Master Thesis Quantitative Finance and Actuarial Science
Tilburg University

# Valuation of insurance products using a Normal Inverse Gaussian distribution


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Tilburg, December 2015


**Abstract:** This thesis tests the Normal Inverse Gaussian (NIG) distribution as an improvement of the normal distribution in a Black-Scholes setting. The NIG distribution is better at catching characteristics such as heavy tails and peakedness than the normal distribution and is an improvement of the normal distribution in terms of historical calibration. A calibration on option prices using the closed-form formula for a European call option price is limited by computation time and instability. Insurance products are valued using the NIG distribution, resulting in unstable but higher prices than the standard Black Scholes ones which can be related to the larger jumps observed in the NIG distribution. The NIG has marginal influence on Markowitz's portfolio theory which only depends on volatility and return; these are similar for the NIG and normal distribution.



**Keywords:** Normal Inverse Gaussian, insurance, valuation, calibration, comparison, Black-Scholes, option pricing, Markowitz



*The author would like to thank the Deloitte Financial Risk Management team for their guidance and useful comments.

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# Chapter 1

# Introduction

Complex financial products, such as derivatives, variable annuities and unit-linked products are introduced on a regular basis by financial institutions. The volatile market and economic crisis in 2008 have driven the regulators of the financial industry to be more focused on the desire for market valuations. Market consistent valuation of assets and liabilities of insurers is part of Pillar I of the new regulatory framework introduced by the European Commission; Solvency II (Sol (2014)).
Insurance companies offer complex investment-linked insurance products, which are created to cover certain risks for policyholders. These products are often hard to value, which can cause an unforeseen mismatch on the balance sheet in times of stress (periods with a lot of claims, e.g. natural disasters, economic crisis etc.). These complex products are illustrated using examples.

Two main products that are underwritten by insurers are Variable Annuities and unit-linked products (AAG (2012)). These products are very similar; both contracts let the buyer pay a premium, which is invested in a fund. The final payoff depends on the value of the fund during the contract period and pre-specified guaranteed amount. The difference is that the former is flexible, i.e. guarantees and/or riders can be added, removed or changed during the contract period while the unit-linked contract details are set in stone during the entire contract period. An example of a Variable Annuity and a unit-linked product are respectively the Guaranteed Minimum Accumulation Benefit (Bauer et al. (2007)) and the Asian option (Hull (2003)).

Variable Annuities and unit-linked products are complex or even impossible to price analytically. A Monte Carlo approach is applicable for the pricing of such products. An Economic Scenario Generator (ESG) applies Monte Carlo theory by simulating possible paths to get an estimation of the market value. In short, an ESG is a tool that generates a collection of simulated economic scenarios that represent a distribution of possible economic futures. Interest rate curves, stock prices, fixed income (e.g. bond markets) or currencies are examples of market variables that are often simulated using an ESG.

Scenarios can both be created in a real-world and risk-neutral setting. In a risk-neutral setting investors are risk-neutral. Here, two investments with the same expected return are equally attractive, even if one is perfectly safe (e.g. a bank) and one is very risky (e.g. stocks) and thus assumes no risk-premia. In this universe, the value of a volatile asset is its expected future value discounted to the present value using the risk-free rate. A more in-depth analysis of the risk-neutral adjustment is discussed in Holton (2005). A real-world ESG allows one to create future scenarios, this helps in choosing an optimal portfolio (e.g. a return-risk portfolio).
Both settings are very functional for risk management at insurers and are used for Asset Liability Management (ALM). ERM (2014) discusses ESGs more extensively. Chapter 5 includes more detail and describes an implementation of the ESG.

To create scenarios using an ESG, the calibration of the underlying model is crucial. Poor model choice or poorly estimated parameters may result in inaccurate future scenarios and poor decision-making with possibly catastrophic results. There are two main forms of calibration, one based on historical data that reflects real-world scenarios and one based on option data that provides implied parameters, i.e. parameters that reflect the expectations of investors in the market. These implied parameters are risk-neutral because they are extracted from market observed prices and are therefore used for the valuation of products.

Black-Scholes (BS) is the best known model for the estimation of option prices and the simulation of future stock prices. Unfortunately, some assumptions in the BS model have been proven to be incorrect (Cont (2001)). Multiple adjustments to the standard BS model have been made in previous literature, e.g. stochastic volatility (Heston (1993)), jump diffusion (Kou (2002)) and/or a short rate model. These characteristics allow future scenarios to better reflect historical data. However some assumptions are seldom changed and differ a lot from empirically observed results. One of these assumptions is that stock returns are lognormally distributed.
The NIG distribution is discussed in current literature as a better fit due to the possibility of having heavier tails and a higher kurtosis than a normal distribution. The distribution is discussed in Chapter 2. Fernandes (2012) shows the historical fit of the NIG distribution to S&P 500 data.
This thesis discusses the possibility of a Normal Inverse Gaussian (NIG) distribution as an improvement to the normal distribution, suggesting that in a real-world setting the NIG distribution is a more adequate fit to stock returns. In a risk-neutral setting the application in a BS setting for the valuation of insurance products is tested.

The main question this thesis answers is whether a Normal Inverse Gaussian distribution performs better than a normal distribution if used within a Black Scholes environment. This thesis starts with a chapter that introduces the Normal Inverse Gaussian distribution; its main properties as well as its usefulness in the process to improve prediction of stock prices.
A well known part of the Black-Scholes concept in the literature is the derivation of a closed-form formula for the price of an option. Such a formula can also be derived for the Normal Inverse Gaussian distribution and is done in the following chapter.
To test whether the NIG performs better than a normal distribution for the prediction of asset returns, a calibration is performed. Initially, this is done by fitting the NIG distribution to historic asset returns, a calibration on option data is performed to obtain implied and risk-neutral parameters. The calibration results are compared to the calibration results of the standard Black Scholes model.
The next chapter uses the results of the calibration in an Economic Scenario Generator to create future economic scenarios. Economic scenarios are created based on both risk-neutral and real-world parameters. The former is used for the pricing of insurance products, while the latter is used for optimal portfolio (risk-return) theory. This thesis tries to answer if the NIG model is an improvement of the BS model regarding the valuation of complex products and/or optimal portfolio theory. This assists in answering the main question.

# Chapter 2

# Normal Inverse Gaussian distribution

The NIG was initially introduced by Barndorff-Nielsen and Halgreen (1977) and was denoted as a member of the generalized hyperbolic distributions. A well known phenomenon in finance is that the returns on most financial assets are not normally distributed but have semi-heavy tails and a higher kurtosis. Eberlein and Keller (1995), Eberlein (2001) and Prause (1999) showed that medium-tailed generalized hyperbolic family of distributions provide a more adequate fit to stock returns observed in the market and use it in an option pricing model.
The main question of this thesis is whether the Normal Inverse Gaussian distribution, hereafter called the NIG distribution, performs better than the normal distribution in predicting asset returns and valuing insurance products. This chapter introduces the NIG distribution using its characteristics, e.g. the distribution functions and its moments. These play critical roles and are used for the calibration of the parameters and the simulation of the NIG-Lévy process. This chapter is intended to get the reader acquainted with the NIG distribution.

## 2.1 Density functions

This section introduces the density functions of the NIG distribution.

### 2.1.1 Probability Density Function (PDF)

The probability density function of the NIG distribution is given by

$$\mathrm{nig}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\gamma+\beta(x-\mu)} \tag{2.1}$$

Here $x$ is an element of the real numbers $\mathbb{R}$, $\gamma^2 = \alpha^2 - \beta^2$ and $K_\lambda$ the modified Bessel function of the third kind with index $\lambda$. The modified Bessel function of the third kind is equal to

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty u^{\lambda-1}e^{-\frac{1}{2}x(u^{-1}+u)}du$$

More information about the Bessel function can be found in Barndorff-Nielsen et al. (200). The NIG distribution is bounded by the following parameter constraints: $\delta > 0$, $\alpha \geq 0$, $\alpha^2 > \beta^2$ and $\mu \in \mathbb{R}$. Every parameter in the NIG distribution has its own function; $\alpha$ determines the shape,

$\beta$ the skewness, $\mu$ the location and $\delta$ the spread of the distribution. A change in each parameter can be seen in Figure 2.1. A 'standard' NIG probability density function (PDF) is chosen with parameters $\alpha = 75, \beta = 0, \delta = 0.01$ and $\mu = 0$. The density in each subfigure is created by changing one parameter; this parameter is denoted in the corresponding legend. Remark that the PDF is symmetric around $\mu$ provided that $\beta = 0$.

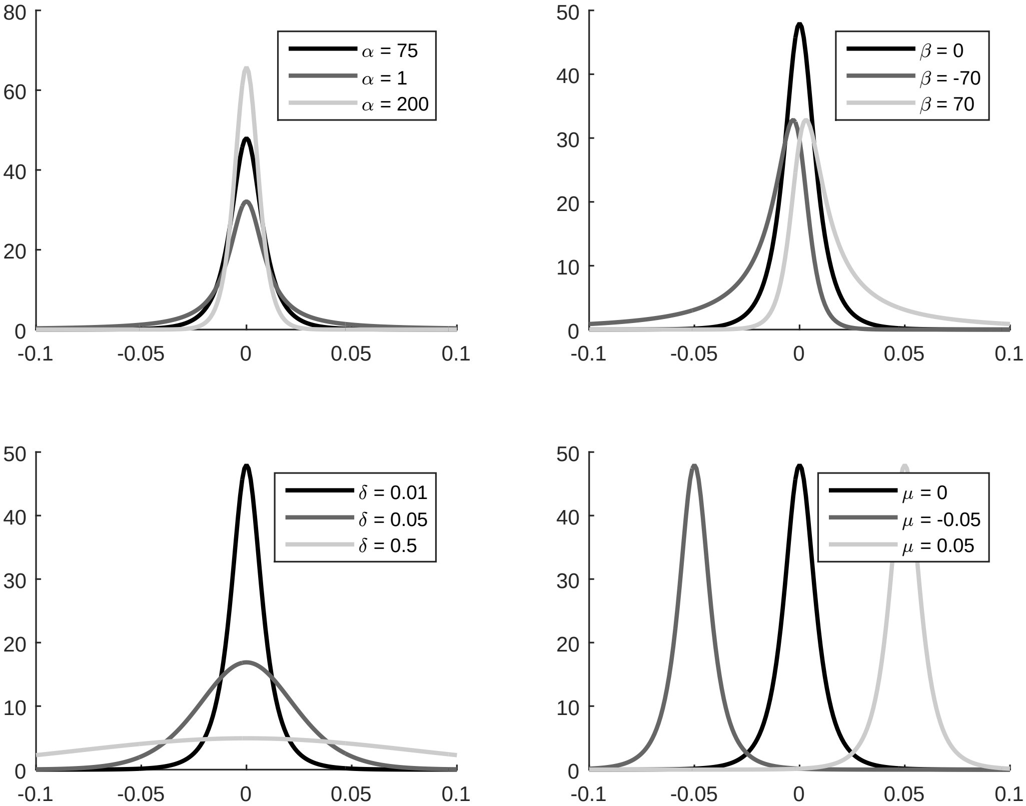


Figure 2.1: Influence of a change in parameters. All 'standard' parameters are characterized by $\alpha = 75, \beta = 0, \delta = 0.01$ and $\mu = 0$. The changed parameter is given in the legend.

### 2.1.2 Cumulative Distribution Function (CDF) & Survivor function

The PDF is used to create the cumulative distribution function (CDF) by taking the integral from $-\infty$ to $x$. The CDF is denoted by NIG and given by

$$\mathrm{NIG}(x; \alpha, \beta, \delta, \mu) = \int_{-\infty}^{x} \mathrm{nig}(y; \alpha, \beta, \delta, \mu) dy \tag{2.2}$$

The survivor function is equal to $1 - \mathbb{P}[X \leq x]$ where $\mathbb{P}[X \leq x]$ is the definition of the CDF and therefore represents the integral from $x$ to $\infty$. It is denoted by $\overline{NIG}$ and given by

$$\overline{\mathrm{NIG}}(x; \alpha, \beta, \delta, \mu) = \int_{x}^{\infty} \mathrm{nig}(y; \alpha, \beta, \delta, \mu) dy \tag{2.3}$$

## 2.2 Moment Generating Function (MGF) and moments

The moment generating function (MGF) for the NIG distribution is given as $\mathbb{E}[e^{tX}]$ where $X \sim \text{nig}(\alpha, \beta, \delta, \mu)$. This function is derived in Barndorff-Nielsen (1978) and given by

$$\text{MGF}(z) = e^{\mu z + \delta(\gamma - \sqrt{\alpha^2 - (\beta+z)^2})}$$

Central moments, i.e. the mean, variance, skewness and kurtosis, of the NIG distribution can be calculated using the MGF[1] and are respectively given by

$$\mathbb{E}[X] = \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{2.4}$$

$$\text{Var}[X] = \frac{\delta\alpha^2}{(\alpha^2 - \beta^2)^{\frac{3}{2}}} \tag{2.5}$$

$$\text{Skew}[X] = \frac{3\beta}{\alpha\sqrt{\delta\gamma}} \tag{2.6}$$

$$\text{Kurt}[X] = \frac{3(1 + 4\beta^2)/\alpha^2}{\delta\gamma} \tag{2.7}$$

Table 2.1 show the sensitivity of the first four central moments with respect to the underlying parameters. This table is introduced by Krog Saebo (2009) and is used for illustrative purposes only. The idea of this table is to interpret the underlying parameters of the NIG distribution better as they do not affect the distribution as straightforward as in a normal distribution. For example an increase in $\beta$ would lead to an increase of the mean, variance, skewness and kurtosis and a decrease in $\beta$ to a decrease of the mean and the skewness but to an increase of the variance and kurtosis.

Table 2.1: Changes in NIG moments with respect to changes in parameters.

| | | $\alpha$ | | $\beta$ | | | | $\delta$ | | $\mu$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Var | Kurt | $\mathbb{E}$ | Var | Skew | Kurt | Var | Kurt | $\mathbb{E}$ |
| Change | $\uparrow$ | $\downarrow$ | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |
| | $\downarrow$ | $\uparrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\uparrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |

Now remark that if $X \sim \text{nig}(x; \alpha, \beta, \delta, \mu)$, then $Y = e^X$ is a log-NIG random variable with mean given by

$$\mathbb{E}[e^X] = MGF(1) = e^{\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2})}$$

The form of the MGF shows that the NIG distribution is closed under convolutions[2]. For the NIG distribution, this means that if $X_1$ and $X_2$ are two independent random variables with NIG densities respectively given by $\text{nig}(x; \alpha, \beta, \delta_1, \mu_1)$ and $\text{nig}(x; \alpha, \beta, \delta_2, \mu_2)$, then $X_1 + X_2 \sim \text{nig}(x; \alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$. The NIG distribution is also proven to be infinitely divisible[3] by Barndorff-Nielsen et al. (200).
Now if $X \sim \text{nig}(\alpha, \beta, \delta, \mu)$ and $Y = aX + b$ then

$$Y \sim \text{nig}(\alpha/a, \beta/a, a\delta, a\mu + b)$$

The NIG distribution was constructed by Barndorff-Nielsen (1977) as a mixing distribution of the In-

[1] These characteristics are respectively calculated by $\mathbb{E}[X] = \text{MGF}'(0) = \mu$, $\text{Var}[X] = \text{MGF}''(0) - \mu^2 = \mu_2$, the derivation of the skewness and kurtosis get more complex and are therefore not shown.

[2] **Closed under convolution:** If $X_1, X_2$ are independent, identically distributed variables. Then the distribution $X = X_1 + X_2$ has the same distribution.

[3] The distribution of a real-valued random variable $x$ is infinitely divisible if for every $n \in \text{N}$, there exists a sequence of independent, identically distributed variables $(X_1, X_2, \ldots, X_n)$ such that $(X_1 + X_2 + \cdots + X_n)$ has the same distribution as $X$.

verse Gaussian (see Appendix A). Thus let $X$ be a NIG distributed random variable, the conditional given $W = w$ is then $N(\mu + \beta w, w)$ where $W$ is Inverse Gaussian distributed $(IG(\delta, \sqrt{\alpha^2 - \beta^2}))$ . This is also described by Jorgensen (1982).

# Chapter 3

# Black Scholes option pricing

This chapter introduces the concept of arbitrage-free option pricing. Initially, the standard Black Scholes call option pricing model will be described. Hereafter, the non-Gaussian Black Scholes option pricing model for the NIG distribution is derived. For more extensive derivations of these option pricing models the reader is referred to Hull (2003) and Schoutens (2003).

## 3.1 Gaussian Black Scholes option pricing

The famous Black Scholes analytic formula for an option was introduced by Black and Scholes (1973). This section will briefly discuss their research and gives a full derivation of the Black Scholes equation for European call option pricing using the principle of risk-neutral valuation.

A Black-Scholes economy needs to be defined. In the Black-Scholes economy there are two assets, a risky stock $S$ and a riskless bond $B$ that are driven by the Stochastic Differential Equations (SDE)

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r_t B_t dt \end{aligned}$$

The lognormal distribution also plays a crucial role and although it is standard in the literature, Appendix B gives some basic properties and calculations needed for the derivation of the Black-Scholes option price. Note that if Itô's Lemma[1] is applied to $\ln S_t$ where $S_t$ follows the Brownian Motion described above, then

$$d \ln S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

Giving the solution to the SDE

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

Since $W_t$ is normally $N(0,t)$ distributed, $\ln S_t$ is normally distributed with mean $\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Thus by using the properties of the lognormal distribution as discussed in Appendix B, $S_t$ follows a lognormal distribution with mean $S_0 \exp(\mu t)$ and variance $S_0^2 \exp(2\mu t)(\exp(\sigma^2 t) - 1)$.

[1] **Informal definition:** Assume $X_t$ is a drift-diffusion process, that satisfies $dX_t = \mu_t dt + \sigma_t dW_t$ then $df = (\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2})dt + \sigma_t \frac{\partial f}{\partial x} dW_t$ where $f$ is a twice-differentiable scalar function and $W_t$ a Wiener process.

The introduction already briefly discussed a risk-neutral universe where investors are equally attracted to an investment which is perfectly safe as to an investment that is risky as long as their expected return is equal. The Fundamental Theorem of Asset Pricing[2] is applicable in a risk-neutral universe and the arbitrage-free price of a derivative is given in terms of an expectation under a risk-neutral probability measure. If this is an European-type contingent payment $f(S_T)$, it has an arbitrage-free price $C_t$ given by

$$C_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} f(S_T)|F_t] \tag{3.1}$$

where $\mathbb{Q}$ is a probability measure, equivalent to $\mathbb{P}$, such that the discounted asset $e^{-rt}S_t$ is a $(F_t, \mathbb{Q})$ martingale[3] for all $t$. Thus, a risk-neutral measure $\mathbb{Q}$ is needed such that the discounted stock price using the bond as numéraire under $\mathbb{Q}$ is a martingale. The following SDE can be used for this property.

$$dS_t = r_t S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

Where $W_t^{\mathbb{Q}} = W_t + \frac{\mu - r}{\sigma} t$. Then under $\mathbb{Q}$, at $t = 0$, the stock price $S_t$ follows a lognormal distribution with mean $S_0 \exp(r_t t)$ and variance $S_0^2 \exp(2r_t t)(\exp(\sigma^2 t) - 1)$.

The European call price $C_{T-t}$ under $\mathbb{Q}$ and with constant interest rates is then

$$\begin{aligned}
C_{T-t} &= e^{-r(T-t)} \mathrm{E}^{\mathbb{Q}}[(S_T - K)^+ |F_t] \\
&= e^{-r(T-t)} \int_K^\infty (S_T - K) dF(S_T) \\
&= e^{-r(T-t)} \int_K^\infty S_T dF(S_T) - e^{-r(T-t)} K \int_K^\infty dF(S_T)
\end{aligned}$$

Remember that $S_t$ follows a lognormal distribution. Thus from Appendix B, the first integral can be calculated as follows

$$\begin{aligned}
\int_K^\infty S_T dF(S_T) =& \mathrm{E}^{\mathbb{Q}}[S_T | S_T > K] \\
=& L_{S_T}(K) \\
=& \exp\left(\ln S_t + (r - \frac{\sigma^2}{2})(T-t) + \frac{\sigma^2 (T-t)}{2}\right) * \\
& \Phi\left(\frac{-\ln K + \ln S_t + (r - \frac{\sigma^2}{2})(T-t) + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
=& S_t e^{r(T-t)} \Phi(d_1)
\end{aligned}$$

[2]**Fundamental Theorem of Asset Pricing:** The market specified by some real-world probability measure $\mathbb{P}$ is free of arbitrage, if and only if, given any numéraire N, there is a measure $\mathbb{Q}_N$ (a $\mathbb{Q}$-measure depending on N) which is equivalent to $\mathbb{P}$, and which is such that all relative price processes are $\mathbb{Q}_N$-martingales (Schumacher (2003)). The importance of the FTAP is that it guarantees absence of arbitrage.

[3]**Martingale:** A martingale is a stochastic process (i.e., a sequence of random variables) $X_1, X_2, X_3, ...$ that satisfies for any time n, $\mathbb{E}[|X_n|] < \infty$ , $\mathbb{E}[X_{n+1}|X_1, ..., X_n] = X_n$.

The second integral is now using the CDF of the lognormal distribution from Appendix B

$$\begin{aligned}
\int_K^\infty dF(S_T) &= 1 - F(K) \\
&= 1 - \Phi\left(\frac{\ln K - \ln S_t - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \Phi(-d_2) \\
&= \Phi(d_2)
\end{aligned}$$

Thus

$$C_{T-t} = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.2}$$

Where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

And

$$\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T-t} \\
&= \frac{\ln(S_t/K) + (r - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

## 3.2 NIG Black Scholes option pricing

Every infinitely divisible distribution generates a Lévy process where the increments are infinitely divisible distributed. Since the NIG is infinitely divisible, a NIG Lévy process (a process with independent and stationary NIG-distributed increments) can be created. In short, the NIG process is an extension of the Brownian Motion that allows for finite dimensional distributions with heavier tails. The NIG Lévy process is defined as follows.

