get that there is a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = (132)$ and $A^m = (3)(12)$ and there are coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that $\forall\ k \in \{1, \ldots, m\}$, we have that $S^k \in E(A^{k-1}, A^k)$ and that $(3)(12) \succ_i A^{k-1}$ for all $i \in S^k$. Thus, for the nonempty coalition S^1 it must hold that $(3)(12) \succ_i (132)$ for all $i \in S^1$. Note that $(132) \sim_1 (3)(12)$, that $(3)(12) \succ_2 (132)$ and that $(132) \succ_3 (3)(12)$. Thus, we get that $S^1 = \{2\}$. Agent 2 can only deviate from (132) to $(1)(2)(3)$, thus we have that $A^1 = (1)(2)(3)$. For the nonempty coalition S^2, it must hold that $(3)(12) \succ_j (1)(2)(3)$ for all $j \in S^2$. Note that $(1)(2)(3) \succ_1 (3)(12)$, $(3)(12) \succ_2 (1)(2)(3)$ and that $(1)(2)(3) \sim_3 (3)(12)$. Thus, we have that $S^2 = \{2\}$. Note that agent 2 can only move from $(1)(2)(3)$ to $(1)(2)(3)$. Hence, agent 1 is never willing to trade with 2, thus we can never have a sequence starting at (132) and ending at $(3)(12)$ such that the conditions in Definition 4.1 are satisfied. Thus, we can conclude that $(3)(12) \not\gg (132)$. △

Example 6.4 shows that $\gg$ and $\gtrsim_a$ are not the same.

6.2 Strong antisymmetric farsighted core

In Section 5, we defined the strong farsighted core and we showed that either $SFCO = \emptyset$ or $SFCO = \{A^*\}$. We define the farsighted core with respect to $\gtrsim_a$, which we call the strong antisymmetric farsighted core. Define the **indirect antisymmetric weak dominance correspondence** as the correspondence $f_{\gtrsim_a} : X \to 2^X$ such that

$$f_{\gtrsim_a}(A) = \{A\} \cup \{A' \in X \mid A' \gtrsim_a A\}.$$

Definition 6.5 (Strong antisymmetric farsighted core)**.**
Let (N, P) be a housing matching model. The **strong antisymmetric farsighted core** $SAFCO$ of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not indirectly antisymmetrically weakly dominated:

$$SAFCO = \{A \in X \mid f_{\gtrsim_a}(A) = \{A\}\}.$$

In Kawasaki (2010), it is shown that when the preferences over the initial items are not necessarily strict, each allocation is indirectly antisymmetrically weakly dominated by a top trading cycle allocation. Because we have strict preferences over the initial items, the top trading cycle allocation is unique. From Theorem 4.11, we get that A^* indirectly dominates all $A \in X \setminus \{A^*\}$ and from Remark 6.3, we know that indirect dominance implies indirect antisymmetric weak dominance. Hence, we have the following corollary.

Corollary 6.6. *For all housing matching models (N, P), each $A \in X \setminus \{A^*\}$ is indirectly antisymmetrically weakly dominated by A^* in $\mathcal{E}(N, P)$.*

For the housing matching model (N, P) as in Example 6.2, we showed that A^* is not indirectly antisymmetrically weakly dominated. This result can be generalized to all housing matching models. In Kawasaki (2010), it is also shown that top trading cycle allocations are not indirectly antisymmetrically weakly dominated. Since we have strict preferences of the agents over the items, we have that A^* is not indirectly antisymmetrically weakly dominated. We write down the proof given in Kawasaki (2010) in the context of our social environment corresponding to the housing matching model (N, P).

Theorem 6.7. *For all housing matching models (N, P), the permutation matrix A^* is not indirectly antisymmetrically weakly dominated in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. We know that A^* is the top trading cycle permutation matrix. From the top trading cycle algorithm, we get that N is partitioned into T disjoint coalitions $S(tc^\tau)$, i.e. $N = \bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, with tc^τ a top trading cycle for $N \setminus \left(\bigcup_{1 \leq r \leq \tau - 1} S(tc^r) \right)$. Recall that $A^* = \prod_{\tau=1}^{T} tc^\tau$.

Let $A' \in X \setminus \{A^*\}$. We show that $A' \not\gtrsim_a A^*$ by a proof by contradiction. Suppose that there exist a sequence of permutation matrices $A^0, \ldots, A^m \in X$ with $A^0 = A^*$ and $A^m = A'$ and a sequence of coalitions $S^1, \ldots, S^m \in 2^N \setminus \{\emptyset\}$, such that for all $k \in \{1, \ldots, m\}$ conditions (1), (2) and (3) in Definition 6.1 hold.

First, we show that $A'_i = A^*_i$ for all $i \in S(tc^1)$. For coalition $S^1 \in E(A^*, A^1)$, we have two cases: $S(tc^1) \cap S^1 = \emptyset$ and $S(tc^1) \cap S^1 \neq \emptyset$. If $S(tc^1) \cap S^1 = \emptyset$, then by condition (2) in the definition of our effectivity correspondence, Definition 2.8, we get that $A^1_i = A^*_i$ for all $i \in S(tc^1)$.

Suppose that $S(tc^1) \cap S^1 \neq \emptyset$. Let $i \in S(tc^1) \cap S^1$. Because $i \in S^1$, we get by condition (2) of Definition 6.1 that $A' \succsim_i A^*$. Because $i \in S(tc^1)$, we have that according to A^* agent i gets his most preferred item, thus it holds that $A^* \succsim_i A'$. Hence, we get that $A' \sim_i A^*$. Then by condition (3) in Definition 6.1, we get that $A^1_i = A^*_i$. Thus, for all $i \in S(tc^1) \cap S^1$, it holds that $A^1_i = A^*_i$. We show that $S(tc^1) \subseteq S^1$. Suppose that $A^*_{ij} = 1$, then we have that $j \in S(tc^1)$ by definition of a top trading cycle and that $j \in S^1$ by condition (1) of Definition 2.8. Thus, we have that $j \in S(tc^1) \cap S^1$. If we repeat this, we get that $S(tc^1) \subseteq S^1$. Thus, we have that $A^1_i = A^*_i$ for all $i \in S(tc^1)$.

Thus, in both cases we have that $A^1_i = A^*_i$ for all $i \in S(tc^1)$. In other words, we have that $tc^1 \in C(A^1)$.

For coalition $S^2 \in E(A^1, A^2)$, we also have two cases: $S(tc^1) \cap S^2 = \emptyset$ and $S(tc^1) \cap S^2 \neq \emptyset$. If $S(tc^1) \cap S^2 = \emptyset$, then with $tc^1 \in C(A^1)$ we get by condition (2) in Definition 2.8 that $A^2_i = A^1_i$ for all $i \in S(tc^1)$. In the case that $S(tc^1) \cap S^2 \neq \emptyset$, we also have that $A^2_i = A^1_i$ for all $i \in S(tc^1)$. To see this, let $i \in S(tc^1) \cap S^2$. Then by condition (2) in Definition 6.1, we have that $A' \succsim_i A^1$ and by definition of tc^1 we have that $A^* \succsim_i A'$. Note that $A^*_i = A^1_i$, since $i \in S(tc^1)$. Hence, we get that $A' \sim_i A^1$. Thus, by condition (3) of Definition 6.1 we get that $A^1_i = A^2_i$. By the same reasoning as for S^1, we get that $S(tc^1) \subseteq S^2$ and that $A^1_i = A^2_i$ for all $i \in S(tc^1)$.

Thus, in both cases we have that $A^*_i = A^1_i = A^2_i$ for all $i \in S(tc^1)$. If we repeat above reasoning for the other coalitions $S^3, \ldots, S^m$, we get that

$$A^*_i = A^1_i = A^2_i = \cdots = A^{m-1}_i = A'_i \text{ for all } i \in S(tc^1).$$

Thus, we have that $tc^1 \in C(A^k)$ for all $k \in \{1, \ldots, m\}$.

Secondly, we show that $A'_i = A^*_i$ for all $i \in S(tc^2)$. We already know that according to all outcomes $A^0, \ldots, A^m$ each agent in $S(tc^1)$ always gets his most preferred item, which belongs to an agent in $S(tc^1)$. Hence, each agent in $S(tc^2)$ can only receive items from the agents in $N \setminus S(tc^1)$. According to A^*, each agent in $S(tc^2)$ gets his most preferred item of the remaining items belonging to the agents in $N \setminus S(tc^1)$. Hence, if we use the same reasoning as above, we get that $A^*_i = A'_i$ for all $i \in S(tc^2)$.

If we repeat this for the other coalitions $S(tc^3), \dots, S(tc^T)$, then we get that $A^*_i = A'_i$ for all $i \in \bigcup_{1 \le \tau \le T} S(tc^\tau) = N$. Hence, we have that $A' = A^*$, which contradicts the fact that $A' \in X \setminus \{A^*\}$. Thus, A^* is not indirectly antisymmetrically weakly dominated in $\mathcal{E}(N, P)$. □

For all housing matching models, we can conclude from Corollary 6.6 and Theorem 6.7, that A^* indirectly antisymmetrically weakly dominates all permutation matrices $A \in X \setminus \{A^*\}$, while it is not indirectly antisymmetrically weakly dominated. Hence, we get the following corollary.

Corollary 6.8. *For all housing matching models (N, P), we have that $SAFCO = \{A^*\}$.*

As in Section 5, we could give the definitions a vNM farsighted stable set, a consistent set and a DEM farsighted stable set with $\gg_a$ instead of $\gg$ and $f_{\gg_a}$ instead of $f_{\gg}$. We show that it is only relevant to look at the definition of a consistent set with respect to $\gg_a$.

In Kawasaki (2010), a farsighted vNM stable set with respect to $\gg_a$ is defined. Note that with Corollary 6.6 and Theorem 6.7, we have that the proofs of Theorem 5.10, Theorem 5.11, Theorem 5.25 and Theorem 5.27 still hold when we replace $\gg$ by $\gg_a$ and $f_{\gg}$ by $f_{\gg_a}$. Thus, we have the following result.

Corollary 6.9. *For all housing matching models (N, P), we have that with respect to $\gg_a$ the unique vNM farsighted set of $\mathcal{E}(N, P)$ and the unique DEM farsighted stable set of $\mathcal{E}(N, P)$ are both equal to $\{A^*\}$.*

6.3 Largest consistent set

In this section, we look at the largest consistent set with respect to indirect antisymmetric weak dominance as defined in Definition 6.1. We give the consistent set with respect to $\gg_a$ the name consistent $\gg_a$ set. Again, we need to have a definition of deterrence of deviations.

Definition 6.10 ($(\mathcal{A}, \gg_a)$-deterrence of deviations)**.**
Let (N, P) be a housing matching model, let $\mathcal{A} \subseteq X$ and let $A \in X$. We say that A satisfies **$(\mathcal{A}, \gg_a)$-deterrence of deviations** if for all $A' \in X$ and for all $S \in E(A, A')$, there exists $A'' \in \mathcal{A}$ such that the following two conditions hold:

(1) either $A'' = A'$ or $A'' \gg_a A'$,

(2) there exists $i \in S$ with $A \succ_i A''$ or for all $j \in S$ we have that $A \succsim_j A''$.

Note that the only difference between $(\mathcal{A}, \gg)$-deterrence of deviations and $(\mathcal{A}, \gg_a)$-deterrence of deviations is that in Definition 5.13, we have that $A'' \gg A'$ and that in Definition 6.10, we have that $A'' \gg_a A'$. Hence, the intuition of $(\mathcal{A}, \gg_a)$-deterrence of deviations is Remark 5.15 with $\gg_a$ instead of $\gg$.

Definition 6.11 (Consistent set with respect to $\gtrsim_a$)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A} \subseteq X$ is a **consistent** $\gtrsim_a$ **set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) if $A \in \mathcal{A}$, then A satisfies $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations,

(2) for all $A \notin \mathcal{A}$ we have that A does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations.

