The use of mortality bonds to hedge the longevity risk of pension funds

Wilbur Damen

Master Thesis
Econometrics and Operations Research
Quantitative Finance and Actuarial Sciences

Supervisors
Tilburg University
Dr. J.J.J. Segers
Dr. A.M.B. de Waegenaere

Watson Wyatt Brans & Co
R. van Dam M.Sc. AAG

September 2006
Contents

Abstract ix
Preface xi

1 Introduction 1
1.1 Problem formulation .......................... 2
1.2 Structure of the thesis ........................ 2

2 Pensions 5
2.1 Three pillars ................................. 5
2.2 Pension suppliers ............................. 6
2.3 Pension schemes .............................. 7
2.4 Indexation .................................. 10
2.5 Conclusions ................................ 10

3 FTK 11
3.1 Introduction ................................ 11
3.2 Realistic value ............................... 12
3.2.1 Assets, equity capital, and loan capital .... 12
3.2.2 Liabilities ................................. 13
3.3 Conclusions ................................. 14

4 Mortality-Linked Securities 17
4.1 Introduction ................................. 17
4.2 Mortality bonds .............................. 18
4.2.1 Classical Mortality Bonds ................. 18
4.2.2 Zero-Coupon Mortality Bonds ............. 18
4.2.3 Deferred Mortality Bonds .................. 19
4.2.4 Principal-at-Risk Mortality Bonds ......... 19
4.2.5 The BNP Paribas Longevity Bond ......... 20
4.3 Mortality Swaps ............................. 22
4.3.1 One-Payment Mortality Swaps .......... 22
4.3.2 Vanilla Mortality Swaps .................. 23
4.3.3 Possible extensions ....................... 23
4.4 Mortality Futures ........................... 23
4.5 Mortality Options ........................................ 23
4.6 Choice of mortality indices ............................... 24
4.7 Conclusions ................................................. 24

5 Stochastic mortality ..................................... 27
5.1 Introduction ................................................ 27
5.2 Notation .................................................... 27
5.3 The Lee-Carter model ..................................... 28
5.4 Estimation of the parameters ............................. 29
5.5 Extrapolation of $\kappa$ ................................. 30
5.6 Applying the stochastic mortality model ............ 30
5.6.1 Data .................................................... 30
5.6.2 Results from the SVD composition .............. 31
5.6.3 Extrapolation of $\kappa$ ............................. 32
5.6.4 Computing the force of mortality for each age ... 36
5.7 Conclusions ............................................... 37

6 Some fictitious pension funds ......................... 39
6.1 Introduction ............................................... 39
6.2 Set-up of the pension scheme ......................... 39
6.3 Fund compositions ....................................... 41

7 Results ......................................................... 45
7.1 Introduction ............................................... 45
7.2 The standardized method ............................... 46
7.3 Investment in mortality bonds ......................... 50
7.4 Conclusions ............................................... 58

8 Conclusions and recommendations .................. 61
8.1 Conclusions ............................................... 61
8.2 Recommendations for further research .......... 62

A The Singular Value Decomposition .................... 65

B Results for $\alpha, \beta$, and $\kappa$ ....................... 67

C Variance computation ................................... 69

D Dutch pension terms and their English translation 71

E M-files ........................................................ 73
E.1 fundsandkappa.m ....................................... 73
E.2 Costfunction.m ........................................... 76

F Optimal bond allocations ............................... 79
List of Figures

1.1 Life expectancy at age of 65 for males (dotted blue line) and females (solid pink line). ......................... 1

2.1 Relation between employer, employee and pension supplier ........ 7

2.2 Back service costs and coming service costs in a final pay plan ... 8

2.3 Yearly accrual (bars) and average accrual (line) of the average pay plan example .................................. 10

4.1 Payoffs of a classical mortality bond ............................. 18

4.2 Payoffs of a deferred mortality bond, payoffs start at time \( \tau \) ... 19

4.3 Payoffs of a principal-at-risk mortality bond ....................... 20

4.4 Connections between the parties involved with the BNP Paribas Longevity Bond ........................................ 21

4.5 Expected payoffs of the BNP Paribas Longevity Bond .......... 22

5.1 \( \alpha \) for males (dotted blue line) and females (solid pink line) .... 31

5.2 \( \beta \) for males (dotted blue line) and females (solid pink line) .... 32

5.3 Change in \( \kappa \), denoted by \( \Delta \), for males (left) and females (right) . 33

5.4 Actual and fitted \( \kappa \) for males and females (1960-2003) ........... 35

5.5 Realized and forecasted \( \kappa \), including 95\% confidence bound for males (blue) and females (red) 1960 – 2050 ....... 36

6.1 Salary per age .................................................. 40

6.2 Accrued pension rights per age .................................. 41

6.3 Graphical representation of the eight fund populations; lower blue part corresponds to males, upper purple part to females; fund numbers are displayed in the top right corners ............ 42

7.1 Term structure of interest rates (February 28, 2006) ......... 45

7.2 Experienced death coefficients for males (dotted blue line) and females (solid pink line) ......................... 47

7.3 Actuarial factors for males (dotted blue line) and females (solid pink line) ............................................. 48
List of Tables

2.1 Yearly numbers of the average pay plan example .......................... 9

5.1 Ljung Box statistics with corresponding p-values for males and females .................................................. 34

5.2 Estimated parameters of the ARIMA(2,1,0) model for males and females .................................................. 35

6.1 Compositions and characteristics of the fictitious pension funds . 43

7.1 Results of the standardized method for the eight funds ................. 49

7.2 Mortality bonds: characteristics, mean, standard deviation, and prices in euros for several values of the risk parameter .............. 53

7.3 Exact probability levels of the capital needed according to the standardized method (odd funds mainly consist of females and even funds mainly of males) ............................................. 55

7.4 Total capital needed according to the internal method, if no investment takes place and difference with the standardized method 56

7.5 Optimal bond allocation in case the risk parameter equals zero .... 57

7.6 Relative differences of the costs of the optimal bond allocations with the costs if no investment takes place, for several values of the risk parameter, and maximal value of the risk parameter for which it is optimal to invest ............................................. 58

B.1 Estimated age-dependent parameters of the Lee-Carter model . 67

B.2 Re-estimated values of the mortality index ................................. 68

D.1 English pension term and corresponding Dutch translations ....... 72

F.1 Characteristics of the mortality bonds and the fund populations . 79

F.2 Optimal bond allocation in case the risk parameter equals 0.1 .... 80

F.3 Optimal bond allocation in case the risk parameter equals 0.5 .... 80

F.4 Optimal bond allocation in case the risk parameter equals one .. 81
Abstract

Longevity risk is the risk that persons live longer than is currently expected. For pension funds the longevity risk is very important since an increase in life expectancy implies an increase of pension payments as well. Hence, to account for unexpected future life expectancy changes, a pension fund must be sufficiently funded. However, this raises the question how to determine whether a particular pension fund is sufficiently funded. As a measure for sufficiency, the financial assessment framework (FTK) that will be effective in the Netherlands as of 2007 prescribes a probability level of 75% with respect to the risks of the liabilities. However, a pension fund does not have to compute this probability level itself; instead it may use a standardized method. This method computes an extra loading to be added to the technical provision at realistic value. The extra loading is computed as a function of the future mortality trend uncertainty ($T_{SO}$) and the possible negative stochastic deviations ($NS_{A}$). However, under certain conditions, a pension fund is allowed to develop and apply an internal method.

In this thesis one possibility for such an internal method is examined and compared with the standardized method. As a criterion to determine the best method, the associated costs are used. The internal method is characterized by an investment in fictitious mortality bonds. The payoff of these bonds will depend on the realization of an underlying survivor index. Since these bonds are not marketed yet, it is not possible to derive unique prices. Therefore the standard deviation pricing principle $\mu + \xi \sigma$ is used for various choices of the parameter $\xi$.

To analyze the internal method, and to be able to compare it with the standardized method, eight fictitious pension funds are used throughout this thesis. The populations of these funds vary across age, gender and size. To determine the payoffs and prices of the bonds and the actual pension payments of the eight pension funds, a stochastic mortality model is used. This model is the well-known Lee-Carter model.

For values of the risk parameter $\xi$ smaller than one, lots of costs can be saved. In particular, pension funds that have a young, large or male population can reduce their costs substantially by an investment in mortality bonds. The cost reduction for funds that have a small, old, or female population is somewhat lower, but it is still significant for small values of the risk parameter.
Preface

This thesis concludes my studies Econometrics and Operations Research, specialization Quantitative Finance and Actuarial Sciences at Tilburg University. The analysis in this thesis has been conducted on behalf of Watson Wyatt Brans & Co, herein known as WWB. WWB forms a strategic alliance with Watson Wyatt Worldwide, which is an internationally leading consultancy agency on the area of human capital and financial management. The Dutch activities mainly consist of advice on actuarial problems, pension law, investment, human capital and insurance. The main office in the Netherlands is located in Amstelveen; other offices are situated in Eindhoven, Nieuwegein, Purmerend, and Rotterdam. WWB offered me the opportunity to write this thesis during an internship at the Rotterdam office, for which I am very grateful.

Of course, I could not have completed this thesis as well as the rest of my studies without the help of other people, whom I would like to thank for this. First of all, I would like to thank my colleagues at the Rotterdam office of WWB for showing interest in my thesis but also for the pleasant working environment they created. Two of them I would like to mention personally: my roommate Linda Gastelaars for all the happy times in “de bibliotheek” and my company supervisor Ronald van Dam for his helpful suggestions and comments on this thesis and for taking time for me despite his busy work schedule.

Next, I would like to thank my university supervisor Johan Segers. With all his helpful suggestions and critical comments he contributed a lot to this thesis. From Tilburg University I would also like to thank Anja de Waegenaeere, who attended several meetings of Johan and myself and took over from Johan when he left Tilburg University.

Finally, many thanks go to my friends and family for all the support I had during my studies. I am very thankful to all my friends, both the “Tilburg” people and the “Wehl” people, for showing interest in my studies and offering me great times during my studies. Furthermore, I would like to thank my family for all the mental, emotional, and financial support I received from them. And last but certainly not least I would like to thank my girlfriend Anneke for always supporting me and for all the wonderful times we have together.

Wilbur Damen
Chapter 1

Introduction

Pension funds supply pensions to their participants. Often these pensions are for life, which implies that the pension payments continue until the death of the participant. Therefore, the amount of capital that is currently needed to fund the pension payments depends, among other things, on the life expectancy of the participants. However, since it is generally known that death probabilities are decreasing over time, life expectancies are increasing. Figure 1.1 displays the evolution of the remaining lifetime of Dutch males and females at the common retirement age of 65.

Figure 1.1: Life expectancy at age of 65 for males (dotted blue line) and females (solid pink line).

\footnote{Data from CBS; results are obtained by taking the average of the life expectancy at age 64.5 and at age 65.5}
It can be deduced from figure 1.1 that life expectancies at the age of 65 over the past 50 years have increased for males by almost 2.5 years and for females by more than five years. In percentages the increase for males is 17% and for females even 37%. Due to these increasing life expectancies, pensions for life are paid out during an increasing number of years as well. Hence, to adequately determine the capital needed to fund the pension benefits, pension funds should take into account the increase of life expectancies. This is also a requirement of the financial assessment framework (FTK), which is discussed later on in this thesis.

However, future life tables are unknown at present, and hence the future evolution of death probabilities and life expectancies are unknown as well. Therefore, it could be that in the future, death probabilities decline at a much faster rate than is currently expected, which implies that people live longer than is expected at present. This is known as longevity risk. To deal with longevity risk, the FTK suggests a standardized method, but also leaves open the possibility that pension funds use their own model, provided that some requirements on the pension fund as well as on the model are satisfied.

1.1 Problem formulation

According to the standardized method the amount of capital a pension funds needs to deal with the problem of longevity risk, is quite large. Therefore, this thesis tries to find an alternative to the standardized method that satisfies the requirements imposed by the FTK. Inspired by the attempt of BNP Paribas to issue a so-called longevity bond\(^{2}\), the alternative will consist of an investment in mortality bonds. As a criterion to determine the best method, the total amount of capital that is needed for each method will be used. This amount of capital will be computed for the current liabilities of a pension fund. Hence, the research question of this thesis can be formulated as:

*Given the current pension rights, which method requires at present the smallest amount of capital, provided that the requirements imposed by the FTK are met?*

1.2 Structure of the thesis

The analysis of this thesis starts by briefly introducing the Dutch pension system in chapter 2. In particular, some terms and concepts that will be used in the analysis are discussed. Next, chapter 3 provides a very rough overview of the FTK. Furthermore, the implications of using the standardized method to determine the liabilities of pensions funds at realistic value are discussed in this chapter.\(^{2}\)

---

\(^{2}\)See subsection 4.2.5 The BNP Paribas Longevity Bond
1.2. *STRUCTURE OF THE THESIS*

Chapters 4 and 5 introduce the ingredients to develop an alternative method to determine the liabilities at realistic value. This method will be based on an investment in mortality-linked securities. Therefore, chapter 4 contains an overview of some possible mortality products. In chapter 5 the Lee-Carter model is introduced and corresponding parameters are estimated. This mortality model can be used to describe the future evolution of survival probabilities. To compare the standardized method and the method of investing in mortality products, eight fictitious pension funds are considered, which are introduced in chapter 6. Chapter 7 then determines for each fictitious pension fund the costs from both methods and compares these methods. Finally, in chapter 8 conclusions are drawn and some possible topics for further research are given.
CHAPTER 1. INTRODUCTION
Chapter 2

Pensions

A pension can be regarded as an insurance against outliving one’s money: during a person’s active life, premiums are being paid and when the person is not active anymore, the person (or the partner in case of a so-called survivor’s pension) receives benefits. In this chapter a rough overview is given of the pension system in the Netherlands³.

2.1 Three pillars

Dutch pensions consist of three parts; these parts are known as pillars. Within each of these pillars, three different pension forms can be distinguished:

- old age pensions: payments starting at the moment the insured person reaches the retirement age (mostly 65) and ending at the moment of the insured’s death;
- survivor’s pensions: payments made to the insured’s partner beginning at the moment of death of the insured and ending at the moment of death of the partner;
- disability pensions: payments to disabled persons ending at the retirement age.