**Definition 3.2.1.** Let $(\Omega, F, (F_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. An adapted càdlàg (a function that is right continuous and has left limits everywhere) $\mathbb{R}$-valued process $X = X(t)_{t\geq 0}$ with $X(0) = 0$ is a NIG Lévy process if $X(t)$ has independent and stationary increments distributed as $\text{nig}(x; \alpha, \beta, \delta t, \mu t)$.

Prause (1999) discusses the NIG-Lévy process in more detail. In short the NIG process is a jump process with the following characteristics:

$$\begin{aligned}
\text{E}[X_t] &= \mu t + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} t \\
\text{Var}[X_t] &= \frac{\delta\alpha^2}{(\alpha^2 - \beta^2)^{3/2}} t \\
\text{Skew}[X_t] &= \frac{3\beta}{\alpha\sqrt{\delta t\gamma}} \\
\text{Kurt}[X_t] &= \frac{3(1 + 4\beta^2)/\alpha^2}{\delta t\gamma}
\end{aligned}$$

This suggests that the expectation and variance increase with $t$, the volatility with $\sqrt{t}$, and the skewness and kurtosis decrease over time with respectively $\sqrt{t}$ and $t$. It is very similar to a Brownian

Motion, with an apparent continuity, but paths that are composed of an infinite number of small jumps. This makes it a very natural model for asset returns.
A general exponential model for asset prices using a NIG process is proposed by Barndorff-Nielsen (1998). This so-called exponential NIG-Lévy model for asset prices is defined as

$$S_t = S_0 e^{X_t} \tag{3.3}$$

where $t > 0$ and $X_t$ a $(F_t, \mathbb{P})$-NIG Lévy process. A plot of different paths of the NIG-Lévy model can be found in Figure 3.1.

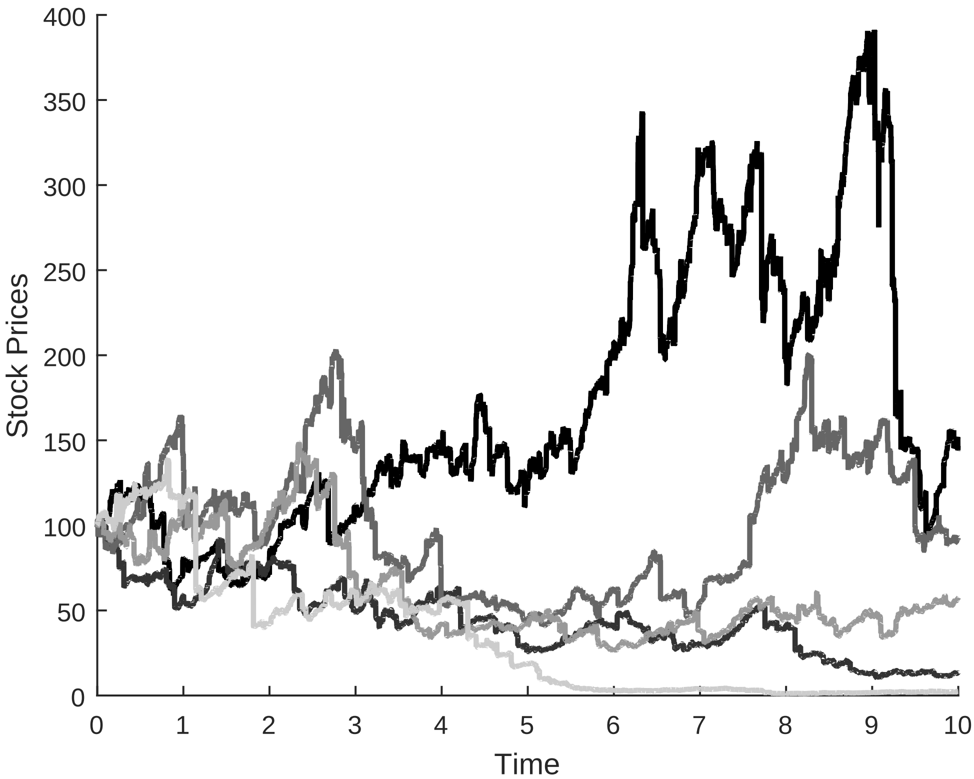

Figure 3.1: Five paths of the NIG-Lévy model plotted over 10 years with an initial stock price of 100 with parameters: $\alpha = 9$, $\beta = -4.5$, $\delta = 1.18$ and $\mu = 0$. These choices for the parameters are taken from Krog Saebo (2009).

Now assume there is a market with a risky asset defined as in equation (3.3) and a risk-free asset of the form $B_t = e^{rt}$, i.e. a similar risk-free asset as in the standard Black-Scholes economy. Then the Fundamental Theorem of Asset Pricing still applies and the arbitrage-free price of a derivative is given in terms of an expectation under a risk-neutral probability measure. Eberlein and Jacod (1997) showed that such NIG-Lévy models are incomplete and that the no arbitrage approach alone does not suffice to value contingent claims as well as that there exist an infinite number of different equivalent martingale measures. Schoutens (2003) found a particular measure for which the model becomes arbitrage-free and a closed form can be obtained. This measure is called the mean-correcting martingale measure. Yao et al. (2011) showed that although this measure cannot be equivalent to a physical probability for a pure jump Lévy process, a European call option price under this measure is arbitrage free. The mean-correcting martingale measure is described in Theorem 3.2.2 and applies the Fundamental Theorem of Asset Pricing to the NIG-Lévy process.

**Theorem 3.2.2.** *Let $S_t$ be the exponential NIG-Lévy price process defined as $S_t = S_0 e^{Z_t}$ (3.3) with parameters $[\alpha, \beta, \delta t, \mu t]$. One possible arbitrage-free price of a European-type contingent pay-off $f(S_T)$ at time $t$ is given by*

$$C_t = \mathbb{E}^{\mathbb{Q}_{\theta^*}}[e^{-r(T-t)} f(S_T)|F_t]$$

*where* $\mathbb{Q}_{\theta^*}$ *is an equivalent martingale measure under which* $S_t$ *is an exponential NIG-Lévy process with parameters* $[\alpha, \beta, \delta t, (\mu + \theta^*)t]$ *and the value of* $\theta^*$ *is given by*

$$\theta^* = r - \mu + \delta[\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}]$$

*This specific equivalent martingale measure is the so-called mean-correcting martingale measure and is initially described and proved by Schoutens (2003).*

Under this measure, the drift parameter $\mu$ becomes $\mu + \theta^*$ (therefore the name mean-correcting). Theorem 3.2.2 shows that a Black-Scholes formula can be derived for a European-type contingent claim.
It is important to notice that in the NIG market model all parameters are changed when moving from the real world to the risk-neutral world, this in contrast to the standard (GBM) market model where the volatility remains the same.

In the setting described in Theorem 3.2.2 the parameter $\mu + \theta^*$ is the drift and the condition $(\beta + 1)^2 < \alpha^2$ needs to be added. A European call with strike $K$ (i.e. $f(S_T, K) = (S_T - K)_+$) can then be derived as follows:

$$\begin{aligned} C_{T-t} &= \mathbb{E}^{\mathbb{Q}_{\theta^*}}[e^{-r(T-t)} \max[S_T - K, 0]] \\ &= \mathbb{E}^{\mathbb{Q}_{\theta^*}}[e^{-r(T-t)} \max[S * e^{Z_{T-t}} - K, 0]] \end{aligned}$$

Note that $S * e^{Z_{T-t}} - K > 0$ if and only if $Z_{T-t} > \log(K/S)$. $C_{T-t}$ can then be rewritten to

$$\begin{aligned} C_{T-t} = &\int_{\log(K/S)}^{\infty} e^{-r(T-t)}(S * e^z - K) * \mathrm{nig}(z; \alpha, \beta, \delta(T-t), (\mu+\theta^*)(T-t))dz \\ =&S * \int_{\log(K/S)}^{\infty} e^{-r(T-t)} e^z \mathrm{nig}(z; \alpha, \beta, \delta(T-t), (\mu+\theta^*)(T-t))dz- \\ &e^{-r(T-t)} K \int_{\log(\frac{K}{S})}^{\infty} \mathrm{nig}(z; \alpha, \beta, \delta(T-t), (\mu+\theta^*)(T-t))dz \\ =&S * \int_{\log(K/S)}^{\infty} e^{-r(T-t)+z+\delta\gamma(T-t)+\beta(z-(\mu+\theta^*)(T-t))} * \\ &\frac{K_1(\alpha\sqrt{\delta^2(T-t)^2 + (z-(\mu+\theta^*)(T-t))^2})}{\sqrt{\delta^2(T-t)^2 + (z-(\mu+\theta^*)(T-t))^2}} dz- \\ &e^{-r(T-t)} * K * \overline{\mathrm{NIG}}(\log(K/S); \alpha, \beta, \delta(T-t), (\mu+\theta^*)(T-t)) \end{aligned}$$

We now define

$$X(z) = \frac{K_1(\alpha\sqrt{\delta^2(T-t)^2 + (z-(\mu+\theta^*)(T-t))^2})}{\sqrt{\delta^2(T-t)^2 + (z-(\mu+\theta^*)(T-t))^2}}$$

Now remember from theorem 3.2.2 that

$$\theta^* = r - \mu + \delta[\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}]$$

Which leads to

$$r = \theta^* + \mu - \delta[\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}]$$

Filling this in the formula for the European call option gives

$$\begin{aligned}C_{T-t} =& S * \int_{\log(K/S)}^{\infty} e^{z-(\theta^*+\mu-\delta[\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}])(T-t)+\delta(T-t)\sqrt{\alpha^2-\beta^2}+\beta(z-\theta^*(T-t))} \\ & * X(z)dz - e^{-r(T-t)} * K * \overline{\mathrm{NIG}}(\log(K/S);\alpha,\beta,\delta(T-t),(\mu+\theta^*)(T-t)) \\ =& S * \int_{\log(K/S)}^{\infty} X(z) * e^{\delta(T-t)\sqrt{\alpha^2-(\beta+1)^2}+(\beta+1)(z-(\mu+\theta^*)(T-t))}dz- \\ & e^{-r(T-t)} * K * \overline{\mathrm{NIG}}(\log(K/S);\alpha,\beta,\delta(T-t),(\mu+\theta^*)(T-t))\end{aligned}$$

Resulting in

$$\begin{aligned}C_{T-t} =& S * \overline{\mathrm{NIG}}(\log(K/S);\alpha,\beta+1,\delta(T-t),(\mu+\theta^*)(T-t))- \\ & e^{-r(T-t)} * K * \overline{\mathrm{NIG}}(\log(K/S);\alpha,\beta,\delta(T-t),(\mu+\theta^*)(T-t))\end{aligned} \tag{3.4}$$

Which is a non-Gaussian Black-Scholes option price based on the NIG Lévy model. Godin et al. (2009) wrote a research paper that derives the same closed form solution using a distortion operator.

# Chapter 4

# Calibration

Calibration is by some called an ‘art’ instead of a ‘science’. This is because calibration is dependent on assumptions made, e.g. starting values, objective functions, optimization methods etc. It is therefore important to interpret, discuss and compare the results obtained. Calibration is important for this thesis, because generating economic scenarios using poorly calibrated models results in values of insurance products reflecting the wrong underlying risk and return. This thesis starts off with the calibration of the normal and NIG distribution based on historic data. Hereafter, a so-called implied calibration is performed on option data.

## 4.1 Data description

The S&P 500, or the Total Return Standard & Poor’s 500 index is used as dataset for the historical calibration. First of all, an index is chosen because insurers often choose indices as underlying asset for their products. The S&P 500 reflects the American stock market based on the market capitalization (price of a stock * number of stocks) of the 500 largest companies having their common stock issued on the NYSE or the NASDAQ. The S&P 500 is one of the most commonly tracked equity indices and is considered to be the best representation of the U.S. stock market as a whole. Over 7.8 trillion USD is benchmarked to the index, and it captures approximately 80% coverage of available market capitalization. Because the S&P 500 is in theory a continuously changing portfolio of the underlying stocks, it also pays out dividends. To take dividend pay-out into account, total return data is used; this is data that adjusts for dividends by making the assumption that dividends are used to reinvest in the S&P 500. The S&P500 Total Return Index (SPXT) is a representation of this phenomenon, and is used as data in this thesis for the calibration on historical events. The total return index goes back to January 4th, 1988 until October 21st, 2015. The price of this index and the histograms of the annual, monthly and daily return are plotted in Figure 4.1. The return figures show a normal distribution that fits the returns as well as possible. Note that indeed the properties discussed earlier in this thesis also apply on the data; the returns show a higher peak with heavier (negative) tails.

Option data on the S&P500 index will be used for the implied parameter calibration and is extracted from Bloomberg. The ‘valuation’ date is chosen to be September 30th, 2015 and these option prices are shown in Table E.1. The other option tables used for the intertemporal out of sample testing (i.e. testing of the model on different dates) are not shown but are also extracted from Bloomberg. As risk-free rate the USD Overnight Indexed Swap (OIS) is chosen because it reflects the US risk-free rate. The curve on the ‘valuation’ date is shown in Figure F.1. Note that this curve shows the spot rates, but since both the NIG and BS option price model are based on continuous interest rates, the following transformation is used

$$r_T^c = \log(1 + r_T^s)$$

where $r_T^c$ is the continuous interest rate and $r_T^s$ the spot rate with maturity T.

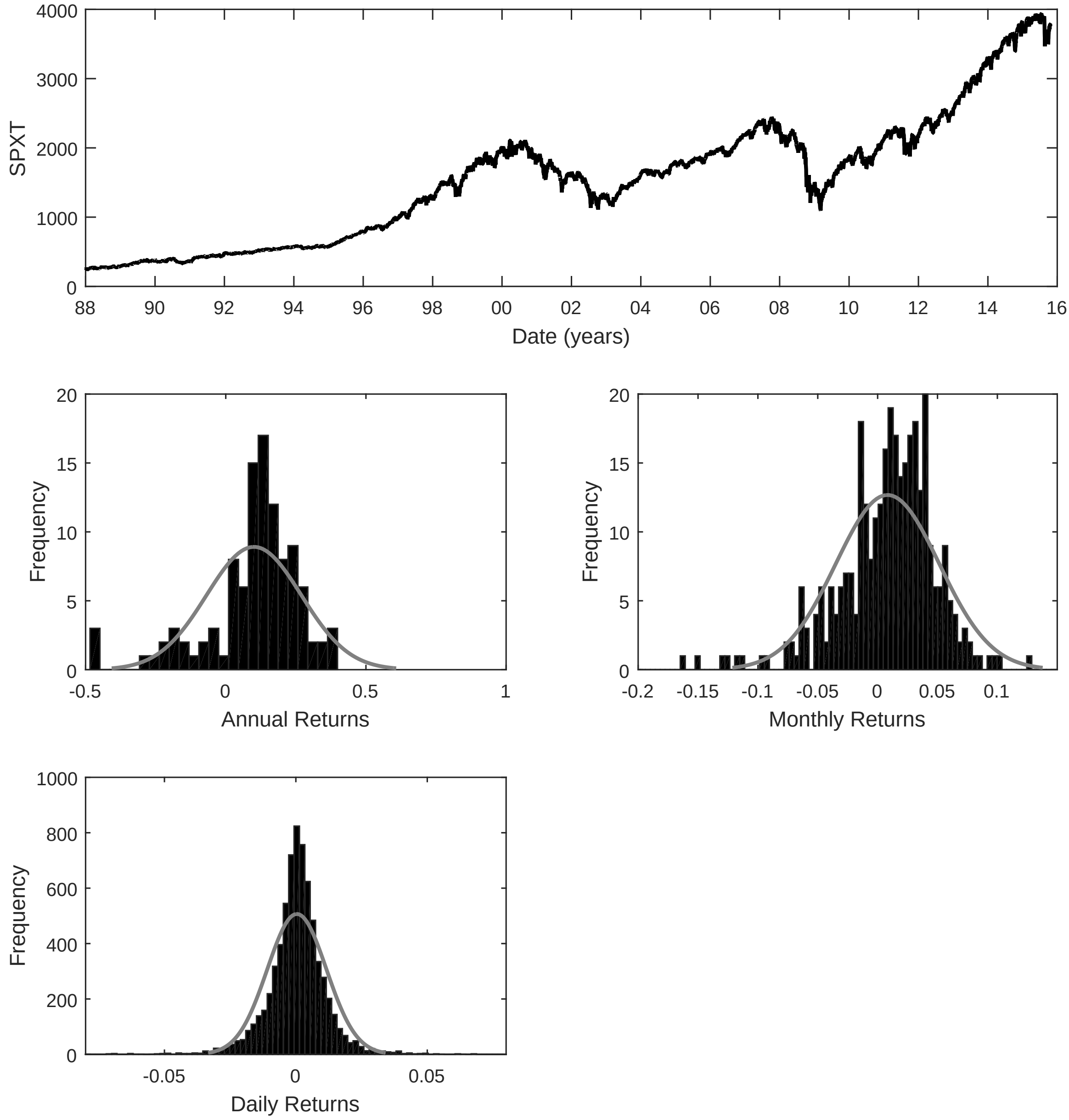


Figure 4.1: Characteristics of the S&P 500 Total Return data

## 4.2 Historical Black-Scholes model calibration

This section provides a methodology for the calibration of the lognormal distribution on historic asset returns, and are used as a best estimate for future returns. The issue of calibration on historical data is that historic events do not necessarily represent the future. To minimize this risk, some ways are proposed to test the calibrated model; back testing (or out of sample) and forward testing. The former tests the calibrated model on an earlier time period. The latter is closely related to back testing but instead the calibration is tested on future results. The time span of these methods are represented in Figure 4.2.

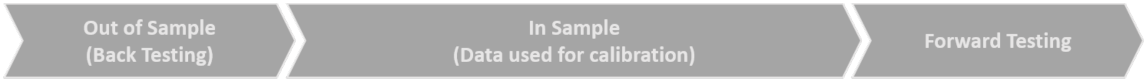


Figure 4.2: How to test a calibration based on historical data

The S&P500 Total Return Index goes from January 4th, 1988 until October 21st, 2015. The in sample historical calibration uses 15 years of data from January 1st, 1995 until December 31st, 2009. The back testing and forward testing can then respectively be done on January 4th, 1988 until December 31st 1994 and January 1st, 2010 until October 21st, 2015.

### 4.2.1 Method of Moments Estimation (MME)

The two moments used for the calibration of the normal distribution using Method of Moments Estimation is fairly straightforward. The first two central moments are

$$\begin{aligned} \mathrm{E}[X] &= \mu, \\ \mathrm{Var}[X] &= \sigma^2 \end{aligned}$$

Implying that the MME estimates are

$$\begin{aligned} \hat{\mu} &= \mathrm{E}[X], \\ \hat{\sigma} &= \sqrt{\mathrm{Var}[X]} \end{aligned}$$

i.e. $\mu$ is an indication of the mean and $\sigma$ of the standard deviation.

Since daily prices of the S&P500 Total Return index are extracted from Bloomberg, the first step is to calculate the daily log-returns.

$$r_j = \log(P_i/P_j)$$

Where $r_j$ is the return on day $j$ and $P_i$ the value of the Total Return Index on day $i$. The calibration is performed in the next subsections.

#### In sample calibration

Using the daily log-returns, the daily mean and standard deviation can be derived on the in sample data set (January 1st, 1995 – December 31st, 2009).

$$\begin{aligned} \bar{x}_{daily} &= 3.1117 * 10^{-4}, \\ s_{daily} &= 0.0128 \end{aligned}$$

Where $\bar{x}$ and $s^2$ are the sample mean and variance. The model that is created in this thesis is based on annual properties. Using the assumption that a year consists of approximately 252 trading days, the following estimation for the annual characteristics of the in sample data is obtained. The models are based on annual parameters.

$$\bar{x}_{annual} = 252 \times \bar{x}_{daily} = 0.0784 = \hat{\mu},$$
$$s_{annual} = \sqrt{252} \times s_{daily} = 0.2034 = \hat{\sigma}$$

As a check, the exact annual return and volatility are calculated and are respectively equal to 0.0774 and 0.2159 suggesting that a slight difference is created by going from daily to annual returns and volatility. The Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) statistics (Appendix C and D) are calculated to check whether the model is an adequate fit to the empirical data. The KS-test statistic is equal to 0.11 against a critical value of 0.34 and the AD-test statistic is equal to 0.67 against a critical value of 0.71 suggesting that the model is an adequate fit of the in sample data at the 95% significance level.

**Out of sample calibration**

The model is now back and forward tested; the KS test statistics are respectively equal to 0.17 and 0.22 compared to the corresponding critical values of 0.48 and 0.56. Here the null hypothesis is not rejected suggesting an acceptable fit on a 95% significant level. However the AD-test statistics are given by 0.69 and 1.07 against critical values of 0.65 and 0.61 rejecting the null hypothesis and thus suggesting that the model isn't a good fit on the back and forward data. As discussed in Appendix D, the AD-statistic is adjusted for the fact that the maximum distance between two CDF's as calculated in the KS-statistics almost never appears in the tails.

The results are compared to the NIG calibrated model in a later section of this chapter using CDF plots.