Define the **consistent** $\gtrsim_a$ **correspondence** as the correspondence $g_{\gtrsim_a} : 2^X \to 2^X$ such that

$$g_{\gtrsim_a}(\mathcal{A}) = \{A \in X \mid A \text{ satisfies } (\mathcal{A}, \gtrsim_a)\text{-deterrence of deviations}\}.$$

Then the following holds: $\mathcal{A}$ is a consistent $\gtrsim_a$ set if and only if $\mathcal{A} = g_{\gtrsim_a}(\mathcal{A})$. Note that for all housing matching models (N, P), we have that $\emptyset$ is a consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$.

Definition 6.12 (Largest consistent set with respect to $\gtrsim_a$)**.**
Let (N, P) be a housing matching model. The **largest consistent** $\gtrsim_a$ **set** of $\mathcal{E}(N, P)$, denoted by $\mathbb{A}_{\gtrsim_a}$, is the union of all consistent $\gtrsim_a$ sets:

$$\mathbb{A}_{\gtrsim_a} = \bigcup_{\mathcal{A} = g_{\gtrsim_a}(\mathcal{A})} \mathcal{A}.$$

For the housing matching model, as in Example 5.20, we showed that there exist at least two consistent $\gtrsim$ sets that do not contain A^*. In the following example, we show for this specific housing matching model that these two sets are not consistent $\gtrsim_a$ sets.

Example 6.13 (Example 6.2 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 6.2:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}.$$

Recall from Example 5.20, that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are both consistent $\gtrsim$ sets of $\mathcal{E}(N, P)$. We show that $\mathcal{A}' = \{(132)\}$ and $\mathcal{A}'' = \{(123)\}$ are not consistent $\gtrsim_a$ sets of $\mathcal{E}(N, P)$.

First, we show that $\mathcal{A}' = \{(132)\}$ is not a consistent $\gtrsim_a$ set. We show that (132) does not satisfy $(\mathcal{A}', \gtrsim)$-deterrence of deviations. Hence, we need to show that there exist $A' \in X$ and $S \in E(A, A')$, such that $(132) \neq A'$ and $(132) \not\gtrsim_a A'$, or such that we have that $(132) \succsim_i (132)$ for all $i \in S$ and $(132) \succ_j (132)$ for at least one $j \in S$. Since we have that $(132) \sim_i (132)$ for all $i \in N$, we need to show that there exists $A' \in X$, such that $(132) \neq A'$ and $(132) \not\gtrsim_a A'$. From Example 6.2, we know that $(132) \not\gtrsim_a (1)(23)$. This shows that (132) does not satisfy $(\mathcal{A}', \gtrsim)$-deterrence of deviations. Thus, $\mathcal{A}'$ is not a consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$.

Since from Example 6.2, we also know that $(123) \not\gtrsim_a (1)(23)$, one can show, in a similar way, that $\mathcal{A}''$ is not a consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$. △

The result in Example 6.13 that the two sets without A^* are not consistent $\gtrsim_a$ sets can be generalized to all housing matching models and to all sets without A^*. In Proposition 4.26, we showed that each nonempty consistent set must contain A^*. With Theorem 6.7, we know that A^* is not indirectly antisymmetrically weakly dominated. Hence, if we replace $\gg$ by $\gtrsim_a$ the proof of Proposition 4.26 is still valid, thus we get the following corollary.

Corollary 6.14. *For all housing matching models* (N, P)*, we have that each nonempty set* $\mathcal{A} \subseteq X \setminus \{A^*\}$ *is not a consistent* $\gtrsim_a$ *set of* $\mathcal{E}(N, P)$*.*

In Theorem 5.18, we showed for all housing matching models (N, P), that $\{A^*\}$ is a consistent $\gtrsim$ set of $\mathcal{E}(N, P)$. From Corollary 6.6, we know that for each $A' \in X \setminus \{A^*\}$ it holds that $A^* \gtrsim_a A'$. Hence, the proof of Theorem 5.18 remains valid if we replace $\gtrsim$ by $\gtrsim_a$. Thus, we get the following corollary.

Corollary 6.15. *For all housing matching models* (N, P)*, the set* $\mathcal{A} = \{A^*\}$ *is a consistent* $\gtrsim_a$ *set of* $\mathcal{E}(N, P)$*.*

From Corollary 6.15, we get that $\{A^*\} = g_{\gtrsim_a}(\{A^*\})$. Since $g_{\gtrsim_a} : 2^X \to 2^X$ is isotonic, we have that for all $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ it holds that $\{A^*\} \subseteq g_{\gtrsim_a}(\mathcal{A})$. In other words, A^* satisfies $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations for all $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$.

In Example 4.33, we showed for a specific housing matching model that there exists a consistent set that contained more than A^*. In the following example, we show that this set is not a consistent $\gtrsim_a$ set.

Example 6.16 (Example 4.33 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 4.33:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

Recall that $A^* = (132)$ and recall that in Example 4.33, we showed that

$$\mathcal{A} = \{(132), (1)(23), (2)(13), (3)(12)\}$$

is a consistent set. We determine that $\mathcal{A}$ is not a consistent $\gtrsim_a$ set by showing that there exists $A \in \mathcal{A}$ that does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations. Hence, we need to show that $\exists\, A \in \mathcal{A}$, for which there exist $A' \in X$ and $S \in E(A, A')$, such that $\forall\, A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\gtrsim_a A'$, or we have that $A'' \succsim_i A$ for all $i \in S$ and $A'' \succ_j A$ for at least one $j \in S$.

Let $A = (1)(23) \in \mathcal{A}$ and let $A' = A^* = (132)$, then we have that $E(A, A') = \{N\}$. We know that A^* is not indirectly antisymmetrically weakly dominated, hence for each $A'' \in \mathcal{A} \setminus \{A^*\}$ we have that $A'' \not\gtrsim_a A^*$. Suppose that $A^* = A''$. Note that A^* weakly dominates A, since $N \in E(A, A^*)$, $(132) \succ_1 (1)(23)$, $(132) \sim_2 (1)(23)$ and $(132) \succ_3 (1)(23)$. Thus, $(1)(23) \in \mathcal{A}$ does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations. We can conclude that $\mathcal{A} = \{(132), (1)(23), (2)(13), (3)(12)\}$ is not a consistent $\gtrsim_a$ set. △

The result in Example 6.16 that $A \in \mathcal{A}$, that is weakly dominated by A^*, does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations can be generalized to all housing matching models (N, P) and to all $A \in X$ that are weakly dominated by A^*.

Proposition 6.17. *For all housing matching models (N, P), we have that the following holds. Let $\mathcal{A} \subseteq X$ be a nonempty consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$. If A^* weakly dominates $A \in X \setminus \{A^*\}$, then $A \notin \mathcal{A}$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$ be a nonempty consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$. We give a proof by contradiction. Suppose that there exists $A \in \mathcal{A}$ such that A^* weakly dominates A.

We show that A does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations. Hence, we need to show that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $\forall\, A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\gtrsim_a A'$, or we have that $A'' \succsim_i A$ for all $i \in S$ and $A'' \succ_j A$ for at least one $j \in S$. From Corollary 6.14, we know that each nonempty consistent $\gtrsim_a$ set must contain A^*, thus we have that $A^* \in \mathcal{A}$.

Let $A' = A^*$. Note that from Theorem 6.7, we know that A^* is not indirectly antisymmetrically weakly dominated. Hence, for all $A'' \in \mathcal{A} \setminus \{A^*\}$ we have that $A'' \neq A^*$ and $A'' \not\gtrsim_a A^*$. Now, let $A'' = A^*$. Recall that A is weakly dominated by A^*. Thus, according to Definition 3.1(2) there exists $S \in E(A, A^*)$, such that $A^* \succsim_i A$ for all $i \in S$ and $A^* \succ_j A$ for at least one $j \in S$.

We can conclude that A does not satisfy $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations. Thus, we showed that if A^* weakly dominates $A \in X \setminus \{A^*\}$, then we have that $A \notin \mathcal{A}$ for any nonempty consistent $\gtrsim_a$ set of $\mathcal{E}(N, P)$. □

From Corollary 6.14 and Proposition 6.17, we know that each nonempty consistent $\gtrsim_a$ set must contain A^* and can only consist of permutation matrices that are not weakly dominated by A^*. Thus, for each housing matching model (N, P) such that A^* weakly dominates all other permutation matrices, we have that $\{A^*\}$ is the unique nonempty consistent $\gtrsim_a$ set.

From Example 3.20, we know that there exists a housing matching model such that A^* does not weakly dominate all other permutation matrices. In the following example, we look at whether for this housing matching model there exists a consistent $\gtrsim_a$ set that contains more than A^*.

Example 6.18 (Example 6.4 continued)**.**
Let $n = 3$ and let the preference matrix be as in Example 6.4:

$$P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Recall that $A^* = (1)(23)$. Note that for all $A \in X \setminus \{A^*\}$ we have that $E(A, A^*) = \{\{2, 3\}, N\}$. Suppose that $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ is a consistent $\gtrsim_a$ set.

Recall that A^* only weakly dominates $(1)(2)(3)$. From Proposition 6.17, we know that each consistent $\gtrsim_a$ set cannot contain $(1)(2)(3)$. Thus, we have that $(1)(2)(3) \notin \mathcal{A}$.

Secondly, we determine whether $\mathcal{A}$ can contain $A = (123)$. In other words, we need to determine whether (123) satisfies $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations. Note that for $A' = A'' = A^*$ we have that $(123) \succ_2 (1)(23)$. Hence, so far it seems that (123) satisfies $(\mathcal{A}, \gtrsim_a)$-deterrence of deviations.

Now, let $A' = (1)(2)(3)$ and $S = \{1\} \in E(A, A')$. We need to determine whether $\exists\, A'' \in \mathcal{A}$, such that either $A'' = A'$ or $A'' \geqq_a A'$, and such that $A \succsim_1 A''$. For $A'' = A^*$, we have that $A^* \succ_1 (123)$. Note that $A' = (1)(2)(3) \notin \mathcal{A}$. Thus, for all $A'' \in \mathcal{A} \setminus \{A^*\}$ we have that $A'' \neq A'$. Also note that $f_{\geqq_a}((1)(2)(3)) = \{(1)(2)(3), (1)(23)\} = \{A', A^*\}$. Thus, for all $A'' \in \mathcal{A} \setminus \{A^*\}$ we also have that $A'' \not\geqq_a A'$. Hence, we cannot find $A'' \in \mathcal{A}$ that satisfies the two conditions. We can conclude that $(123) \notin \mathcal{A}$.

Note that with the same reasoning as above and $A^* \succ_1 A$ for all

$$A \in \{(123), (132), (2)(13), (3)(12)\},$$

we get that $\{(132), (2)(13), (3)(12)\} \cap \mathcal{A} = \emptyset$. Hence, the set $\{A^*\}$ is the only nonempty consistent $\geqq_a$ set of $\mathcal{E}(N, P)$. △

In Example 6.18, we showed for a housing matching model such that A^* does not weakly dominate all other permutation matrices, that $\{A^*\}$ is the only nonempty consistent $\geqq_a$ set. This result can be generalized to all housing matching models. Recall from Corollary 6.14, that each nonempty consistent $\geqq_a$ set must contain A^* and from Corollary 6.15, that $\{A^*\}$ is a consistent $\geqq_a$ set. Hence, for each housing matching model we only need to show that each set that contains A^* and at least one other permutation matrix is not a consistent $\geqq_a$ set. For the proof of this theorem, it can be helpful to note the following.

Example 6.19 (Example 6.18 continued)**.**
Note that the first top trading cycle in Example 6.18 is (1) and that the second top trading cycle is (23). We also have that (1)(2)(3) weakly dominates each permutation matrix $A \in \{(123), (132), (2)(13), (3)(12)\}$ and that (1)(23) weakly dominates (1)(2)(3). △

Theorem 6.20. *For all housing matching models (N, P), each set $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$ is not a consistent $\geqq_a$ set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A} \subseteq X$ with $A^* \in \mathcal{A}$ and $|\mathcal{A}| > 1$. Let $A \in \mathcal{A} \setminus \{A^*\}$. We show that A does not satisfy $(\mathcal{A}, \geqq_a)$-deterrence of deviations. In other words, we need to show that $\exists\, A' \in X$ and $\exists\, S \in E(A, A')$, such that $\forall\, A'' \in \mathcal{A}$, we have that $A'' \neq A'$ and $A'' \not\geqq_a A'$, or we have that $\forall\, i \in S$ it holds that $A'' \succsim_i A$ and that $\exists\, j \in S$ with $A'' \succ_j A$.