The government is responsible for the pensions in the first pillar: benefits in this pillar are the so-called basic provisions; every inhabitant of the Netherlands is entitled to these provisions and the payments are not influenced by a person’s labor history. Examples of pension payments in the first pillar are social securities such as the AOW and the ANW. WAO-benefits⁴, which are paid for by the employers, are also part of the first pillar.

³In this chapter some Dutch names and abbreviations will be used to describe the pension system in the Netherlands. For a description, although in Dutch, of these terms, one is referred to Pensioengids 2006 [6].
⁴As of 2006 the WAO is replaced by the WIA.
The pension payments in the second pillar, which is the most important pillar for pension funds and therefore also for this thesis, are the responsibility of employer and employee: by working, an employee builds up pension commitments, which are supplied by the employer and are held in trust at a pension supplier. The way in which pension commitments are formed depends on the particular pension scheme. However, all schemes have in common that payments start when the employee stops working (i.e., he or she retires, dies or becomes disabled).

The third pillar consists of individual agreements on top of the benefits of the first two pillars. Typically, these agreements are insurance contracts and can be divided into two categories: annuity contracts and endowment contracts. The first type is characterized by a stream of periodic cash flows, whereas the latter type is characterized by a lump sum.

Since this thesis is mainly concerned with the longevity risk pension funds are exposed to, the focus of this thesis will be on old age pensions from the second pillar.

2.2 Pension suppliers

In the previous section it was explained that the pension commitments of the second pillar are placed at a pension supplier. It is prohibited that the employer acts as the supplier. In this way the employer cannot use the provision for pension commitments for other objectives. For instance, this implies that the pension commitments of the employee are protected in case of bankruptcy of the employer. In the Netherlands there are many of these pension suppliers; these suppliers can be divided into four different categories:

- industry-wide pension funds: pension funds for employees in a specific branch of industry (for instance ABP, which is the Dutch pension fund for civil servants);
- professional pension funds: pension funds for specific professions such as dentists or notaries;
- company pension funds: pension funds for employees of a specific company (examples of Dutch funds with company pension funds are Philips and Ahold);
- insurance companies (must have a license to supply pensions).

The employer pays pension premia to the pension supplier and provides pension commitments to the employees. These premia are computed in such a way that the pension supplier will be able to pay the (future) pension benefits to the employees. Hence, the relation between employer, employee and pension supplier can be symbolized by a triangle (Pensioengids 2006 [6]).

---

5 In the remainder of this thesis pension suppliers will be denoted by pension funds.
2.3 Pension schemes

The amount of pension benefits that a participant of a pension fund receives depends on the pension scheme the participant belongs to. In the Netherlands most pension schemes are defined benefit (DB) schemes, which implies that the participants receive pension rights and the employer pays premiums for these rights. Hence, the employer bears the risk, since the premiums that are paid must be enough to pay out the promised pension rights. The accrued pension rights depend on the participant’s pensionable salary, which is defined as the actual salary minus the offset. Several agreements to determine these rights are allowed in the Netherlands, but in practice the most common ones are the final pay plan and the average pay plan. In the final pay plan, the amount of pension benefits that a participant receives depends on the last earned salary; hence the pensionable salary corresponds to the difference of the last earned salary and the offset. The exact computation of the pension rights is different for each scheme, but it can be generalized by

\[
pension\ rights = accrual\ percentage \times pensionable\ salary \times years\ of\ service,\tag{2.1}
\]

where the accrual percentage denotes the amount of pension rights that is obtained in exchange for one year of labor and is maximized at 2% by fiscal law. Typically in a final pay plan a salary increase affects the accrued pension rights. This follows from the fact that a change in last earned salary also changes the pensionable salary and therefore the past pension rights as well. These extra costs are known as back service costs (BS). The costs originating from the increasing years of service are the coming service costs (CS). Example 1 clarifies a final pay plan.

---

Footnote 6: The offset corresponds to the part of salary over which no pension rights are built, because one receives an old age pension from the government (AOW).
Example 1 Consider a participant in a final pay plan. Suppose the parameters of the scheme are

- offset: €15,000;
- accrual percentage: 2%.

The characteristics of the participant are

- years of service equal to 19;
- a salary of €40,000.

The pension rights are then computed as

\[
2\% \times (€40,000 - €15,000) \times 19 = €9,500.
\]

In the twentieth service year the salary increases with €2,000 to €42,000 and the offset does not change. The new pension rights are

\[
2\% \times (€42,000 - €15,000) \times 20 = €10,800.
\]

The BS are the costs to increase the pension rights by

\[
2\% \times (€42,000 - €25,000 - (€40,000 - €25,000)) \times 19 = €760.
\]

Then the CS correspond to the costs to increase the pension rights by

\[€1,300 - €760 = €540.\]

Figure 2.2 graphically represents these costs; the amounts correspond to the areas of the rectangles.

Figure 2.2: Back service costs and coming service costs in a final pay plan
2.3. PENSION SCHEMES

An average pay plan is characterized by the fact that the yearly accrual in pension rights depends upon the pensionable salary in the corresponding year. Already built up rights do not change anymore. Hence, the yearly accrual can be formulated as

\[
\text{yearly accrual} = \text{accrual percentage} \times \text{pensionable salary}, \tag{2.2}
\]

and the total pension rights can be computed by summing all yearly accruals. The only costs are the costs to finance the yearly accrual. The accrual percentage for an average pay plan in the Netherlands is bounded above by 2.25%. Since most Dutch pension schemes are average pay plans, this type of pension plan will be considered in this thesis. As an illustration of an average pay plan, consider example 2.

**Example 2** Consider an average pay plan that is characterized by

- an accrual percentage equal to 2%;
- the offset of €12,000, which is assumed to be constant over time.

During five years a particular participant receives these incomes: €30,000, €32,000, €35,000, €40,000 and €48,000. In table 2.1 the corresponding pensionable salaries and yearly accruals are given, and figure 2.3 displays these results.

<table>
<thead>
<tr>
<th>year</th>
<th>salary</th>
<th>offset</th>
<th>pensionable salary</th>
<th>yearly accrual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>€30,000</td>
<td>€12,000</td>
<td>€18,000</td>
<td>€360</td>
</tr>
<tr>
<td>2</td>
<td>€32,000</td>
<td>€12,000</td>
<td>€20,000</td>
<td>€400</td>
</tr>
<tr>
<td>3</td>
<td>€35,000</td>
<td>€12,000</td>
<td>€23,000</td>
<td>€460</td>
</tr>
<tr>
<td>4</td>
<td>€40,000</td>
<td>€12,000</td>
<td>€28,000</td>
<td>€560</td>
</tr>
<tr>
<td>5</td>
<td>€48,000</td>
<td>€12,000</td>
<td>€36,000</td>
<td>€720</td>
</tr>
</tbody>
</table>

| total | €2,500 |

Table 2.1: Yearly numbers of the average pay plan example

An alternative to a DB scheme is a defined contribution scheme (DC). In such schemes the employer pays premia to the pension supplier and these premia are used as pension capital for the participants. However this cannot be regarded as a promise: no rights are assigned and the participants are just entitled to their share of the total premia paid. This share must be used for pension objectives; the total amount of pension benefits at the retirement age depends on the actual size of the share at that time. Hence, in contrast to a DB scheme, the participants take the risk, because capital is assigned rather than rights. The execution of pension payments may, depending on the particular pension scheme, be done either by the pension supplier itself or by an insurance company.
CHAPTER 2. PENSIONS

Pensions are built up during one’s working life and are paid out afterwards. In the period between building up and paying out pensions it is likely that price inflation will occur. Price inflation decreases the purchasing power of the pensioners, which is considered to be undesirable. Therefore, often indexation is offered: the pension benefits or, before the retirement age, rights are increased by a percentage derived from a price index. Indexation can be either conditional or unconditional. In the Netherlands most schemes offer conditional indexation. In this thesis however indexation aspects will be ignored, since the focus will be on the current pension rights.

2.5 Conclusions

In the Dutch pension system pensions consist of three pillars. Within each pillar several forms of pensions can be distinguished. The focus of this thesis will be on the old age pensions of the second pillar, since pension funds are possible suppliers for pensions of the second pillar. To be able to supply these pensions, capital is needed. However, the exact amount of capital needed has not been specified yet. The remainder of this thesis is concerned with this topic.
Chapter 3

FTK

3.1 Introduction

In the Netherlands the supervising authority on the area of pension funds and insurance companies is De Nederlandsche Bank (DNB). Its task is to monitor these institutions concerning the fulfillment of their commitments to their clients (such as pension fund participants and insurance policyholders). However, not the DNB, but the boards of the institutions are responsible for the conducts of business: the boards should ensure sufficient solvability and continuity, which implies the knowledge and control of the largest risks. Yearly, these institutions must transparently and reliably report their financial situation and their risks to DNB. Evaluating such reports requires an appropriate framework. Therefore, in October 2004, DNB presented the document “Financial assessment framework consultation document”. This document provides a thorough overview of the Financial Assessment Framework, in Dutch abbreviated by FTK, which will be effective in the Netherlands as of 2007. This framework can be used to evaluate the reports in a transparent and comparable way (DNB, 2004[7]). Another important property of the FTK is that it can be considered to be flexible: DNB intends to evaluate it frequently and both new insights and future developments may lead to future adjustments (DNB, 2004[7]).

The basic principle of the FTK is that institutions can either use a standardized method to evaluate their position and determine their risks or they may develop their own method, which must be approved by DNB. However, regardless of the method used, the report about the financial position and the risks should consist of the following three parts:

- Realistic value: assesses whether there is an adequate capital funding,
based on realistic value, of the liabilities the institution has entered into.

- Solvency test: determines if the amount of actual solvency at the reporting date as well as one year after that date is adequate.

- Continuity analysis: tests the capital funding on a long-term horizon and tries to give insight into the financial structure of the institution.

For the analysis of this thesis, only the part on realistic value is of interest. Therefore, the remainder of this chapter elaborates on this part. For the contents of the other parts, one is referred to the document issued by DNB.

3.2 Realistic value

To be able to determine the financial position and the risks, correct valuation methods must be used. Currently, nationally and internationally, many developments regarding valuation methods take place; DNB has tried to take into account the most recent developments. One such development, resulting from the new IAS/IFRS\(^9\), is valuation at realistic value, since this is important for the continuity of an institution. Until the FTK, only the assets of an institution were to be valued at realistic value, but now also the liabilities as well as equity capital and outside capital must be valued at realistic value. Because in general longevity risk arises from the liabilities, in the remainder of this section little attention is paid to the assets as well as equity and outside capital, and more attention is paid to the liabilities.

3.2.1 Assets, equity capital, and loan capital

The realistic value of assets is determined as the value of the asset in a deal between well-informed and independent parties that are willing to trade (DNB, 2004, [7]). Hence, three situations may occur:

- The asset is traded at a regular market: the realistic value is the current price of the asset.

- The asset is not traded, but comparable assets are: the realistic value can be derived from the prices of the comparable assets, correcting for dissimilarities.

- The asset is not traded, nor are there comparable assets: the asset must be valued using a model based on economic assumptions. These assumptions must be in line with regular market prices.

According to DNB (2004) [7], equity capital is a residual item on the balance sheet that can be derived when all other items are valued at realistic value.\(^9\)International Accounting Standards/International Financial Reporting Standards, the new international accounting framework.
3.2. REALISTIC VALUE

Loan capital arises if foreseeable obligations for payments are associated with the financing instrument. Within the FTK, the valuing of loan capital is similar to the valuing of assets.

3.2.2 Liabilities

Although no official guidelines about valuing the liabilities at realistic value already exist, the FTK prescribes the provision for pension and insurance commitments to be determined at realistic value. Also, the provision is bounded from below: for pension funds the minimal value corresponds to the realistic value of the paid up pension commitments if the participation in the pension scheme were ended. If the provision would be below this minimal value, the continuity of the pension fund would be at risk. To determine the provision at realistic value, it must be computed prospectively: the provision must be the most realistic expectation of the cash flows resulting from the liabilities. In addition an extra loading that is in conformity with the market must be added to the provision. This loading is meant to cover inevitable risks, such as model uncertainty and parameter risk; longevity risk is a great part of the inevitable risks. The adding of this loading can be justified by the fact that well informed parties at a financial market also would use a similar loading. Together, the sum of the expected cash flows and the extra loading are called the technical provision at realistic value.

The expectation of the liabilities must be computed as the present value of the expected cash flows resulting from the pension agreements. Realistic assumptions (for instance about death probabilities and future transfers of value) and expected future developments (demographical, social, technological, medical etc.) in these assumptions must be taken into account when determining the expected cash flows. The present value corresponds to the realistic value of replicating assets. Since in general the realistic value of assets is computed using the term structure of interest rates, the liabilities must be valued using the term structure as well.

For the determination of the amount of extra loading that is required to cover inevitable risks, no widely accepted methods exist. Therefore, DNB prescribes an approximation. This approximation is defined as the difference between the expected value and the value corresponding to a probability level of 75% with respect to the risks of the liabilities (DNB, 2004 [7]). However, determining the value of this approximation is quite difficult and therefore DNB has developed a standardized method. For pension funds this standardized method needs two parameters:

- The weighted average age of the fund population, denoted by $\bar{x}$. This average age is weighted by the most realistic expectation of the provision.
- The number of participants $n$.

Furthermore, a distinction is made between pension agreements with and without survivor’s pensions, due to a diversification effect. This diversification
effect can be seen as follows: if life expectancy increases (decreases), the present value of survivor’s pension decreases (increases).