### 4.2.2 Maximum Likelihood Estimation (MLE)

Remember the PDF of a normal distribution to be

$$f_x(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

The likelihood function

$$L(\mu, \sigma^2; x_1, ..., x_n) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

The log-likelihood is then equal to

$$l(\mu, \sigma; x_1, ..., x_n) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

It follows automatically from the first order conditions that the MLE estimators are equal to

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

This suggests that the MLE estimate for $\mu$ is the same as the MME estimate, the $\sigma$ MLE estimate is fairly close to the MME one, the difference is that this one is not a sample variance (i.e. divided by $n$ instead of $(n-1)$).

**In sample calibration**

The estimates obtained on in sample data are

$$\hat{\mu} = 0.0774,$$
$$\hat{\sigma} = 0.2159$$

The results differ slightly from the in sample MME calibration due to the fact that already annualized data is used for the MLE calibration whereas daily returns were used for the MME calibration. The difference is created by the assumption that a year consists of 252 trading days and by the marginal difference between the sample and normal variance. Resulting in a KS-test statistic of 0.12 against a critical value of 0.34 and an AD-test statistic of 0.63 against a critical value of 0.71 again suggesting that the MLE calibrated normal distribution is an acceptable fit to the observed empirical in sample data at a 95% significance level.

**Out of sample**

The KS-test statistics for the back and forward testing are respectively 0.18 and 0.22 against critical values of respectively 0.48 and 0.56. However the AD-test statistics are respectively 0.75 and 1.10 against the critical values 0.65 and 0.61 rejecting the the null hypothesis on a 95% significance level thus suggesting that the model is not a good fit to the empirical back and forward data.

## 4.3 Historical Normal Inverse Gaussian model calibration

The same data as well as the same testing approach as in the previous section is used to make the comparison between the two models.

### 4.3.1 Method of Moments Estimation (MME)

In (2.4) - (2.7) the first four central moment of the NIG distribution are shown. The idea of the MME calibration is to solve these four equations for the four parameters of the NIG distribution. Karlis (2002) solves these equations for their parameters resulting in

$$\hat{\gamma} = \frac{3}{s\sqrt{3\bar{\gamma}_2 - 5\bar{\gamma}_1^2}}, \tag{4.1}$$

$$\hat{\beta} = \frac{\bar{\gamma}_1 s \hat{\gamma}^2}{3}, \tag{4.2}$$

$$\hat{\delta} = \frac{s^2\hat{\gamma}^3}{\hat{\beta}^2 + \hat{\gamma}^2}, \tag{4.3}$$

$$\hat{\mu} = \bar{x} - \hat{\beta}\frac{\hat{\delta}}{\hat{\gamma}}, \tag{4.4}$$

$$\hat{\alpha} = \sqrt{\hat{\gamma}^2 + \hat{\beta}^2} \tag{4.5}$$

Where $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the sample skewness and kurtosis. Note that the moments do not exist for $3\bar{\gamma}_2 < 5\bar{\gamma}_1^2$.

#### In sample calibration

The daily mean, variance, skewness and kurtosis of the log-returns can be derived on the in sample data set (January 1st, 1995 — December 31st, 2009).

$$\begin{aligned}
\bar{x}_{daily} &= 3.1117 \times 10^{-4}, \\
s^2_{daily} &= 1.6424 \times 10^{-4}, \\
\bar{\gamma}_{1,daily} &= -0.1994, \\
\bar{\gamma}_{2,daily} &= 11.1521
\end{aligned}$$

The change to annual characteristics is made again such that annual parameter estimates are obtained.

$$\begin{aligned}
\bar{x}_{annual} &= 252 \times \bar{x}_{daily} = 0.0784, \\
s^2_{annual} &= 252 \times s^2_{daily} = 0.0414, \\
\bar{\gamma}_{1,annual} &= \frac{\bar{\gamma}_{1,daily}}{\sqrt{252}} = -0.0126, \\
\bar{\gamma}_{2,annual} &= \frac{\bar{\gamma}_{2,daily}}{252} = 0.0443
\end{aligned}$$

The equations in (4.1) until (4.5) are now solved resulting in the following MME estimates

$$\hat{\alpha} = 40.6157, \tag{4.6}$$

$$\hat{\beta} = -1.4037, \tag{4.7}$$

$$\hat{\delta} = 1.6780, \tag{4.8}$$

$$\hat{\mu} = 0.1364 \tag{4.9}$$

The Kolmogorov Smirnov statistic can now be derived and a test statistic of 0.11 against a critical value of 0.34 and an AD-test statistic of 0.66 against a critical value of 0.76 suggest that the MME calibrated NIG model is an adequate fit to the in sample empirical data at a 95% significance level. In Figure 4.3 the CDF of the empirical data is plotted against the MME calibrated NIG CDF as well as the MME calibrated normal CDF.

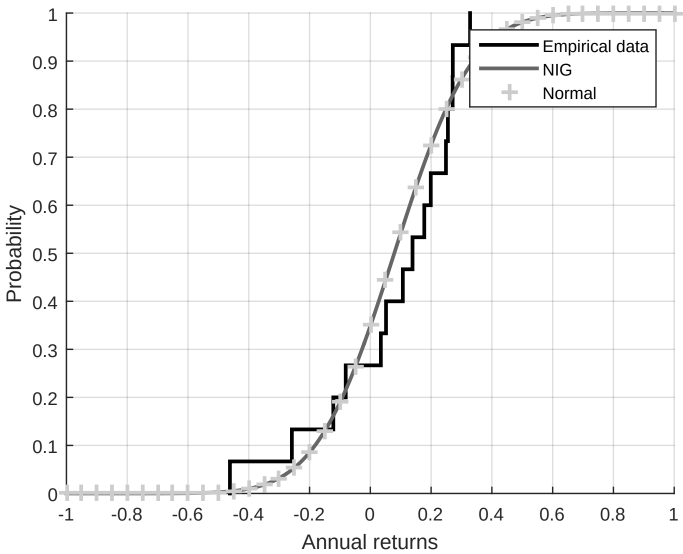


Figure 4.3: In sample empirical CDF vs. in sample MME calibrated NIG CDF and normal CDF

Note that the normal and NIG CDF are on top of each other. The difference between the two CDF's is computed and is marginal suggesting that the MME calibrated NIG distribution is similar to the MME calibrated normal distribution. The result that the test results of these models were similar is therefore not surprising.

**Out of sample**

The results are now tested on out of sample data. In Figure 4.4 the empirical CDF of the back and forward data is plotted against the MME calibrated NIG and normal CDF. The test statistics of the Kolmogorov-Smirnov test for the back and forward data are respectively 0.17 and 0.22 against critical values of 0.48 and 0.56. However the AD-test statistics are given by 0.68 and 1.07 against critical values of 0.76 and 0.76 suggesting an adequate fit on the back testing data, however the forward test is rejected on a 95% significance level.

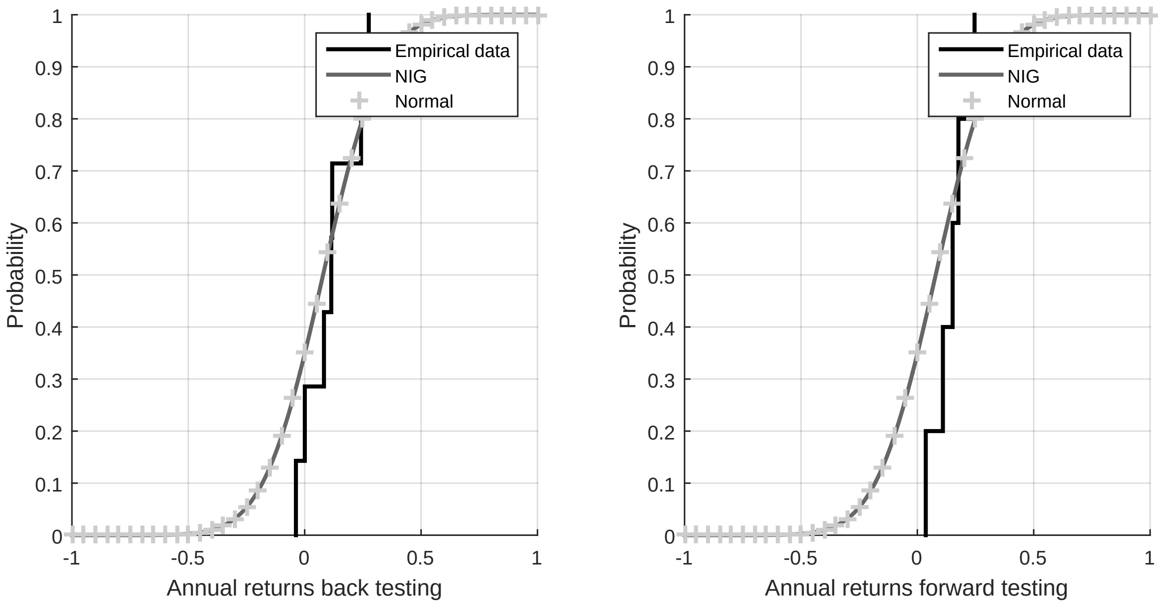


Figure 4.4: Out of sample empirical CDF vs. in sample MME calibrated NIG CDF and normal CDF.

### 4.3.2 Maximum Likelihood Estimation (MLE)

Karlis (2002) describes a new type of algorithm for the Maximum Likelihood Estimation of the NIG distribution. His idea is briefly discussed for illustrative purposes. Note that given a random sample of size $n$ from a NIG$(\alpha, \beta, \delta, \mu)$ distribution the log-likelihood function is given by

$$L = -n\log(\pi) + n\log(\alpha) + n(\delta\gamma - \beta\mu) - \frac{1}{2}\sum_{i=1}^{n}(1 + ((x_i - \mu)/\delta)^2) + \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} K_1(\delta\alpha(1 + ((x_i - \mu)/\delta)^2)^{1/2})$$

The first order conditions of this function involve the Bessel function, which makes direct maximization a difficult problem. The following Lemma is stated in Karlis (2002).

**Lemma 4.3.1.** One of the likelihood equations for ML estimation of the NIG distribution reduces to the first moment equation that equates the sample mean with the theoretical one.

**Proof.** The derivative of the log-likelihood with respect to $\beta$ is equal to

$$\frac{\partial L}{\partial \beta} = -n\mu + \sum_{i=1}^{n} x_i - \frac{n\delta\beta}{\gamma} = 0$$
$$\Leftrightarrow \bar{x} = \mu + \delta(\beta/\gamma)$$

The right-hand side of this equation is the theoretical mean, which proves the lemma given in Karlis (2002).
This property is going to be helpful for the EM algorithm described in Karlis (2002), because one parameter can be updated easily.

The EM algorithm created in Dempster et al. (1977) is an algorithm for MLE used on data that contains missing values. Every iteration consists of an expectation step followed by a maximization step, it is therefore called the EM-algorithm.
Now suppose that the true data $Y_i = (X_i, Z_i)$ consists of an observable part $X_i$ and an unobservable part $Z_i$. The estimation becomes straightforward if $Z_i$ is observable. During the E-step one computes the expectations of the unobservable part given the current values of the parameters, then during the M-step one maximizes the complete log-likelihood using the expectations computed in the E-step.

Remember from Chapter 2 that if $X$ is NIG distributed then $X$ given $Z = z$ is $N(\mu + \beta z, z)$ distributed where $Z$ Inverse Gaussian IG$(\delta, \sqrt{\alpha^2 - \beta^2})$ distributed. Assume that $z$ is known, then the log-likelihood L is given by

$$L = \text{constant} - \sum_{i=1}^{n} \frac{(x_i - \mu - \beta z_i)^2}{2z_i}$$

This is the log-likelihood of a normal distribution with parameters $\mu+\beta z$ as mean and $z$ as variance. By calculating the first order conditions, the estimating equations are obtained

$$\frac{\partial L}{\partial \mu} = \sum_{i=1}^{n} \frac{\mu}{z_i} + n\beta - \sum_{i=1}^{n} \frac{x_i}{z_i} = 0,$$
$$\frac{\partial L}{\partial \beta} = n\mu + \beta \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} x_i = 0$$

Solving this system leads to the estimates

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(x_i/z_i) - \bar{x}\sum_{i=1}^{n} 1/z_i}{n - \bar{z}\sum_{i=1}^{n} 1/z_i},$$
$$\hat{\mu} = \bar{x} - \hat{\beta}\bar{z}$$

Therefore if the values of $z_i$ are known the ML estimates are very easy to derive. Note that if these are unknown it is important to estimate the quantities $z$ and $1/z$. This means that for the NIG distribution where the $z$ is unknown, the E-step is the computation of the conditional expectation of the sufficient statistic for the Inverse Gaussian distribution, and are equal to $\sum Z_i$ and $\sum Z_i^{-1}$ (Bain and Engelhardt (2000)). Thus the algorithm for the MLE of the NIG distribution consists of calculating E$[z_i^{-1}|x_i, \theta^{(k)}]$ in step $k$ where $\theta^{(k)}$ are the current values of the parameters. The M-step then updates the parameters using the expectations of the sufficient statistics, derived at the E-step.

The EM-algorithm is described in Karlis (2002) as follows:
*E-step:* Let $\theta = (\alpha, \beta, \delta, \mu)$ be the vector of parameters estimated. Given the values of the parame-

ters after the $k$th iteration, say $\theta^{(k)}$, calculate the pseudovalues $s_i$ and $w_i$ as

$$s_i = \mathrm{E}[z_i|x_i, \theta^{(k)}] = \frac{\delta^{(k)}\phi^{(k)}(x_i)^{1/2}}{\alpha^{(k)}} \frac{K_0(\delta^{(k)}\alpha^{(k)}\phi^{(k)}(x_i)^{1/2})}{K_1(\delta^{(k)}\alpha^{(k)}\phi^{(k)}(x_i)^{1/2})}$$
$$w_i = \mathrm{E}[z_i^{-1}|x_i, \theta^{(k)}] \frac{\alpha^{(k)}}{\delta^{(k)}\phi^{(k)}(x_i)^{1/2}} \frac{K_{-2}(\delta^{(k)}\alpha^{(k)}\phi^{(k)}(x_i)^{1/2})}{K_{-1}(\delta^{(k)}\alpha^{(k)}\phi^{(k)}(x_i)^{1/2})}$$

For $i = 1, ..., n$ and where $\phi(x)^{(k)} = 1 + [(x - \mu^{(k)}]/\delta^{(k)}]^2$.
*M-step:* Update the parameters using the pseudovalues calculated at the E-step. Calculate $\hat{M} = \sum_{i=1}^{n} s_i/n$ and $\hat{\Lambda} = n(\sum_{i=1}^{n}(w_i - \hat{M}^{-1}))^{-1}$ and then update as

$$\begin{aligned}
\delta^{(k+1)} &= \hat{\Lambda}^{1/2}, \\
\gamma^{(k+1)} &= \delta^{(k+1)}/\hat{M}, \\
\beta^{(k+1)} &= \frac{\sum_{i=1}^{n} x_i w_i - \bar{x}\sum_{i=1}^{n} w_i}{n - \bar{s}\sum_{i=1}^{n} w_i}, \\
\mu^{(k+1)} &= \bar{x} - \beta^{(k+1)}\bar{s}, \\
\alpha^{(k+1)} &= [(\gamma^{(k+1)})^2 + (\beta^{(k+1)})^2]^{1/2}
\end{aligned}$$

Where $\bar{s} = \sum_{i=1}^{n} s_i/n$. Repeat these steps until some convergence criterion is satisfied.
Note that the updates for $\mu$ and $\beta$ are based on the Maximum Likelihood Estimator of the regression coefficient of the regression $\mathrm{E}[x_i] = \mu + \beta z_i$ with $\mathrm{Var}[x_i] = z_i$. For a more detailed explanation of this method, the reader is referred to Dempster et al. (1977), and Karlis (2002).

**In sample calibration**

This EM algorithm for the ML estimation is performed in MATLAB on 15 years of in sample data. Resulting in the following parameters

$$\begin{aligned}
\alpha &= 14.3626, \\
\beta &= -10.8998, \\
\delta &= 0.2107, \\
\mu &= 0.3228
\end{aligned}$$

These parameters in itself do not say much, therefore the central moments are calculated using equation (2.4) until (2.7).

$$\begin{aligned}
\bar{x}_{MLE} &= 0.0774, \\
s^2_{MLE} &= 0.0531, \\
\bar{\gamma}_{1,MLE} &= -1.6220, \\
\bar{\gamma}_{2,MLE} &= 3.5152
\end{aligned}$$

These central moments are compared to the empirical ones, showing a significant difference for most of the central moments. The KS test statistic calculated for the MLE calibrated NIG model against the in sample empirical data is equal to 0.10 against a critical value of 0.34, the AD test statistic is equal to 0.22 against a critical value of 0.76 suggesting an adequate fit at a significance level of 95%. Figure 4.5 shows the normal and NIG CDF plotted against the empirical CDF.

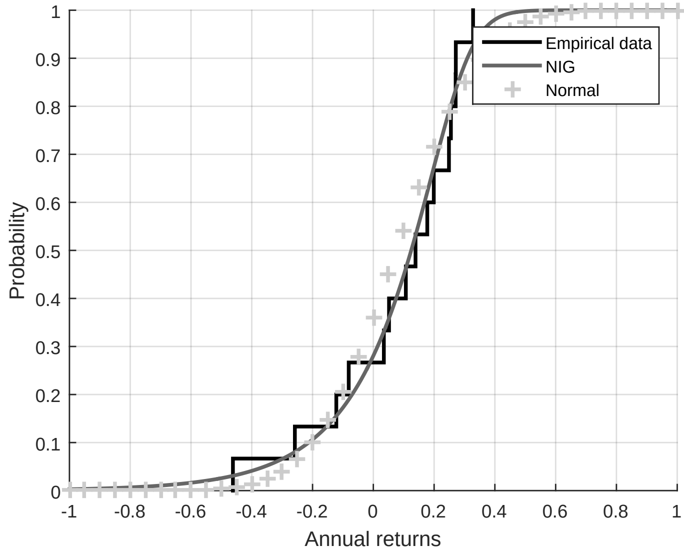

Figure 4.5: In sample empirical CDF vs. MLE calibrated in sample NIG-CDF and normal CDF

### Out of sample

The MLE results are now tested on out of sample data. For the back and forward testing, the KS test statistics are equal to 0.23 and 0.22 against critical values of 0.48 and 0.56. The AD test statistics are equal to 0.50 and 0.69 against critical values of 0.76 and 0.76 suggesting an adequate fit at a 95% significance level. Figure 4.6, which shows the MLE calibrated normal CDF, MLE calibrated NIG CDF and empirical CDF, and the results of the statistical tests suggest that the MLE calibrated NIG model resembles the empirical data best.

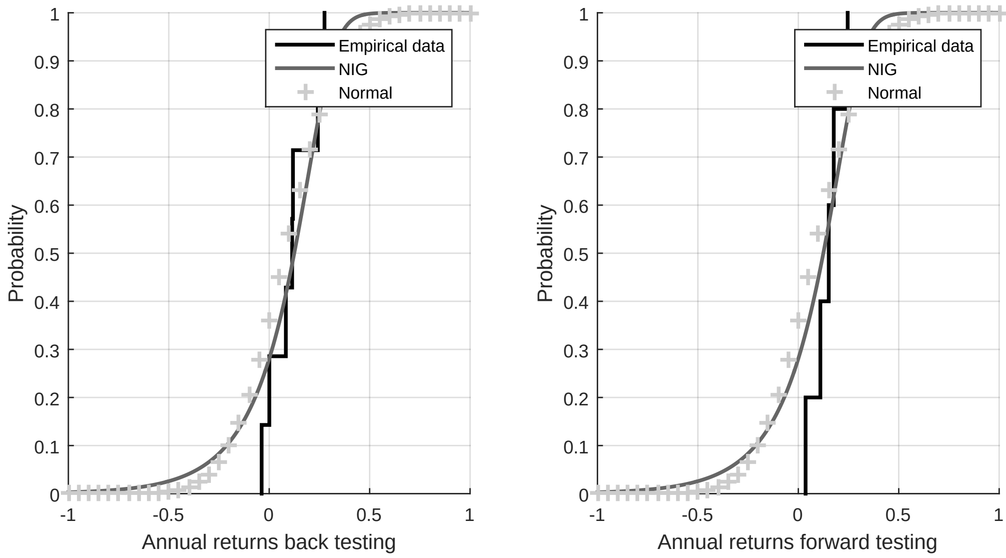

Figure 4.6: Out of sample empirical CDF vs. out of sample MLE carlibrated NIG CDF and normal CDF

## 4.3.3 Comparison historical calibration

The historical calibration of the normal distribution is compared to the NIG one. As a short review, all results of the KS and AD tests are shown in Table 4.1. The KS test statistic did not give a good picture of which model is best as all models seemed to be an adequate fit on all data sets at a 95% significance level. However the AD-test statistic showed that the MLE calibrated NIG model performed significantly better than the normal distribution as it was an adequate fit at a 95% significance level on the forward and back testing data whereas the standard BS models were not. It also outperformed the MME calibrated NIG model as this model was rejected on the back testing data. The difference between the CDFs is shown in Figure 4.5 and Figure 4.6.

Table 4.1: Results Kolmorogov - Smirnov and Anderson-Darling tests

| | Kolmogorov - Smirnov | | | Anderson - Darling | | |
|---|---|---|---|---|---|---|
| | In sample | Back | Forward | In sample | Back | Forward |
| MME BS | PASS | PASS | PASS | PASS | FAIL | FAIL |
| MLE BS | PASS | PASS | PASS | PASS | FAIL | FAIL |
| MME NIG | PASS | PASS | PASS | PASS | PASS | FAIL |
| MLE NIG | PASS | PASS | PASS | PASS | PASS | PASS |

## 4.4 Implied calibration

This section provides the methodology for the calibration of the normal distribution and the NIG distribution by fitting these to option prices observed in the market. The calibration uses the closed-form formulas for European call options derived in Chapter 3. Krog Saebo (2009) calibrated the NIG distribution on call option prices for his MSc. thesis. He uses numerical path optimization whereas this thesis uses the closed-form formula for a Vanilla European option to estimate the implied parameters. The advantage of using a closed-form formula is that these are more exact estimators while numerical path optimization depends on a certain confidence interval and thus an estimation error term.