From Theorem 3.12, we know that there exists $A' \in X$, such that $A \in X \setminus \{A^*\}$ is weakly dominated by A' in $\mathcal{E}(N, P)$ for coalition $S = \bigcup_{1 \leq r \leq s} S(tc^r) \in E(A, A')$, with $s \in \{1, \dots, T\}$ the first index such that $tc^s \notin C(A)$ and with $tc^1, \dots, tc^s$ in the cycle decomposition of A'.

For allocation $A'' \in \mathcal{A}$, we have two cases: either $tc^1, \dots, tc^s$ are all part of the cycle decomposition of A'' or there exists $r \in \{1, \dots, s\}$ such that tc^r is not in the cycle decomposition of A''.

For each $A'' \in \mathcal{A}$ with $\{tc^1, \dots, tc^s\} \subseteq C(A'')$, we have that $A'' \sim_i A$ for all $i \in \bigcup_{1 \leq r \leq s-1} S(tc^r)$, $A'' \succsim_i A$ for all $i \in S(tc^s)$ and $A'' \succ_j A$ for at least one $j \in S(tc^s)$.

For each $A'' \in \mathcal{A}$ such that $\exists\, r \in \{1, \dots, s\}$ with $tc^r \notin C(A'')$, we have that $A'' \neq A'$. We show that $A'' \not\geqq_a A'$. Note that $\{tc^1, \dots, tc^s\} \subseteq C(A')$, i.e. $tc^1, \dots, tc^s$ are all in the cycle decomposition of A'. By using the proof of Theorem 6.7 with A' instead of A^* and $\bigcup_{1 \leq r \leq s} S(tc^r)$ instead of $\bigcup_{1 \leq \tau \leq T} S(tc^\tau)$, we can conclude that for each $A''' \in X$, which indirectly antisymmetrically weakly dominates A', it must hold that $\{tc^1, \ldots, tc^s\} \subseteq C(A''')$. Thus, we can conclude that $A'' \not\geqq_a A'$.

Thus, for all $A'' \in \mathcal{A}$ it holds that, we have that $A'' \neq A'$ and $A'' \not\geqq_a A'$, or we have that $\forall\, i \in S$ it holds that $A'' \succsim_i A$ and that $\exists\, j \in S$ with $A'' \succ_j A$. Hence, $A \in \mathcal{A}$ does not satisfy $(\mathcal{A}, \geqq_a)$-deterrence of deviations. We can conclude that the set $\mathcal{A}$ is not a consistent $\geqq_a$ set of $\mathcal{E}(N, P)$. □

From Corollary 6.14, Corollary 6.15 and Theorem 6.20, we get that $\mathcal{A} = \{A^*\}$ is the unique nonempty consistent $\geqq_a$ set for all housing matching models. Thus, we have the following result.

Corollary 6.21. *For all housing matching models (N, P), we have that $\mathbb{A}_{\geqq_a} = \{A^*\}$.*

7 Horizon-K Farsightedness

In Section 3, we looked at different solution concepts under the assumption that all agents are myopic. In Section 4, Section 5 and Section 6, we studied stability concepts under the assumption that all agents are fully farsighted.

One can also wonder what happens when agents can only look two or three steps ahead. We denote the degree of farsightedness by K, which represents the number of steps agents can look ahead. For the intermediate case between myopia and full farsightedness, two models have been developed: horizon-K farsightedness by Herings et al. (2019) and level-K farsightedness by Herings and Khan (2022).

Herings et al. (2019) introduced the concept of a horizon-K farsighted set to study the influence of a limited degree of farsightedness on the stability of networks. In Herings et al. (2019), the dominance correspondence of an outcome contains all the end outcomes of a sequence from that outcome. A set is a horizon-K farsighted set if it satisfies horizon-K deterrence of external deviations, horizon-K external stability and minimality as defined in Herings et al. (2019).
A set satisfies horizon-K deterrence of external deviations if each deviation from any outcome A inside the set to an outcome A' outside the set is deterred by the credible threat of ending in another outcome A'', which compared to A is not strictly preferred by all agents in the deviating coalition. With a credible threat, we mean that A'' is such that either A'' can be reached from A' by a sequence of a length smaller than or equal to $K-2$ and A'' belongs to the set or A'' can be reached from A' by a sequence of a length equal to $K-1$ and there does not exist a sequence of a length smaller than $K-1$ starting at A' and ending at A''.
A set satisfies horizon-K external stability if from each outcome outside the set there is a finite sequence, which consists of sequences of outcomes of length smaller than or equal to K, leading to an outcome inside the set. Minimality means that there does not exist a proper subset of the set that satisfies horizon-K deterrence of external deviations and horizon-K external stability. Herings et al. (2019) proved that a horizon-K farsighted set always exists and that the horizon-1 farsighted set is unique. Note that for $K \geq 2$ a horizon-K farsighted set does not have to be unique.

In the context of network games, Herings and Khan (2022) introduced the concept of a level-K stable set and the concept of heterogeneity in the degree of foresight. In Herings and Khan (2022), the dominance correspondence of an outcome contains all the outcomes which are considered to be a deviation from that outcome. Hence, they define when a deviation from an outcome to another outcome is considered to be a level-K deviation.

In comparison to the horizon-K farsightedness in Herings et al. (2019), level-K farsightedness in Herings and Khan (2022) is defined in an inductive way. In order to define a level-K deviation, one need to know what the level-$(K-k)$ deviations are for $k \in \{1, \dots, K-1\}$. A deviation from an outcome A to an outcome A' is a level-K deviation, if there exists a coalition S which can move from A to A' and if there exists a sequence of outcomes $A, A', A^2, \dots, A^{K'}$ starting at A and ending at another outcome A'' with either a length of K, i.e. $K' = K$, or a length of at most $K-1$, i.e. $K' \leq K-1$, such that $A^{K'} \succ_i A$ for all $i \in S$, the kth induced deviation, for any $k \in \{1, \dots, K'-1\}$, is a level-$(K-k)$ deviation and such that if $K' \leq K-1$, then there does not exist a level

$(K - K')$-deviation from $A^{K'}$.
A level-K stable set is a set that satisfies deterrence of external deviations, iterated external stability and minimality as defined in Herings and Khan (2022). Deterrence of external deviations means that all level-K deviations from any outcome inside the set are part of the set. Iterated external stability says that from each outcome outside the set there exists a finite sequence of level-K deviations to an outcome inside the set. Minimality means that there does not exist a proper subset of the set that satisfies deterrence of external deviations and iterated external stability. Herings and Khan (2022) showed that a level-K stable set always exists and that it is unique.

In Section 4, we defined indirect dominance in the sense that A' indirectly dominates A, if there is a finite sequence of permutation matrices starting from A and ending to A' such that the conditions of Definition 4.1 are satisfied. Hence, this is more similar to the horizon-K farsighted set of Herings et al. (2019). Thus, under the assumption that all agents are horizon-K farsighted, we study the core, the vNM stable set and the horizon-K farsighted set of Herings et al. (2019) in the context of our social environment corresponding to the housing matching model (N, P). We only do this with respect to strict dominance, because the results of the stability concepts with respect to indirect weak dominance, which we studied in Section section 5, are undesirable.

7.1 Horizon-K strict dominance

In the context of networks, a farsighted improving path of length smaller or equal to K from one network to another is defined in Herings et al. (2019). We give a similar definition in the context of our social environment corresponding to the housing matching model (N, P).

Definition 7.1 (Horizon-K strict dominance)**.**
Let (N, P) be housing matching model, let $K \in \mathbb{N}$ and let $A, A' \in X$ be two different permutation matrices. The permutation matrix A' **horizon-K strictly dominates** A in $\mathcal{E}(N, P)$, denoted by $A' \gg_K A$, if there $\exists\ K' \in \mathbb{N}$ with $K' \leq K$, such that there is a sequence of permutation matrices $A^0, \dots, A^{K'} \in X$ with $A^0 = A$ and $A^{K'} = A'$ and there are coalitions $S^1, \dots, S^{K'} \in 2^N \setminus \{\emptyset\}$, such that $\forall\ k \in \{1, \dots, K'\}$ the following two conditions hold:

(1) $S^k \in E(A^{k-1}, A^k)$,

(2) $A' \succ_i A^{k-1}$ for all $i \in S^k$.

Remark 7.2. Note that horizon-1 strict dominance is equal to strict dominance, that for each $K \in \mathbb{N}$ horizon-K strict dominance implies indirect dominance and that horizon-∞ strict dominance is equal to indirect dominance.

From Theorem 4.6, we know that A^* is not indirectly dominated, hence with Remark 7.2, we get the following corollary.

Corollary 7.3. *For all housing matching models (N, P), we have for each $K \in \mathbb{N}$ that A^* is not horizon-K strictly dominated in $\mathcal{E}(N, P)$.*

Like in Section 4 and in Herings et al. (2019), we can define a dominance correspondence with respect to horizon-K strict dominance. Define the **horizon-K dominance correspondence** as the correspondence $f_K : X \to 2^X$ such that $f_K(A)$ denotes the subset of X that contains A and all the permutation matrices that horizon-K strictly dominate A. In other words,

$$f_K(A) = \{A\} \cup \{A' \in X \mid A' \gg_K A\}.$$

Note that for all $A \in X$, we have that $f_1(A) = f(A)$, that $f_\infty(A) = f_\gg(A)$ and that $\forall\ K, K' \in \mathbb{N}$ with $K' \leq K$ it holds that $f_{K'}(A) \subseteq f_K(A)$. Define $f_K^2(A)$ as the set of permutation matrices that can be reached from A by at most two consecutive horizon-K strict dominations:

$$f_K^2(A) = \{A'' \in X \mid \exists\ A' \in X \text{ such that } A' \in f_K(A) \text{ and } A'' \in f_K(A')\}$$

Define $f_K^k(A)$ as the set of permutation matrices that can be reached from A by at most k consecutive horizon-K strict dominations and $f_K^{\mathbb{N}}(A)$ as the set of permutation matrices that can be reached from A by a finite number of horizon-K strict dominations:

$$f_K^{\mathbb{N}}(A) = \bigcup_{k \in \mathbb{N}} f_K^k(A).$$

Definition 7.4 (Horizon-K farsighted core)**.**
Let (N, P) be a housing matching model and let $K \in \mathbb{N}$. The **horizon-K farsighted core** $KFCO$ of $\mathcal{E}(N, P)$ is defined as the set of permutation matrices that are not horizon-K strictly dominated:

$$KFCO = \{A \in X \mid f_K(A) = \{A\}\}.$$

Note that with $f_1(A) = f(A)$ for all $A \in X$ we get that the horizon-1 farsighted core is the core and that with $f_\infty(A) = f_\gg(A)$ for all $A \in X$ we get that the horizon-∞ farsighted core is equal to the farsighted core. From Corollary 7.3, we get the following result.

Corollary 7.5. *For all housing matching models (N, P), we have for all $K \geq 1$ that $A^* \in KFCO$.*

Note that $f_K(A)$ is the set that contains A and all the permutation matrices that horizon-K strictly dominate A and that $f_K^k(A)$ is the set that contains A and all the permutation matrices that can be reached from A by a sequence of at most k consecutive horizon-K strict dominations. To denote the difference, we look at an example.