The risk loading consists of two parts: the TSO (toekomstige sterfgetrenddonzerheid) and the NSA (negatief stochastische afwijking). The TSO-part adds a loading for the uncertainty arising from the trend in death probabilities, and is computed by

\[ TSO = \left[ 2 + \frac{9}{40} \max \{ p - \bar{x}; 0 \} \right] \%, \]  
(3.1)
in case there are no survivor’s pensions, and by

\[ TSO = \left[ 2 + \frac{4}{40} \max \{ p - \bar{x}; 0 \} \right] \%, \]  
(3.2)
if there are survivor’s pensions. In these two formulations \( p \) denotes the retirement age.

The NSA-part adds a loading to adjust for disadvantageous stochastic deviations and the corresponding expression is, in case of no survivor’s pensions:

\[ NSA = \left[ \frac{60}{\sqrt{n}} \right] \%. \]  
(3.3)

In case of survivor’s pensions the NSA equals:

\[ NSA = \left[ \frac{40}{\sqrt{n}} \right] \%. \]  
(3.4)

Together, the TSO and the NSA determine the required risk loading according to the relation:

\[ extra \ loading = \sqrt{TSO^2 + NSA^2}. \]  
(3.5)

### 3.3 Conclusions

By introducing the FTK, DNB has developed a framework that can be used by pension funds and insurance companies to reliably and transparently report their financial situation as well as the risks these institutions are exposed to. The reporting framework consists of three parts. Together, these three parts give insight in the current situation, the situation one year into the future and the situation on a long-term horizon. To determine these situations, standardized methods as well as internal methods may be used.

This thesis concentrates on the current position, which implies that the liabilities as well as the assets are valued at realistic value. Under the standardized method, the liabilities are computed as the sum of the most realistic expectation of the future cash flows and an extra loading, which is a function of the TSO and the NSA. As can be deduced from formulas (3.1) to (3.5), this loading can be very substantial, especially for pension funds that have a small number of participants or that have a young population.
3.3. CONCLUSIONS

However, in general pension funds prefer to maintain minimal loadings. Therefore, it is interesting to investigate the possible use of internal methods to cover the inevitable risks and to determine the capital that is needed by applying these methods. In the next chapter mortality-linked securities will be introduced. The payoff of such products depends on realized mortality indices and therefore these products could be used to (partially) cover the inevitable risks.
Chapter 4

Mortality-Linked Securities

4.1 Introduction

Pension funds are exposed to longevity risk. As a possibility to hedge this risk the use of mortality-linked securities has recently been proposed by some authors, see, for instance, Blake et al. (2006) [2], Blake and Burrows (2001) [1], and Cox et al. (2000) [4]. Although these products can have several forms, their payoff always depends on an underlying survivor index. These products can be interesting for two types of investors in particular:

- investors that are exposed to mortality risks and want to hedge these risks, for example, pension funds;
- investors that are searching for products that have a low correlation with standard products in order to diversify their portfolio.

This section gives a partial overview of the wide range of mortality products that can be thought of, as well as the corresponding cash flow structures. Some firsts attempts to market these products have already been made; most securities\(^\text{10}\) however still only exist in theory. These mortality-linked securities can be divided into four categories:

- mortality bonds;
- mortality swaps;
- mortality futures;
- mortality options.

Most attention will be paid to mortality bonds, since this type of product will be used in this thesis.

\(^{10}\)Strictly speaking, not all products are actually securities (Blake et al., 2006 [2]). For the sake of simplicity however, this will be ignored in this thesis.
4.2 Mortality bonds

In general, a bond consists of two parts: the principle, also known as the face value, which is repaid at the maturity date of the bond, and the coupons, paid periodically. The coupons are often, but not always, a proportion of the face value. Hence, mortality bonds, sometimes also referred to as longevity bonds, can have various forms, since both the face value and the coupons can be made mortality dependent. In the remainder of this section some of the most obvious forms that can be thought of for mortality bonds are discussed, following Blake et al. (2006) [2].

4.2.1 Classical Mortality Bonds

Among the first to introduce mortality-linked securities were Blake and Burrows with their 2001 article “Survivor Bonds: Helping to Hedge Mortality Risk” [1]. In this paper they recommend the issue of so-called survivor bonds, from now on denoted as classical mortality bonds. The structure of such bonds is fairly simple: beforehand a reference population is specified and at every future payment date the bond pays the proportion of a fixed amount. This proportion equals the fraction of survivors in the reference population. The fixed amount can be regarded as the principle, although the principle itself will not be repaid. The payments of the bonds continue until the death of the last person remaining in the reference population. Hence, this bond has a face value equal to zero and the coupon payments and maturity date are mortality-linked. In figure 4.1 the development of the coupon payments of a classical mortality bond over time are displayed. These payoffs are assumed to equal the fraction of survivors in the reference population and are represented by bars.

![Figure 4.1: Payoffs of a classical mortality bond](image)

4.2.2 Zero-Coupon Mortality Bonds

A zero-coupon bond (“zero”) is a bond that provides no coupon payments; at maturity, it solely pays the face value. A zero is often created by stripping a
4.2. MORTALITY BONDS

Figure 4.2: Payoffs of a deferred mortality bond, payoffs start at time $\tau$

regular bond$^{11}$, and next by selling the separate strips. Hence, to create zero-coupon mortality bonds, the classical mortality bond, described above, can be stripped which results in bonds paying an amount depending on the surviving proportion of the reference population at a specific point in time. Hence, the payoff of a zero-coupon mortality bond can be represented by a single bar in figure 4.1. The advantage of zeros over regular bonds lies in the larger number of trading and hedging strategies. This bigger possibility stems from the ability to adjust the portfolio of zeros to the needs of the portfolio holder (i.e., a pension fund) by buying and selling zeros. However, this flexibility will likely be offset by the illiquidity of the market, since most bonds will be purchased on a buy-and-hold basis as mentioned by Blake et al. (2006) [2].

4.2.3 Deferred Mortality Bonds

The greater part of the longevity risk pension funds are exposed to originates from currently unknown future trends in mortality. Therefore, classical mortality bonds, of which the payments start immediately, contain several payments that are hardly subject to longevity risk. A deferred mortality bond might solve this problem. Such bonds do not start paying right away; instead the mortality-linked payments commence at a future point in time, which explains the supplement ‘deferred’. Figure 4.2 displays the payoffs of a deferred mortality bond that start at the future date $\tau$.

4.2.4 Principal-at-Risk Mortality Bonds

In general, a principal-at-risk mortality bond is a coupon-paying bond of which the principal, i.e., the amount the investor receives at maturity, may not be paid out (fully), in some circumstances, depending on the realization of some

$^{11}$ Instead of stripping an existing bond, an issue of only zeros is also possible.
underlying mortality index. The events, in which the principal is at risk, can be either extremely high mortality or extremely low mortality. One such bond already exists: the Swiss Re mortality bond, issued by the reinsurance agency Swiss Re. The principal is not paid out in case of an extremely high value of a mortality index based on mortality figures of five different countries: the United Kingdom, United States, Germany, Japan, and Canada. The payoffs of a fictitious principal-at-risk mortality bond are displayed in figure 4.3. The question mark indicates that the final payoff is at risk.

4.2.5 The BNP Paribas Longevity Bond

Despite the huge potential demand and the simple structure it was not until November 2004 that an issue of a mortality bond was announced\textsuperscript{12}. Together with the European Investment Bank (EIB), which is the financing institution of the European Union, BNP Paribas came up with the (BNP Paribas) Longevity Bond. The EIB acts as the issuer and BNP Paribas structures and manages the bond.

The bond has a maturity of 25 years and is based on English and Welsh mortality data for a cohort of males aged 65 in 2003. These data are produced and published by the Office for National Statistics (ONS)\textsuperscript{13}. Each year, the proportion of survivors in the cohort until that year will be determined from these data. Coupons are paid annually and these yearly coupon payments equal the corresponding proportion of £50 million. These coupons are hardly at risk since the issuer, EIB, has a very high credit rating (AAA by Standard & Poor’s and Aaa1 by Moody).

Hence, pension funds purchasing the Longevity Bond can, partially, hedge their longevity risk exposure. By selling the bond the EIB takes over the longevity risk. The EIB swaps the mortality linked, fixed interest rate, obligations into mortality risk free, floating interest rate, coupons with BNP Paribas.

\textsuperscript{12}However, one should keep in mind that the translation of a theoretical idea into a practical framework takes time and comes with many difficulties.

\textsuperscript{13}www.statistics.gov.uk
Finally, BNP Paribas transfers the received longevity risk to PartnerRe by taking reinsurance. PartnerRe is a leading global reinsurer based in Bermuda. Figure 4.4 shows the connections between the different parties.

The numbers in figure 4.4 indicate the three different cash flow exchanges associated with the BNP Paribas Longevity Bond. Transaction (1) is the issue of the bond by the EIB. Investors, of which likely the bulk will be pension funds, buy the bond and obtain future random cash flows, which are longevity linked. The swap between the EIB and BNP Paribas, described in the previous paragraph, is given by (2). BNP Paribas has to pay a premium for the reinsurance contract with PartnerRe and receives payments in case the number of survivors is (much) higher than expected.

The price of the BNP Paribas Longevity Bond is based on a straightforward computation: it is the sum of the expected, discounted, cash flows. Hence, determining the price of the bond requires the knowledge of two input parameters: one to discount of the cash flows and one to compute the expectation of these. As a discount rate the London Inter-Bank Offered Rate (Libor) minus 0.35% is used. This 0.35% can be regarded as some sort of risk premium.

The cash flows depend on the number of survivors in the cohort each year. Therefore, the expected cash flows must be based on the expected number of survivors, which can be calculated using a mortality table that takes into account future trends in mortality improvement. Such tables are constructed by the Government Actuary’s Department (GAD) for various populations of the United Kingdom. The price of the Longevity Bond is obtained by applying the GAD’s projection of future mortality to the cohort of English and Welsh males aged 65 in 2003. Figure 4.5 represents the expected payoffs of the BNP Paribas Longevity Bond.

However, demand for this bond was very low and it was decided not to issue the bond. There are a number of possible reasons why pension funds were not interested in buying the bond:

- the reference population is not appropriate for an entire fund: the bond is solely based on 65 year old males, whereas in general funds have both male and female participants and have participants of varying ages. Furthermore, it is commonly known that fund participants have above-average life expectancies;

- the bond has a maturity of only 25 years, in contrast to the liabilities of a pension fund.
CHAPTER 4. MORTALITY-LINKED SECURITIES

Next to these shortcomings of the bond, also the price could have decreased the demand: the risk premium of 0.35% is quite substantial\textsuperscript{14}. Hence, in total it can be concluded that the bond provides a small hedge for a high price, which could explain the low demand.

4.3 Mortality Swaps

A swap can be described as an agreement between two parties to exchange one or more future cash flows of which at least one is random. By imposing a mortality component on such random cash flow, a mortality swap can be created. Whereas products such as bonds require the existence of a liquid market and are involved with relatively high transaction costs, swaps do not, since swaps typically are over-the-counter contracts. Moreover, swaps can be tailor-made which results in more flexibility. Therefore, mortality swaps are very promising hedge instruments regarding longevity risk and some first trades have already been reported (Blake et al., 2006 [2]). Just like bonds, swaps exist in many different forms and therefore also many different forms of mortality swaps can be constructed. Below two basic types of mortality swaps are explained.

4.3.1 One-Payment Mortality Swaps

The most basic type of swap is the exchange at a predetermined time $t$ of one single floating rate payment for one single preset amount. By making the floating rate payments mortality dependent, a so-called one-payment mortality swap is created. Let $S(t)$ denote the floating rate that is based on a survivor

\textsuperscript{14}In fact, the risk premium is even larger, since the Libor is a short rate. In general short rates are below long rates and therefore the long-term coupon payments are discounted at a discount rate that is far below the corresponding long term discount rate.
index, which must be paid by agent 1, and \( C(t) \) the preset amount that must be paid by agent 2. Then the payoff at time \( t \) of agent 1 is expressed by \( C(t) - S(t) \), and for agent 2 the payoff is the opposite: \( S(t) - C(t) \).

### 4.3.2 Vanilla Mortality Swaps

A vanilla mortality swap can be regarded as a collection of one-payment mortality swaps: the parties exchange a floating rate for a constant rate over a predetermined number of time periods. Suppose \( T \) denotes the number of time periods. Furthermore, at each point in time \( t; t = 1, \cdots, T \), agent 1 receives the preset amount \( C(t) \) and agent 2 receives the floating rate \( S(t) \). Hence the payoff of agent 1 at time \( t \) is given by \( C(t) - S(t) \) and the total payoff for this agent is computed by the sum of all payoffs:

\[
\text{total payoff}_1 = \sum_{t=1}^{T} (C(t) - S(t)) = \sum_{t=1}^{T} C(t) - \sum_{t=1}^{T} S(t).
\]

Obviously, for agent 2 the payoff is again opposite to the payoff of agent 1.

### 4.3.3 Possible extensions

The two basic types of swaps described above can be extended. For example, both parties could receive floating rates. These floating rates can be both mortality dependent, however it could also be that one floating rate depends on a totally different quantity such as for example a stock index. Furthermore, \( S(t) \) was assumed to correspond to a survivor index, but other mortality figures can be used as well. Examples of other possible mortality-linked figures are mortality spreads or the other mortality products described in this chapter.

### 4.4 Mortality Futures

The basic principle of a future is that at time \( t \) an agreement is made to deliver a product at maturity date \( T \) for a price that is determined at time \( t \). Such product can have many different forms: future contracts that involve the delivery of stocks exist, but also for goods such as potatoes or metals, future contracts are being traded. Hence, it seems logical that also mortality futures will be traded in the future. Likely, these futures will have an underlying, i.e., the product that will be delivered, that corresponds to a mortality-linked security that is traded, \( S(t) \).

### 4.5 Mortality Options

By buying an option an investor buys the right to buy (in case of a call option) or sell (in case of a put option) a product at a predetermined price, known as the strike price. Mostly, the product is not actually exchanged between the
parties, only the difference between the actual price, $S$, and the strike price, $K$. Hence this, non-linear, payoff is given by $\max\{K - S; 0\}$ in case of a put option and by $\max\{S - K; 0\}$ in case of a call option. Through their structure, options can very well be used as hedge instruments, since upward potential is maintained, whereas downside risks are covered. Therefore, also mortality options have great potential. Underlying mortality products can be mortality bonds, mortality swaps or other mortality products.