The procedure for a calibration on option prices is clearly explained by Hendriks (2012). This is cited here, with minor changes to make it applicable for the model in this thesis: "Let the parameters that define the model be defined as a parameter vector called $\hat{\sigma}$ for the normal distribution or $\hat{\kappa} = (\alpha, \beta, \delta)'$ for the NIG distribution. A vector of real numbers for $\sigma$ or $\kappa$ such that the prices generated by the model given that the vector match the market prices 'best' (what 'best' means here could be objectively determined or subjectively determined, or a combination of both) need to be found. The outcome for $\sigma$ and $\kappa$, is denoted as $\hat{\sigma}$ and $\hat{\kappa}$. As the process is iterative, a starting vector, denoted by $\hat{\sigma}_0$ and $\hat{\kappa}_0$ is required. The iterative process is stopped when the relative change in the objective function is less than a pre-specified minimum change. The iteration is run in MATLAB[1]. The algorithm search can end up in a local optimum. Different initial points from which the iteration starts, starting vectors, can lead to different local optima."

### 4.4.1 Starting values

Picking a starting value is a crucial part of the optimization using iterations. Even though the implied parameter calibration is based on a risk-neutral setting, the MLE values found in the

[1] The *fminsearch* option is used for the NIG calibration, this optimization function is chosen for its speed. Constraints are added by penalizing results that are infeasible, e.g. $\alpha < 0$ etc. For the standard Black-Scholes, *fmincon* is used because otherwise MATLAB tries to fill in negative volatilities during the iteration, which are infeasible.

previous section are used as starting values. The starting values chosen in this thesis are therefore

$$\hat{\sigma}_0 = 0.2159,$$
$$\hat{\kappa}_0 = [\hat{\alpha}_0 \quad \hat{\beta}_0 \quad \hat{\delta}_0] = [14.3626 \quad -10.8998 \quad 0.2107]$$

### 4.4.2 Objective functions

The calibration is performed on matrices of option prices. The principle of this calibration is that the prices obtained by the model are compared to prices in the market. A minimization process is performed on an objective function which uses both the model prices as well as the market prices as input. There are different objective functions that can be used as optimization tool. Four different objective functions are compared in this thesis:

- The sum of squared differences (SSD),
- The root mean squared error (RMSE),
- The adjusted root mean square error (ARMSE),
- The average relative percentage error (ARPE)

For the calibration of the NIG distribution these objective functions are defined as follows

$$\text{SSD} = \sum_i \sum_j (C_{i,j} - \widehat{C_{i,j}}(\alpha, \beta, \delta))^2 \tag{4.10}$$

$$\text{RMSE} = \sqrt{\frac{1}{I * J} \sum_i \sum_j (C_{i,j} - \widehat{C_{i,j}}(\alpha, \beta, \delta))^2} \tag{4.11}$$

$$\text{ARMSE} = \sqrt{\frac{1}{I * J} \sum_i \sum_j \left( \frac{C_{i,j} - \widehat{C_{i,j}}(\alpha, \beta, \delta)}{C_{i,j}} \right)^2} \tag{4.12}$$

$$\text{ARPE} = \frac{1}{I * J} \sum_i \sum_j \frac{|C_{i,j} - \widehat{C_{i,j}}(\alpha, \beta, \delta)|}{C_{i,j}} \tag{4.13}$$

Where $i$ is the index of the vector of strike prices and $j$ the index for the vector of all the maturity times observed in the market. $I$ and $J$ are respectively the length of these vectors. $C_{i,j}$ is the option price observed in the market and $\widehat{C_{i,j}}$ the estimated option price using the closed-form option formulas derived in equation (3.2) and (3.4). For the normal distribution the $\alpha, \beta$ and $\delta$ are interchanged with $\sigma$.

Note that the RMSE is a monotonic transformation of the SSD, and should therefore by definition have the same optimum. The usefulness of adding both is to change the landscape of the optimization process and thereby getting a better feeling whether these objective functions end up in a local or global optimum.
The main differences between (4.10)-(4.11) and (4.12)-(4.13) is that the latter look at relative option prices, i.e. the absolute difference between option price and estimated value is 'punished' less for higher option prices.

Another objective function called the Maximum Relative Error (MRE) is given by

$$\text{MRE} = \max \left( \frac{|C_{i,j} - \widehat{C_{i,j}}|}{C_{i,j}} \right)$$

Which will not be used as objective function to calibrate, but is used to check the calibration results.

The MRE has the ability to show large misfits in the calibration.

### 4.4.3 Computation issues NIG PDF

The NIG PDF consists of two parts in which one gets very large ($e^{\delta\gamma+\beta(x-\mu)}$) and one (Bessel function: $K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})$ gets very small when $\alpha$ and $\delta$ get large. If this is calculated in MATLAB, they respectively become infinity and zero. In the calibration this happens for high maturities, since they directly influence the $\delta$ component. If the PDF is then calculated, 'NaN' is obtained in MATLAB. This can be solved by rewriting the PDF by taking the exponential of the log of the Bessel function. Remember from 2.1 that the Bessel function is equal to

$$K_1(x) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}x(u^{-1}+u)}du$$

Note that the minimum of $u^{-1}+u$ is equal to 2 on the interval $[0,\infty)$. This results that the minimum of $-\frac{1}{2}x(u^{-1}+u)$ is equal to $-x$. The Bessel function is then rewritten to

$$K_1(x) = e^{-x}\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}x(u^{-1}+u)+x}du$$

Here the minimum of $-\frac{1}{2}x(u^{-1}+u)+x$ is equal to zero. This changes the issue of the Bessel function becoming small quickly that zero is obtained is moved to the $e^{-x}$ part. By taking the log of this function, the following formula for the log-Bessel function is obtained

$$\log K_1(x) = -x + \log\left(\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}x(u^{-1}+u)+x}du\right)$$

Here $x$ and thus $\alpha\sqrt{\delta^2+(x-\mu)^2}$ in the PDF of the NIG has to become extremely large to obtain $-\infty$ within MATLAB. The NIG PDF implemented in MATLAB then becomes

$$\text{nig}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\frac{e^{\log K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})+\delta\gamma+\beta(x-\mu)}}{\sqrt{\delta^2+(x-\mu)^2}}$$

This solves the issue where the two parts $K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})$ and $e^{\delta\gamma+\beta(x-\mu)}$ become 0 and $\infty$ so quickly and have much higher computational extreme values. Both the original (using the Bessel function in MATLAB) and this rewritten PDF are used for the optimization and calculation of objective functions because the rewritten PDF demands much more computation power than the original function. The rewritten PDF is denoted by adding '(l)' to the corresponding objective function.

### 4.4.4 Results

The calibration on option prices can be tested out of sample in two different ways. The first one is based on the performance of estimating option prices with higher and lower strikes and maturities than the ones calibrated on. The other one is based on time; how stable is the calibration over time (i.e. test the parameters found on other dates than the valuation date).

**In sample**

As mentioned before in Section 4.1 the 'valuation' date for the implied calibration is September 30th, 2015. The in sample data used for this calibration is highlighted in Table E.1. A range of

strike prices of 100 above and below the current stock price (1920.03) as well as maturities from one day until one year are chosen for the calibration. The results of the calibration are found in Table 4.2. Note that the exact NIG calibration is thus more complex, demands more computation time and has some computation issues as describes above which could lead to minor accuracy errors.
Table 4.2 can be interpreted as follows; in the upper (NIG) part, the calibrated NIG model parameters are given for the objective function defined at the top of each column. The annual first four central moments are then calculated which are useful for a better interpretation of the model. Finally, the values of the objective functions are given for each set of calibrated parameters. The lower (normal) part of the table is set up in a similar way for the calibration of the normal distribution. Note that to calculate the annual characteristics the risk-free spot rate for a maturity of one year is used (0.278%).
Remark that the mean returns of the NIG models are negative whereas the mean return is expected to be approximately equal to the risk-free rate. This is caused by the fact that these moments are based on log-returns. Fortunately the drift term is still positive; when large returns occur, which happen during simulations, log-returns can give significant differences from exact returns. An example to illustrate this property: assume an asset A has a price of 100 at time $t = 0$, and grows to 200 at $t = 1$, the log-return is then $\log(200/100) = 69.3\%$ whereas the exact return is 100%. In Chapter 5, it is shown that the stock price has a positive drift over time even though the mean log-return is negative.

In Table 4.2 the cells highlighted show which calibrated model gives the lowest objective function. Naturally, these cells should be on the diagonal because the model is created by minimizing the corresponding objective function. The ARMSE(l) calibrated model does not give the lowest value for the ARMSE(l) objective function suggesting that it stopped the optimization process in a local optimum. This is also seen in the annual characteristics which differ a lot from the other models especially for the skewness and kurtosis coefficient. All the other models seem to give similar annual characteristics.
Compare the NIG objective function values to the normal ones; the NIG objective function values are lower, in specific the MRE ones suggesting that the model captures the skew that is observed in option prices better. Finally, the model based on a certain objective function using the rewritten PDF has different parameters than the model based on the same objective function but with the MATLAB Bessel function. However, the calculated objective functions are similar, it shows that the calibration is not stable and that there are possibly a lot of local optima, which are around or close to the global optimum.

**Out of sample**

The out of sample calibration is performed on the complete table as given in Appendix E. Looking at the results in Table 4.3, 'Not a Number' are the results obtained by the computation issues discussed in Section4.4.3.

The standard Black Scholes ARMSE model obtains the lowest value for all objective functions. The NIG models are less stable in this respect; the SSD/RMSE model perform best for the ARMSE(l), ARPE(l) and MRE(l) objective function whereas the ARMSE(l) model performs best on the SSD(l) and RMSE(l) objective function. The latter is remarkable because that model is significantly different from the other models as well as that it was not performing best for the in sample objective function on which it was calibrated. The NIG models have lower objective function values than the standard Black-Scholes ones assuming that the same objective function is used for the calibration. The next section tests intertemporal out of sample performance of the models.

Table 4.2: Results of the in sample model calibration on September 30th, 2015. The highlighted cells show the model with the lowest objective function.

| | SSD | SSD(l) | RMSE | RMSE(l) | ARMSE | ARMSE(l) | ARPE | ARPE(l) |
|---|---|---|---|---|---|---|---|---|
| **NIG parameter calibration** | | | | | | | | |
| $\hat{\alpha}$ | 442.1144 | 1747.9 | 442.1144 | 1747.9 | 175.1068 | 3198.6 | 188.2765 | 510.3508 |
| $\hat{\beta}$ | -416.4074 | -1721.1 | -416.4074 | -1721.1 | -78.0191 | 278.443 | -110.3336 | -410.7061 |
| $\hat{\delta}$ | 0.5468 | 0.3018 | 0.5468 | 0.3018 | 3.6836 | 87.6626 | 3.0955 | 3.2968 |
| **Annual characteristics based on log-returns** | | | | | | | | |
| $\bar{\mu}$ | -0.0132 | -0.0132 | -0.0132 | -0.0132 | -0.0118 | -0.0111 | -0.0126 | -0.0126 |
| $\bar{s}$ | 0.1805 | 0.1804 | 0.1805 | 0.1804 | 0.1712 | 0.1665 | 0.1758 | 0.1757 |
| $\bar{\gamma}_1$ | -0.3135 | -0.3079 | -0.3135 | -0.3079 | -0.0556 | 0.0005 | -0.0809 | -0.0764 |
| $\bar{\gamma}_2$ | 0.1310 | 0.1264 | 0.1310 | 0.1264 | 0.0041 | 0.0000 | 0.0087 | 0.0078 |
| **Calculated objective functions NIG** | | | | | | | | |
| SSD | 1744.2 | NaN | 1744.2 | NaN | 3480.6 | NaN | 2551.1 | NaN |
| SSD(l) | 1748.1 | 1740.6 | 1748.1 | 1740.6 | 3480.6 | 5285.2 | 2551.1 | 2562.9 |
| RMSE | 2.4652 | NaN | 2.4652 | NaN | 3.4825 | NaN | 2.9814 | NaN |
| RMSE(l) | 2.468 | 2.4627 | 2.468 | 2.4627 | 3.4825 | 4.2913 | 2.9814 | 2.9883 |
| ARMSE | 0.1531 | NaN | 0.1531 | NaN | 0.0677 | NaN | 0.0709 | NaN |
| ARMSE(l) | 0.1531 | 0.1543 | 0.1531 | 0.1543 | 0.0677 | 0.0722 | 0.0709 | 0.071 |
| ARPE | 0.057 | NaN | 0.057 | NaN | 0.0439 | NaN | 0.0422 | NaN |
| ARPE(l) | 0.0571 | 0.0572 | 0.0571 | 0.0572 | 0.0439 | 0.0492 | 0.0422 | 0.0421 |
| MRE(l) | 0.1665 | 0.1685 | 0.1665 | 0.1685 | 0.3137 | 0.2991 | 0.3651 | 0.3709 |
| **BS parameter calibration** | | | | | | | | |
| | SSD | | RMSE | | ARMSE | | ARPE | |
| $\hat{\sigma}$ | 0.1764 | | 0.1764 | | 0.1667 | | 0.1698 | |
| **Calculated objective functions BS** | | | | | | | | |
| SSD | 3136.6 | | 3136.6 | | 5417.3 | | 4190.4 | |
| RMSE | 3.3059 | | 3.3059 | | 4.3446 | | 3.8211 | |
| ARMSE | 0.0995 | | 0.0995 | | 0.0723 | | 0.0755 | |
| ARPE | 0.2296 | | 0.2296 | | 0.2222 | | 0.219 | |
| MRE | 0.6192 | | 0.6192 | | 0.3023 | | 0.3769 | |

### Intertemporal out of sample

In Figure 4.7, six graphs show the results of the intertemporal out of sample performance of the models found using implied calibration. The lines show the ratio NIG objective function over the same BS objective function for a given model e.g. the darkest line in the first figure is the SSD objective function of the NIG model divided by the SSD objective function of the standard BS model using the SSD/RMSE model for both the NIG as well as the BS over time. This is done for the dates September 25th until October 5th 2015, in this period there are seven trading days; three days before 'valuation' date and three days after. Note that the NIG objective functions are based on the rewritten PDF because others could give 'NaN' or incorrect numbers due to the computation issues described before. However, the models found using the original PDF are used to calculate the objective function, e.g. the ratio NIG ARMSE(l) divided by BS ARMSE(l) for the ARPE model is shown in the third figure but the NIG ARMSE divided by BS ARMSE for the ARPE model is not shown.

The ratio is over time fairly stable for all comparable models. First of all, note that the MRE ratio is between 0.1 and 0.5 suggesting that the NIG models are better at predicting the skew of the option prices and therefore have lower extreme outliers. The ARPE objective functions performs better in for NIG models as well. Finally, the SSD, RMSE and ARMSE objective functions are more volatile for different models. Within the SSD/RMSE and SSD(l)/RMSE(l) models, the NIG

Table 4.3: Results of out of sample model testing on September 30th, 2015. The highlighted cells show the model with the lowest objective function.

| Calculated objective functions NIG | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | SSD | SSD(l) | RMSE | RMSE(l) | ARMSE | ARMSE(l) | ARPE | ARPE(l) |
| SSD | NaN | NaN | NaN | NaN | NaN | NaN | NaN | NaN |
| SSD(l) | 72,931 | 72,354 | 72,931 | 72,354 | 51,433 | 42,079 | 67,825 | 67,922 |
| RMSE | NaN | NaN | NaN | NaN | NaN | NaN | NaN | NaN |
| RMSE(l) | 9.3401 | 9.3031 | 9.3401 | 9.3031 | 7.8436 | 7.0947 | 9.0072 | 9.0137 |
| ARMSE | NaN | NaN | NaN | NaN | NaN | NaN | NaN | NaN |
| ARMSE(l) | 0.8011 | 0.8038 | 0.8011 | 0.8038 | 1.105 | 1.07 | 1.2085 | 1.2149 |
| ARPE | NaN | NaN | NaN | NaN | NaN | NaN | NaN | NaN |
| ARPE(l) | 0.2125 | 0.2129 | 0.2125 | 0.2129 | 0.2431 | 0.2316 | 0.2675 | 0.2685 |
| MRE(l) | 9.3002 | 9.3321 | 9.3002 | 9.3321 | 12.3364 | 12.3402 | 13.4781 | 13.5748 |

| Calculated objective functions BS | | | | |
|---|---|---|---|---|
| | SSD | RMSE | ARMSE | ARPE |
| SSD | 77,827 | 77,827 | 42,338 | 50,641 |
| RMSE | 9.6485 | 9.6485 | 7.1165 | 7.783 |
| ARMSE | 1.459 | 1.459 | 1.0759 | 1.1919 |
| ARPE | 0.5586 | 0.5586 | 0.4825 | 0.5061 |
| MRE | 75.2112 | 75.2112 | 67.1212 | 67.3886 |

performs better than the standard BS model but for the ARMSE, ARMSE(l), ARPE, ARPE(l) models they move around the ratio of 1 suggesting a similar performance to the BS model.

## 4.5 Case study

This section calibrates the same models but focuses on a more practical approach. Insurers tend to use at the money options for model calibration because these are more liquid. Plus they often want to value products with longer maturities and it is therefore convenient to include options that have longer maturities for the calibration. The option data on the 'valuation' date found in Table E.1 is based on strike prices, the strike price of 1925 (with a underlying stock price of 1920.03) is therefore chosen as at the money because of the availability of option prices. The difference of 0.26% in price compared to exact at the money options is decided to be small enough. The same starting values and objective functions as in the previous section are used.

### 4.5.1 In sample

In Table 4.5 the models and the calculated in sample (on the eleven at the money options) objective functions are shown. The first remarkable result is that the SSD and SSD(l) models, which in theory should give the same value, show large differences. Tests are performed on this result and are shown in Table 4.4. The price for the options with the longest maturity, December 15th, 2017, is lower than the price of an option with a shorter maturity. This is infeasible and leads to arbitrage due to the time value of an option, suggesting that the Bessel function becomes zero before $e^{\delta\gamma+\beta(x-\mu)}$ becomes infinity and thus does not become 'NaN', but has a significant impact on the objective functions.

Keep in mind for the rest of this section that the models calibrated with the PDF's calculated using MATLAB's Bessel function reflect non-optimal solutions. Now note that if the annual characteristics in Table 4.5 is compared to these in Table 4.2, the analytic mean and standard deviation are fairly

Table 4.4: Option prices using the closed-form formula for the NIG option price computed for different models based on the NIG's PDF using MATLAB's Bessel function compared to the rewritten PDF as described in Section 4.4.3.

| | **Bessel** | 10/16/15 | 01/15/16 | 03/18/16 | 12/16/16 | 01/20/17 | 06/16/17 | 12/15/17 |
|---|---|---|---|---|---|---|---|---|
| **SSD** | MATLAB | 25.68 | 68.39 | 86.94 | 143.95 | 149.99 | 173.77 | 0 |
| | Computed | 25.68 | 68.39 | 86.94 | 143.95 | 149.99 | 173.77 | 200.76 |
| **ARMSE** | MATLAB | 26.65 | 70.81 | 89.98 | 148.83 | 155.06 | 179.54 | 130.91 |
| | Computed | 26.65 | 70.81 | 89.98 | 148.83 | 155.06 | 179.54 | 207.29 |
| **ARPE** | MATLAB | 26.22 | 69.95 | 88.92 | 147.16 | 153.34 | 177.59 | 171.81 |
| | Computed | 26.22 | 69.95 | 88.92 | 147.16 | 153.34 | 177.59 | 205.09 |

close to each other, whereas the skewness and kurtosis are significantly different. This suggests that the in and out of the money options translate into the skewness and kurtosis of the distribution when calibrating the NIG model. The standard BS model does not differ much from the broad calibration done in the previous section. This time the diagonal objective functions are the lowest, which is desirable.

### 4.5.2 Out of sample

The out of sample results of the case study model calibration on the 30th of September can be found in Table 4.6. The difference with Table 4.3 is that no 'NaN' are obtained for the SSD, RMSE, and ARMSE models. This is due to the fact that the calibration is performed on the longest maturity, but remember that in Table 4.4 it was shown that these objective functions can still represent poor results due to computation limits. The SSD(l)/RMSE(l) NIG models perform best on most objective functions for the objective functions where the Bessel function is calculated analytically as discussed in Section 4.4.3.

### 4.5.3 Intertemporal out of sample

The results are similar to the broadly calibrated method in the previous section. The MRE and ARPE objective functions of the NIG models still show a better performance over time than the standard BS models. The other objective functions are fairly similar to the BS model ones.