Example 7.6. Let $n = 3$ and let the preference matrix be

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{pmatrix}.$$

Then we have that $A^* = (123)$. Note that $f((1)(23)) = \{(1)(23), (3)(12)\}$, $f((2)(13)) = \{(2)(13), (1)(23)\}$ and that $f((3)(12)) = \{(3)(12), (2)(13)\}$. Thus, we have that

$$f^2((1)(23)) = f^2((2)(13)) = f^2((3)(12)) = \{(1)(23), (2)(13), (3)(12)\}.$$

With the following sequences

$$\begin{aligned}
(1)(23) &\xrightarrow[S^1=\{2\}]{} (1)(2)(3) \xrightarrow[S^2=(1)(2)(3)]{} (123)\\
(2)(13) &\xrightarrow[S^1=\{3\}]{} (1)(2)(3) \xrightarrow[S^2=(1)(2)(3)]{} (123)\\
(3)(12) &\xrightarrow[S^1=\{1\}]{} (1)(2)(3) \xrightarrow[S^2=(1)(2)(3)]{} (123)
\end{aligned}$$

we get that $(123) \in f_2((1)(23))$, $(123) \in f_2((2)(13))$ and that $(123) \in f_2((3)(12))$. Hence, we have that f_1^2 and f_2 are not the same. △

In Theorem 4.11, we showed that A^* indirectly dominates all $A \in X \setminus \{A^*\}$ by a construction in which all the cycles in $C(A) \setminus C(A^*)$ of a length greater than 1 are decomposed, one at a time, until allocation A^ℓ is reached, such that either $A^\ell = A^*$ or $C(A^\ell) \setminus C(A^*)$ only contains cycles of length 1, and then the cycles in $C(A^*) \setminus C(A^\ell)$ are formed, one at a time.

For each housing matching model (N, P) with $n = 3$, we have that the above construction is done in at most two steps, namely at most one step for each of the two parts of the construction. Hence, for each housing matching model with three agents, we have that A^* horizon-2 strictly dominates all other permutation matrices. In the following example, we look at whether A^* horizon-2 strictly dominates all $A \in X \setminus \{A^*\}$, for a housing matching model with four agents.

Example 7.7. Let $n = 4$, $N = \{1, 2, 3, 4\}$ and let the preference matrix be given as

$$P = \begin{pmatrix} -2 & 0 & -1 & -3 \\ 0 & -3 & -1 & -2 \\ 0 & -2 & -3 & -1 \\ -3 & -1 & 0 & -2 \end{pmatrix}.$$

Note that X contains $4! = 24$ permutation matrices and that $A^* = (12)(34)$. Hence, we have that $C(A^*) = \{(12), (34)\}$.

First, we show that A^* horizon-2 strictly dominates all

$$A \in \{(1)(2)(3)(4), (1)(2)(34), (3)(4)(12)\}.$$

Recall that strict dominance is equal to horizon-1 strict dominance and that horizon-1 strict dominance implies horizon-2 strict dominance. Note that $A^* \succ_i (1)(2)(3)(4)$ for all $i \in N$ and that $N \in E((1)(2)(3)(4), A^*)$. Hence, with Definition 3.1(1) and the above, we get that $A^* \gg_2 (1)(2)(3)(4)$. Also note that $A^* \succ_i (1)(2)(34)$ for all $i \in \{1, 2\}$ with $\{1, 2\} \in E((1)(2)(34), A^*)$ and that $A^* \succ_j (3)(4)(12)$ for all $j \in \{3, 4\}$ with $\{3, 4\} \in E((3)(4)(12), A^*)$. Thus, we also have that $A^* \gg_2 (1)(2)(34)$ and that $A^* \gg_2 (3)(4)(12)$.

Now, we show that A^* horizon-2 strictly dominates $A = (13)(24)$. Note that $C(A) \setminus C(A^*) = \{(13), (24)\}$, hence according to Lemma 4.9 there exist $i \in \{1, 3\}$ such that $A^* \succ_i A$ and $j \in \{2, 4\}$ such that $A^* \succ_j A$. We have that $(12)(34) \succ_1 (13)(24)$, $(13)(24) \succ_3 (12)(34)$, $(12)(34) \succ_2 (13)(24)$ and that $(12)(34) \succ_4 (13)(24)$. Hence, if we take $S^1 = \{1, 2, 4\}$ and $A^1 = (1)(2)(3)(4)$, then we have that $S^1 \in E(A, A^1)$ and that $A^* \succ_i A$ for all $i \in S^1$. Note that $A^* \succ_j A^1$ for all $j \in N$ and that $N \in E(A^1, A^*)$. Let $S^2 = N$, then with the following sequence

$$(13)(24) \xrightarrow[S^1=\{1,2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$$

and Definition 7.1 we can conclude that A^* horizon-2 strictly dominates A.

In Table 18, the following construction is used to show that A^* horizon-2 strictly dominates all

$$A \in X \setminus \{(1)(2)(3)(4), (1)(2)(34), (3)(4)(12)\}.$$

Note that $C(A) \setminus C(A^*)$ is the set of cycles that are in the cycle decomposition of A and are not in the cycle decomposition of A^*. From Lemma 4.9, we know in particular for all $c_A \in C(A) \setminus C(A^*)$ with $|S(c_A)| > 1$, that there exists $i \in S(c_A)$ such that $A^* \succ_i A$. Hence, let S^1 be the set of agents that prefer A^* to A and that are contained in some $S(c_A)$ with $c_A \in C(A) \setminus C(A^*)$ and $|S(c_A)| > 1$. Note that

$$A \in X \setminus \{(1)(2)(34), (3)(4)(12)\}$$

implies that $C(A) \cap C(A^*) = \emptyset$. Let $A^1 = (1)(2)(3)(4)$, then with $C(A) \cap C(A^*) = \emptyset$ and $A \neq (1)(2)(3)(4)$, we have that $S^1 \in E(A, A^1)$. Note that $A^* \succ_j A^1$ for all $j \in N$ and that $N \in E(A^1, A^*)$. Let $S^2 = N$, then with the following sequence

$$A \xrightarrow[S^1]{} A^1 \xrightarrow[S^2]{} A^*$$

we get that A^* horizon-2 strictly dominates all

$$A \in X \setminus \{(1)(2)(3)(4), (1)(2)(34), (3)(4)(12)\}.$$

Thus, we have that $A^* \gg_2 A$ for all $A \in X \setminus \{A^*\}$.

Sequence	$E(A, A^1)$ and $E(A^1, A^*)$	Preference
$(1)(2)(3)(4) \xrightarrow[S^1=N]{} (12)(34)$	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^1$
$(1)(2)(34) \xrightarrow[S^1=\{1,2\}]{} (12)(34)$	$\{1,2\} \in E((1)(2)(34), (12)(34))$	$(12)(34) \succ_i (1)(2)(34)\ \forall\ i \in S^1$
$(1)(3)(24) \xrightarrow[S^1=\{2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,4\} \in E((1)(3)(24), (1)(2)(3)(4))$	$(12)(34) \succ_i (1)(3)(24)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1)(4)(23) \xrightarrow[S^1=\{2,3\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,3\} \in E((1)(4)(23), (1)(2)(3)(4))$	$(12)(34) \succ_i (1)(4)(23)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(2)(3)(14) \xrightarrow[S^1=\{1,4\}]{} (1)(2)(3)(4) \xrightarrow[S^1=N]{} (12)(34)$	$\{1,4\} \in E((2)(3)(14), (1)(2)(3)(4))$	$(12)(34) \succ_i (2)(3)(14)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(2)(4)(13) \xrightarrow[S^1=\{1\}]{} (1)(2)(3)(4) \xrightarrow[S^1=N]{} (12)(34)$	$\{1\} \in E((2)(4)(13), (1)(2)(3)(4))$	$(12)(34) \succ_1 (2)(4)(13)$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(3)(4)(12) \xrightarrow[S^1=\{3,4\}]{} (12)(34)$	$\{3,4\} \in E((3)(4)(12), (12)(34))$	$(12)(34) \succ_i (3)(4)(12)\ \forall\ i \in S^1$
$(13)(24) \xrightarrow[S^1=\{1,2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,2,4\} \in E((13)(24), (1)(2)(3)(4))$	$(12)(34) \succ_i (13)(24)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(14)(23) \xrightarrow[S^1=N]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$N \in E((14)(23), (1)(2)(3)(4))$	$(14)(23) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1)(234) \xrightarrow[S^1=\{2,3\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,3\} \in E((1)(234), (1)(2)(3)(4))$	$(12)(34) \succ_i (1)(234)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1)(243) \xrightarrow[S^1=\{2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,4\} \in E((1)(243), (1)(2)(3)(4))$	$(12)(34) \succ_i (1)(243)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(2)(134) \xrightarrow[S^1=\{1\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1\} \in E((2)(134), (1)(2)(3)(4))$	$(12)(34) \succ_1 (2)(134)$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(2)(143) \xrightarrow[S^1=\{1,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,4\} \in E((2)(143), (1)(2)(3)(4))$	$(12)(34) \succ_i (2)(143)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(3)(124) \xrightarrow[S^1=\{1,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,4\} \in E((3)(124), (1)(2)(3)(4))$	$(12)(34) \succ_i (3)(124)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(3)(142) \xrightarrow[S^1=\{2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,4\} \in E((3)(142), (1)(2)(3)(4))$	$(12)(34) \succ_i (3)(142)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(4)(123) \xrightarrow[S^1=\{1,3\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,3\} \in E((4)(123), (1)(2)(3)(4))$	$(12)(34) \succ_i (4)(123)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(4)(132) \xrightarrow[S^1=\{2\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2\} \in E((4)(132), (1)(2)(3)(4))$	$(12)(34) \succ_2 (4)(132)$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1234) \xrightarrow[S^1=\{1,3\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,3\} \in E((1234), (1)(2)(3)(4))$	$(12)(34) \succ_i (1234)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1243) \xrightarrow[S^1=\{1,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,4\} \in E((1243), (1)(2)(3)(4))$	$(12)(34) \succ_i (1243)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1324) \xrightarrow[S^1=\{1,2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{1,2,4\} \in E((1324), (1)(2)(3)(4))$	$(12)(34) \succ_i (1324)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1342) \xrightarrow[S^1=\{2\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2\} \in E((1342), (1)(2)(3)(4))$	$(12)(34) \succ_2 (1342)$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1423) \xrightarrow[S^1=N]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$N \in E((1423), (1)(2)(3)(4))$	$(12)(34) \succ_i (1423)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$
$(1432) \xrightarrow[S^1=\{2,4\}]{} (1)(2)(3)(4) \xrightarrow[S^2=N]{} (12)(34)$	$\{2,4\} \in E((1432), (1)(2)(3)(4))$	$(12)(34) \succ_i (1432)\ \forall\ i \in S^1$
	$N \in E((1)(2)(3)(4), (12)(34))$	$(12)(34) \succ_i (1)(2)(3)(4)\ \forall\ i \in S^2$

Table 18: Illustration that each $A \in X \setminus \{(12)(34)\}$ is horizon-2 strictly dominated by $A^* = (12)(34)$.

△

In Example 7.7, we showed for a specific housing matching model with four agents that A^* horizon-2 strictly dominates all other permutation matrices. This gives rise to the conjecture that the proof of Theorem 4.11 can be shortened to at most two steps by using the construction described in Example 7.7.

Theorem 7.8. *For all housing matching models (N, P), the permutation matrix A^* horizon-2 strictly dominates any $A \in X \setminus \{A^*\}$ in $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $A \in X \setminus \{A^*\}$. We need to show that $A^* \gg_2 A$. In other words, we need to show that there exists $K' \in \{1, 2\}$, such that there is a sequence of permutation matrices $A^0, A^1, A^{K'} \in X$ with $A^0 = A$ and $A^{K'} = A^*$ and there are coalitions $S^1, S^{K'} \in 2^N \setminus \{\emptyset\}$, such that $\forall\ k \in \{1, K'\}$ we have that $S^k \in E(A^{k-1}, A^k)$ and that $A^* \succ_i A^{k-1}$ for all $i \in S^k$.