4.6 Choice of mortality indices

In the previous four sections several possible mortality-linked securities have been discussed. These securities have payoffs that depend on realized mortality indices. However, little attention was paid to the specific choice of index although this is very important. If the used mortality indices are unreliable or inaccessible, investors will probably be very reluctant to invest in the mortality products. Hence, the mortality indices must have some properties. Blake et al. ([2]) postulate several such requirements on the mortality index. For instance,

- potential investors must have as much relevant information as possible in order to be able to quantify the risks they will be exposed to;
- the mortality index must be computed in such a way that integrity is maintained.

However, even with these requirements a mortality index can have several drawbacks:

- mortality rates are often published yearly and there is a substantial time lag;
- by using either smoothed or crude death rates discrepancies can occur;
- the used mortality index may cause a hedge failure for an investor. This might happen in case the reference population of the mortality index does not match the investor’s needs, which is known as basis risk.

Hence, institutions that are planning to issue mortality-linked securities should choose the underlying mortality index carefully. Investors that want to invest in these securities should take into account the fact that the used mortality index may not exactly match their needs.

4.7 Conclusions

Although at present little mortality-linked products are traded, there is a huge potential demand for such products. After all, many different positions can be created by investing in these securities, which can be interesting for several
types of investors. Such investors can be pension funds or insurance companies, but also other investors that are interested in products that have low correlation with ‘regular’ products. However, the market for mortality products is still very nascent, and comes with many difficulties. For instance, a correct understanding of the possible outcomes underlying mortality index is needed. Therefore, the next chapter introduces a stochastic mortality model, which can be used to model the underlying mortality index.
Chapter 5

Stochastic mortality

5.1 Introduction

In the previous chapter several mortality-linked securities have been introduced. The payoff of these products depends on the realization of an underlying mortality index. Hence, to be able to examine the use of these products and to determine prices for these products, the evolution on the mortality index must be quantified. Many different models to model mortality have been suggested in the literature. However, this thesis does not aim at fully describing mortality; instead it aims at studying and evaluating the possible actions a pension fund can undertake in order to hedge longevity risk. Therefore only one single mortality model will be considered and used throughout the analysis of this thesis. This mortality model, proposed by Lee and Carter (1992) [10], is widely accepted and has been used as a starting point for many other models.

However, before specifying the model, first some relevant notation will be provided. Next, the Lee-Carter model will be explained and corresponding parameters will be estimated by making use of a dataset on Dutch mortality rates.

5.2 Notation

Before defining the model, it is convenient to describe some notation regarding mortality figures that are commonly used in actuarial science.

First of all, the probability that a person aged $x$ during year $t$ reaches age $x + 1$ (i.e., the survival probability) is denoted by $p_{x,t}$. The one-year death probability of a person aged $x$ during year $t$, $q_{x,t}$, is then, of course, computed as

$$q_{x,t} = 1 - p_{x,t}. \quad (5.1)$$

However, probabilities are between zero and one which is difficult to model. Therefore, often the force of mortality $\mu_{x,t}$ is being considered, which is the instantaneous mortality rate for a person aged $x$ at time $t$. To derive the force
of mortality, first note that the probability that a person aged $x$ at time $t$ dies before the age of $x + \Delta x$ is given by

$$P(x < T < x + \Delta x \mid T > x) = \frac{F_x(x + \Delta x) - F_x(x)}{1 - F_x(x)}; \quad (5.2)$$

where $T$ denotes the age of death and $F_x$ the probability distribution for a person aged $x$.

In case $\Delta x$ becomes infinitely small, an expression for $\mu$ can be found:

$$\mu_{x,t} = \lim_{\Delta x \to 0} \frac{F_x(x + \Delta x) - F_x(x)}{\Delta x(1 - F_x(x))} = \frac{F_x'(x)}{1 - F_x(x)}. \quad (5.3)$$

By assuming that $\mu$ is constant within in a year, i.e.,

$$\mu_{x,t+t} = \mu_{x,t} \quad \text{for } 0 \leq \tau < 1, \text{ and integer } t, \quad (5.4)$$

it can be deduced that the following expression holds:

$$p_{x,t} = \exp \left( -\mu_{x,t} \right). \quad (5.5)$$

A maximum likelihood estimator (MLE) for $\mu$ can be found by computing the ratio of the number of deaths of age $x$ at year $t$, $D_{x,t}$, and the risk exposure $E_{x,t}$, which is the number of person years from which $D_{x,t}$ occurred (Brouhns et al., 2002 [3]). Therefore $\mu$ could be larger than one, since theoretically $D_{x,t}$ may be larger than $E_{x,t}$. This implies that the force of mortality is a number between zero and infinity and is therefore not bounded by an upper limit in contrast with a probability and is more easy to model.

### 5.3 The Lee-Carter model

To model the force of mortality, the model suggested by Lee and Carter (1992) [10] is very suitable, due to its powerful and elegant approach (Brouhns et al., 2002 [3]). They propose a log-bilinear model in which the logarithm of the force of mortality for age $x$ during year $t$ is explained from two age-specific parameters $\alpha$ and $\beta$ and one time-dependent parameter $\kappa$. This time-varying parameter $\kappa$ is constant across ages. In symbols the model is given by

$$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \quad (5.6)$$

where the error terms $\epsilon_{x,t}$ are assumed to be independent and identically distributed with mean zero and variance $\sigma^2$. The parameter $\alpha_x$ can be regarded as the general pattern of mortality (i.e., the differences in mortality between ages), $\kappa_t$ denotes general mortality improvement during year $t$ and $\beta_x$ reflects the sensitivity of age $x$ with respect to mortality changes over time.

---

15 Consider for example a closed group of $N$ people all reaching age $x$ at the first day of year $t$. Suppose that everybody dies simultaneously halfway year $t$. Then the force of mortality is computed by $N/(0.5 \times N) = 2$ which is larger than one
5.4 Estimation of the parameters

Although the setting in (5.6) appears to be a simple regression formulation, it is not, since there are no exogenous variables at the right-hand side. Furthermore, the parameters cannot uniquely be determined. For instance, suppose the vector \((a, b, k)\) is a solution to (5.6), then also the vectors \((a, bc, k/c)\) and \((a - bc, b, k + c)\) are solutions to (5.6) for any non-zero \(c\) (Lee and Carter, 1992 [10]). Therefore standard regression techniques cannot be used to find the parameters of (5.6) and alternative procedures have to be followed. First, to establish the uniqueness of the parameters the following two restrictions are imposed:

\[
\sum_{x} \beta_x = 1, \quad (5.7)
\]

and

\[
\sum_{t} \kappa_t = 0. \quad (5.8)
\]

The choice for restriction (5.7) implies that the sensitivity at different ages can easily be compared. Furthermore, by imposing restriction (5.8) it is easy to find a formulation for \(x\): since ignoring the error terms, it holds that

\[
\sum_{x} \ln x_t = T \sum_{x} \alpha_x + \sum_{x} \beta_x \kappa_t \Rightarrow T \alpha_x = \sum_{t} \ln \mu_{x,t} - \beta_x \sum_{t} \kappa_t \Rightarrow \alpha_x = \frac{1}{T} \sum_{t} \ln \mu_{x,t},
\]

where \(T\) denotes the number of data points.

Thus \(\alpha_x\) can be estimated by the average over time of \(\ln \mu_{x,t}\). The parameters \(\beta_x\) and \(\kappa_t\) are somewhat more complicated to retrieve. Applying the singular value decomposition (SVD, explained in Appendix A) to \(\ln \mu_x - \alpha_x\) yields unique least squares solutions for \(\beta_x\) and \(\kappa_t\) by using restrictions (5.7) and (5.8). The mortality index \(\kappa_t\) is then given by the first column of \(U\), and the solution for \(\beta_x\) can be determined from the first row of \(V^T\), where \(U\) and \(V^T\) are matrices that result from the SVD.

The next step is to re-estimate \(\kappa_t\) in order to match the actual number of total deaths in year \(t\) to the expected number of total deaths in year \(t\), taking the estimated coefficients \(\alpha_x\) and \(\beta_x\) as given. Hence \(\kappa_t\) must be solved from

\[
\sum_{x} D_{x,t} = \sum_{x} E_{x,t} \exp \left( \alpha_x + \beta_x \kappa_t \right). \quad (5.10)
\]

Expression (5.10) does not have an analytical solution, but it can be solved numerically. Note that this solution is unique, since for positive \(\beta_x\) it holds that

\[
\lim_{\kappa_t \to -\infty} \sum_{x} E_{x,t} \exp \left( \alpha_x + \beta_x \kappa_t \right) = 0, \quad (5.11)
\]

and
\[
\lim_{\kappa_t \to \infty} \sum_x E_{x,t} \exp (\alpha_x + \beta_x \kappa_t) = \infty, \quad (5.12)
\]
and
\[
\frac{\partial}{\partial \kappa_t} \sum_x E_{x,t} \exp (\alpha_x + \beta_x \kappa_t) = \sum_x E_{x,t} \beta_x \exp (\alpha_x + \beta_x \kappa_t), \quad (5.13)
\]
which implies that it is a strictly increasing function with outcome range \((0, \infty)\).

Since the first estimates of \(\kappa_t\) are based on logarithms, large differences between actual and estimated death numbers may occur. The re-estimation procedure of \(\kappa_t\) described above avoids these undesirable irregularities. Furthermore, such a re-estimation procedure yields an indirect estimate of \(\kappa_t\), which can be used to include years for which age-specific mortality data are missing (Lee, 2000 [9]).

### 5.5 Extrapolation of \(\kappa\)

The reason the stochastic mortality model was introduced in this thesis is that future mortality rates should be predicted. Therefore \(\kappa_t\) should be extrapolated since the age-specific parameters \(\alpha_x\) and \(\beta_x\) are assumed to be constant over time. As is commonly accepted, people tend to live longer and mortality rates, and thus also \(\kappa_t\), decline. Hence \(\kappa_t\) will not be stationary and therefore a model to extrapolate \(\kappa_t\) should focus on the change in \(\kappa_t\), denoted by

\[
\Delta_t = \kappa_t - \kappa_{t-1}, \quad (5.14)
\]

rather than on the actual value of \(\kappa_t\) itself. The type of model that can be used will depend on the realizations of \(\Delta_t\); for example, if the changes appear to be uncorrelated, a simple model of the form \(\Delta_t = \mu + \epsilon_t\) can be used; fluctuating changes can be modeled by autoregressive models like \(\Delta_t = \mu + \gamma (\Delta_{t-1} - \mu) + \epsilon_t\). To judge these models, the estimated values of \(\epsilon_t\) can be used: if these values indicate that the sequence \(\epsilon_t\) can be regarded as so-called ‘white noise’\(^\text{16}\), the model is acceptable; if not the model should be adjusted in order to obtain a sequence \(\epsilon_t\) that does look like ‘white noise’.

### 5.6 Applying the stochastic mortality model

#### 5.6.1 Data

In order to derive the probability distribution of the remaining lifetimes of the participants of the pension funds that are considered in this thesis, a stochastic mortality model should be constructed. The required statistics, i.e., yearly exposure and yearly deaths per age, are available on the internet\(^\text{17}\) for many

\(^{16}\) \(\epsilon_t\) is white noise if \(E(\epsilon_t) = 0\) and \(E(\epsilon_t \epsilon_s) = 0\) for \(t \neq s\).

\(^{17}\) www.mortality.org
5.6. APPLYING THE STOCHASTIC MORTALITY MODEL

different countries. Dutch data are available from 1850 to 2003. In the computations only data starting 1960 are considered, because these data are not affected by wars and, most importantly, before 1960 there were very few very old people, which implies that mortality data for the very old are unreliable. To keep the number of parameters to be estimated under control, age classes of five years have been used.

Hence, first the model that was specified in setting (5.6) is applied to these data, which provides expressions for the logarithms of the forces of mortality for the different age classes and different years; next the forces of mortality will be interpolated in order to obtain values for every age.

5.6.2 Results from the SVD composition

From the data described above it is possible to estimate the parameters of the model described in (5.6). First a maximum likelihood estimator (MLE) for the force of mortality $\mu_{x,t}$ is computed by taking the ratio of $D_{x,t}$ over $E_{x,t}$. Next, the parameters $\alpha_x$ can easily be computed by taking the average over time of the logarithm of the force of mortality; values for males and females are provided in table B.1 in appendix B and are displayed in figure 5.1. These results show that $\alpha_x$ increases with age, which makes sense since higher values for $\alpha$ indicate higher values for $\mu$ and therefore also higher values for the death probabilities since relation (5.1) implies that $q_{x,t} = 1 - \exp(-\mu_{x,t})$. For each age category the $\alpha$ for males is significantly higher than the $\alpha$ for females which expresses the fact that in general males have higher death probabilities. The parameters $\beta_x$ and $\kappa_t$ can be found by applying the Singular Value Decomposition; the values for $\beta_x$ are in table B.1 in appendix B and are graphically represented in figure 5.2. Indeed the coefficients of $\beta_x$ for both males and females sum to one as was imposed by restriction (5.7).

As can be derived from figure 5.2 the sensitivity of death probabilities with respect to changes over time, which is measured by the parameter $\beta$, differs between males and females. This difference can be explained from the progression
regarding the treatment and prevention of heart and vascular diseases. Van der Meulen (2004) [11] shows that in the Netherlands the number of deaths caused by these diseases substantially decreased the past decades, in contrast to other causes of death. According to De Jong (2005) [8] the number of Dutch males dying from heart and vascular diseases is relatively high at ages between 40 and 75, which explains the shape of $\beta_x$ for males. For females the shape of the sensitivity is quite opposite, although this can be explained from the progression of medical science on the area of heart and vascular diseases as well. However, until the age of 70, in the Netherlands the female deaths caused by cancer outnumbers the female deaths caused by heart and vascular diseases. As noted by De Jong (2005) [8], the death probability from cancer for Dutch females has not decreased the past decades. Hence, the sensitivity at these ages is relatively low. However, at older ages, heart and vascular diseases are the main cause of death for Dutch females, which explains the high values of $\beta_x$ for ages between 70 and 85.