Table 4.5: Results of the implied model calibration based on at the money options on September 30th, 2015. The highlighted cells show the model with the lowest objective function.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **NIG parameter calibration** | | | | | | | | |
| | SSD | SSD(l) | RMSE | RMSE(l) | ARMSE | ARMSE(l) | ARPE | ARPE(l) |
| $\hat{\alpha}$ | 105.5652 | 697.269 | 105.5652 | 697.2685 | 101.7264 | 827.3449 | 107.4867 | 667.431 |
| $\hat{\beta}$ | -6.2154 | -258.34 | -6.2154 | -258.337 | -1 | 599.9123 | -30.3376 | 271.2529 |
| $\hat{\delta}$ | 2.987 | 15.3976 | 2.987 | 15.3976 | 3.0937 | 8.0127 | 2.8329 | 15.1326 |
| **Annual characteristics based on log-returns** | | | | | | | | |
| $\bar{\mu}$ | -0.0114 | -0.0110 | -0.0114 | -0.0110 | -0.0124 | -0.0121 | -0.0121 | -0.0121 |
| $\bar{s}$ | 0.1687 | 0.1660 | 0.1687 | 0.1660 | 0.1744 | 0.1722 | 0.1728 | 0.1724 |
| $\bar{\gamma}_1$ | -0.0100 | -0.0111 | -0.0100 | -0.0111 | -0.0017 | 0.0322 | -0.0495 | 0.0127 |
| $\bar{\gamma}_2$ | 0.0001 | 0.0002 | 0.0001 | 0.0002 | 0.0000 | 0.0014 | 0.0033 | 0.0002 |
| **Calculated objective functions NIG** | | | | | | | | |
| SSD | 127.1074 | NaN | 127.1074 | NaN | 264.9671 | NaN | 188.8029 | NaN |
| SSD(l) | 300.2407 | 253.229 | 300.2407 | 253.2293 | 652.1302 | 491.4144 | 494.5529 | 496.3989 |
| RMSE | 3.993 | NaN | 3.993 | NaN | 4.9079 | NaN | 4.1429 | NaN |
| RMSE(l) | 5.2244 | 4.798 | 5.2244 | 4.798 | 7.6996 | 6.6839 | 6.7052 | 6.7177 |
| ARMSE | 0.0596 | NaN | 0.0596 | NaN | 0.05 | NaN | 0.0523 | NaN |
| ARMSE(l) | 0.0632 | 0.0683 | 0.0632 | 0.0683 | 0.0592 | 0.0562 | 0.0594 | 0.0564 |
| ARPE | 0.0423 | NaN | 0.0423 | NaN | 0.0376 | NaN | 0.0371 | NaN |
| ARPE(l) | 0.0487 | 0.0528 | 0.0487 | 0.0528 | 0.0472 | 0.0435 | 0.0456 | 0.0435 |
| MRE(l) | 0.0702 | 0.0545 | 0.0702 | 0.0545 | 0.105 | 0.0934 | 0.0933 | 0.0938 |
| **BS parameter calibration** | | | | | | | | |
| | SSD | | RMSE | | ARMSE | | ARPE | |
| $\hat{\sigma}$ | 0.1657 | | 0.1657 | | 0.1725 | | 0.1725 | |
| **Calculated objective functions BS** | | | | | | | | |
| SSD | 252.9151 | | 252.9151 | | 494.2799 | | 494.1421 | |
| RMSE | 4.795 | | 4.795 | | 6.7033 | | 6.7227 | |
| ARMSE | 0.0687 | | 0.0687 | | 0.0565 | | 0.0565 | |
| ARPE | 0.2311 | | 0.2311 | | 0.2088 | | 0.2088 | |
| MRE | 0.0534 | | 0.0534 | | 0.0938 | | 0.0938 | |

Table 4.6: Out of sample results for the implied model calibration based on at the money options on September 30th, 2015. The highlighted cells show the model with the lowest objective function.

| Calculated objective functions NIG | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | SSD | SSD(l) | RMSE | RMSE(l) | ARMSE | ARMSE(l) | ARPE | ARPE(l) |
| SSD | 195,380 | NaN | 195,380 | NaN | 47,698 | NaN | NaN | NaN |
| SSD(l) | 46,685 | 40,207 | 46,685 | 40,207 | 68,100 | 61,856 | 57,499 | 60,839 |
| RMSE | 15.2873 | NaN | 15.2873 | NaN | 7.5535 | NaN | NaN | NaN |
| RMSE(l) | 7.4728 | 6.935 | 7.4728 | 6.935 | 9.0255 | 8.6018 | 8.2933 | 8.5307 |
| ARMSE | 1.1325 | NaN | 1.1325 | NaN | 1.3795 | NaN | NaN | NaN |
| ARMSE(l) | 1.1329 | 1.0239 | 1.1329 | 1.0239 | 1.3815 | 1.3771 | 1.1845 | 1.3299 |
| ARPE | 0.246 | NaN | 0.246 | NaN | 0.2882 | NaN | NaN | NaN |
| ARPE(l) | 0.2445 | 0.2234 | 0.2445 | 0.2234 | 0.295 | 0.2916 | 0.2586 | 0.2836 |
| MRE(l) | 13.2001 | 11.6189 | 13.2001 | 11.6189 | 16.7452 | 16.9221 | 13.5299 | 16.063 |

| Calculated objective functions BS | | | | |
|---|---|---|---|---|
| | SSD | RMSE | ARMSE | ARPE |
| SSD | 40,258 | 40,258 | 60,184 | 60,184 |
| RMSE | 6.9394 | 6.9394 | 8.4847 | 8.4847 |
| ARMSE | 1.0398 | 1.0398 | 1.2978 | 1.2978 |
| ARPE | 0.4751 | 0.4751 | 0.5273 | 0.5273 |
| MRE | 67.042 | 67.042 | 67.6495 | 67.6495 |

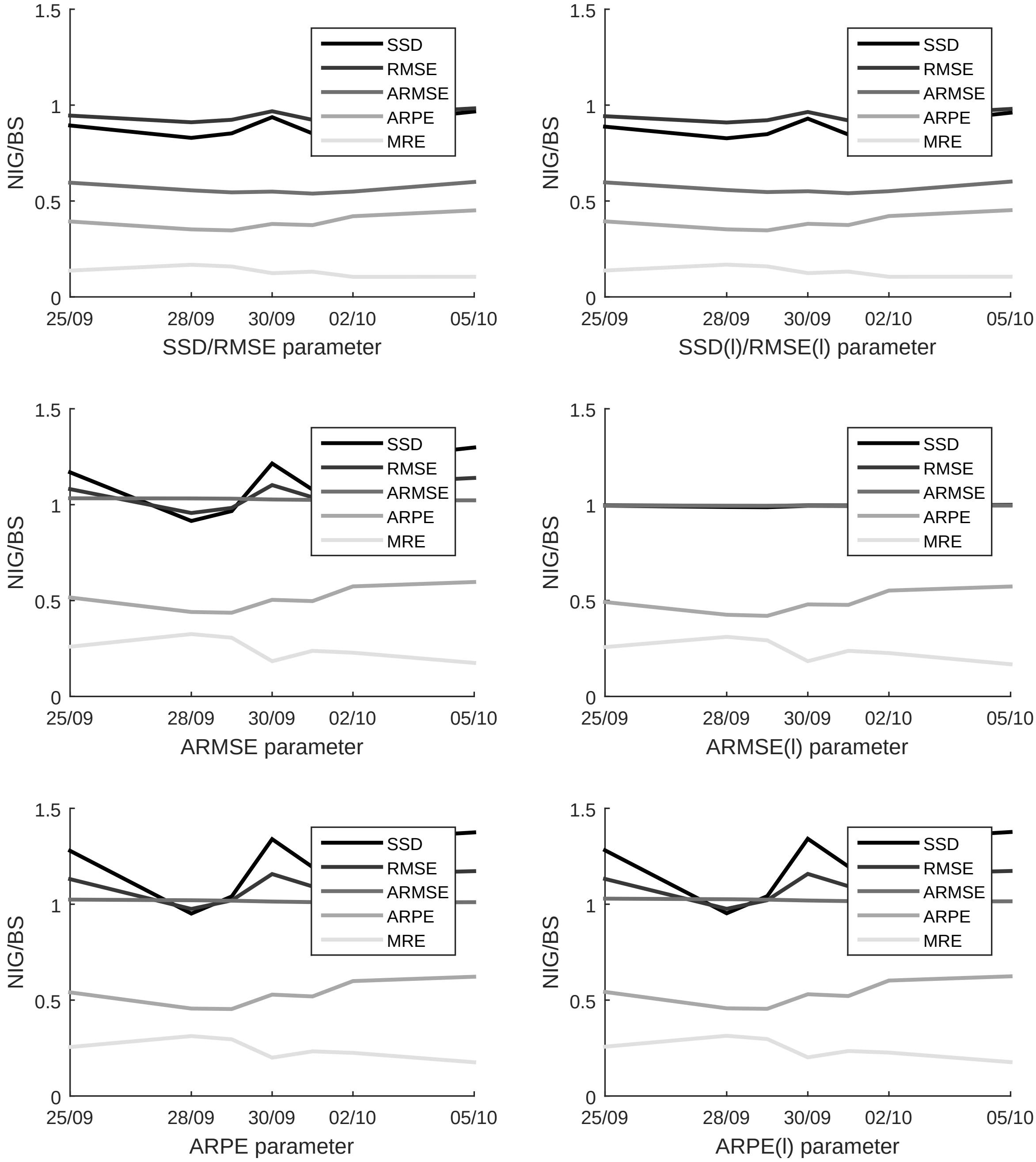


Figure 4.7: The results of the out of sample objective function of the NIG models divided by the same objective function of the BS models based on the same calibration method. This is done for each model and for each objective function over time. Data goes from September 25th, 2015 until October 5th, 2015 where September 26 and 27 and October 3 and 4 are weekend days and do not give results.

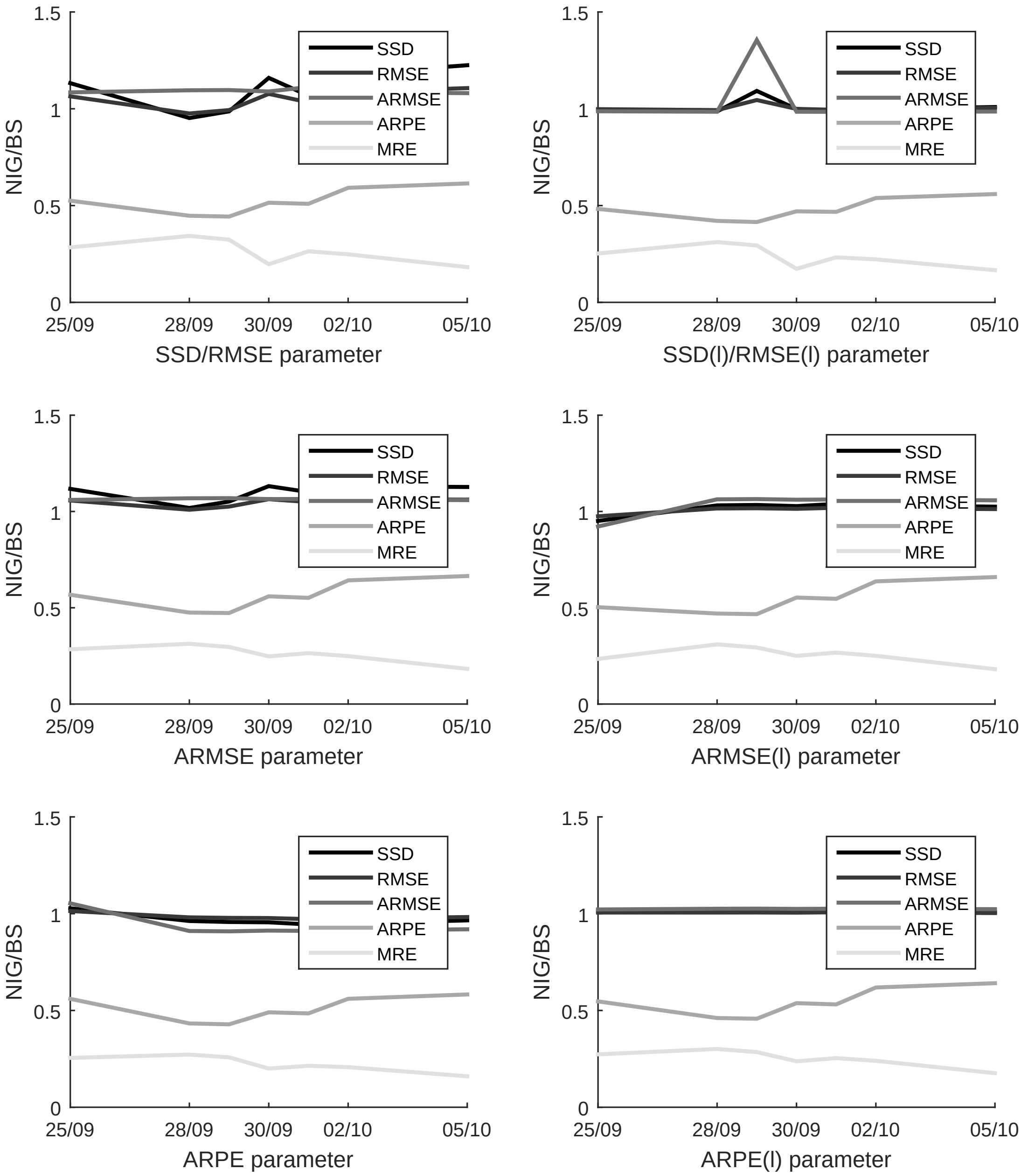


Figure 4.8: The results of the out of sample objective function for NIG model calibrated on at the money options is divided by the same objective function of the BS model using parameters found by the same calibration method. Data goes from September 25th, 2015 until October 5th, 2015 where September 26 and 27 and October 3 and 4 are weekend days and do not give results.

# Chapter 5

# Economic Scenario Generator

The models calibrated in the previous chapter are used in this chapter to simulate asset prices in both a real-world and a risk-neutral setting. In the introduction, the idea of an economic scenario generator was already briefly discussed. This will be done more extensively in this chapter.

## 5.1 Idea of an ESG

The main reason to use an economic scenario generator to price complex financial products is because it is hard or impossible to create a closed-form formula or replicating portfolio that reflects the right market price. The most important reason for this in insurance contracts is that these often contain options that are path-dependent, meaning that the cash-flows depend on how the asset prices moved before the expiration date and not just where it is at the end. Often an insurance contract includes multiple options, to value the entire contract one should find a closed-form formula or a replicating portfolio for each one of these options to be able to price this contract. An Economic Scenario Generator is then in many cases more convenient to implement.

An Economic Scenario Generator generates, using Monte Carlo simulation, different scenarios. The aim of this simulation is to simulate a model that represents reality (either the real-world, or risk-neutral reality) as well as possible. This can then be used to backward calculate the present value of the insurance contract or to test a portfolio.

A key element in creating these scenarios is that they should be risk-neutral if used for the valuation of products. The Fundamental Theorem of Asset Pricing is mentioned before and proves that to price a product in such a way that it is free of arbitrage possible that the expectation in a risk-neutral setting should be taken. This translates to an Economic Scenario Generator by simulating risk-neutral scenarios and taking the present value of the average of all the claims in each individual setting. Note that each path generated in a risk-neutral setting can also be a path in a real-world setting. The difference is not in the path but in the probability assigned to these paths (distribution of the scenarios).

## 5.2 Implementation of the ESG

The steps to simulate Monte Carlo are discussed in Benth et al. (2007) and is implemented in MATLAB in Kienitz and Wetterau (2012). The simulation algorithm for a NIG$(\alpha, \beta, \mu, \delta)$-distributed variable $X$ is initially considered by Rydberg (1997). It can be set-up as follows:

- Sample $Z$ from $\text{IG}(\delta^2, \alpha^2 - \beta^2)$
- Sample $Y$ from $\text{N}(0, 1)$
- Return $X = \mu + \beta Z + \sqrt{Z}Y$

The sampling of $Z$ is not straightforward, it consists of creating a random variable $V$ which is $\chi^2$ distributed with one degree of freedom, i.e. the square of a standard normal distribution. Then $W$ needs to be computed as

$$W = \varepsilon + \frac{\varepsilon^2 V}{2\delta^2} - \frac{\varepsilon}{2\delta^2}\sqrt{4\varepsilon\delta^2 V + \varepsilon^2 V^2}$$

Then

$$Z = W * 1_{\{U_1 \leq \frac{\varepsilon}{\varepsilon + W}\}} + \frac{\varepsilon^2}{W} * 1_{\{U_1 > \frac{\varepsilon}{\varepsilon + W}\}}$$

Where $U_1$ is uniformly distributed and $\varepsilon = \delta/\sqrt{\alpha^2 - \beta^2}$.
This simulation is given in MATLAB in Kienitz and Wetterau (2012) and will be used in this thesis for the simulation of the Normal Inverse Gaussian distribution.
This MATLAB code is given but not proven by any tests that it works properly. This thesis dedicates a subsection to the testing of this MATLAB code.

The implementation of an ESG for the standard Black-Scholes model is standard in the literature and is not given.

### 5.2.1 Testing of the Monte Carlo simulation of the NIG distribution

Assume the parameters $\alpha = 9.2214, \beta = -4.5964, \delta = 1.1783$ and a constant risk-free rate of $r = 1.92\%$. These parameters are chosen are found by Krog Saebo (2009) in his calibration of the NIG-process. $(\mu + \theta)$ can then be calculated using these parameters resulting in $\mu + \theta = 0.6048$. The characteristics for the returns can be calculated analytically using equations 2.4-2.7 for the $\text{NIG}(\alpha, \beta, \delta T, (\mu + \theta)T)$ distribution. This is done for $T = 1$ until $T = 10$ and given in Table 5.1.

Table 5.1: Table of analytically calculated characteristics of a $\text{NIG}(\alpha, \beta, \delta T, (\mu + \theta)T)$ distribution with respect to the time horizon T where $\alpha = 9.2214, \beta = -4.5964, \delta = 1.1783$ and a constant risk-free rate of $r = 1.92\%$.

| | T=1 | T=2 | T=3 | T=4 | T=5 | T=6 | T=7 | T=8 | T=9 | T=10 |
|---|---|---|---|---|---|---|---|---|---|---|
| E[$X$] | -0.0727 | -0.1454 | -0.2181 | -0.2908 | -0.3635 | -0.4362 | -0.5089 | -0.5816 | -0.6543 | -0.7270 |
| Var[$X$] | 0.1961 | 0.3922 | 0.5884 | 0.7845 | 0.9806 | 1.1767 | 1.3728 | 1.5690 | 1.7651 | 1.9612 |
| Skew[$X$] | -0.4872 | -0.3445 | -0.2813 | -0.2436 | -0.2179 | -0.1989 | -0.1842 | -0.1723 | -0.1624 | -0.1541 |
| Kurt[$X$] | 3.6350 | 3.3175 | 3.2117 | 3.1587 | 3.1270 | 3.1058 | 3.0907 | 3.0794 | 3.0706 | 3.0635 |

The next step is to simulate this distribution using the MATLAB code of Kienitz and Wetterau (2012). The number of simulations is important in this step since a higher number of simulations will result in a more accurate answer. The characteristics above can only be obtained by creating a sample using simulation. A sample of samples is therefore needed to create the confidence interval that is demanded for completeness. Therefore one thousand simulations of one thousand paths are calculated (implying a total of one million paths). Using the Weak Law of Large Numbers the mean characteristics are normally distributed when using a sample that is large enough. This implies that the 95% confidence interval can be derived as

$$P[\bar{X} - \Phi^{-1}(0.975) * \sigma/\sqrt{n} \leq \mu \leq \bar{X} + \Phi^{-1}(0.975) * \sigma/\sqrt{n}] = 0.95$$
$$P[\bar{X} - 1.96 * \sigma/\sqrt{n} \leq \mu \leq \bar{X} + 1.96 * \sigma/\sqrt{n}] = 0.95$$

Where $\bar{X}$ is the sample mean (e.g. the mean of the all the simulated means, variances, skewnesses and kurtosises), $\Phi$ the cumulative of the standard normal distribution, $\sigma$ the sample variance and $n$ the number of simulations done. $\mu$ is the 'real' mean of the characteristic estimated.

These paths are simulations of the underlying stock price whereas the characteristics calculated in Table 5.1 are based on log-returns. The log-return for each simulation and time step is therefore calculated. On these returns the confidence interval for the characteristics are calculated using MATLAB's standard formulas resulting in Table 5.2 for the lower bound and Table 5.3 for the upper bound.

Table 5.2: Table of the lower bound of the 95%confidence interval for the characteristics of a NIG$(\alpha, \beta, \delta T, (\mu + \theta)T)$ distribution using Monte Carlo simulations with respect to time horizon T where $\alpha = 9.2214, \beta = -4.5964, \delta = 1.1783$ and a constant risk-free rate of $r = 1.92\%$.

| | T=1 | T=2 | T=3 | T=4 | T=5 | T=6 | T=7 | T=8 | T=9 | T=10 |
|---|---|---|---|---|---|---|---|---|---|---|
| E$[X]$ | -0.0734 | -0.1467 | -0.2198 | -0.2926 | -0.3659 | -0.4386 | -0.5114 | -0.5843 | -0.6567 | -0.7298 |
| Var$[X]$ | 0.1956 | 0.3910 | 0.5869 | 0.7829 | 0.9770 | 1.1727 | 1.3683 | 1.5643 | 1.7589 | 1.9541 |
| Skew$[X]$ | -0.4874 | -0.3460 | -0.2859 | -0.2483 | -0.2224 | -0.2042 | -0.1901 | -0.1767 | -0.1661 | -0.1569 |
| Kurt$[X]$ | 3.5811 | 3.2856 | 3.1888 | 3.1425 | 3.1026 | 3.0826 | 3.0698 | 3.0593 | 3.0472 | 3.0403 |

Table 5.3: Table of the upper bound of the 95% confidence interval for the characteristics of a NIG$(\alpha, \beta, \delta T, (\mu + \theta)T)$ distribution using Monte Carlo simulations with respect to time horizon T where $\alpha = 9.2214, \beta = -4.5964, \delta = 1.1783$ and a constant risk-free rate of $r = 1.92\%$.

| | T=1 | T=2 | T=3 | T=4 | T=5 | T=6 | T=7 | T=8 | T=9 | T=10 |
|---|---|---|---|---|---|---|---|---|---|---|
| E$[X]$ | -0.0716 | -0.1442 | -0.2168 | -0.2891 | -0.3621 | -0.4343 | -0.5069 | -0.5795 | -0.6516 | -0.7244 |
| Var$[X]$ | 0.1969 | 0.3934 | 0.5902 | 0.7873 | 0.9825 | 1.1793 | 1.3760 | 1.5728 | 1.7685 | 1.9649 |
| Skew$[X]$ | -0.4742 | -0.3346 | -0.2751 | -0.2382 | -0.2124 | -0.1939 | -0.1800 | -0.1666 | -0.1560 | -0.1468 |
| Kurt$[X]$ | 3.6328 | 3.3174 | 3.2176 | 3.1681 | 3.1261 | 3.1054 | 3.0924 | 3.0815 | 3.0694 | 3.0628 |

One can see that the analytic values are in the 95% confidence interval of the simulation for both short and longer maturities. This implies that the MATLAB code of the Monte-Carlo simulation of the Normal Inverse Gaussian distribution created by Kienitz and Wetterau (2012) is performing well and is used in this thesis.