Take $A^0 = A$. The first step is to construct $A^1 \in X$ by partially decomposing A into cycles consisting of one agent. From Lemma 4.9, we know for all $c_A \in C(A) \setminus C(A^*)$, that there exists $i \in S(c_A)$ such that $A^* \succ_i A$. In other words, in each coalition of agents, which forms a trading cycle in A, but not in A^*, there exists an agent that strictly prefers A^* to A.

Let $S^1 \in 2^N$ be defined as

$$S^1 = \left\{ i \in N \;\middle|\; \begin{array}{l} \exists\ c_A \in C(A) \setminus C(A^*) \text{ such that } |S(c_A)| > 1, \\ i \in S(c_A) \text{ and } A^* \succ_i A \end{array} \right\}.$$

In other words, coalition S^1 consists of all the agents that strictly prefer A^* to A and are in a coalition of more than one agent which forms a trading cycle in A, but not in A^*.

If $S^1 = \emptyset$, then we have that $K' = 1$ and we continue with the second step. Note that $C(A) \setminus C(A^*) = \emptyset$ if and only if $A = A^*$. Since $A \in X \setminus \{A^*\}$, we have that $S^1 = \emptyset$ means that $\forall\ c_A \in C(A) \setminus C(A^*)$, we have that $|S(c_A)| = 1$. In other words, $S^1 = \emptyset$, if each trading cycle of A which is not equal to any top trading cycle consists of one agent.

If $S^1 \neq \emptyset$, then we define $\phi : S^1 \to S^1$ as the bijection $\phi(i) = i$. From Lemma 2.11, we know that there exists $A^1 \in X$, such that $S^1 \in E(A, A^1)$ and $A^1_{i\phi(i)} = 1$ for all $i \in S^1$. Thus, we have that $(i) \in C(A^1)$ for all $i \in S^1$. By definition of S^1, we have that $A^* \succ_i A$ for all $i \in S^1$.
There are two cases: $A^* = A^1$ and $A^* \neq A^1$. In the former case, we have that $K' = 1$. Thus, with the above we already showed that A^* horizon-2 strictly dominates A. In the latter case, we have that $K' = 2$ and we continue with the second step.

The second step is to construct $A^* = A^{K'} \in X$ by using the decomposition in the first step. Let $C(A^*) \setminus C(A^{K'-1}) = \{c^1, \ldots, c^h\}$ be the set of cycles that are in the cycle decomposition of A^* and are not in the cycle decomposition of $A^{K'-1}$. Suppose that for all $q \in \{1, \ldots, h\}$ we have that $c^q = (c^q_1 \cdots c^q_{\ell^q})$ with $c^q_{\ell^q} = c^q_0$. Define

$$S^{K'} = \bigcup_{1 \leq q \leq h} S(c^q)$$

and define $\phi : S^{K'} \to S^{K'}$ as the following bijection:

$$\phi(c^q_p) = c^q_{p-1}\ \forall\ p \in \{1, \ldots, \ell^q\} \text{ and } \forall\ q \in \{1, \ldots, h\}.$$

Then from Lemma 2.11, we know that there exists $A^{K'} \in X$, such that $S^{K'} \in E(A^{K'-1}, A^{K'})$ and $A^{K'}_{i\phi(i)} = 1$ for all $i \in S^{K'}$. In other words, we have that $\{c^1, \ldots, c^h\} \subseteq C(A^{K'})$. Recall that condition (2) of Definition 2.8 is: $\forall\ i \in N \setminus S^{K'}$ such that $S(c^i_{A^{K'-1}}) \cap S^{K'} = \emptyset$, it holds that: if $A^{K'-1}_{ij} = 1$, then we have that $A^{K'}_{ij} = 1$. Note that for each $c_{A^{K'-1}} \in C(A^*) \cap C(A^{K'-1})$, we have that $S(c_{A^{K'-1}}) \cap S^{K'} = \emptyset$. Hence, for all $c_{A^{K'-1}} \in C(A^*) \cap C(A^{K'-1})$, we have that $c_{A^{K'-1}} \in C(A^{K'})$.

Hence, we have that

$$C(A^*) = \Big(C(A^*) \cap C(A^{K'-1})\Big) \bigcup \Big(C(A^*) \setminus C(A^{K'-1})\Big) \subseteq C(A^{K'}).$$

Since, we have that

$$\bigcup_{c_{A^*} \in C(A^*)} S(c_{A^*}) = N = \bigcup_{c_{A^{K'}} \in C(A^{K'})} S(c_{A^{K'}}),$$

we get that $C(A^{K'}) = C(A^*)$. Thus, with $C(A^*) \setminus C(A^{K'}) = \emptyset$, we have that $A^{K'} = A^*$. In other words, the cycle decomposition of $A^{K'}$ consists of the cycles $c^1, \ldots, c^h$ and of all the cycles $c \in C(A^*) \cap C(A^{K'-1})$, i.e. the cycles that are in the cycle decomposition of A^* and are in the cycle decomposition of $A^{K'-1}$. From Theorem 4.11, we know that $A^* \succ_i A^{K'-1}$ for all $i \in S^{K'}$. Hence, it holds that $A^* \gg_2 A$.

Thus, we have that A^* horizon-2 strictly dominates any $A \in X \setminus \{A^*\}$ in $\mathcal{E}(N, P)$. □

From Theorem 7.8, we know that $A^* \in f_2(A)$ for all permutation matrices A not equal to the top trading cycle permutation matrix. With the fact that $f_{K'}(A) \subseteq f_K(A)$ for all $K, K' \in \mathbb{N}$ with $K' \leq K$ and for all $A \in X$, we get the following corollary.

Corollary 7.9. *For all housing matching models (N, P), we have for all $K \geq 2$, that $A^* \in f_K(A)$ for all $A \in X \setminus \{A^*\}$.*

From Corollary 7.5, we know for all housing matching models that $A^* \in KFCO$ for all $K \geq 2$. Thus, with Corollary 7.9, we get the following result.

Corollary 7.10. *For all housing matching models (N, P), we have for all $K \geq 2$ that the horizon-K farsighted core is equal to $\{A^*\}$.*

7.2 Horizon-K von Neumann-Morgenstern stable set

For all housing matching models, we proved in Section 3 that each von Neumann-Morgenstern stable set contains the core and in Section 4 that the unique farsighted vNM stable set is equal to $\{A^*\}$. In this section, we look at the von Neumann-Morgenstern stable set under the assumption that all agents are horizon-K farsighted.

Definition 7.11 (Horizon-K von Neumann-Morgenstern stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A}_K \subseteq X$ of permutation matrices is a **horizon-K von Neumann-Morgenstern (vNM) stable set** of $\mathcal{E}(N, P)$ if it satisfies the following two conditions:

(1) **internal stability**: $\forall\, A \in \mathcal{A}_K$ we have that $f_K(A) \cap \mathcal{A}_K = \{A\}$,

(2) **external stability**: $\forall\, A \notin \mathcal{A}_K$ it holds that $f_K(A) \cap \mathcal{A}_K \neq \emptyset$.

With the fact that $f_1(A) = f(A)$ for all $A \in X$, we can conclude that the definitions of a horizon-1 vNM stable set and a vNM stable set are equivalent. Also with the fact that $f_\infty(A) = f_\gg(A)$ for all $A \in X$, we can conclude that the definitions of a horizon-∞ vNM stable set and a farsighted vNM stable set are equivalent. Hence, from now of on, we focus on horizon-K vNM stable sets for $2 \leq K < \infty$.

Note that for a set $\mathcal{A}_K$ with $|\mathcal{A}_K| = 1$ the internal stability condition is automatically satisfied. Hence, if we want to know whether a set $\mathcal{A}_K$ with $|\mathcal{A}_K| = 1$ is a horizon-K vNM stable set, we only need to check whether it satisfies the external stability condition. With the help of Corollary 7.9 we show that for all housing matching models the unique horizon-K vNM stable set is $\{A^*\}$ for all $K \geq 2$.

Theorem 7.12. *For all housing matching models (N, P), we have for all $K \geq 2$, that $\mathcal{A}_K = \{A^*\}$ is the unique horizon-K vNM stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $K \geq 2$. First, we show that $\mathcal{A}_K = \{A^*\}$ is a horizon-K vNM stable set. Note that $|\mathcal{A}_K| = 1$, thus we have that $\mathcal{A}_K$ satisfies internal stability. From Corollary 7.9, we know that $A^* \in f_K(A)$ for all $A \in X \setminus \{A^*\}$. This shows that $\mathcal{A}_K$ also satisfies external stability. Hence, $\mathcal{A}_K = \{A^*\}$ is a horizon-K vNM stable set of $\mathcal{E}(N, P)$.

Now we show that $\{A^*\}$ is the unique horizon-K vNM stable set. Suppose that $\mathcal{A}'_K \subseteq X$ is a horizon-K vNM stable set with $\mathcal{A}'_K \neq \{A^*\}$. There are two cases: $A^* \in \mathcal{A}'_K$ with $|\mathcal{A}'_K| > 1$ and $A^* \notin \mathcal{A}'_K$. In the former case we have that there exists $A \in \mathcal{A}'_K \setminus \{A^*\}$. From Corollary 7.9, we know that $A^* \in f_K(A)$, thus it holds that $A^* \in f_K(A) \cap \mathcal{A}'_K$. This gives a contradiction with the internal stability of $\mathcal{A}'_K$.

In the latter case we have that $A^* \notin \mathcal{A}'_K$. From Corollary 7.3, we know that $f_K(A^*) = \{A^*\}$. Hence, it holds that $f_K(A^*) \cap \mathcal{A}'_K = \{A^*\} \cap \mathcal{A}'_K = \emptyset$. This gives a contradiction with the external stability of $\mathcal{A}'_K$. We can conclude that if $\mathcal{A}'_K$ is a horizon-K vNM stable set, then it must hold that $\mathcal{A}'_K = \{A^*\}$. Thus, we have that $\{A^*\}$ is the unique horizon-K vNM stable set of $\mathcal{E}(N, P)$. □

7.3 Horizon-K farsighted stable set

In the literature, there are more definitions of a set to be stable. In Herings et al. (2019), a horizon-K farsighted set of networks is defined as a minimal set that satisfies horizon-K deterrence of external deviations and horizon-K external stability.

In our social environment corresponding to the housing matching model (N, P), horizon-K deterrence of external deviations says that each deviation from A to an arbitrary permutation matrix A' outside the set is deterred by the credible threat of ending up at another permutation matrix A'' that is not strictly preferred to A for at least one agent in the deviating coalition.

Now, we explain what we mean with a credible threat. Consider the deviation from A to A'. There are two cases $A'' \in f_{K-1}(A') \setminus f_{K-2}(A')$ and $A'' \in f_{K-2}(A')$. In the former case going from A to A'' takes exactly K steps. This threat is credible, since each agent has a horizon-K farsightedness. In the latter case we have that going from A to A'' takes at most $K-1$ steps. Hence, the threat is only credible when A'' is also an element of the set.

In this thesis, horizon-K external stability, as defined in Herings et al. (2019), is referred to as horizon-K iterated external stability. It means that from each permutation matrix outside the set there exists a sequence of horizon-K strict dominations to some permutation matrix inside the set.

We give the definition of a horizon-K farsighted set of Herings et al. (2019) in the context of our social environment corresponding to the housing matching model (N, P) and we call it a horizon-K farsighted stable set. In the definition below, we use $f_0(A) = \{A\}$ and $f_{-1}(A) = \emptyset$ for all permutation matrices $A \in X$.

Definition 7.13 (Horizon-K farsighted stable set).
Let (N, P) be a housing matching model and let $K \in \mathbb{N}$. A set $\mathcal{A}_K \subseteq X$ is a **horizon-K farsighted stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three conditions:

(1) **horizon-K deterrence of external deviations**: $\forall\ A \in \mathcal{A}_K$, $\forall\ A' \notin \mathcal{A}_K$ and $\forall\ S \in E(A, A')$, there exists $A'' \in (f_{K-2}(A') \cap \mathcal{A}_K) \bigcup (f_{K-1}(A') \setminus f_{K-2}(A'))$ such that $\exists\ i \in S$ with $A \succsim_i A''$,

(2) **horizon-K iterated external stability**: $\forall\ A \notin \mathcal{A}_K$ it holds that $f_K^{\mathbb{N}}(A) \cap \mathcal{A}_K \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}'_K \subsetneq \mathcal{A}_K$ that satisfies (1) and (2).