Next, these estimates of $\alpha_x$ and $\beta_x$ are used to re-estimate $\kappa_t$ to match the actual number of deaths to the expected number of deaths. These results for $\kappa_t$ are displayed in table B.2 in appendix B. The results show that $\kappa_t$ decreases over time which indicates that mortality declines over time, since lower values for $\kappa_t$ yield lower values for $\mu_{x,t}$ and therefore also lower mortality rates.

### 5.6.3 Extrapolation of $\kappa$

The fitted values for $\kappa_t$ are displayed in table B.2 in appendix B for males and females. To extrapolate $\kappa_t$ the changes $\Delta_t$ should be modeled. In figure 5.3 these changes are presented. This figure seems to show a delta that fluctuates around a (negative) equilibrium value. These deviations with respect to the equilibrium value occur systematically: if at a certain point in time the $\Delta_t$ is below the equilibrium value, it is generally above the equilibrium value the next point...
5.6. APPLYING THE STOCHASTIC MORTALITY MODEL

Figure 5.3: Change in $\kappa$, denoted by $\Delta$, for males (left) and females (right) in time. This systematic variety can be interpreted by noting that if during a certain year mortality improves significantly above average ($\kappa_t$ decreases more than average which implies a $\Delta_t$ below the equilibrium value), the next year mortality can hardly improve since a lot of improvement capacity is used the year before. Hence a model that seems suitable is an autoregressive model $ARIMA(N,1,0)$, which is given by

$$
\Delta_t = \mu + \sum_{i=1}^{N} \gamma_i (\Delta_{t-i} - \mu) + \epsilon_t,
$$

where the number of time lags $N$ is minimized under the condition that the sequence $\epsilon_t$ describes a white noise process. The null hypothesis that a series can be regarded as white noise can be tested by the Ljung Box statistic, defined by

$$
Q = T(T+2) \sum_{i=1}^{p} \frac{r_k^2}{T-k},
$$

The parameter $T$ denotes the number of time periods, which in this case equals 43 (since there are 44 years, there are 43 changes). In expression (5.16) the term $r_k$ corresponds to the correlogram, which is computed as

$$
r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})},
$$

where $y_t$ is the variable under investigation, in this case $\Delta_t$, and $\bar{y}$ its mean.

Under the null hypothesis the statistic $Q$ is asymptotically $\chi^2$-distributed with $p$ degrees of freedom. Hence, critical values for the test whether the series $\epsilon_t$ is white noise are given by the 5% quantile of the $\chi^2$-distribution with $p$ degrees
of freedom. The number of degrees of freedom \( p \) should not be chosen too small, since this may cause high-order serial correlation to be unnoted; however \( p \) may also not be chosen too large, since this may result into significant correlations at high order lags caused by insignificant correlations at lower lags. In Table 5.1 Ljung Box statistics and \( p \)-values are presented for both males and females and for two values of \( N \). The number of degrees of freedom varies from one to ten. Corresponding critical values of the \( \chi^2 \)-distribution are displayed as well.

<table>
<thead>
<tr>
<th>degrees of freedom</th>
<th>critical value (5%)</th>
<th>( N = 1 )</th>
<th>( N = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>males</td>
<td>females</td>
</tr>
<tr>
<td>1</td>
<td>3.841</td>
<td>0.180</td>
<td>0.671</td>
</tr>
<tr>
<td>2</td>
<td>5.991</td>
<td>0.920</td>
<td>0.631</td>
</tr>
<tr>
<td>3</td>
<td>7.815</td>
<td>0.960</td>
<td>0.045</td>
</tr>
<tr>
<td>4</td>
<td>9.488</td>
<td>9.083</td>
<td>0.106</td>
</tr>
<tr>
<td>5</td>
<td>11.070</td>
<td>10.440</td>
<td>0.107</td>
</tr>
<tr>
<td>6</td>
<td>12.592</td>
<td>12.990</td>
<td>0.072</td>
</tr>
<tr>
<td>7</td>
<td>14.067</td>
<td>12.990</td>
<td>0.072</td>
</tr>
<tr>
<td>8</td>
<td>15.507</td>
<td>13.222</td>
<td>0.104</td>
</tr>
<tr>
<td>9</td>
<td>16.919</td>
<td>13.256</td>
<td>0.151</td>
</tr>
<tr>
<td>10</td>
<td>18.307</td>
<td>18.014</td>
<td>0.055</td>
</tr>
</tbody>
</table>

Table 5.1: Ljung Box statistics with corresponding \( p \)-values for males and females

As the results of Table 5.1 show, a model that explains \( \Delta_t \) by just one autoregressive term, i.e., \( N \) equals one, is not sufficient, since for three degrees of freedom, the value of \( Q \) is for both males and females larger than its critical value of 7.815. However, if the number of autoregressive terms equals two, the null hypothesis is not rejected since all the test statistics are smaller than the corresponding critical values. Hence, the model for \( \Delta_t \) is given by

\[
\Delta_t = \mu + \gamma_1 (\Delta_{t-1} - \mu) + \gamma_2 (\Delta_{t-2} - \mu) + \epsilon_t, \quad \epsilon_t \text{ white noise}
\]

(5.18)
### 5.6. Applying the Stochastic Mortality Model

<table>
<thead>
<tr>
<th></th>
<th>males</th>
<th>females</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$-0.13736$</td>
<td>$-0.14705$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$-0.36430$</td>
<td>$-0.31831$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>$-0.22341$</td>
<td>$-0.29835$</td>
</tr>
</tbody>
</table>

Table 5.2: Estimated parameters of the ARIMA(2,1,0) model for males and females

In table 5.2 the estimated values for $\mu$, $\gamma_1$ and $\gamma_2$ are shown. Indeed these values indicate a negative average ($\mu$) and values for $\gamma$ that are negative, which indicates that an above-average mortality improvement is followed by a below-average mortality improvement.

Substituting the estimated values in expression (5.18) and applying this to the observed values for $\kappa$ of 1963 until 2003 yields fitted values for the mortality index $\kappa$, assuming the error term to be zero. Figure 5.4 displays these estimated values as well as the realized values. The fit appears to be quite acceptable.

![Figure 5.4: Actual and fitted $\kappa$ for males and females (1960-2003)](image)

Furthermore, it is possible to construct both extrapolated expected values for $\kappa_t$ and confidence bounds. The expected values follow from similar computations as are used to determine the fitted values. To define confidence bounds, variances are needed. To derive these variances, first note that

$$\kappa_t = \kappa_0 + \sum_{i=1}^{t} \Delta_i,$$

(5.19)

where $\kappa_0$ denotes the last observed realization of $\kappa$; in this dataset, $\kappa_0$ corresponds to the $\kappa$ of 2003. Hence, computing the variance of $\kappa$ boils down to
computing the variance of the sum over all \( \Delta \)'s. From elementary probability theory it follows that

\[
\text{var} \left( \sum_{i=1}^{t} \Delta_i \right) = \sum_{i=1}^{t} \sum_{j=1}^{t} \text{cov} (\Delta_i, \Delta_j),
\]

(5.20)

where \( \text{cov} (\Delta_i, \Delta_i) \) equals \( \text{var} (\Delta_i) \). In appendix C it is described how these covariances can be computed. The extrapolation has been undertaken for years until 2120. Results for years until 2050 are in figure 5.5.

![Figure 5.5: Realized and forecasted \( \kappa \), including 95% confidence bound for males (blue) and females (red) 1960 – 2050](image)

5.6.4 Computing the force of mortality for each age

In the previous subsection forecasts for \( \kappa_t \) were computed. Applying these forecasts as well as the parameter estimates for \( \alpha_x \) and \( \beta_x \) displayed in table B.1 to the model specified in (5.6) yields forecasts for the (logarithm of the) force of mortality for the different age classes. However, age-specific forces of mortality are needed. Hence, the estimates should be interpolated in order to arrive at such age-specific values for \( \mu_{x,t} \). It is assumed that the estimated forces of mortality per class correspond to the force of mortality of the ‘average’ person in the class and that these ‘average’ persons are aged equal to the average of the minimum age and the maximum age per class. For instance, persons belonging to class 30-34 are on average \((30 + 34)/2 = 32\) years old. Persons belonging to age class 95+ are assumed to be 100 years of age on average. For ages unequal
5.7. CONCLUSIONS

to the average, the forces of mortality can be found by applying interpolation techniques. As interpolation method a linear variant is used. For example, the force of mortality for a person aged 30 can be computed as follows:

\[ \ln \mu_{30,t} = 0.4 \cdot \ln \mu_{27,t} + 0.6 \cdot \ln \mu_{32,t}. \] (5.21)

5.7 Conclusions

The Lee-Carter model imposes a fairly simple yet powerful setting to describe the force of mortality. The parameters of this model that result from the SVD-analysis are in line with other studies concerning Dutch mortality. Furthermore, the model, that has been derived to model mortality changes, very well fits the observed mortality changes. Hence, it can be concluded that the stochastic mortality model discussed in this chapter can very well be used in order when assessing the effects of future mortality developments.
Chapter 6

Some fictitious pension funds

6.1 Introduction

In the previous two chapters possible mortality products have been discussed as well as a stochastic mortality model that can be used to evaluate and price these products. Together, these products and the mortality model can be used to develop an internal model to cover the inevitable risks that pension funds face. However, a pension fund may also use the standardized method discussed in chapter 3. In order to evaluate and compare the effectiveness and prices of these methods, they will be applied to some fictitious pension funds. The pension scheme, which is identical for all funds, is chosen to be relatively simple, since a choice for such simple pension plans makes it possible to fully concentrate on the most important issue in this thesis, namely longevity risk. The set-up of this chapter is as follows: first the pension scheme, which is identical for each fund, will be discussed. Secondly, the compositions of the funds will be provided.

6.2 Set-up of the pension scheme

Recall from chapter 1 that longevity risk occurs when the participants of a pension fund live longer than is (currently) expected. Hence, the biggest part of longevity risk with pension funds is caused by old-age pensions, since the expected benefits of other types of pensions, for example survivor’s pensions, will be small if participants live longer than expected. Therefore, the pension scheme that is used in this thesis consists solely of an old-age agreement. The retirement age is, in line with many real-life pension schemes in the Netherlands\textsuperscript{18}, set at 65.

\textsuperscript{18}Currently, in the Netherlands as well as in other countries, debates are going on to increase the pension age due to the aging population. However, in this thesis the retirement age is
The pension scheme is an average pay plan: each year an active participant’s pension right increases according to formula (2.2). In conformity with many Dutch pension schemes, it is assumed that participants start building up pension rights at the age of 25 and end building up these rights at the retirement age of 65. At the starting age of 25 participants, both males and females, are supposed to have a salary equal to €30,000. This income more or less corresponds to the average income of the Dutch population\footnote{Source www.cbs.nl. In reality the average income of males is higher than the average income of females. For the sake of simplicity this will be ignored in this thesis.}. To account for career developments, it is assumed that these wages increase according to the so-called 3-2-1-0-model. This model implies that between the ages of 25 and 35 the salary increase, due to career improvements, is yearly 3%, between 35 and 45 wages increase yearly with 2%, between 45 and 55 with 1% and after the age of 55 no salary increases take place. Figure 6.1 displays the salary distribution across ages.

To compute the total pension rights accrual for each participant, an accrual percentage of 1.75% is assumed as well as an offset equal to €14,000. By assuming that in the past the pension rights were computed by using the same figures, the pension rights for each participant can be found. Figure 6.2 shows these accrued pension rights.

Note that after the retirement age of 65 the pension rights are constant as there is no further accrual. For simplicity it is assumed that the pension benefits are totally paid out at the end of each year, whereas in reality pension benefits are mostly paid out on a monthly basis.
Since this thesis aims at hedging the current pension rights against longevity risk, the analysis focuses on these pension rights; future accrual is ignored as well as indexation aspects. This boils down to considering the participants to be deferreds\textsuperscript{21}.

6.3 Fund compositions

To illustrate the effects of the possibilities pension funds have regarding longevity risk, eight different fictitious funds are used. The reason eight funds are investigated rather than one single fund lies in the fact that the standardized method to determine the technical provision at realistic value is based on two specific elements: the weighed average age of the fund population ($\bar{x}$ in equation (3.1)) and the number of participants ($n$ in equation (3.3)). Therefore, it is useful to consider several funds by varying these two specific elements. Furthermore, since in general the mortality improvements for males and females are not identical, it is interesting to vary the proportions of males and females. Table 6.1 shows some characteristics of the fictitious pension funds that are used throughout the analysis in this thesis\textsuperscript{22}, and figure 6.3 graphically illustrates the compositions of the fund populations. For simplicity, every participant is assumed to reach the next age at the beginning of the year.

\textsuperscript{21} However, the analysis can be extended to a situation in which indexation is incorporated and participants are treated as active.