### 5.2.2 Simulation and interpretation of calibrated models

This section is dedicated to showing paths of the real-world calibrated NIG models and are held against the empirical data. Characteristics are already shown in the previous chapter. Therefore, a QQ-plot will be used in this section to illustrate the goodness of fit. The risk-neutral paths are not plotted because these cannot be compared to empirical data. Remember that risk-neutral paths are similar to real-world ones, but that the probability assigned to each scenario is changed. Risk-neutral simulations are useful in the next section where these are used to value insurance products.

In Figure 5.1, one can see the 20th, 50th and 80th quantile paths of the scenarios generated of the S&P500 starting at September 30th, 2015 using the MME and MLE calibrated NIG models and using the same seed[1]. Since these models are calibrated on historical data these correspond to real world quantile scenarios. Both figures are simulated for a period of ten years using days as time step. The MLE quantile plots are slightly wider than the MME plot.
In Figure 5.2 a QQ-plot of the annual mean log-returns for 100,000 simulations are plotted against the empirical quantiles of annual log-returns as in Figure 4.1 (including all the data available). The mean of every 5th quantile is plotted, i.e. the 5th quantile, 10th quantile until the 95th quantile.

[1]This means random numbers are controlled and will give the same random values.

Both the MME and MLE calibrated NIG model show very similar results for the QQ-plot. The extremes fit better on the line for the MLE calibrated model for all quantiles (Note that QQ-plot y-axis for the MLE calibrated model shows much smaller steps).

In Figure 5.3 the quantile paths and QQ-plot for the MME/MLE calibrated BS model is shown. Since the MME and MLE calibrated model both have the same parameters, only one QQ-plot and one figure of the quantile paths is shown. The smallest quantiles (5 and 10th) in the QQ-plot differ are significantly different from the empirical data set.

As shown in Chapter 4, the MLE calibrated NIG model seems to have the best fit based on QQ-plots. The quantile paths are however fairly similar for all models.

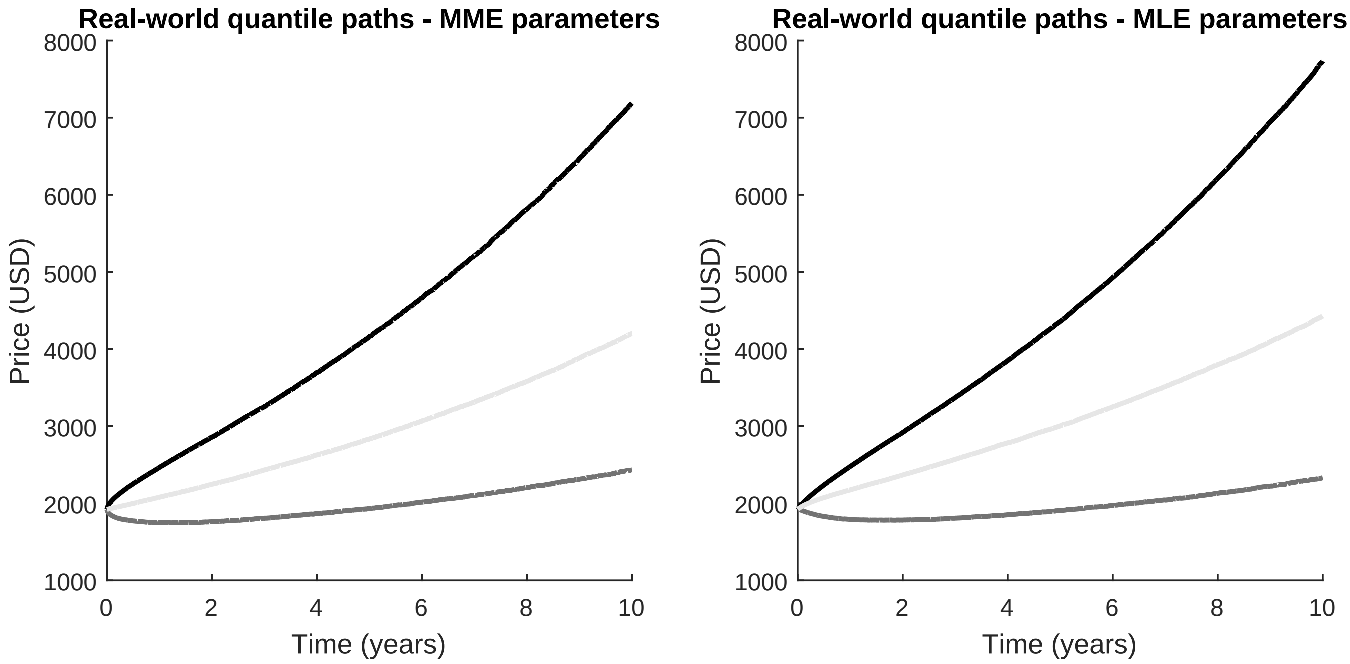


Figure 5.1: 20th, 50th and 80th quantile path of real-world scenarios using NIG models

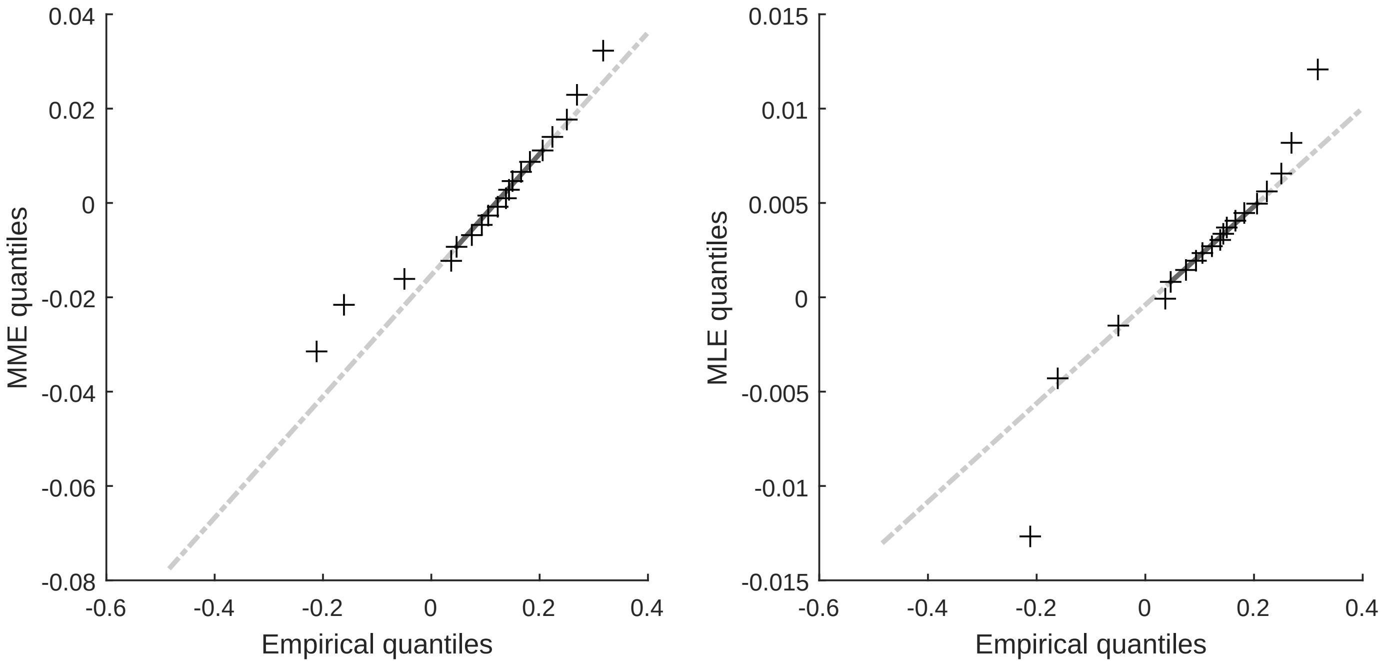


Figure 5.2: QQ-plots of the annual mean log-returns using NIG models versus the empirical annual log-returns

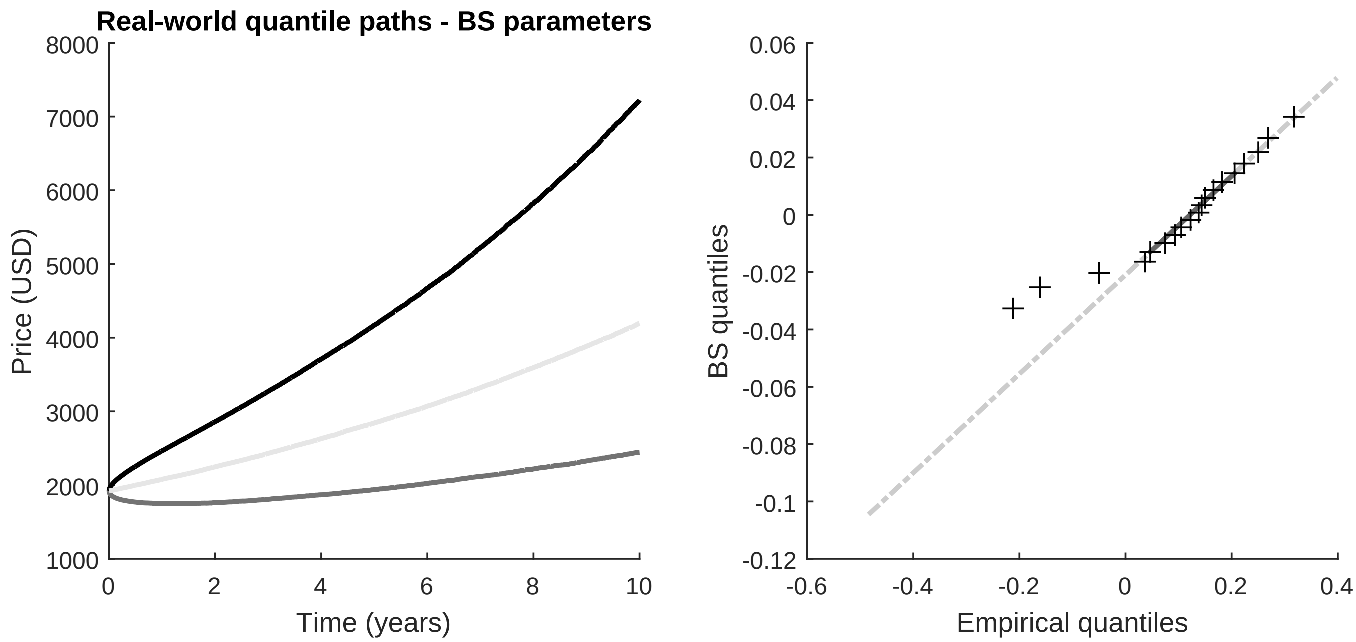


Figure 5.3: 20th, 50th and 80th quantile path of real-world scenarios and QQ-plot of annual log-returns using BS MME/MLE models

# Chapter 6

# Valuation

A lot of different insurance products are complex and therefore do not have a closed-form formula for the no-arbitrage price. Typically, there are European-style products, i.e. products that offer a payoff at a certain given maturity, but there are also products that exchange cash flows between the insured and the insurer during the period of the contract. Intuitively, the former is more easy to price; using a similar approach as the calculation of a European option introduced by Black and Scholes (1973). For less standard products without a closed-form formula for the price, Monte Carlo simulations, i.e. an economic scenario generator, is often used. For more complex products where cash flows are already interchanged during the period of the contract, Monte Carlo simulations are a necessity.

This thesis discusses two products, that were already discussed in the introduction. These are the Variable Annuity and the unit-linked product. Examples of a Variable Annuity and unit-linked products are respectively the GMAB and Asian Option. This chapter starts off with the description of these products in more detail. Then a value of some example products using the risk-neutral models calibrated in Chapter 4 will be derived, i.e. four cases - broadly calibrated NIG and BS models and at the money calibrated NIG and BS models.

## 6.1 Product description

This section describes the specific insurance products discussed in this thesis and intends to give a clear overview of what these products are and why they are issued by insurers.

### 6.1.1 Guaranteed Minimum Accumulation Benefit (GMAB)

A Variable Annuity is a deferred, fund-linked annuity contract, often with a single premium payment up-front. This single premium is then invested in one or a bucket of assets. Insureds often choose a certain bucket based on their risk profile. Variable Annuities are more flexible than unit-linked products, they offer the option to add, remove or change riders and guarantees during the period of the contract. There are two standard Variable Annuity products given out by insurers; the Guaranteed Minimum Death Benefit (GMDB) and the Guaranteed Minimum Life Benefit (GMLB). Since the GMAB was chosen as a product to value in this thesis, only GMLB will be discussed in more detail. To give a little bit of background about the GMDB; it is a product that pays out a death benefit to a person stated in the contract if the insured dies within the deferment period. More information about these products in found in Bauer et al. (2007).

#### Guaranteed Minimum Life Benefits (GMLB)

GMLBs were offered for the first time in the 1990s, and have two main forms; the Guaranteed Minimum Accumulation Benefit (GMAB) and the Guaranteed Minimum Income Benefit (GMIB). These products are very similar in the sense that they both offer a guaranteed benefit at the end of the deferment period. The GMIB offers the policyholder the choice to annuitize with a pre-specified guaranteed amount. The next section discusses the GMAB.

#### Guaranteed Minimum Accumulation Benefit (GMAB)

The GMAB is actually the simplest form of GMLB. Here, the customer is entitled to a minimal account value at the end of the contract period. This can be extended by the fact that if the fund reaches a pre-specified level, the new guaranteed amount becomes that value, i.e. a number of click-levels can be chosen such that the guaranteed amount is increased when the fund reaches a certain level.

### 6.1.2 Asian option

An Asian option is standard in the current literature and this thesis will therefore not go into full detail. The global idea is that the payoff of an Asian option is determined by the average underlying price over a pre-specified period. It then pays out the maximum of zero and the average of the underlying price minus the strike price at the end of the period. More information can be found in Hull (2003). An Asian option is an example of a unit-linked product, because a premium is paid and the final pay-off depends on an underlying fund. Asian options do not offer the flexibility of a unit-linked product to change contract details during the period.

## 6.2 Product valuation

This section will discuss the valuation of three products based on the NIG models compared to the BS models.

#### Guaranteed Minimum Accumulation Benefit (GMAB)

The value of a product depends on the assumptions made, for the GMAB the following assumptions are made

- The contract's starting date is September 30, 2015,
- The contract's maturity is 10 years,
- The insured makes an initial payment of USD 1,000,
- The fund is invested for 100% in US equity, i.e. S&P500,
- The click-levels are respectively USD 1,000, USD 1,250, USD 1,500, USD 1,750 and USD 2,000 based on daily closing prices

Even though a Variable Annuity is flexible in changing contract terms, this will not be included in the assumptions made for the valuation. This is because in practice an insured will pay the difference in market price if contract details are changed and is thus not of importance for the valuation of the product. In Figure 6.1 the theoretical click-levels, underlying investment amount and pay-off are shown for a similar contract between September 1st, 2005 until August 31st, 2015.

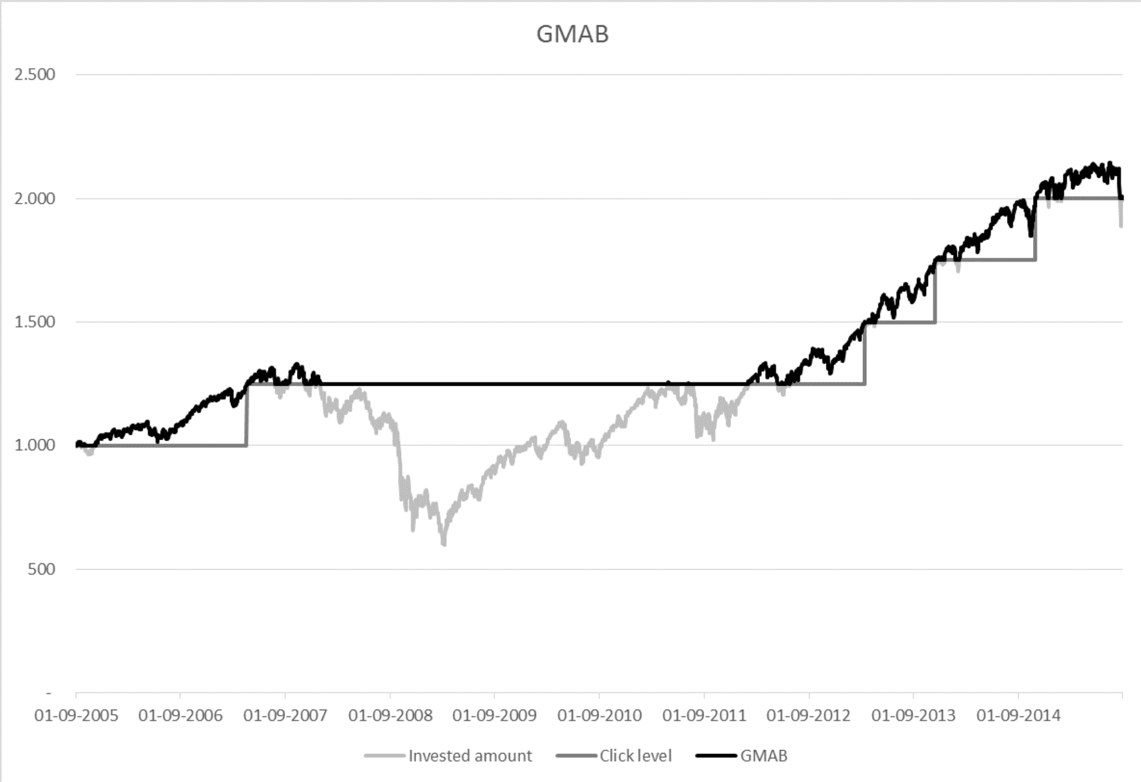


Figure 6.1: Guaranteed Minimum Accumulation Benefit illustrated using an example

### Asian option

The value of a product depends on the assumptions made, for the Asian option the following assumptions are made

- The contract's starting date is September 30, 2015,
- The contract's maturity is 5 years,
- Based on US equity, i.e. S&P500,
- The average over daily closing prices,
- Strike price is equal to the S&P500 price at September 30, 2015

In Figure 6.2 the average price, real price and theoretical pay-off are shown for a similar contract between September 1st, 2010 until August 31st, 2015.


Figure 6.2: S&P500, average price and pay-off of the Asian option. Note that the latter is plotted using a secondary axis.

### Unit-linked product

The value of a product depends on the assumptions made, for the unit-linked product the following assumptions are made

- The contract's starting date is September 30, 2015,
- The contract's maturity is 15 years,
- The insured makes annual payments of USD 1,000,
- The fund is invested for 100% in US equity, i.e. S&P500,
- The guaranteed rate is an annualized 0%

In Figure 6.3 the real price, guaranteed rate and theoretical pay-off are shown for a similar contract between September 1st, 2000 until August 31st, 2015.

The price of these products is now computed for the different calibrated BS and NIG models. These are the models calibrated in Chapter 4, i.e. these are all based on the valuation date of September 30th 2015. The broad calibration is the calibration done on the large amount of options, i.e. including in and out of the money options. The other calibration is the one done as a case study based on only at the money options. The results are shown in Table 6.1 and Table 6.2 and for the calibration based on at the money option prices in Table 6.3 and Table 6.4. These results have also been plotted in Figure 6.4, 6.5 and 6.6.

The price is calculated by subtracting the present value of the payout by the insurer at maturity by the present value of the payments made by the insured during the contract period. Using the ESG, one can simulate S&P500's future paths and get an estimation of the final value of the fund. This can then be discounted to receive the present value of the amount invested. The present value of

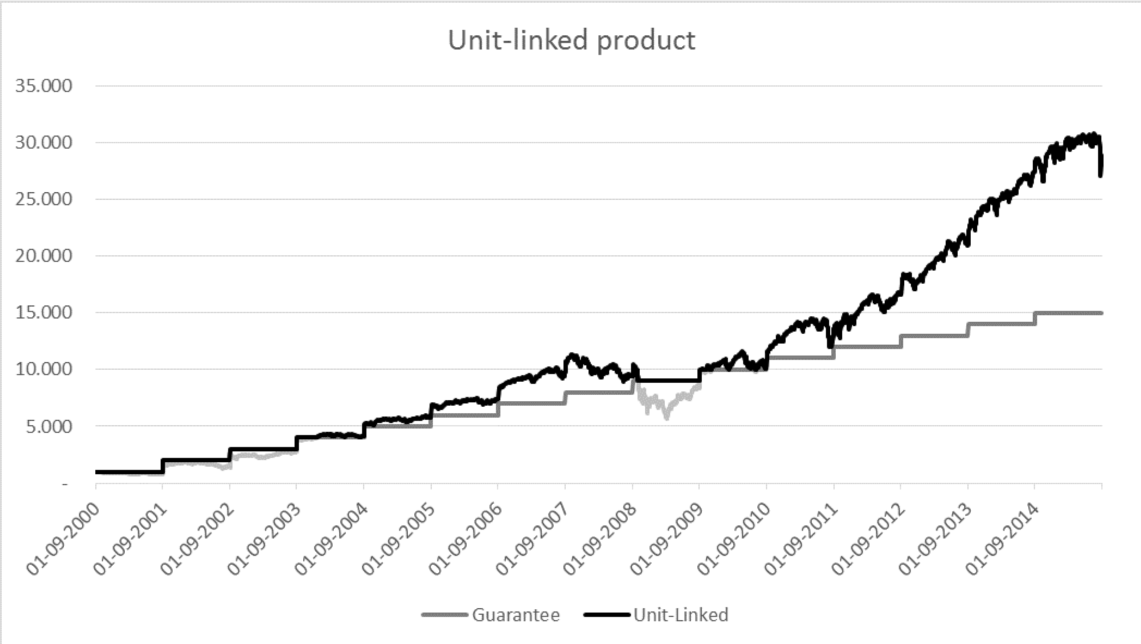


Figure 6.3: S&P500, guaranteed rate and pay-off of the unit-linked product.

the payments is more straightforward by multiplying the payment with the correct discount factor resulting in a present value of all payments.