Note that $\emptyset$ does not satisfy horizon-K iterated external stability for all $K \in \mathbb{N}$. Hence, for all $K \in \mathbb{N}$, we have that each set $\mathcal{A}_K \subseteq X$ with $|\mathcal{A}_K| = 1$ satisfies minimality.

First, we look at horizon-∞ farsighted stable sets. In Section 4, we defined the DEM farsighted stable set, which is the farsightedly stable set of Herings et al. (2010), which is a generalized concept of the pairwise farsightedly stable set of Herings et al. (2009). Note that with $f_{\gg}(A) = f_\infty(A)$ for all $A \in X$, we get that horizon-∞ deterrence of external deviations is equal to condition (1) in Definition 4.34. Recall that condition (2) in Definition 4.34 is: $\forall\ A \notin \mathcal{A}$ it holds that $f_{\gg}(A) \cap \mathcal{A} \neq \emptyset$. Note that horizon-$\infty$ iterated external stability is: $\forall\ A \notin \mathcal{A}_K$ it holds that $f_\infty^{\mathbb{N}}(A) \cap \mathcal{A}_K \neq \emptyset$. In Herings et al. (2019), it is shown, with $f_{\gg}(A) = f_\infty(A) \subseteq f_\infty^{\mathbb{N}}(A)$ for all $A \in X$, that for every DEM farsighted stable set $\mathcal{A}$, there is a set $\mathcal{A}_\infty \subseteq \mathcal{A}$ such that $\mathcal{A}_\infty$ is a horizon-∞ farsighted stable set. In Theorem 4.35, we showed for all housing matching models that $\mathcal{A} = \{A^*\}$ is the unique DEM farsighted stable set. Hence, with the fact that $\emptyset$ cannot be a horizon-∞ farsighted stable set, we get that $\mathcal{A}_\infty = \{A^*\}$ is the unique horizon-∞ farsighted stable set for all housing matching models.

Now, we focus on horizon-1 farsighted stable sets. With $f_0(A) = \{A\}$ and $f_{-1}(A) = \emptyset$ for all $A \in X$, we can simplify horizon-1 deterrence of external deviations. The simplification of Definition 7.13 for $K = 1$ is given below.

Definition 7.14 (Horizon-1 farsighted stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A}_1 \subseteq X$ is a **horizon-1 farsighted stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three conditions:

(1) **horizon-1 deterrence of external deviations**: $\forall\ A \in \mathcal{A}_1$, $\forall\ A' \notin \mathcal{A}_1$ and $\forall\ S \in E(A, A')$, we have that $\exists\ i \in S$ with $A \succsim_i A'$,

(2) **horizon-1 iterated external stability**: $\forall\ A \notin \mathcal{A}_1$ it holds that $f_1^{\mathbb{N}}(A) \cap \mathcal{A}_1 \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}_1' \subsetneq \mathcal{A}_1$ that satisfies (1) and (2).

In the context of networks the following results are known in the literature. In Herings et al. (2019), it is shown that the pairwise myopically stable set as defined in Herings et al. (2009) is the horizon-1 farsighted set. Also, the myopic stable set as defined in Demuynck et al. (2019a) is equal to the pairwise myopically stable set. In the context of our social environment corresponding to the housing matching model (N, P), we have the following result.

Lemma 7.15. *For all housing matching models (N, P), the myopic stable set of $\mathcal{E}(N, P)$ is the unique horizon-1 farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model. We already know from Demuynck et al. (2019a) that there exists a unique myopic stable set. Thus, it is sufficient to show that the definitions of a myopic stable set and a horizon-1 farsighted stable set are equivalent.

First, we show that horizon-1 deterrence of external deviations is condition (1) of Definition 3.22. Note that condition (1) in Definition 7.14 means that each permutation matrix inside $\mathcal{A}_1$ is not strictly dominated by any permutation matrix outside $\mathcal{A}_1$. In other words, $\forall\ A \in \mathcal{A}_1$ we must have that $f(A) \subseteq \mathcal{A}_1$, which is condition (1) in Definition 3.22.

Recall that $f_1(A) = f(A)$ for all $A \in X$. Thus, we also have that horizon-1 iterated external stability is equal to condition (2) in Definition 3.22. Note that both the myopic stable set and a horizon-1 farsighted stable set must satisfy minimality in the sense that there does not exist a proper subset that satisfies the other two conditions. This shows that the definitions of a myopic stable set and a horizon-1 farsighted set are equivalent. Thus, the myopic stable set is the unique horizon-1 farsighted stable set. □

For each housing matching model, we can conclude from Theorem 3.8, Lemma 3.23 and Lemma 7.15, that the horizon-1 farsighted stable set contains A^*. This result can be generalized to all horizon-K farsighted stable sets with $K \in \mathbb{N}$.

In the context of networks Herings et al. (2019) proved that the set of horizon-K pairwise stable networks, the set of networks that are not horizon-K pairwise dominated, is a subset of each horizon-K farsighted set. In the context of our social environment of housing matching model (N, P), we know from Corollary 7.10, that $KFCO = \{A^*\}$ for all $K \geq 2$. Hence, we get the following result.

Lemma 7.16. *For all housing matching models (N, P), we have for all $K \in \mathbb{N}$ that each horizon-K farsighted stable set of $\mathcal{E}(N, P)$ must contain A^*.*

Proof. Let (N, P) be a housing matching model and let $K \in \mathbb{N}$. Suppose that $\mathcal{A}_K \subseteq X \setminus \{A^*\}$ is a horizon-K farsighted stable set of $\mathcal{E}(N, P)$. From Corollary 7.3, we know that A^* is not horizon-K strictly dominated, hence it holds that $f_K(A^*) = \{A^*\}$. This gives us that $f_K^{\mathbb{N}}(A^*) = \{A^*\}$, thus we have that $f_K^{\mathbb{N}}(A^*) \cap \mathcal{A}_K = \emptyset$. Hence, we can conclude that $\mathcal{A}_K$ does not satisfy the horizon-K iterated external stability condition. Hence, each horizon-K farsighted stable set of $\mathcal{E}(N, P)$ must contain A^*. □

From now of on, we focus on horizon-K farsighted stable sets for $K \geq 2$. From Herings et al. (2019), we know that a horizon-K farsighted set always exists, but that it does not have to be unique.

From Lemma 7.16, we know for all housing matching models that each set $\mathcal{A}_K \subseteq X \setminus \{A^*\}$ cannot be a horizon-K farsighted stable set for all $K \in \mathbb{N}$. With the help of Corollary 7.9 we show, for all housing matching models, that $\mathcal{A}_K = \{A^*\}$ is the unique horizon-K farsighted stable set for each $K \geq 3$.

Theorem 7.17. *For all housing matching models (N, P), we have for all $K \geq 3$ that the set $\mathcal{A}_K = \{A^*\}$ is the unique horizon-K farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model and let $K \geq 3$. First, we show that $\mathcal{A}_K = \{A^*\}$ satisfies conditions (2) and (3) in Definition 7.13. Note that $|\mathcal{A}_K| = 1$, hence $\mathcal{A}_K$ satisfies minimality. From Corollary 7.9, we know that $A^* \in f_K(A)$ for all $A \in X \setminus \{A^*\}$. Hence, we have that $A^* \in f_K^{\mathbb{N}}(A) \cap \mathcal{A}_K$ for all $A \in X \setminus \{A^*\}$. This shows that $\mathcal{A}_K$ also satisfies horizon-K iterated external stability.

The proof that $\mathcal{A}_K$ satisfies horizon-K deterrence of external deviations depends on whether $K = 3$ or $K \geq 4$. Let $K = 3$, then we need to show that $\mathcal{A}_3 = \{A^*\}$ satisfies horizon-3 deterrence of external deviations. In other words, we need to show that $\forall\, A' \in X \setminus \{A^*\}$ and $\forall\, S \in E(A^*, A')$, there exists $A'' \in (f_1(A') \cap \{A^*\}) \bigcup (f_2(A') \setminus f_1(A'))$ such that $\exists\, i \in S$ with $A^* \succsim_i A''$. Let $A' \in X \setminus \{A^*\}$ and let $S \in E(A^*, A')$. From Corollary 7.9, we know that $A^* \in f_2(A')$. There are two cases: $A^* \in f_1(A')$ and $A^* \notin f_1(A')$. In the former case we have that $A^* \in f_1(A') \cap \{A^*\}$ and in the latter case we have that $A^* \in f_2(A') \setminus f_1(A')$. Hence, in both cases we have that $A'' = A$ is a credible threat with $A^* \sim_i A''$ for all $i \in N$. Thus, $\mathcal{A}_3 = \{A^*\}$ satisfies horizon-3 deterrence of external deviations.

Now, we show that $\mathcal{A}_K = \{A^*\}$ satisfies horizon-K deterrence of external deviations for $K \geq 4$. Let $K \geq 4$, then we need to show that $\forall\, A' \in X \setminus \{A^*\}$ and $\forall\, S \in E(A^*, A')$, we have that there exists $A'' \in (f_{K-2}(A') \cap \{A^*\}) \bigcup (f_{K-1}(A') \setminus f_{K-2}(A'))$ such that $\exists\, i \in S$ with $A^* \succsim_i A''$. From Corollary 7.9, we know that $A^* \in f_{K'}(A')$ for all $K' \geq 2$ and for all $A' \in X \setminus \{A^*\}$. We know that $K \geq 4$, thus it holds that $K' = K - 2 \geq 2$. Hence, we get that $A^* \in f_{K-2}(A')$ for all $A' \in X \setminus \{A^*\}$. Thus, we have that $A'' = A^* \in f_{K-2}(A') \cap \{A^*\}$ with $A^* \sim_i A''$ for all $i \in N$. Thus, $\mathcal{A}_K = \{A^*\}$ satisfies horizon-K deterrence of external deviations.

This shows that $\mathcal{A}_K = \{A^*\}$ is a horizon-K farsighted stable set of $\mathcal{E}(N, P)$ for each $K \geq 3$. With the minimality condition, we can conclude for all $K \geq 3$ that each set

$\mathcal{A}'_K \subseteq X$ with $A^* \in \mathcal{A}'_K$ and $|\mathcal{A}'_K| > 1$ is not a horizon-K farsighted stable set of $\mathcal{E}(N, P)$. From Lemma 7.16, we already know that each set that does not contain A^* is not a horizon-K farsighted stable set of $\mathcal{E}(N, P)$ for all $K \geq 3$. Thus, for all $K \geq 3$, we can conclude that $\mathcal{A}_K = \{A^*\}$ is the unique horizon-K farsighted stable set of $\mathcal{E}(N, P)$. □

Now, we focus on horizon-2 farsighted stable sets. Recall that $f_0(A) = \{A\}$ for all $A \in X$. Note that for all sets $\mathcal{A}_2 \subseteq X$ and for all permutation matrices $A' \notin \mathcal{A}_2$, it holds that $f_0(A') \cap \mathcal{A}_2 = \{A'\} \cap \mathcal{A}_2 = \emptyset$. Hence, we can simplify horizon-2 deterrence of external deviations. The simplification of Definition 7.13 for $K = 2$ is given below.

Definition 7.18 (Horizon-2 farsighted stable set)**.**
Let (N, P) be a housing matching model. A set $\mathcal{A}_2 \subseteq X$ is a **horizon-2 farsighted stable set** of $\mathcal{E}(N, P)$ if it satisfies the following three conditions:

(1) **horizon-2 deterrence of external deviations**: $\forall\ A \in \mathcal{A}_2$, $\forall\ A' \notin \mathcal{A}_2$ and $\forall\ S \in E(A, A')$, there exists $A'' \in f_1(A') \setminus \{A'\}$ such that $\exists\ i \in S$ with $A \succsim_i A''$,

(2) **horizon-2 iterated external stability**: $\forall\ A \notin \mathcal{A}_2$ it holds that $f_2^{\mathbb{N}}(A) \cap \mathcal{A}_2 \neq \emptyset$,

(3) **minimality**: there is no proper subset $\mathcal{A}'_2 \subsetneq \mathcal{A}_2$ that satisfies (1) and (2).