\textsuperscript{22} Note that due to rounding differences the participant percentages for each fund may not sum to 100%.
Figure 6.3: Graphical representation of the eight fund populations; lower blue part corresponds to males, upper purple part to females; fund numbers are displayed in the top right corners.
### Fund Compositions

#### Table 6.1: Compositions and characteristics of the fictitious pension funds

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>average age</td>
<td>36.8</td>
<td>36.8</td>
<td>33.7</td>
<td>33.7</td>
</tr>
<tr>
<td>number of participants</td>
<td>75,000</td>
<td>75,000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>males/females</td>
<td>40/60</td>
<td>60/40</td>
<td>30/70</td>
<td>70/30</td>
</tr>
<tr>
<td>ages 25-34(%)</td>
<td>33,196(44%)</td>
<td>33,196(44%)</td>
<td>290(58%)</td>
<td>290(58%)</td>
</tr>
<tr>
<td>ages 35-44(%)</td>
<td>29,323(39%)</td>
<td>29,323(39%)</td>
<td>190(38%)</td>
<td>190(38%)</td>
</tr>
<tr>
<td>ages 45-54(%)</td>
<td>10,605(14%)</td>
<td>10,605(14%)</td>
<td>20(4%)</td>
<td>20(4%)</td>
</tr>
<tr>
<td>ages 55-64(%)</td>
<td>1,195(2%)</td>
<td>1,195(2%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
<tr>
<td>ages 65-74(%)</td>
<td>428(1%)</td>
<td>428(1%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
<tr>
<td>ages 75-85(%)</td>
<td>253(0%)</td>
<td>253(0%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>fund</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>average age</td>
<td>66.6</td>
<td>66.6</td>
<td>68.9</td>
<td>68.9</td>
</tr>
<tr>
<td>number of participants</td>
<td>75,000</td>
<td>75,000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>males/females</td>
<td>33/67</td>
<td>67/33</td>
<td>40/60</td>
<td>60/40</td>
</tr>
<tr>
<td>ages 25-34(%)</td>
<td>1,667(2%)</td>
<td>1,667(2%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
<tr>
<td>ages 35-44(%)</td>
<td>3,271(4%)</td>
<td>3,271(4%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
<tr>
<td>ages 45-54(%)</td>
<td>6,147(8%)</td>
<td>6,147(8%)</td>
<td>0(0%)</td>
<td>0(0%)</td>
</tr>
<tr>
<td>ages 55-64(%)</td>
<td>15,966(21%)</td>
<td>15,966(21%)</td>
<td>139(28%)</td>
<td>139(28%)</td>
</tr>
<tr>
<td>ages 65-74(%)</td>
<td>26,553(35%)</td>
<td>26,553(35%)</td>
<td>250(50%)</td>
<td>250(50%)</td>
</tr>
<tr>
<td>ages 75-85(%)</td>
<td>21,396(29%)</td>
<td>21,396(29%)</td>
<td>111(22%)</td>
<td>111(22%)</td>
</tr>
</tbody>
</table>

Table 6.1: Compositions and characteristics of the fictitious pension funds
CHAPTER 6. SOME FICTITIOUS PENSION FUNDS
Chapter 7

Results

7.1 Introduction

In this chapter the technical provision at realistic value for the liabilities of each of the eight fictitious pension funds is determined. The aim of this thesis is to develop a method that minimizes this technical provision based on the current accrued pension rights. Such a method must fit into the regulations of the DNB, which are expressed in the FTK, and were discussed in chapter 3. One such requirement is that future cash flows are discounted using the term structure of interest rates. For the sake of simplicity, the term structure of interest rates that will be used in this analysis is kept constant at the level of February 28, 2006 as determined by the DNB based upon the swapcurve\textsuperscript{23}. Figure 7.1 displays this term structure.

\begin{center}
\includegraphics[width=0.5\textwidth]{term_structure.png}
\end{center}

Figure 7.1: Term structure of interest rates (February 28, 2006)

\textsuperscript{23}See www.dnb.nl for a full explanation.
As discussed before, DNB allows two methods to determine the technical provision: a standardized method and an internal method. Hence, first the standardized method will be applied to the pension funds. Subsequently, an internal method will be developed and implemented. Both these methods require an amount of capital that is needed at present to cover the future liabilities sufficiently.

To determine the best method for each pension fund, the costs of the methods are used as a criterion. Hence, the objective is to find the method with minimal costs, which implies that the optimal costs are given by

$$\text{cost}_{\text{opt}} = \min \{\text{costs standardized method}; \text{costs internal method}\}.$$  \hspace{1cm} (7.1)

### 7.2 The standardized method

To derive the technical provision at realistic value using the standardized method, first the expectation of the future cash flows that arise from the liabilities must be computed prospectively. This implies that expected changes in mortality should be taken into account. Furthermore, the fact that in general insured people (i.e., the participants of pension funds) have above-average life expectancies should also be incorporated into the computations. At WWB these two factors are included in the so-called Brans mortality table. The death probabilities in this mortality table vary across age, gender and year. To derive these probabilities, two ingredients are used:

- The prognosis established by the CBS (the Dutch statistics bureau) for future death probabilities until the year of 2050.
- Experienced death coefficients.

These experienced death coefficients are the result of a study undertaken by KPMG Brans & Co (the predecessor of WWB) in the nineties of the previous century. This study compared the observed death figures at twenty pension funds during the period 1990-1995 with the number of deaths predicted using the death probabilities of the GBM/V 85/90 mortality table. The experienced death coefficients were determined, both for males and females, as the quotient of actual deaths and predicted deaths:

$$c_x = \frac{q^{\text{actual}}_x}{q^{\text{predicted}}_x},$$  \hspace{1cm} (7.2)

where $q_x$ denotes the death probability and $c_x$ the experienced death coefficient for age $x$. Figure 7.2 displays the coefficients that resulted from this study.

The fact that the coefficients are smaller than or equal to one implies that pension fund participants face smaller death probabilities than the total population. Therefore, it can be concluded that insured people indeed have above-average life expectancies.

---

24 Mortality table for the total Dutch population, constructed by the Dutch society of actuaries.
Combining the Brans mortality table and the term structure of interest rates, displayed in figure 7.1, the expectation of the future discounted cash flows of each fund can be computed. The expectation, denoted by \( VPV_{\text{tot}} \), equals the sum of the individual expectations \( VPV_i \). These individual expectations can be computed using an actuarial factor. This factor corresponds to the present value of one euro pension benefit to be paid yearly to an individual as soon as this individual has reached the retirement age of 65. Hence, it can be regarded as the discounted value of future pension payments. The factor, for persons younger than the retirement age denoted by \( 65-x\alpha_x \), and for persons that have reached the retirement age by \( \alpha_x \), depends on three elements:

- the age \( x \) of the individual;
- future death probabilities for the individual: \( q_{x,t} \) which denotes the one-year death probability of a person aged \( x \) in year \( t \);
- the appropriate discount rate, which can be derived from the term structure of interest rates.

The factor is computed by

\[
65-x\alpha_x = \sum_{k=65-x}^{\text{MaxAge}-x} \nu_k \cdot k \cdot p_x
\]

\[
\alpha_x = \sum_{k=0}^{\text{MaxAge}-x} \nu_k \cdot k \cdot p_x
\]  \hspace{1cm} (7.3)

\( 25 \) \( VPV \) (Voorziening pensioenverplichtingen) is the Dutch term for provision for pension commitments.
where $v_k$ denotes the discount rate for $k$ years from now.

For ages 25 to 85 these factors are displayed in figure 7.3.

![Figure 7.3: Actuarial factors for males (dotted blue line) and females (solid pink line)](image)

As can be seen from figure 7.3 the factors for females are greater than those for males, which can be explained from the fact that deathprobabilities are smaller for females than for males. Furthermore, for persons below the retirement age of 65 the factors increase with age, because older persons have to survive a smaller number of years to reach the age of 65 than younger persons. After the age of 65, the factors decrease since some payments already have been made and therefore the factor corresponds to a smaller expected number of future pension payments.

Next, the expectation of the sum of the future pension payments can be computed by

$$ V_P V_i = accrued \text{ pension rights}_i \times actuarial \text{ factor}_i, \quad (7.4) $$

and the total expectation, $V_P V_{tot}$, can be obtained by summing all $V_P V_i$’s:

$$ V_P V_{tot} = \sum_{i=1}^{n} V_P V_i, \quad (7.5) $$

where $n$ denotes the total number of participants in the pension fund.

Under the standardized method, an extra loading must be added to the total expectation in order to cover inevitable risks, as was discussed in chapter 3. Since the pension scheme under investigation does not contain survivor’s pensions, the extra loading can be determined according to formulas (3.1), (3.3), and (3.5):

$$ extra \text{ loading} = \sqrt{TSO^2 + NSA^2}, $$
7.2. **THE STANDARDIZED METHOD**

where

\[ TSO = \left[ 2 + \frac{9}{40} \max\{p - \bar{x}; 0\} \right] \% , \]

and

\[ NSA = \left[ \frac{60}{\sqrt{n}} \right] \% . \]

The retirement age \( p \) for participants of the pension scheme is 65. The weighted average age \( \bar{x} \) is computed by :

\[ \bar{x} = \frac{\sum_{i=1}^{n} \frac{VPV_i}{VPV_{tot}} \times age_i}{VPV_{tot}}. \quad (7.6) \]

Results for the eight funds are in table 7.1.

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>weighted average age ( \bar{x} )</td>
<td>46.3</td>
<td>46.3</td>
<td>38.5</td>
<td>38.5</td>
</tr>
<tr>
<td>fund population</td>
<td>75,000</td>
<td>75,000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>males/females</td>
<td>40/60</td>
<td>60/40</td>
<td>30/70</td>
<td>70/30</td>
</tr>
<tr>
<td>( VPV_{tot} (\times \€1,000) )</td>
<td>2,180,337</td>
<td>2,149,588</td>
<td>7,994</td>
<td>7,828</td>
</tr>
<tr>
<td>TSO(%)</td>
<td>6.20%</td>
<td>6.22%</td>
<td>7.95%</td>
<td>7.96%</td>
</tr>
<tr>
<td>NSA(%)</td>
<td>0.22%</td>
<td>0.22%</td>
<td>2.68%</td>
<td>2.68%</td>
</tr>
<tr>
<td>total loading(%)</td>
<td>6.20%</td>
<td>6.22%</td>
<td>8.39%</td>
<td>8.40%</td>
</tr>
<tr>
<td>realistic value(( \times \€1,000 ))</td>
<td>2,315,615</td>
<td>2,283,347</td>
<td>8,665</td>
<td>8,485</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>fund</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>weighted average age ( \bar{x} )</td>
<td>68.6</td>
<td>68.6</td>
<td>68.5</td>
<td>68.5</td>
</tr>
<tr>
<td>fund population</td>
<td>75,000</td>
<td>75,000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>males/females</td>
<td>33/67</td>
<td>67/33</td>
<td>40/60</td>
<td>60/40</td>
</tr>
<tr>
<td>( VPV_{tot} (\times \€1,000) )</td>
<td>14,335,245</td>
<td>14,001,927</td>
<td>111,714</td>
<td>110,110</td>
</tr>
<tr>
<td>TSO(%)</td>
<td>2.00%</td>
<td>2.00%</td>
<td>2.00%</td>
<td>2.00%</td>
</tr>
<tr>
<td>NSA(%)</td>
<td>0.22%</td>
<td>0.22%</td>
<td>2.68%</td>
<td>2.68%</td>
</tr>
<tr>
<td>total loading(%)</td>
<td>2.01%</td>
<td>2.01%</td>
<td>3.35%</td>
<td>3.35%</td>
</tr>
<tr>
<td>technical provision at realistic value(( \times \€1,000 ))</td>
<td>14,623,665</td>
<td>14,283,640</td>
<td>115,453</td>
<td>113,795</td>
</tr>
</tbody>
</table>

Table 7.1: Results of the standardized method for the eight funds

These results show that, especially for the ‘young’ funds 1 to 4, the extra loading is quite large. This implies that the amount of capital needed to sufficiently cover the future cash flows, i.e., the technical provision at realistic value, is also quite large. Therefore, it is useful to consider other methods that possibly require less capital. The next section explores such an alternative method.
7.3 Investment in mortality bonds

In chapter 4, several possible mortality-linked securities were introduced. The payoff of such securities depends on the realization of some underlying mortality index. Hence, mortality-linked securities appear to be very suitable hedge instruments for pension funds, since the liabilities of pension funds and the payoff of the securities are correlated. This section examines the use of mortality bonds as hedge instruments by pension funds. Although also other types of securities, such as mortality options or swaps, could be used, only bonds are investigated, for two reasons:

- the structure of bonds is easier than the structure of most other securities;
- a first attempt to market a mortality bond has already been made\(^\text{26}\), whereas for other mortality securities no such attempts have been made.

In this section twelve hypothetical mortality bonds are used. The structure of these bonds is comparable to the structure of the BNP Paribas Longevity Bond, however some of the shortcomings of that bond are removed. The assumptions on the bonds, as well on the market they will be brought to, are discussed below:

- The payoffs of the bonds depend on cohorts of the Dutch population.
- Six bonds are based on male cohorts and six are based on female cohorts.
- For each gender, six different cohorts have been determined; the difference between the cohorts is the year of birth. These years of birth are: 1931, 1941, 1951, 1961, 1971, and 1981. Hence, in 2006, these cohorts are aged 75, 65, 55, 45, 35, and 25 respectively.
- Coupon payments start in the year the cohort reaches the age of 65 and end at the moment the total cohort has died. It is assumed that these coupons are paid out at the end of each year. Hence, for the cohorts aged 65 and 75 payments will be made already at the end of 2006; the bonds based on the younger cohorts are deferred mortality bonds.
- The coupon amounts are proportional to the survival rates of the cohort: suppose that until year \( t \) 80\% of cohort \( i \) has survived, the payoff in year \( t \) will be 80\% of the face value. The face value is assumed to be €1,000.
- These survival rates are computed from future mortality figures that will be available in the future from the electronic database of the CBS, called Statline\(^\text{27}\).

\(^{26}\) See subsection 4.2.5 “The BNP Paribas Longevity Bond”

\(^{27}\) These mortality figures can be obtained from Statline, available on www.cbs.nl, by subsequently selecting: mens en maatschappij; bevolking; sterfte, doodsoorzaken en euthanasie; sterfte; sterfte: leeftijd, burgerlijke staat; sterfte mannen/sterfte vrouwen; sterfte per 1000 mannen/vrouwen.
7.3. INVESTMENT IN MORTALITY BONDS

- The bonds are issued by some creditworthy institution, for example the Dutch government or the EIB, which implies that the payoffs are hardly at risk.

- It is assumed that these bonds are marketed on a frictionless market, although short selling will not be allowed.