## 6.3 Interpretation

The estimated prices are unfortunately highly unstable for both the NIG and BS models. However this is not surprising due to the fact that the calibration was already unstable for the calibration on different objective functions.

The broadly calibrated NIG models give in general a slightly higher value for the products than the broadly calibrated BS models, with the ARMSE(l) NIG model as an exception. But remember that this model ended up in a local optimum, with significantly different characteristics than the other models. The higher values are explained by the fact that even though the mean and variance do not differ much, the more negative skewness and higher kurtosis lead to more extreme jumps in asset returns. This is valuable for option-like products such as insurance products.

The models that are calibrated on at the money options do not give significantly higher prices for the NIG model than for the BS model. In Chapter 4, it was already shown that even though the mean and variance of the calibration based on at the money options was almost equal to the mean and variance of the broadly calibrated NIG model, the skewness and kurtosis were significantly different and close to zero.

An interesting result is that the BS prices are fairly unstable as well, based on different objective functions used in the calibration. The BS model is widely used in practice and it is therefore critical to know what an impact a choice of objective function in the calibration can mean for estimated prices of insurance products.

Table 6.1: 95% confidence interval of prices of products described in Chapter 6 using the broadly calibrated NIG model.

| | SSD/RMSE | | SSD(l)/RMSE(l) | | ARMSE | | ARMSE(l) | | ARPE | | ARPE(l) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| GMAB | $ 233.49 | $ 239.64 | $ 239.47 | $ 245.61 | $ 222.68 | $ 228.64 | $ 213.92 | $ 219.70 | $ 233.11 | $ 239.27 | $ 232.43 | $ 238.56 |
| Asian option | $ 186.93 | $ 190.61 | $ 185.09 | $ 188.73 | $ 180.00 | $ 183.62 | $ 174.27 | $ 177.80 | $ 182.28 | $ 185.98 | $ 182.50 | $ 186.20 |
| Unit-linked product | $ 1,018.89 | $ 1,037.96 | $ 1,021.71 | $ 1,040.83 | $ 942.00 | $ 960.61 | $ 892.26 | $ 910.38 | $ 990.86 | $ 1,010.02 | $ 980.17 | $ 999.23 |

Table 6.2: 95% confidence interval of prices of insurance products using the broadly calibrated BS model.

| | SSD/RMSE | | ARMSE | | ARPE | |
|---|---|---|---|---|---|---|
| GMAB | $ 227.61 | $ 233.96 | $ 216.81 | $ 222.71 | $ 221.89 | $ 227.82 |
| Asian option | $ 181.91 | $ 185.65 | $ 174.94 | $ 178.49 | $ 177.18 | $ 180.79 |
| Unit-linked product | $ 1,001.77 | $ 1,021.18 | $ 901.74 | $ 919.18 | $ 934.40 | $ 953.00 |



Table 6.3: 95% confidence interval of prices of insurance products using the at the money options to calibrate the NIG model.

| | SSD/RMSE | | SSD(l)/RMSE(l) | | ARMSE | | ARMSE(l) | | ARPE | | ARPE(l) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| GMAB | $ 214.14 | $ 220.23 | $ 214.56 | $ 220.36 | $ 230.86 | $ 236.98 | $ 230.75 | $ 236.86 | $ 227.61 | $ 233.71 | $ 226.88 | $ 232.94 |
| Asian option | $ 177.92 | $ 181.51 | $ 173.84 | $ 177.33 | $ 182.10 | $ 185.81 | $ 180.19 | $ 183.88 | $ 180.28 | $ 183.94 | $ 179.12 | $ 182.82 |
| Unit-linked product | $ 915.99 | $ 934.38 | $ 892.69 | $ 910.75 | $ 979.18 | $ 998.39 | $ 956.77 | $ 975.69 | $ 956.19 | $ 975.04 | $ 964.61 | $ 983.56 |

Table 6.4: 95% confidence interval of prices of insurance products using the at the money options to calibrate the BS model.

| | SSD/RMSE | | ARMSE | | ARPE | |
|---|---|---|---|---|---|---|
| GMAB | $ 206.54 | $ 212.42 | $ 229.33 | $ 235.40 | $ 229.82 | $ 235.91 |
| Asian option | $ 175.33 | $ 178.87 | $ 178.93 | $ 182.58 | $ 179.71 | $ 183.39 |
| Unit-linked product | $ 876.43 | $ 894.42 | $ 952.17 | $ 971.01 | $ 964.46 | $ 983.43 |

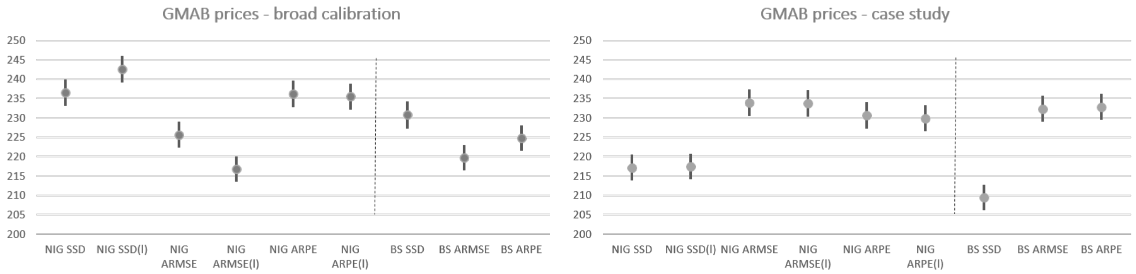


Figure 6.4: Confidence interval of GMAB prices

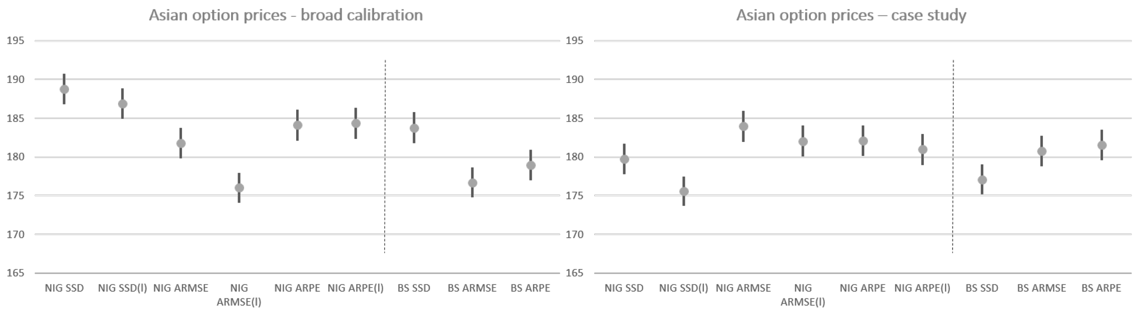


Figure 6.5: Confidence interval of Asian option prices

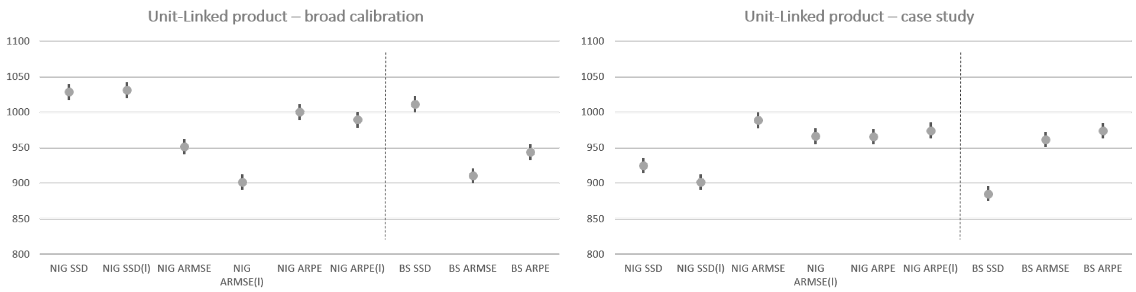


Figure 6.6: Confidence interval of Unit-Linked product prices

# Chapter 7

# Optimal portfolio

This chapter tests the NIG against the normal distribution in the context of optimal portfolio theory. Markowitz's theory will be used as guideline for this optimal portfolio. Markowitz (1991) described an optimal portfolio theory which is now standard in the literature. Markowitz's optimal portfolio theory only depends on risk and return, translated into expected standard deviation and return. Obviously, every MME calibration gives similar portfolio weights, since the sample standard deviation and return are used as best estimates for the future. An MLE calibration may however result in different portfolio weights.

## 7.1 Data

The data for this chapter is chosen in such a way that the entire world economy is taken into account. This can be done by taking the MSCI ACWI and MSCI Frontier Markets indices. However, these can be split up in smaller indices; the MSCI ACWI is the combination of the MSCI World and MSCI Emerging Markets indices, and these are a combination of smaller regional indices as shown in Figure 7.1. These 9 smaller indices are

1. MSCI North America
2. MSCI Europe & Middle East
3. MSCI Pacific
4. MSCI Emerging Markets Latin America
5. MSCI Emerging Markets EMEA
6. MSCI Emerging Markets Asia
7. MSCI Frontier Markets Latin America and Caribbean
8. MSCI Frontier Markets EMEA
9. MSCI Frontier Markets Asia

These will be used as different equity indices, one can invest in. The total return is used for all the indices, the data is chosen such that the all nine total return indices have data, this means from May 26th, 2010 until September 30th, 2015. The risk free is the same as used in the calibration in Chapter 4, i.e. the USD OIS curve given in Figure F.1.


Figure 7.1: Set up of MSCI indices to cover most of the world economy

## 7.2 Calibration

Thus, as shown in the previous section, nine data sets are used. Unfortunately, not all of them have been in use or tracked for a long time. Markowitz modern portfolio theory is a mathematical formulation of the concept of diversification in investing, aiming to select a collection of assets that finds a lower overall risk than any other combination of assets with the same expected return.
Thus, by finding the moments; historical return, historical standard deviation and historical covariance matrix one can create a first Markowitz portfolio. The second approach fits the MLE calibrated NIG using the data available and then calculates the return and standard deviation. The model described in this thesis does not allow for the calibration of a covariance matrix, and therefore the same historical covariance matrix is used as for the MME approach.

In Table 7.1 the expected return and standard deviation of the historical calibration on the nine MSCI indices covering the world economy are shown. Remember that the MME calibration is equal for the NIG and BS models and that the MLE calibration for the BS model is equal to the MME calibration, therefore in Table 7.1 the only separation shown is between the MME and the MLE calibration. Here the former is thus the MME calibration of both the BS and NIG model and the latter is the MLE calibration for the NIG model.
Minor differences are observed in the expected return and standard deviation between the MME and MLE calibrated models. The difference are in the skewness and kurtosis, however these are not shown because Markowitz's portfolio theory does not take these into account.

Table 7.1: Table of historical calibrated annual returns and standard deviation on the MSCI indices based on the MME, both for BS and NIG (these are equal) as well as the NIG MLE calibrated returns and standard deviation.

| | NA | EME | Pacific | EM LAC | EM EMEA | EM Asia | FM LA | FM EMEA | FM Asia |
|---|---|---|---|---|---|---|---|---|---|
| $\mu_{MME}$ | 12.03% | 7.92% | 5.37% | -8.40% | -1.49% | 4.08% | 2.09% | 5.06% | 2.13% |
| $\sigma_{MME}$ | 15.06% | 19.88% | 16.02% | 21.73% | 21.76% | 16.64% | 33.67% | 9.20% | 12.68% |
| $\mu_{MLE}$ | 12.03% | 7.92% | 5.37% | -8.40% | -1.49% | 4.08% | 2.09% | 5.06% | 2.13% |
| $\sigma_{MLE}$ | 15.27% | 20.08% | 15.89% | 21.76% | 21.75% | 16.62% | 33.83% | 9.04% | 12.28% |

## 7.3 Markowitz portfolio

From Table 7.1 results that the optimal portfolio of both models do not differ much. The efficient frontiers of both models are plotted in Figure 7.2. Indeed the frontiers resemble to each other and the optimal risky portfolios both have a Sharpe ratio (return divided by risk) of 0.9019.

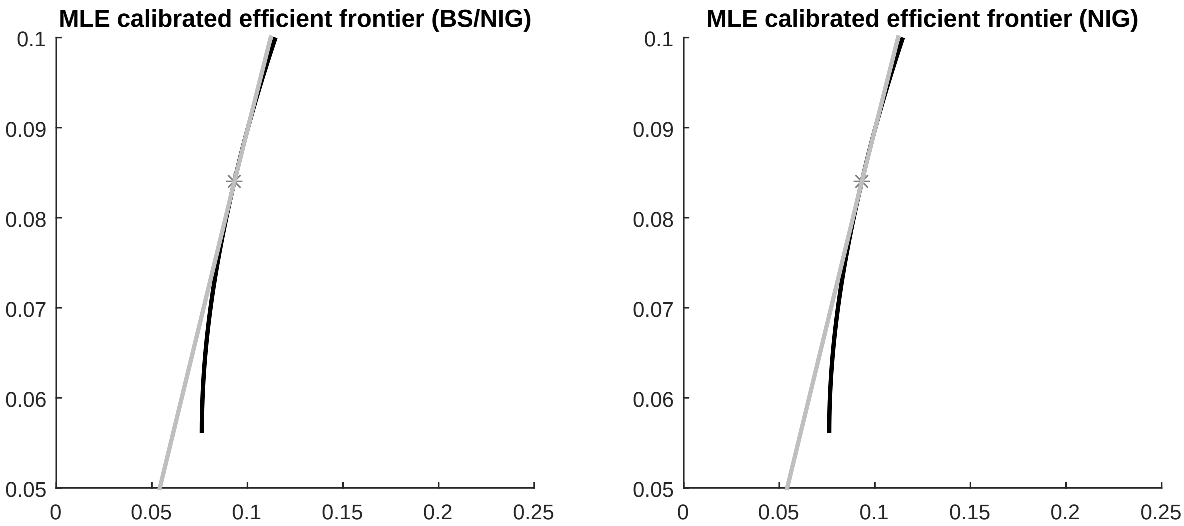


Figure 7.2: Efficient frontiers using MME and MLE calibrated returns and standard deviation.

The weights of the optimal risky portfolios, i.e. the portfolios that gives the risk and return corresponding to the stars in Figure 7.2 are given by

$$w_{MME} = \begin{bmatrix} 0.4891 \\ 0 \\ 0.0400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.4431 \\ 0.0278 \end{bmatrix}, \qquad w_{MLE} = \begin{bmatrix} 0.4891 \\ 0 \\ 0.0400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.4431 \\ 0.0278 \end{bmatrix}$$

Where the weight in position $i$ is the index as defined in section 7.1. Thus even though the analytic standard deviation of the MME (BS/NIG) calibration against the MLE (NIG) calibration show marginal differences, the weights of both optimal risky portfolios are equal.

## 7.4 Testing of a Markowitz portfolio

This section uses the weights of the optimal risky portfolio found in the previous section and simulates the assets using these weights. Since the MME and MLE calibrated weights are equal the portfolio evolution can be simulated for three different hypothetical worlds; a MME/MLE calibrated BS world, a MME calibrated NIG world and a MLE calibrated NIG world. Figure 7.3 shows the results, unsurprisingly all paths are similar and show marginal differences. The figure plots the mean, and the 95% confidence interval paths of the optimal risky portfolio if 1 USD would be invested according to the optimal portfolio weights.

This result suggests that for optimal portfolio theory in a Markowitz setting the NIG model does

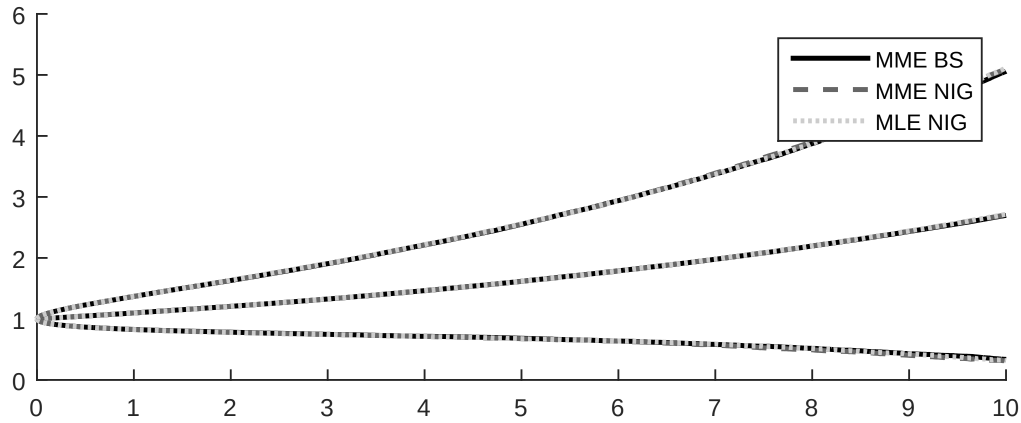


Figure 7.3: Portfolio paths when investing 1 USD according to the optimal risky Markowitz portfolio and simulating these in different worlds. The upper and lower lines are the 95% confidence interval, the middle lines are the mean.

not differ much from the results obtained using a BS model. This is probably caused by the fact that Markowitz only takes into account the risk and return of a portfolio and these are very similar for historically calibrated NIG and BS models.
However, an optimal portfolio theory that takes into account more moments, e.g. the assumption that investors are reluctant to large negative jumps in portfolio returns, could result in different results for the NIG and BS model.

# Chapter 8

# Conclusion

This thesis tested whether the NIG distribution is better at the prediction of asset returns and the valuation of insurance products than the normal distribution. The NIG distribution appears to have properties that are empirically observable in asset returns, i.e. heavy tails and a higher peakedness. A closed-form formula for an arbitrage-free price of a European call option assuming that asset returns are NIG distributed within a Black-Scholes setting can be derived.

The core of this thesis is in the calibration of the NIG distribution, the historical calibration performs well using the NIG distribution with a significant improvement in fitting the distribution to the empirical data based on Maximum Likelihood Estimation.
The NIG and normal distribution can also be calibrated on option prices, obtaining so-called implied parameters. One has to be careful with the interpretation of this calibration because the NIG closed-form option price formula depends on three parameters whereas the standard BS depends on one. Thus, by definition the NIG is expected to fit observed option prices better than the normal distribution. A modeling error could occur when a function is too closely fit to the data, such that the model tries to explain some degree of error or random noise in the data, this is called overfitting. Overfitting can be observed by (intertemporal) out of sample testing. The Maximum Relative Error (MRE) and Average Relative Percentage Error (ARPE) objective functions are the two objective functions that are significantly lower for all in sample and (intertemporal) out of sample NIG models compared to the BS models while all other objective functions are fairly similar for both models. The lower MRE suggests that the NIG model performs better at catching the skew observed in option prices. Both models are highly unstable, with large differences in parameter values if calibrated on another objective function. The NIG model thus seems to have some minor advantages over the BS model in the estimation of option prices. However the NIG model is more complex, demands more computation time and has computation issues that could lead to minor accuracy errors.

An Economic Scenario Generator was created based on the NIG and normal distributions. These ESGs performed well and the analytic characteristics matched the characteristics of the simulated paths.
The ESG was used to create risk-neutral paths with implied calibrated NIG and BS models. As expected, the prices of insurance products were unstable as a result of the unstable calibration results. The values for the insurance products were however slightly higher for the NIG distribution than for the normal distribution which is related to the characteristic of more extreme values in the NIG model having a direct impact on products with guarantees.
The ESG was also used to create real-world paths using historical calibrated models. The impact of the NIG distribution on Markowitz's optimal portfolio theory was marginal, primarily because this theory depends solely on the mean and standard deviation.

There is not one clear answer to the main research question formulated in the introduction. The NIG distribution fits historical asset returns better than the normal distribution. For the implied calibration the NIG model catches the skew observed in option prices better, but it is more complex, demands more computation time and has computation issues that could lead to inaccuracy.
The unstable implied calibration leads to problems in the valuation of insurance products. The price depends significantly on the parameters chosen as input. The prices of insurance products based on the NIG model are slightly higher due to the fact that the NIG model allows for more extreme values. The complexity and computation heavy properties of the NIG model are a trade-off against the more simple and faster BS model for the valuation of insurance products.
The BS model is recommended in a Markowitz's optimal portfolio setting, here both the BS and NIG models result in the same optimal portfolio. The normal distribution is however more easier to interpret and explain since this model is solely dependent on risk and return just like Markowitz's theorem. However, an optimal portfolio theory that takes into account more moments, e.g. the assumption that investors are reluctant to large negative jumps in portfolio returns, could result in different results for the NIG and BS model. The NIG model could be an improvement of the BS model in such a optimal portfolio setting.