Because of condition (1) in Definition 7.18, we have that the core must be a subset of each horizon-2 farsighted stable set.

Lemma 7.19. *For all housing matching models (N, P), it holds that each horizon-2 farsighted stable set $\mathcal{A}_2 \subseteq X$ of $\mathcal{E}(N, P)$ must contain the core, i.e. $CO \subseteq \mathcal{A}_2$.*

Proof. Let (N, P) be a housing matching model and let $\mathcal{A}_2 \subseteq X$ be a horizon-2 farsighted stable set of $\mathcal{E}(N, P)$ with $CO \nsubseteq \mathcal{A}_2$. Hence, there exists $A' \in CO \setminus \mathcal{A}_2$. Since $A' \in CO$, we have that $f_1(A') = f(A') = \{A'\}$. Thus, it holds that $f_1(A') \setminus \{A'\} = \emptyset$. Since $A' \notin \mathcal{A}_2$ and $\mathcal{A}_2$ satisfies horizon-2 deterrence of external deviations, we must have for all $A \in \mathcal{A}_2$ and for all $S \in E(A, A')$, that there exists $A'' \in f_1(A') \setminus \{A'\}$ such that there is an agent $i \in S$ with $A \succsim_i A''$. Since $f_1(A') \setminus \{A'\} = \emptyset$ this statement is false. Hence, $\mathcal{A}_2$ does not satisfy horizon-2 deterrence of external deviations. This gives a contradiction. Thus, we can conclude that each horizon-2 farsighted stable set of $\mathcal{E}(N, P)$ must contain the core. □

From the proof of Lemma 7.19, we know that each set that satisfies horizon-2 deterrence of external deviations must contain the core. Thus, the core satisfies minimality. From Lemma 7.19, we also get that each horizon-2 farsighted stable set contains the core. Hence, if the core satisfies conditions (1) and (2) in Definition 7.18, we can conclude with the minimality condition that CO is the unique horizon-2 farsighted stable set. In the following example, we show for a specific housing matching model that the core is the unique horizon-2 farsighted stable set.

Example 7.20 (Example 6.16 continued)**.**
Let $n = 3$ and let the preference matrix P be as in Example 6.16:

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

Note that $A^* = (132)$. Recall that $f((1)(2)(3)) = X$, $f((1)(23)) = \{(1)(23), (2)(13)\}$, $f((2)(13)) = \{(2)(13), (3)(12)\}$, $f((3)(12)) = \{(3)(12), (1)(23)\}$ and that $f((123)) = \{(123), (132)\}$. Hence, we have that $CO = \{A^*\}$. We show that the core satisfies the three conditions of Definition 7.18. Note that $|CO| = 1$, hence the core automatically satisfies minimality. From Theorem 7.8, we know that $A^* \in f_2(A) \subseteq f_2^{\mathbb{N}}(A)$ for all $A \in X \setminus \{A^*\}$, hence the core also satisfies horizon-2 iterated external stability.

Thus, we need to show that the core satisfies horizon-2 deterrence of external deviations. In other words, we need to show that $\forall\ A' \in X \setminus \{A^*\}$ and $\forall\ S \in E(A^*, A')$, there exists $A'' \in f_1(A') \setminus \{A'\}$ such that $\exists\ i \in S$ with $A^* \succsim A''$. Note that $f_1(A) = f(A)$ for all $A \in X$. Let $A' = (1)(23)$, then we have that $f_1(A') \setminus \{A'\} = \{(2)(13)\}$ and that $E(A^*, A') = \{\{2,3\}, N\}$. Suppose that coalition $\{2,3\}$ considers the deviation from A^* to A'. After this deviation, coalition $\{1,3\}$ can deviate from A' to $(2)(13)$, but then agent 2 is worse off then at A^*. Hence, the deviation from A^* to A' is deterred. In Table 19, it is shown that each deviation from A^* to any $A' \in X \setminus \{A^*\}$ is deterred by a credible threat.

$A' \in X \setminus \{A^*\}$	$E(A^*, A')$	$A'' \in f_1(A') \setminus \{A'\}$	$\exists\ i \in S$ with $A^* \succsim_i A''$
(1)(2)(3)	$2^N \setminus \{\emptyset\}$	A^*	$A^* \sim_i A''\ \forall\ i \in N$
(1)(23)	$\{\{2,3\}, N\}$	(2)(13)	$(132) \succ_2 (2)(13)$
(2)(13)	$\{\{1,3\}, N\}$	(3)(12)	$(132) \succ_3 (3)(12)$
(3)(12)	$\{\{1,2\}, N\}$	(1)(23)	$(132) \succ_1 (1)(23)$
(123)	$\{N\}$	A^*	$A^* \sim_i A''\ \forall\ i \in N$

Table 19: Illustration that the core satisfies horizon-2 deterrence of external deviations.

Thus, the core is the unique horizon-2 farsighted stable set of $\mathcal{E}(N, P)$. △

The result in Example 7.20 that $\mathcal{A}_2 = \{A^*\}$ is the unique horizon-2 farsighted stable set, can be generalized to all housing matching models such that $CO = \{A^*\}$.

Theorem 7.21. *For all housing matching models (N, P) such that $CO = \{A^*\}$, we have that the set $\mathcal{A}_2 = \{A^*\}$ is the unique horizon-2 farsighted stable set of $\mathcal{E}(N, P)$.*

Proof. Let (N, P) be a housing matching model such that $CO = \{A^*\}$. First, we show that $\mathcal{A}_2 = \{A^*\}$ is a horizon-2 farsighted stable set. Note that $\mathcal{A}_2$ satisfies minimality, since $|\mathcal{A}_2| = 1$. From Theorem 7.8, we know that $A^* \in f_2(A) \subseteq f_2^{\mathbb{N}}(A)$ for all $A \in X \setminus \{A^*\}$. Hence, $\mathcal{A}_2$ also satisfies horizon-2 iterated external stability.

We show that $\mathcal{A}_2 = \{A^*\}$ satisfies horizon-2 deterrence of external deviations by a proof by contradiction. Hence, suppose that $\mathcal{A}_2$ does not satisfy horizon-2 deterrence of external deviations. In other words, $\exists\ A' \in X \setminus \{A^*\}$ and $\exists\ S \in E(A^*, A')$, such that $\forall\ A'' \in f_1(A') \setminus \{A'\}$ we have that $A'' \succ_i A^*$ for all $i \in S$. Note that $f_1(A') \setminus \{A'\} \neq \emptyset$, since $CO = \{A^*\}$ and $A' \in X \setminus \{A^*\}$. There are two cases: $A^* \in f_1(A') \setminus \{A'\}$ and $A^* \notin f_1(A') \setminus \{A'\}$. In the former case we get a contradiction by taking $A'' = A^*$, since then we have that $A^* \sim_i A''$ for all $i \in N$.

In the latter case, we get that $\exists\, A'' \in f_1(A') \setminus \{A'\}$ such that $A'' \neq A^*$ and $A'' \succ_i A^*$ for all $i \in S$. Note that $A'' \in f_1(A') \setminus \{A'\}$ means that there exists $S^2 \in E(A', A'')$ such that $A'' \succ_j A'$ for all $j \in S^2$. Hence, with $S^1 = S \in E(A^*, A')$ and $A'' \succ_i A^*$ for all $i \in S$, we can conclude with the following sequence

$$A^* \xrightarrow[S^1=S]{} A' \xrightarrow[S^2]{} A''$$

that $A'' \gg_2 A^*$. This contradicts the fact that A^* is not horizon-2 strictly dominated in $\mathcal{E}(N, P)$. Hence, we can conclude that $\mathcal{A}_2$ satisfies horizon-2 deterrence of external deviations. Thus, we get that $\mathcal{A}_2 = \{A^*\}$ is a horizon-2 farsighted stable set of $\mathcal{E}(N, P)$.

We know that $CO = \{A^*\}$. Hence, from Lemma 7.19, we get that each horizon-2 farsighted stable set contains A^*. We already showed that $\{A^*\}$ satisfies conditions (1) and (2) in Definition 7.18. Hence, with the minimality condition we can conclude that $\mathcal{A}_2 = \{A^*\}$ is the unique horizon-2 farsighted stable set of $\mathcal{E}(N, P)$. $\square$

8 Conclusion

In Section 3, we studied the core, the von Neumann-Morgenstern stable set and the myopic stable set of Demuynck et al. (2019a) under the assumption that agents are myopic. We found the results given in Table 20.

		Myopic stability concepts, $\mathcal{A} \subseteq X$ is		
		core	vNM stable set	myopic stable set
dominance	strict	$A^* \in CO$	$CO \subseteq \mathcal{A}$	$CO \subseteq \mathcal{A}$
	weak	$SCO = \{A^*\}$	$A^* \in \mathcal{A}$	strong core

Table 20: Overview of the results under the assumption that all agents are myopic.

Under the assumption that agents are fully farsighted, we studied the farsighted core, the farsighted vNM stable set, the largest consistent set of Chwe (1994) and the DEM farsighted stable set, as given in Herings et al. (2010), with respect to three different definitions of indirect dominance. In Section 4, Section 5 and Section 6, we studied these four solution concepts with respect to indirect dominance, with respect to indirect weak dominance and with respect to indirect antisymmetric weak dominance, respectively. The results can be found in Table 21.

		Fully farsighted stability concepts, $\mathcal{A} \subseteq X$ is			
		farsighted core	vNM stable set	largest consistent set	DEM farsighted stable set
dominance	$\gg$	$FCO = \{A^*\}$	$\mathcal{A} = \{A^*\}$	$A^* \in \mathcal{A}$	$\mathcal{A} = \{A^*\}$
	$\ggcurly$	$SFCO = \emptyset$ or $SFCO = \{A^*\}$	$A^* \notin \mathcal{A}$ or $\mathcal{A} = \{A^*\}$	$A^* \in \mathcal{A}$	$A^* \notin \mathcal{A}$ or $\mathcal{A} = \{A^*\}$
	$\ggcurly_a$	$SAFCO = \{A^*\}$	$\mathcal{A} = \{A^*\}$	$\mathcal{A} = \{A^*\}$	$\mathcal{A} = \{A^*\}$

Table 21: Overview of the results under the assumption that all agents are fully farsighted. Note that the largest consistent set is denoted by $\mathbb{A}_{\gg}$ in Section 4, by $\mathbb{A}_{\ggcurly}$ in Section 5 and by $\mathbb{A}_{\ggcurly_a}$ in Section 6.

Moreover, we also studied the intermediate case between myopia and full farsightedness, in which agents can look at most K steps ahead. For $K \in \mathbb{N}$, we studied the horizon-K farsighted core, the horizon-K vNM stable set and the horizon-K farsighted stable set, as defined in Herings et al. (2019). As expected, the results depend on the degree of farsightedness of the agents. The extreme cases are myopia, $K = 1$, and full farsightedness, $K = \infty$. As shown in Section 7, the results of the intermediate case depend on whether $K = 2$ or $K \geq 3$. The results are given in Table 22.

		Horizon-K farsighted stability concepts, $\mathcal{A} \subseteq X$ is		
		horizon-K farsighted core	horizon-K vNM stable set	horizon-K farsighted stable set
degree of farsightedness	$K = 1$	core	vNM stable set	myopic stable set
	$K = 2$	$KFCO = \{A^*\}$	$\mathcal{A} = \{A^*\}$	$CO \subseteq \mathcal{A}$
	$K \geq 3$	$KFCO = \{A^*\}$	$\mathcal{A} = \{A^*\}$	$\mathcal{A} = \{A^*\}$
	$K = \infty$	FCO	farsighted vNM stable set	$\mathcal{A} = \{A^*\}$

Table 22: Overview of the results under the assumption that all agents are horizon-K farsighted.