- Since currently no comparable assets are traded, no unique prices for the bonds can be determined due to the absence of market data. Therefore a common pricing principle is used. This pricing principle is a function of the mean, \( \mu_i \), and standard deviation, \( \sigma_i \), of the sum of the expected discounted cash flows of bond \( i \) and is given by

\[
\text{price} = \mu_i + \xi \sigma_i, \quad (7.7)
\]

where the non-negative constant \( \xi \) is a risk parameter. The parameters \( \mu_i \) and \( \sigma_i \) are estimated using the Lee-Carter model that was specified in formulation (5.6). Since mortality bonds are new products and the payoffs are very uncertain, it is likely that the issuers of these bonds want to be sufficiently compensated for the risks they are exposed to. Therefore the choice to incorporate the standard deviation into the pricing principle has been made.

By using these mortality bonds, probably on a buy-and-hold strategy, a pension fund can reduce its longevity risk. However, a total elimination of the longevity risk arising from the liabilities is not possible, since the payoff of the bonds depends on cohorts of the total population rather than the fund population. Still it is possible to partly hedge the liabilities by buying the mortality bonds, although such a hedge requires capital. To determine whether a hedge is interesting for a particular pension fund, the costs are used as a criterion as discussed in the introduction of this chapter.

To determine the costs of a strategy of investing in mortality bonds, first of all, the prices of the bonds must be derived. Equation (7.7) provides these prices. However, whereas \( \mu \) and \( \sigma \) can be determined by using the stochastic mortality model discussed in chapter 5, in the absence of market data, the parameter \( \xi \) cannot be computed, and therefore the value of this parameter must be chosen arbitrarily. In this analysis several values for \( \xi \) are used. By setting \( \xi \) at a specific value a vector of bond prices can be determined. As argued before, investing in the mortality bonds does not provide a complete hedge of the liabilities\(^2\); therefore next to the bonds, capital must be reserved, in case the liabilities exceed the total coupon payments of the bonds that were bought by the pension fund. To determine the amount of capital needed, FTK rules are used.

Recall from section 3.2 that when using an internal method to determine the realistic value of the liabilities a probability level of 75% was crucial. Hence, the

\(^{2}\)Of course, a (nearly) complete hedge can be created by buying a great number of each bond. However, such a strategy can a priori be judged as too expensive.
amount of capital must be sufficient to cover the part of the liabilities that is
not covered by the bond portfolio with a probability of 75%. This implies that
the total costs of a strategy of investing in a bond portfolio are given by:

\[
\text{costs} = \text{product costs} + \text{capital needed},
\]

where the product costs can be computed as

\[
\text{product costs} = \sum_{i=1}^{12} \omega_i P_i
\]

In expression (7.9) \( \omega_i \) denotes the number of bonds \( i \) to buy and \( P_i \) corresponds
to the price of bond \( i \). Hence, \( \omega \) and \( P \) will be referred to as the coefficient
vector and the price vector respectively.

To determine the total costs, first of all the liabilities of the funds as well as
the coupon payments of the mortality bonds are needed. Since one of the as-
sumptions on the bonds was that the payoffs depend on the stochastic mortality
model of chapter 5, it makes sense to derive the liabilities using the same model.
In section 5.6 the parameters of this model were estimated, which can be used
to derive future probability distributions. These future probability distributions
depend on a corresponding trajectory of the mortality index \( \kappa_t \). Equation (5.6)
provides a model for the yearly change of \( \kappa_t \), denoted by \( \Delta_t \):

\[
\Delta_t = \mu + \gamma_1 (\Delta_{t-1} - \mu) + \gamma_2 (\Delta_{t-2} - \mu) + \epsilon_t,
\]

where the error term \( \epsilon_t \) is assumed to be normally distributed with mean zero
and variance \( \sigma^2 \).

Hence, by simulating values for the error term, trajectories for the mortality
index can be found, from which probability distributions can be derived. For
the analysis of this thesis 1,000 of such trajectories have been simulated both
for males and females.

By assuming that the death rates of the cohorts, on which the payoffs of
the mortality bonds are based, exactly match the corresponding death proba-
bilities of the derived probability distributions\(^{29}\), the yearly discounted coupon
payments of each mortality bond can be computed for each trajectory. Based
on these outcomes, statistics such as the mean and standard deviation can be
computed. Next, the pricing principle (7.7) can be used to determine prices of
the bonds for several values of \( \xi \). Table 7.2 displays these prices as well as some
statistics and characteristics of the bonds.

The results in table 7.2 are in line with the actuarial factors displayed in
figure 7.3: the expectation of the payoffs is smaller for males than for females,
increases until the age of 65 and decreases after the age of 65.

\(^{29}\)It makes sense to impose this assumption, since the cohort data are not really computed
from a specific cohort; instead, these data are computed from the total population and take
into account expected changes in future mortality. For more information about cohorts, see
for instance www.gad.gov.uk.
7.3. INVESTMENT IN MORTALITY BONDS

Table 7.2: Mortality bonds: characteristics, mean, standard deviation, and prices in euros for several values of the risk parameter

<table>
<thead>
<tr>
<th>bond</th>
<th>gender</th>
<th>age</th>
<th>mean (€)</th>
<th>standard deviation (€)</th>
<th>price (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>male</td>
<td>25</td>
<td>2,258</td>
<td>346</td>
<td>2,431</td>
</tr>
<tr>
<td>2</td>
<td>male</td>
<td>35</td>
<td>3,225</td>
<td>447</td>
<td>3,448</td>
</tr>
<tr>
<td>3</td>
<td>male</td>
<td>45</td>
<td>4,700</td>
<td>567</td>
<td>4,984</td>
</tr>
<tr>
<td>4</td>
<td>male</td>
<td>55</td>
<td>7,136</td>
<td>629</td>
<td>7,451</td>
</tr>
<tr>
<td>5</td>
<td>male</td>
<td>65</td>
<td>11,466</td>
<td>716</td>
<td>11,824</td>
</tr>
<tr>
<td>6</td>
<td>male</td>
<td>75</td>
<td>7,325</td>
<td>608</td>
<td>7,629</td>
</tr>
<tr>
<td>7</td>
<td>female</td>
<td>25</td>
<td>2,813</td>
<td>427</td>
<td>3,028</td>
</tr>
<tr>
<td>8</td>
<td>female</td>
<td>35</td>
<td>3,968</td>
<td>525</td>
<td>4,231</td>
</tr>
<tr>
<td>9</td>
<td>female</td>
<td>45</td>
<td>5,759</td>
<td>610</td>
<td>6,065</td>
</tr>
<tr>
<td>10</td>
<td>female</td>
<td>55</td>
<td>8,702</td>
<td>787</td>
<td>9,096</td>
</tr>
<tr>
<td>11</td>
<td>female</td>
<td>65</td>
<td>13,514</td>
<td>872</td>
<td>13,950</td>
</tr>
<tr>
<td>12</td>
<td>female</td>
<td>75</td>
<td>9,170</td>
<td>777</td>
<td>9,559</td>
</tr>
</tbody>
</table>

The liabilities of the eight fictitious pension funds are computed by using the derived probability distributions as well. However, these probabilities cannot be used straightaway; they must be adapted in order to correctly reflect the death probabilities of the fund population. As discussed in the previous section, such an adjustment is needed due to the fact that pension fund participants have lower death probabilities. Relation (7.2) is used to transform the death probabilities of the total population into death probabilities for the fund populations. Using the probabilities the yearly liabilities for each fund can be computed by using a binomial distribution: suppose \( N_{x,t,m} \) denotes the number of males of age \( x \) in year \( t \) and \( q_{x,t,m} \) is the corresponding one-year death probability. Then the number of deaths, \( Q_{x,t,m} \), occurring during year \( t \) from the group of males has the following distribution:

\[
Q_{x,t,m} \sim \text{Bin}(N_{x,t,m}; q_{x,t,m})
\]

Of course, for females the number of deaths is distributed analogously.

Hence, by simulating from the distribution given by expression (7.10) the development of the fund population can be simulated. Next, the total amount of pension benefits to be paid to the group can easily be computed by multiplying the number of survivors by the accrued pension rights of that group, in case \( x \) is greater than or equal to the retirement age of 65; in case \( x \) is smaller than 65, no pension benefits will be paid out to the group. Note that the accrued pension rights are equal to the level of 2006 for every year \( t \), since future accrual and indexation aspects are ignored, as explained in section 6.1. Finally, the total amount of pension benefits for the fund in year \( t \) can be computed by summing the benefits over all groups. In Appendix E the m-file fundsandkappa.m that was used to simulate the trajectories as well as the fund simulations is displayed.
Now that both the bond payoffs and the fund liabilities have been determined, the amount of capital needed to satisfy the 75% probability level can be computed for every vector $\omega$.

Let $C_{i,t,s}$ denote the discounted payoff of bond $i$ during year $t$ for mortality index trajectory $s$; then the total discounted payoff $C_{t,s}$ of a bond portfolio with $\omega_i$ bonds $i$ in year $t$ for scenario $s$ can be computed by:

$$C_{t,s}(\omega) = \sum_{i=1}^{12} \omega_i C_{i,t,s}. \quad (7.11)$$

For each scenario $s$, $s = 1, \ldots, 1000$, the discounted liabilities in year $t$ for fund $f$, where $f = 1, \ldots, 8$, are given by $L_{f,t,s}$. Hence, for each simulation $s$, the discounted capital needed by fund $f$ in year $t$, denoted by $K_{f,t,s}$, is computed as the difference of the liabilities and the payoff of the portfolio in year $t$:

$$K_{f,t,s}(\omega) = L_{f,t,s} - C_{t,s}(\omega). \quad (7.12)$$

Equation (7.12) can provide negative values for $K_{f,t,r,s}$ in case the liabilities during a year are smaller than the payoff of the assets in that year. It is assumed that under such circumstances the capital that flows into the pension fund is invested in a risk free asset with return equal to the discount rate. In future years this capital can be used if the liabilities exceed the portfolio payoff. Hence, for each simulation $s$ the corresponding total capital needed can be determined by summing over all years:

$$K_{f,s}(\omega) = \sum_{t=2006}^{2120} K_{f,t,s}(\omega) \quad (7.13)$$

where $K_{f,s}$ denotes the total amount of capital needed in scenario $s$ for fund $f$.

According to the FTK, the capital must be sufficient to cover the liabilities with a probability level of 75%. The amount of capital that follows from equation (7.13) is sufficient to cover the liabilities entirely for each trajectory $s$. Therefore the amount of capital needed to sufficiently cover the liabilities with a probability level of 75% is determined as the amount of capital at which in 75% of the scenarios the liabilities are sufficiently covered: the 75th percent quantile. It is assumed that funds are not allowed to maintain negative amounts of capital. The total costs for vector $\omega$ can then be computed by using equations (7.8) and (7.9) as

$$total costs = \sum_{i=1}^{12} \omega_i P_i + \max \left\{ K_{75\%,f}; 0 \right\}, \quad (7.14)$$

where $K_{75\%,f}$ denotes the 75th percent quantile of $K_f$. The m-file costfunction.m that can be used to compute the costs is contained in Appendix E.

The simulations can also be used to determine the exact probability level at which the required amount of capital according to the standardized method is sufficient to cover the liabilities. In table 7.3 these probability levels are
Table 7.3: Exact probability levels of the capital needed according to the standardized method (odd funds mainly consist of females and even funds mainly of males)

displayed; for each fund the probability level is higher than the required level of 75%; hence, apparently the standardized method leads to a somewhat more prudent probability level than the required level of 75%.

However, the question arises which coefficients vector to choose. An interesting choice for the vector of $\omega$ is the zero-vector, since in that case the product costs are zero. The amount of capital needed can then be regarded as a stack of capital out of which the pension benefits must be paid out. Hence, in that case the internal model is more or less equal to the standardized method. Results in case $\omega$ is the zero-vector are given in table 7.4; this table also displays the difference with the results of the standardized method.

In line with the results of table 7.3 the standardized method requires more capital.

However, the zero-vector is just one out of infinitely many possible choices for $\omega$. The aim of this analysis is to find the strategy with the lowest costs. Therefore the vector $\omega$ must be chosen so that it minimizes the costs for pension fund $f$ for a given price vector $P$. To determine these costs, equations (7.7) to (7.14) can be used. Hence, for fund $f$ the minimization problem is given by

$$\min_{\omega_f} \text{total costs}_f (\omega_f, P),$$

where

$$\text{total costs}_f (\omega_f, P) = \text{product costs}_f (\omega_f, P) + \text{capital costs}_f (\omega_f),$$

$$\omega_f = (\omega_{1,f}, \cdots, \omega_{12,f}),$$

$$P = (P_1, \cdots, P_{12}),$$

$$\text{product costs}_f (\omega_f, P) = \omega_f' P,$$
CHAPTER 7. RESULTS

Table 7.4: Total capital needed according to the internal method, if no investment takes place and difference with the standardized method

<table>
<thead>
<tr>
<th>fund</th>
<th>total capital needed (x €1,000)</th>
<th>difference with standardized method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,300,199</td>
<td>-0.67%</td>
</tr>
<tr>
<td>2</td>
<td>2,241,045</td>
<td>-1.85%</td>
</tr>
<tr>
<td>3</td>
<td>8,602</td>
<td>-0.72%</td>
</tr>
<tr>
<td>4</td>
<td>8,209</td>
<td>-2.25%</td>
</tr>
<tr>
<td>5</td>
<td>14,556,188</td>
<td>-0.46%</td>
</tr>
<tr>
<td>6</td>
<td>14,083,435</td>
<td>-1.40%</td>
</tr>
<tr>
<td>7</td>
<td>114,880</td>
<td>-0.50%</td>
</tr>
<tr>
<td>8</td>
<td>112,357</td>
<td>-1.26%</td>
</tr>
</tbody>
</table>

Table 7.4: Total capital needed according to the internal method, if no investment takes place and difference with the standardized method

\[
capital\ costs_f(\omega_f) = \max\ \{K_{75\%,f}(\omega_f); 0\}, \quad (7.20)
\]

under the short-selling constraint

\[
\omega_{i,f} \geq 0 \ \forall i. \quad (7.21)
\]

The subscripts \( f \) in equations (7.15) to (7.21) refer to the fund for which the minimization is carried out. As can be derived from (7.15) the optimal bond allocation depends on the price vector \( P \), and therefore also on the risk parameter \( \xi \). In table 7.5 the optimal bond allocations in case \( \xi = 0 \) are displayed. Optimal bond allocations for some other values are given in appendix F.