# Chapter 9

# Future research

The NIG distribution has interesting properties for practical applications in finance. The research in this thesis did not give a clear answer to whether the NIG distribution provides a strict improvement of the normal distribution. The NIG distribution contains dynamics that the normal distribution, used primarily in the insurance business, does not. During the research, I encountered several problems. I was unable to solve all these problems, the main questions that could offer interesting insights are shown below.

1. One could derive different closed-form arbitrage-free option price formulas that could be used for the calibration, perhaps resulting in more stable models and thus insurance product prices.
2. One could make the calibration more efficient, either by improving MATLAB's original optimization process, implementing a new one or making the code more efficient.
3. One could test the NIG distribution, not only on insurance products but on other products, perhaps based on another (more stable) calibration.
4. One could use the NIG distribution using another optimal portfolio theory, that also depends on the skewness and kurtosis for example and thus seeing more effect of the NIG distribution than was observed in this thesis.
5. One could extend the NIG model by implementing a jump-diffusion or stochastic volatility component.
6. One could extend the NIG model by implementing correlation structures, e.g. the correlation between the interest rate and the modelled asset or between different assets in one model.

# Appendix A

# Inverse Gaussian distribution

The Inverse Gaussian distribution is an exponential distribution. The fact that extremely large outcomes can occur when almost all outcomes are small is a distinguishing characteristic.

The PDF is given by

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

For $0 < x < \infty$. The expected value is $\mu$ and the variance of the Inverse Gaussian is

$$\mathrm{Var}(x) = \frac{\mu^3}{\lambda}$$

And the skewness and kurtosis are equal to

$$\mathrm{Skew} = 3\sqrt{\frac{\mu}{\lambda}}$$
$$\mathrm{Kurt} = \frac{15\mu}{\lambda}$$

# Appendix B

# Lognormal distribution

This Appendix discusses the Lognormal distribution as descibed in Hull (2003), because it is less standard than the Normal distribution but plays a critical role in the derivation of the standard Black-Scholes option formula. Note that if a random variable $Y \in \mathbb{R}$ follows a normal distribution with mean $\mu$ and variance $\sigma^2$, then $X = e^Y$ follows a lognormal distribution with mean

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2}$$

and variance

$$\mathrm{Var}[X] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

The PDF is given by

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

And the CDF by

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

Where $\Phi(y)$ is the standard normal CDF.

An important property needed for the derivation of the standard Black-Scholes option price is the lognormal conditional expected value, i.e. $L_X(x) = \mathrm{E}[X|X > x]$. Using the PDF of the lognormal distribution, this becomes

$$L_X(K) = \int_K^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln x - \mu}{\sigma})^2} dx$$

Now apply the substitution rule where $y = \ln x \Leftrightarrow x = e^y \Leftrightarrow dx = e^y dy$. Gives

$$\begin{aligned}
L_X(K) &= \int_{\ln K}^{\infty} \frac{e^y}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2} dy \\
&= \int_{\ln K}^{\infty} \frac{e^y}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y^2-2y\mu+\mu^2-2\sigma^2 y)} dy \\
&= \int_{\ln K}^{\infty} \frac{e^y}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-(\mu+\sigma^2))^2+\mu+\frac{1}{2}\sigma^2} dy \\
&= \exp(\mu + \frac{1}{2}\sigma^2)\frac{1}{\sigma}\int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-(\mu+\sigma^2)}{\sigma}\right)^2\right) dy
\end{aligned}$$

Using the scale-location transformation of the standard normal CDF, i.e. $\Phi(-x) = 1 - \Phi(x)$ implies

$$L_X(K) = \exp\left(\mu + \frac{\sigma^2}{2}\Phi\left(\frac{-\ln K + \mu + \sigma^2}{\sigma}\right)\right)$$

# Appendix C

# Kolmogorov-Smirnov statistic

In Bain and Engelhardt (2000) the Kolmogorov-Smirnov (KS) test is discussed. This test is a non-parametric test of the equality of continuous, one-dimensional probability distributions and compares an empirical sample with a modeled one. The KS statistic calculates the maximum distance between the model's CDF and the empirical one. The null distribution is calculated under the null hypothesis that the sample is drawn from the reference distribution.
The two-sample KS test used in this thesis is sensitive to both location and shape differences. Note that the empirical CDF for n i.i.d. observations $X_i$ is defined as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[-\infty,x]}(X_i)$$

Where $I_{[-\infty,x]}$ is an indicator function. The KS statistic for a given CDF is then

$$D_n = \sup_x |F_n(x) - F(x)|$$

Where $\sup_x$ is the supremum of the set. The Kolmogorov-Smirnov test is constructed using the critical values of the Kolmogorov distribution, which can be found in Bain and Engelhardt (2000). The null hypothesis is rejected at level $\alpha$ if

$$D_n > K_\alpha$$

Where $K_\alpha$ is the critical value.

# Appendix D

# Anderson - Darling Test

The Anderson - Darling test tests just like the Kolmogorov - Smirnov test if an empirical datasets fits to a CDF. However, it differs from the K-S test that the probability of scale is arithmetic. The maximum deviation which is calculated in the K-S test is seldom observed in the tails, the A-D test takes this into account. The A-D statistic is equal to

$$A^2 = -\sum_{i=1}^{n}[(2i-1)(\ln F_x(x_i) + \ln[1 - F_x(X_{n+1-i})])/n] - n$$

Where $x_n$ is the largest observed value and $x_1$ the smallest, $F_x(x_i)$ is the CDF of the proposed distribution. The Anderson-Darling test is discussed in Ang and Tang (2006) and the null hypothesis is rejected at level $\alpha$ if

$$A > K_\alpha$$

Where $K_\alpha$ is the critical value.

# Appendix E

# Option tables for calibration

In Table E.1, the Option prices used to calibrate the BS and NIG distributions is shown. The over-time out of sample data is chosen not to put in the Appendix. This data can be found using Bloomberg or a similar system. The light gray area is the in sample data, i.e. the option prices used to calibrate the parameters, these are then tested on the full data table as well as on option prices on other dates.

Table E.1: S&P500 Mid ((Bid + Ask)/2) Monthly Option Prices on September 30th, 2015. On the x-axis the expiration date is listed (MM/DD/'YY) and on the y-axis the different strike prices (K). These are all the monthly options listed in Bloomberg for which a Bid and an Ask price were available. The stock price on September 30th 2015 was 1920.03. The light gray area is the in sample data used for the academically calibration and the dark gray area for the case study, the parameters are then tested on the full matrix and over time.

| | 10/16/'15 | 11/20/'15 | 12/19/'15 | 1/15/'16 | 3/18/'16 | 6/17/'16 | 9/16/'16 | 12/16/'16 | 1/20/'17 | 6/16/'17 | 12/15/'17 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1400 | | | | | | | 514.45 | 517 | 519.2 | 522 | 530.95 |
| 1425 | | | | | 484.95 | 488.75 | 493 | 496.6 | 498.25 | 501.8 | 509.75 |
| 1450 | | | | | 461.8 | 466.5 | 471.5 | 475.65 | 477.55 | 484.75 | 492.1 |
| 1475 | | | | | 438.8 | 444.45 | 450 | 454.95 | 457 | 465.8 | 473.65 |
| 1500 | | | | | 416.05 | 422.7 | 429.1 | 434.5 | 436.8 | 443.1 | 455.6 |
| 1525 | | | | | 393.3 | 401.15 | 408.35 | 414.3 | 416.85 | 423.95 | 436.8 |
| 1550 | | | | | 371.25 | 379.95 | 387.85 | 394.45 | 397.15 | 408.9 | 418.4 |
| 1575 | | | | | 349.2 | 359 | 367.7 | 374.2 | 377.8 | 390.05 | 401.65 |
| 1600 | | | | | 327.3 | 338.4 | 347.85 | 354.95 | 359.35 | 369.35 | 383.5 |
| 1625 | | | | | 306.05 | 318.05 | 328.3 | 336.05 | 340.65 | 354.25 | 365.15 |
| 1650 | | | | | 284.9 | 298.1 | 309.1 | 317.5 | 321.6 | 333.05 | 347.95 |
| 1675 | | | | | 264.45 | 278.5 | 290.3 | 299.25 | 303.55 | 319.55 | 331.75 |
| 1700 | | | | | 244.25 | 259.35 | 271.9 | 281.5 | 285.95 | 302.75 | 317.6 |
| 1725 | | | | | 224.5 | 240.65 | 253.95 | 264.1 | 268.8 | 283.35 | 301.75 |
| 1750 | | | | | 205.25 | 222.5 | 236.45 | 247.15 | 252.05 | 266.4 | 286.3 |
| 1775 | | | | | 186.55 | 204.75 | 219.4 | 230.65 | 235.7 | 250.75 | 270.55 |
| 1800 | | | | | 168.5 | 187.6 | 202.9 | 214.65 | 220.65 | 237.05 | 256.55 |
| 1820 | 102.8 | 119.2 | 129.8 | 138.8 | | | | | | | |
| 1825 | 98.45 | 115.35 | 125.9 | 135.2 | 151.1 | 171.05 | 186.85 | 199.1 | 205.2 | 221.5 | 240.5 |
| 1830 | 94.3 | 111.35 | 122 | 131.6 | | | | | | | |
| 1835 | 90.35 | 107.85 | 118.6 | 128.05 | | | | | | | |
| 1840 | 86.35 | 104.1 | 114.9 | 124.55 | | | | | | | |
| 1845 | 82.2 | 100.45 | 111.45 | 120.85 | | | | | | | |
| 1850 | 78.25 | 96.85 | 107.75 | 117.6 | 134.35 | 155.05 | 171.2 | 183.15 | 188.65 | 207.35 | 226.15 |
| 1855 | 74.4 | 93.25 | 104.5 | 114 | | | | | | | |
| 1860 | 70.75 | 89.75 | 101.3 | 110.85 | | | | | | | |
| 1865 | 67.25 | 86.3 | 97.5 | 107.15 | 124.7 | | | | | | |
| 1870 | 63.45 | 82.85 | 94.4 | 104.25 | | 142.7 | | | | | |
| 1875 | 59.7 | 79.5 | 91.1 | 100.75 | 118.35 | 139.8 | 156 | 170.05 | 174.5 | 194.1 | 212.95 |
| 1880 | 56.2 | 76.2 | 87.9 | 97.75 | | | | | | | |
| 1885 | 52.9 | 72.9 | 84.6 | 94.7 | | | | | | | |
| 1890 | 49.45 | 69.65 | 81.35 | 91.45 | | | | | | | |
| 1895 | 46.2 | 66.5 | 78.3 | 88.55 | | | | | | | |
| 1900 | 43 | 63.4 | 75.35 | 85.5 | 103.2 | 124.95 | 142 | 156 | 161.1 | 180.6 | 200 |
| 1905 | 39.9 | 60.25 | 72.3 | 82.5 | | | | | | | |
| 1910 | 36.9 | 57.35 | 68.95 | 79.5 | | | | | | | |
| 1915 | 34.15 | 54.4 | 66.25 | 76.25 | | | | | | | |
| 1920 | 31.3 | 51.55 | 63.35 | 73.75 | | | | | | | |
| 1925 | 28.6 | 48.75 | 60.8 | 70.95 | 89 | 111 | 128.35 | 142.2 | 148 | 167.55 | 187.4 |
| 1930 | 26.05 | 46.05 | 57.8 | 67.95 | | | | | | | |
| 1935 | 23.65 | 43.35 | 55.1 | 65.2 | | | | | | | |
| 1940 | 21.3 | 40.8 | 52.45 | 62.5 | 80.9 | | | | | | |
| 1945 | 19.15 | 38.3 | 49.85 | 59.9 | | | | | | | |
| 1950 | 17.1 | 35.9 | 47.35 | 57.35 | 75.65 | 97.85 | 115.3 | 129.45 | 134.95 | 155 | 175.45 |
| 1955 | 15.2 | 33.55 | 44.9 | 54.85 | | | | | | | |
| 1960 | 13.45 | 31.3 | 42.6 | 52.45 | | | | | | | |
| 1965 | 11.8 | 29.15 | 40.3 | 50.05 | | | | | | | |
| 1970 | 10 | 27 | 37.95 | 47.7 | | | | | | | |
| 1975 | 9.15 | 25.05 | 35.75 | 45.45 | 63.4 | 85.5 | 102.95 | 117.2 | 121.3 | 142.9 | 163.7 |
| 1980 | 7.5 | 22.6 | 33.7 | 43.25 | | | | | | | |
| 1985 | 6.45 | 21.35 | 31.7 | 41.15 | | | | | | | |
| 1990 | 5.5 | 19.6 | 29.65 | 39.05 | | | | | | | |
| 1995 | 4.675 | 17.95 | 27.8 | 37 | | | | | | | |
| 2000 | 4.05 | 16.35 | 26.05 | 35 | 52.35 | 73.9 | 91.8 | 105.1 | 111.05 | 131.3 | 152.15 |
| 2005 | 4.375 | 14.95 | 24.35 | 33 | | | | | | | |
| 2010 | 2.7 | 13.55 | 22.75 | 31.25 | | | | | | | |
| 2015 | 2.125 | 12.3 | 21.15 | 29.45 | | | | | | | |
| 2020 | 1.85 | 11.1 | 19.55 | 27.75 | | | | | | | |
| 2025 | 1.25 | 10 | 18.15 | 26.15 | 42.35 | 63.25 | 80.4 | 94.2 | 100.1 | 119.95 | 141.45 |
| 2050 | | | | | 33.6 | 53.25 | 69.85 | 84.55 | 89.7 | 110.2 | 130.75 |
| 2075 | | | | | 25.85 | 44.45 | 60.45 | 74.5 | 79.9 | 100 | 120.8 |
| 2100 | | | | | 19.45 | 36.85 | 52.35 | 65.65 | 70.4 | 91.05 | 111.85 |
| 2125 | | | | | 14.3 | 29.85 | 44.35 | 56.8 | 61.85 | 82.2 | 102.1 |
| 2150 | | | | | 10.35 | 23.75 | 37.35 | 49.45 | 54.25 | 73.75 | 92.9 |
| 2175 | | | | | 7 | 18.75 | 29.9 | 42.45 | 47 | 65.95 | 84.85 |
| 2200 | | | | | 5.4 | 14.45 | 24.3 | 36.15 | 40.05 | 58.6 | 77.15 |
| 2225 | | | | | 2.925 | 9.6 | 20.65 | 29.8 | 34.15 | 51.85 | 69.95 |
| 2250 | | | | | 2.05 | 8.05 | 16.5 | 25.15 | 29 | 45.5 | 63.05 |
| 2275 | | | | | 1.025 | 5.85 | 11.9 | 20.55 | 24.45 | 39.7 | 56.8 |
| 2300 | | | | | 0.75 | 4.2 | 10.1 | 17.55 | 20.7 | 34.5 | 51 |
| 2325 | | | | | 0.425 | 2.925 | 6.6 | 14.05 | 16.55 | 29.7 | 45.55 |
| 2350 | | | | | 0.425 | 1.95 | 4.675 | 11.4 | 13.6 | 25.8 | 40.55 |
| 2375 | | | | | 0.275 | 0.95 | 4.5 | 8.6 | 11.9 | 21.65 | 36 |
| 2400 | | | | | 0.175 | 0.75 | 3.325 | 6.8 | 9.05 | 18.35 | 31.85 |
| 2425 | | | | | | 0.5 | | 6.35 | 6.9 | 15.7 | 28.05 |
| 2450 | | | | | | | 1.75 | | | | |

# Appendix F

# Risk-free rates

In Figure F.1, one can find the curve that is used as risk-free. This rate is extracted from the USD Overnight Indexed Swap on September 30th, 2015.

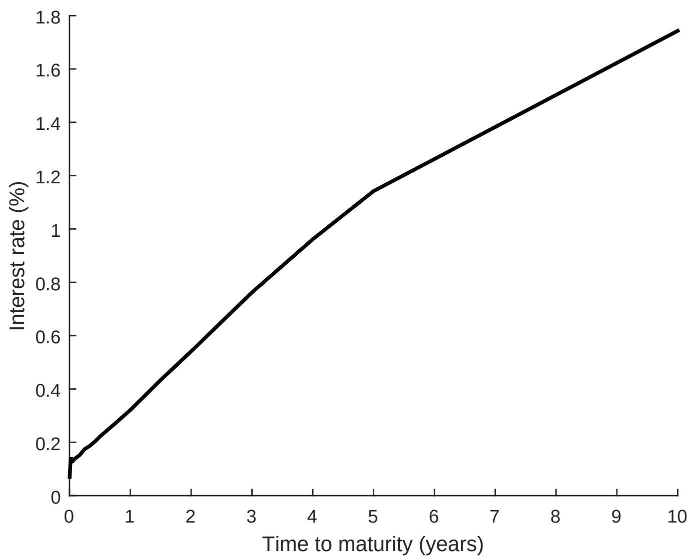


Figure F.1: USD Overnight Indexed Swap curve on September 30th, 2015

# Bibliography


AAG: Universal Life, case study 5. Technical report, Actuarieel Genootschap & Actuarieel Instituut, (2012).

Economic Scenario Generators: a mean to many ends. Technical report, Insurance ERM, (2014). Available at `www.insuranceerm.com`.

Solvency II delegated act. Technical report, European Commission, (2014).

Ang, A.H.S. and Tang, W.H. (2006). *Probability concepts in engineering.* Wiley.

Bain, L.J. and Engelhardt, M. (2000). *Introduction to Probability and Mathematical Statistics.* Duxbury Classic Series.

Barndorff-Nielsen, O.E. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (The Royal Society).*

Barndorff-Nielsen, O.E. (1978). Hyperbolic distributions and distributions on Hyperbolae. *Scandinavian Journal of Statistics, Volume 5, Issue 3.*

Barndorff-Nielsen, O.E. (1998). Processes of Normal Inverse Gaussian type. *Finance and Stochastics Volume 2, Issue 1.*

Barndorff-Nielsen, O.E. and Halgreen, C. (1977). Infinite divisibility of the Hyperbolic and Generalized Inverse Gaussian distributions. *Zeitschrift fur Währscheinlichkeitstheorie und Verwandte Gebiete, Volume 38, Issue 4.*

Barndorff-Nielsen, O.E., Mikosh, T. and Resnick, S. (200). *Lévy Processes — Theory and Applications.* Birkhauser.

Bauer, D., Kling, A. and Russ, J. (2007). A universal pricing framework for Guaranteed Minimum Benefits in Variable Annuities. Available at: `actuaries.org`.

Benth, Fred Espen, Groth, Martin and Kettler, Paul C. (2007). A quasi-Monte Carlo algorithm for the Normal Inverse Gaussian distribution and valuation of financial derivatives. *2000 Mathematics Subject Classification.*

Black, Fischer and Scholes, Myron. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy, Volume 81, Number 3.*

Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical properties. *Quantitative Finance, 1, Sections 1,2 and 8.*

Dempster, A.P., Laird, N.M. and Rubin, D. (1977). Maximum likelihood from incomplete data via the EM algorithm. *J. Roy. Statist. Soc. B 39, 1-38.*

Eberlein, E. (2001). *Application of Generalized Hyperbolic Lévy Motions.* Birkhauser.

Eberlein, E. and Jacod, J. (1997). On the range of option prices. *Finance and Stochastics 1.*

Eberlein, E. and Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli, Volume 1, Number 3.*

Fernandes, J. (2012). Fitting the Normal Inverse Gaussian distribution to the S&P 500 stock return data. *Department of Mathematics, UMass Darthmouth.*

Godin, F., Mayoral, S. and Morales, M. (2009). Contingent claim pricing using a Normal Inverse Gaussian probability distortion operator. *Centre de Recherches Mathematiques, Universite de Montreal.*

Hendriks, B. Calibration and estimation of a correlated multi-factor term structure model, (2012). Written at Tilburg University.

Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of Financial Studies 6(2).*

Holton, G.A. (2005). Fundamental theorem of asset pricing.

Hull, J.C. (2003). *Options, Futures and other derivatives, 5th edition.* Pearson Education International.

Jorgensen, B. (1982). Statistical properties of the Generalized Inverse Gaussian distribution. *Lecture Notes in Statistics, Volume 9.*

Karlis, D. (2002). An EM type algorithm for maximum likelihood estimation of the Normal Inverse Gaussian distribution. *Statistics & Probability Letters 57, 43-52.*

Kienitz, J. and Wetterau, D. (2012). *Financial Modelling: Theory, Implementation and Practice with MATLAB source.* Wiley Finance.

Kou, S.G. (2002). A jump-diffusion model for option pricing. *Management Science 2002 Vol. 48, No.8.*

Krog Saebo, K. Pricing Exotic Options with the Normal Inverse Gaussian market model using Numerical Path Integration, (2009). Written at Norwegian University of Science and Technology.

Markowitz, H.M. (1991). Portfolio selection.

Prause, K. (1999). The Generalized Hyperbolic model: Estimation, financial derivatives and risk measures. *Insitut fur Mathematische Stochastick, Albert-Ludwigs Universitat Freiburg.*

Rydberg, T.H. (1997). The Normal Inverse Gaussian Lévy process: simulation and approximation. *Comm. Statist. Stochastic Models 13(4).*

Schoutens, W. (2003). *Lévy Processes in Finance: Pricing Financial Derivatives.* Wiley series in probability and statistics.

Schumacher, J.M. (2003). *Financial Models.* Tilburg University.

Yao, L., Yang, G. and Yang, X. (2011). A note on the mean correcting martingale measure for geometric Lévy processes.

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