Now, we compare the results given in Table 20, in Table 21 and in Table 22.
First, we compare with respect to the degree of farsightedness of the agents. When agents are myopic, the top trading allocation is a stable outcome, but it is not the only stable outcome. When agents can look two steps ahead, the top trading allocation is a stable outcome and only the horizon-2 farsighted stable set says that more outcomes are considered to be stable. Moreover, if agents can look at least three steps ahead, we can conclude that the top trading cycle allocation is considered to be the only stable outcome.

Secondly, we compare the stability concepts, which we studied within the same degree of farsightedness and the same notion of dominance. When agents are myopic, we can conclude that the three stability concepts that we studied are similar in the sense that A^* is considered to be stable, but differ in the sense that other allocations can be considered stable. In particular, the vNM stable set seems to be a weak concept.

When agents are neither myopic nor fully farsighted, i.e. $1 < K < \infty$, then we make a distinction between the case in which agents are horizon-2 farsighted and the case in which agents are at least horizon-3 farsighted. If all agents are horizon-2 farsighted, then all the three stability concepts that we studied say that A^* is a stable outcome. Moreover, the horizon-2 farsighted core and the horizon-2 vNM stable set say that allocation A^* is the only stable outcome. However, each horizon-2 farsighted stable set says that more outcomes are considered to be stable. If all agents are at least horizon-3 farsighted, the three stability concepts that we studied say that A^* is the only stable outcome.
When agents are fully farsighted, we can conclude that the farsighted core, the farsighted vNM stable set and the DEM farsighted stable set all give the same result.

Now, we compare the largest consistent set with the other farsighted solution concepts. As Chwe (1994) mentioned, the largest consistent set is a weak concept in the sense that it rules out with confidence. With the results given in Table 21, we can conclude that with respect to indirect dominance, this also holds in our housing matching model, since the largest consistent set compared to the other solution concepts contains allocations that are not stable. With respect to indirect antisymmetric weak dominance, we can conclude that all four solution concepts that we studied give the same result, namely that A^* is the only stable outcome.

Thirdly, we give a general comparison. We can conclude that the top trading cycle allocation is part of each stability concept, except when agents are fully farsighted and the notion of dominance is indirect weak dominance. Therefore, we only compare the other results. Each stability concept says that the top trading cycle allocation is stable. Moreover, in most cases A^* is the only stable outcome. While studying these stability concepts, we did expect that the top trading cycle allocation would play a role, but we did not expect that the top trading cycle allocation would play such an overwhelming role in the sense that it is the only stable outcome.

The results of indirect weak dominance, given in Table 21, are undesirable. They arise, because agents who are indifferent between the end state and the current state may be required to move according to the definition of indirect weak dominance. Kawasaki (2010) introduced indirect antisymmetric weak dominance to solve this problem of indirect weak dominance. Indirect antisymmetric weak dominance has an additional restriction compared to indirect weak dominance.
Kawasaki (2019) introduced another solution, which is minimal enforceability. Instead of defining a new type of dominance as in Kawasaki (2010), the effectivity correspondence has an additional restriction, namely the deviating coalition S is the minimal set that satisfies condition (1) in Definition 2.8. Recall that condition (1) in Definition 2.8 says that each agent in the deviating coalition S receives an item belonging to an agent in S. Kawasaki (2019) showed that this minimality condition does not affect the indirect dominance as defined in Klaus et al. (2010) and the indirect antisymmetric weak dominance as defined in Kawasaki (2010).
Therefore, for further research, one could study the effect of minimal enforceability on the results of the stability concepts with respect to indirect weak dominance. With minimal enforceability, we expect that the top trading cycle allocation is not indirectly weakly dominated, while it indirectly weakly dominates all other allocations. Therefore, we expect that the results of the stability concepts in this new setting will be similar to the results with respect to indirect antisymmetric weak dominance.

In this thesis, we studied stability concepts by varying the degree of the farsightedness of the agents and we made the assumption of homogeneity in the sense that all agents had the same degree of farsightedness. Therefore, for further research, one could study the effect of heterogeneity in the degree of farsightedness of the agents on the results of the horizon-K farsighted core, the horizon-K farsighted vNM stable set and the horizon-K farsighted stable set.
In the context of network games, the concept of heterogeneity in the degree of farsightedness is introduced in Herings and Khan (2022). As in Herings and Khan (2022), let K_i denote the degree of farsightedness of agent $i \in N$ and let $\underline{K} = (K_i)_{i \in N} \in \mathbb{N}^N$ represent the degree of foresight of all the agents. We expect that horizon-$\underline{K}$ strict dominance could be defined as in Definition 8.1.

Definition 8.1 (Horizon-$\underline{K}$ strict dominance)**.**
Let (N, P) be housing matching model, let $\underline{K} = (K_i)_{i \in N} \in \mathbb{N}^N$ and let $A, A' \in X$ be two different permutation matrices. The permutation matrix A' **horizon-$\underline{K}$ strictly dominates** A in $\mathcal{E}(N, P)$, denoted by $A' \gg_{\underline{K}} A$, if there exists $K' \in \mathbb{N}$ with $K' \leq \max_{i \in N} K_i$, such that there is a sequence of permutation matrices $A^0, \ldots, A^{K'} \in X$ with $A^0 = A$ and $A^{K'} = A'$ and there are coalitions $S^1, \ldots, S^{K'} \in 2^N \setminus \{\emptyset\}$, such that $\forall\ k \in \{1, \ldots, K'\}$ the following two conditions hold:

(1) $S^k \in E(A^{k-1}, A^k)$,

(2) for all $i \in S^k$, it holds that $A^h \succ_i A^{k-1}$ with $h = \begin{cases} K_i + k - 1 & \text{if } K_i + k - 1 \leq K', \\ K' & \text{if } K_i + k - 1 > K'. \end{cases}$

Note that if all agents have the same degree of farsightedness, i.e. $K_i = K$ for all $i \in N$, then Definition 8.1 is Definition 7.1. This is desirable, since it means that the setting in which all agents have the same degree of foresight is contained within the setting of heterogeneity.

Moreover, we expect that the concept of heterogeneity in the degree of foresight of the agents will lead to new results, since the top trading cycle allocation might not horizon-$\underline{K}$ strictly dominate all permutation matrices. To see this, one can look at the housing matching model (N, P) as in Example 7.20, i.e. $n = 3$ and the preference matrix

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix},$$

and prove that the top trading cycle allocation $A^* = (132)$ does not horizon-$(1, 2, 2)$ strictly dominate the allocation $(2)(13)$. Because of this, we expect that agents with the lowest degree of foresight will determine whether other allocations than A^* are considered to be stable.
Furthermore, whether A^* horizon-$\underline{K}$ strictly dominates all other permutation matrices also seems to depend on whether the agent with the lowest degree of foresight is part of the first top trading cycle. To see this, one can look at the housing matching model (N, P) as in Example 6.13, i.e. $n = 3$ and the preference matrix

$$P = \begin{pmatrix} -2 & 0 & -1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix},$$

and prove that $A^* = (1)(23)$ horizon-$(1, 2, 2)$ strictly dominates all other allocations, but does not horizon-$(2, 1, 2)$ strictly dominate (123).
Moreover, we expect that the top trading cycle allocation A^* is not horizon-$\underline{K}$ strictly dominated, since according to A^* each agent in tc^τ gets his most preferred item of the remaining items in $N \setminus \left(\bigcup_{1 \leq r \leq \tau - 1} S(tc^r) \right)$. Hence, we expect that A^* is considered to be stable.

Therefore, studying the effect of heterogeneity in the degree of foresight of the agents on the stability concepts seems to be interesting.

References

Chartrand, G., Polimeni, A. D., & Zhang, P. (2018). *Mathematical proofs: A transition to advanced mathematics* (Fourth). Pearson. https://www.pearson.com/us/higher-education/program/PGM1763590.html

Chwe, M. S. Y. (1994). Farsighted coalitional stability. *Journal of Economic Theory*, *63*(2), 299–325. https://doi.org/10.1006/jeth.1994.1044

Demuynck, T., Herings, P. J. J., Saulle, R. D., & Seel, C. (2019a). The myopic stable set for social environments. *Econometrica*, *87*(1), 111–138. https://doi.org/10.3982/ECTA14954

Demuynck, T., Herings, P. J. J., Saulle, R. D., & Seel, C. (2019b). Supplement to “the myopic stable set for social environments”. *Econometrica*. https://doi.org/10.3982/ECTA14954

Diamantoudi, E., & Xue, L. (2003). Farsighted stability in hedonic games. *Social Choice and Welfare*, *21*(1), 39–61. http://www.jstor.org/stable/41106544

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (Third). John Wiley & Sons, Inc., Hoboken, NJ.

Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, *69*(1), 9–15. http://www.jstor.org/stable/2312726

Herings, P. J. J., & Khan, A. (2022). Network stability under limited foresight [CentER Discussion Paper Nr. 2022-013]. *CentER, Center for Economic Research*, *2022-013*. http://dx.doi.org/10.2139/ssrn.4123996

Herings, P. J. J., Mauleon, A., & Vannetelbosch, V. J. (2004). Rationalizability for social environments. *Games and Economic Behavior*, *49*, 135–156. https://doi.org/10.1016/j.geb.2003.11.003

Herings, P. J. J., Mauleon, A., & Vannetelbosch, V. J. (2009). Farsightedly stable networks. *Games and Economic Behavior*, *67*, 526–541. https://doi.org/10.1016/j.geb.2008.12.009

Herings, P. J. J., Mauleon, A., & Vannetelbosch, V. J. (2010). Coalition formation among farsighted agents. *Games*, *1*(3), 1–13. https://EconPapers.repec.org/RePEc:gam:jgames:v:1:y:2010:i:3:p:286-298:d:9473

Herings, P. J. J., Mauleon, A., & Vannetelbosch, V. J. (2017). Stable sets in matching problems with coalitional sovereignty and path dominance. *Journal of Mathematical Economics*, *71*, 14–19. https://doi.org/10.1016/j.jmateco.2017.03.003

Herings, P. J. J., Mauleon, A., & Vannetelbosch, V. J. (2019). Stability of networks under horizon-K farsightedness. *Economic Theory*, *68*, 177–201. https://doi.org/10.1007/s00199-018-1119-7

Kawasaki, R. (2010). Farsighted stability of the competitive allocations in an exchange economy with indivisible goods. *Mathematical Social Sciences*, *59*, 46–52. https://doi.org/10.1016/j.mathsocsci.2009.10.006

Kawasaki, R. (2019). Minimal enforceability and indirect domination relations in the Shapley-Scarf economy. *SSRN Electronic Journal*, 24. http://dx.doi.org/10.2139/ssrn.3421952

Kimya, M. (2023). Coalition formation under dominance invariance. *Dynamic Games and Applications*, *13*(4), 480–496. https://doi.org/10.1007/s13235-022-00437-z

Klaus, B., Klijn, F., & Walzl, M. (2010). Farsighted house allocation. *Journal of Mathematical Economics*, *46*, 817–824. https://doi.org/10.1016/j.jmateco.2010.06.006

Mauleon, A., & Vannetelbosch, V. J. (2004). Farsightedness and cautiousness in coalition formation games with positive spillovers. *Theory and Decision*, *56*(3), 291–324. https://doi.org/10.1007/s11238-004-2646-1

Mauleon, A., Vannetelbosch, V. J., & Vergote, W. (2011). von Neumann-Morgenstern farsightedly stable sets in two-sided matching. *Theoretical Economics*, *6*(3), 499–521. https://doi.org/10.3982/TE527

Ray, D., & Vohra, R. (2015). The farsighted stable set. *Econometrica*, *83*(3), 977–1011. http://www.jstor.org/stable/43616959

Roth, A. E., & Postlewaite, A. (1977). Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics*, *4*(2), 131–137. https://doi.org/10.1016/0304-4068(77)90004-0

Shapley, L., & Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, *1*(1), 23–27. https://doi.org/10.1016/0304-4068(74)90033-0

von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton University Press.