As could be expected it is optimal for the funds with the younger populations to mainly invest in the mortality bonds based on the younger cohorts, and for the funds with the older populations to invest mostly in the bonds of the older cohorts. Furthermore, the optimal number of bonds to invest in is larger for the older populations, since older persons have greater accrued pension rights. The optimal bond allocations reflect the distribution of males and females across the funds as well: funds that have a population that mainly consists of females mostly invest in the female mortality bonds and funds with male populations mostly buy male bonds in optimum. However, differences in the optimal number of bonds between funds that solely differ across gender exist. These differences can be explained from the fact that males have lower experienced death coefficients, as shown in figure 7.2. Hence, male pension fund participants have better survival probabilities than female participants, relative to the survival probabilities of the total male and female population respectively. Therefore, to hedge the liabilities of males, more mortality bonds are needed.

In case the risk parameter increases, and therefore the price of the bonds as well, the optimal number of bonds decreases. Apparently in case \( \xi \) increases the
hedge becomes too expensive in comparison with the risk that is covered, and a smaller hedge is better. The decrease is heaviest at the bonds that are based on the young cohorts: these bonds have the largest volatility in comparison with the expected value and are therefore relatively expensive in case of a large value of the risk parameter $\xi$.

By taking the optimal bond positions the costs can be reduced substantially. Table 7.6 shows the relative difference of the optimal bond allocations with the costs if no investment takes place, i.e., the costs given in table 7.4. In case it is optimal not to invest at all, the relative difference is set to 0%. The last column of table 7.6 displays the highest value of $\xi$ for which it is optimal to invest in the mortality bonds.

As can be deduced from table 7.6, improvements are larger for the funds with a large population than for the funds with a small population. This can be explained from the fact that for large funds the number of deaths is less volatile than for small funds (law of large numbers). Hence, the amount of pension payments will have a very high correlation with the payoff of the mortality bonds. For small funds, this correlation will be smaller, since stochastic deviations are more likely to occur. In the standardized method the NSA is used for this fact.

Differences between funds with young populations and funds with old populations exist as well: for the young populations the costs can be reduced more than for the old fund populations. This stems from the fact that young populations are more exposed to future mortality trend uncertainty, reflected in the standardized method by the TSO part. By taking a position in mortality bonds, the future mortality trend can (partially) be hedged. Therefore, pension funds with young populations can profit more from an investment in mortality bonds.

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond</td>
<td>young</td>
<td>large</td>
<td>small</td>
<td>young</td>
<td>large</td>
<td>small</td>
<td>young</td>
<td>large</td>
</tr>
<tr>
<td>1</td>
<td>14,689</td>
<td>19,502</td>
<td>79</td>
<td>209</td>
<td>431</td>
<td>1,112</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>68,348</td>
<td>147,691</td>
<td>510</td>
<td>771</td>
<td>6,749</td>
<td>9,340</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>89,067</td>
<td>36,522</td>
<td>295</td>
<td>627</td>
<td>27,764</td>
<td>32,050</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>20,130</td>
<td>45,684</td>
<td>0</td>
<td>0</td>
<td>76,503</td>
<td>171,798</td>
<td>584</td>
<td>996</td>
</tr>
<tr>
<td>5</td>
<td>5,339</td>
<td>10,035</td>
<td>0</td>
<td>0</td>
<td>156,110</td>
<td>331,074</td>
<td>2,423</td>
<td>2,869</td>
</tr>
<tr>
<td>6</td>
<td>1,982</td>
<td>6,861</td>
<td>0</td>
<td>0</td>
<td>214,067</td>
<td>422,911</td>
<td>1,541</td>
<td>2,385</td>
</tr>
<tr>
<td>7</td>
<td>16,531</td>
<td>8,375</td>
<td>155</td>
<td>47</td>
<td>699</td>
<td>380</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>93,639</td>
<td>53,590</td>
<td>535</td>
<td>436</td>
<td>9,052</td>
<td>3661</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>27,835</td>
<td>71,006</td>
<td>453</td>
<td>275</td>
<td>25,314</td>
<td>21,216</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>94,888</td>
<td>17,655</td>
<td>0</td>
<td>0</td>
<td>122,830</td>
<td>63,973</td>
<td>796</td>
<td>548</td>
</tr>
<tr>
<td>11</td>
<td>11,312</td>
<td>2,833</td>
<td>0</td>
<td>0</td>
<td>295,483</td>
<td>165,646</td>
<td>2,497</td>
<td>1,800</td>
</tr>
<tr>
<td>12</td>
<td>4,708</td>
<td>1,579</td>
<td>0</td>
<td>0</td>
<td>517,830</td>
<td>225,025</td>
<td>2,749</td>
<td>2,006</td>
</tr>
</tbody>
</table>

Table 7.5: Optimal bond allocation in case the risk parameter equals zero.
CHAPTER 7. RESULTS

<table>
<thead>
<tr>
<th>fund</th>
<th>$\xi = 0$</th>
<th>$\xi = 0.1$</th>
<th>$\xi = 0.5$</th>
<th>$\xi = 1$</th>
<th>$\xi = 1.5$</th>
<th>$\xi^{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.12%</td>
<td>6.47%</td>
<td>3.32%</td>
<td>0.76%</td>
<td>0%</td>
<td>1.11</td>
</tr>
<tr>
<td>2</td>
<td>7.55%</td>
<td>6.93%</td>
<td>3.92%</td>
<td>0.93%</td>
<td>0%</td>
<td>1.23</td>
</tr>
<tr>
<td>3</td>
<td>5.76%</td>
<td>5.27%</td>
<td>2.39%</td>
<td>0%</td>
<td>0%</td>
<td>0.94</td>
</tr>
<tr>
<td>4</td>
<td>6.24%</td>
<td>5.63%</td>
<td>2.88%</td>
<td>0.12%</td>
<td>0%</td>
<td>1.03</td>
</tr>
<tr>
<td>5</td>
<td>4.80%</td>
<td>3.97%</td>
<td>0.37%</td>
<td>0%</td>
<td>0%</td>
<td>0.60</td>
</tr>
<tr>
<td>6</td>
<td>5.01%</td>
<td>4.27%</td>
<td>0.65%</td>
<td>0%</td>
<td>0%</td>
<td>0.67</td>
</tr>
<tr>
<td>7</td>
<td>3.17%</td>
<td>2.29%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0.31</td>
</tr>
<tr>
<td>8</td>
<td>3.52%</td>
<td>2.82%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0.38</td>
</tr>
</tbody>
</table>

Table 7.6: Relative differences of the costs of the optimal bond allocations with the costs if no investment takes place, for several values of the risk parameter, and maximal value of the risk parameter for which it is optimal to invest

Furthermore, the difference between genders attracts attention: funds with a population that mainly consists of males can reduce their costs more than the funds of which females form the majority of the fund population. This can be explained from figures 5.4 and 5.5. These figures show that the changes in female mortality are less volatile than the changes in male mortality. Hence, pension funds that have mostly males in their population are exposed to more longevity risk, and therefore these funds can benefit more from a hedge with mortality bonds.

The last column of table 7.6 displays the maximal value of the risk parameter at which investment takes place for each fund. The distribution of this maximal value at the different funds is in line with the distribution of the relative differences: it is largest for the fund with a young, large, and male population, since this fund can reduce its costs the most. The maximal values are at first sight not very high. However, one should keep in mind that in case the risk parameter equals one, the risk premium on the bonds is high since it equals the standard deviation of the cash flows of the bond, which is around 10% for each bond.

7.4 Conclusions

In this chapter results regarding the amounts of capital that is needed to sufficiently fund the current pension rights have been derived for both the standardized method and an internal method. It can be concluded that the results for the internal method are associated with lower costs than the standardized method in case the risk parameter $\xi$ is smaller than the values reported in the last column of table 7.6. The differences are largest for funds that have young populations. Size and gender also matter: large pension funds can reduce their
costs substantially more than small pension funds, and funds that have mainly males in their populations can save more capital than funds of which the majority of the fund population consists of males. In case the risk parameter equals zero, the cost reduction is in the range of 3.17% to 7.55% compared to the internal method without investment; the cost reduction in comparison with the standardized method is even larger. For positive risk parameters the amounts of capital that can be saved are somewhat lower, but are still substantial for values of the risk parameter smaller than one.
Chapter 8

Conclusions and recommendations

This chapter concludes the analysis of this thesis. First, the results that were derived in the analysis will be evaluated and will be used to answer the research question. After this, some possibilities to extend the analysis will be discussed.

8.1 Conclusions

The analysis of this thesis concentrated on the current accrued pension rights of some fictitious pension funds, in particular on the amount of capital that is needed to sufficiently fund these accrued rights. In the new financial assessment framework (FTK) it is described how Dutch pension funds must determine this amount of capital: the amount of capital must be sufficient to match the liabilities with a probability level of 75%. Pension funds may use a standardized method or develop their own internal method to determine this amount of capital.

Under the standardized method, the amount of capital is determined as the most realistic expectation of future cash flows plus an extra loading. In section 7.2 the amount of capital was determined for each pension fund. For the ‘older’ funds this loading was between 2% and 3.4%; for the younger funds this loading was larger: between 6.2% and 8.4%. Since a total provision often amounts billions of euros, these loadings are very substantial. Therefore, the use of an internal method was investigated.

This internal method consisted of an investment in hypothetical mortality bonds and to hold an amount of capital that is needed to fund the difference between liabilities and the payoffs of the mortality bonds. The pricing principle that was used to price these bonds involved a non-negative risk parameter $\xi$. In case no investment in bonds was assumed but the internal method was still used, there were substantial differences in the amounts of capital needed for each funds. These differences ranged from 0.46% up to 2.25%. By investing in
the mortality bonds even less capital was needed, at least for values of $\xi$ that are below the values reported in the last column of table 7.6. The costs that can be saved in comparison with the internal method without investment can amount up to 7.5\% in case the risk parameter equals zero. For values of $\xi$ greater than zero but smaller than one the cost reduction is somewhat smaller but still substantial. The largest cost savings are found at the younger and larger funds. The future mortality improvements of young persons is much more uncertain than the mortality improvements of older persons, which implies that a hedge for young persons has more use. Larger pension funds can reduce their costs more than smaller funds, since the larger the fund population is, the higher is the correlation between the payoff of the mortality bonds and the pension payments, due to the law of large numbers. Furthermore, since the mortality changes for males are more volatile than the mortality changes for females, the greater the proportion of males in the fund population is, the greater the cost reduction is.

Hence it can be concluded that the standardized method proposed by the FTK is an expensive way of dealing with longevity risk. If the mortality bonds that are discussed in this thesis would really exist, and the risk loadings of these bonds are not too high, it would be wise for pension funds to take a position in these mortality bonds, since this may lead to a significant cost reduction.

8.2 Recommendations for further research

To derive the results of this thesis, many choices had to be made. Some of these choices were concerned with model selection; others were needed to create a setting in which an answer to the research question could be formulated. However, other choices could have been made that probably would have lead to different results. Therefore, this section discusses some alternatives that could have been chosen and that can be used in further research.

The analysis of this thesis concentrated solely on the current pension rights, and these rights were assumed not to change over the years. In reality however, pensions rights are typically subject to future accrual, due to future incomes, and are often (conditionally) indexated. Hence, it would be useful to include indexation and future accrual into the analysis, since this is important for the continuity of the pension fund. Among other things, future accrual will then depend on resignation probabilities.

To analyze and compare the standardized method and the internal method, only the costs were used as a criterion. However, pension funds can be considered to be risk averse, which implies that likely they will be prepared to spend a small extra amount of capital in order lose some of the risks they are exposed to. Hence, it would be wise to incorporate the remaining longevity risk as a criterion as well.

In this thesis several mortality products were discussed, although only mortality bonds were used in the analysis. Therefore, it would be interesting to construct other mortality-linked products and to investigate the use of these
products when minimizing the amount of capital needed. However, also when other products are incorporated into the analysis, it still would be wise to include the remaining risk as a criterion as well.

Furthermore, the choice of pricing principle was somewhat arbitrary, since a wide range of pricing principles exists. For instance, several authors, such as Cox and Lin (2004) [5] suggest the use of the Wang transform to price mortality-linked products. Using this transform implies that the cumulative distribution function of the product to be priced is transformed into a new cumulative distribution function by making use of a distortion operator. The mean of this new cumulative distribution function gives a risk-adjusted premium for the product. For more information, one is referred to Wang (2000) [13].

Finally, the stochastic mortality model that was used in this thesis, the Lee-Carter model, can be extended into many different forms. For instance, Lee and Carter (1992) [10] included a dummy variable into the model to account for the Spanish flu pandemic that had a great impact on their data. Furthermore, whereas the Lee-Carter model estimates the parameters using solely the first set of vectors that result from the SVD, Renshaw and Haberman (2002) [12] suggest the use of several of these sets. A totally different approach is the one by Brouhns et al. (2002) [3]. They propose a model in which the number of deaths $D_{x,t}$ has a Poisson distribution with parameter $E_{x,t} \mu_{x,t}$. To model $\mu_{x,t}$ they suggest the same setting as the Lee-Carter model, i.e., relation (5.6), although they determine the parameters by applying maximum likelihood. The last possible extension that is discussed here would be to investigate the correlation of changes in the mortality index for males and females. Likely, these changes are positively correlated, which can have a big impact on the liabilities of pension funds.
Appendix A

The Singular Value Decomposition

Suppose matrix $A$ is a $m \times n$ real matrix with $m > n$. If the matrix of eigenvectors $P$ is not invertible, no eigen decomposition of $A$ can be derived. However, it is possible to derive the so-called Singular Value Decomposition (SVD), which implies that

$$A = UDV^T. \quad (A.1)$$

In equation A.1 $U$ is a $m \times n$ matrix, $D$ a $n \times n$ matrix, and $V$ a $n \times n$ matrix. Furthermore, $D$ is a diagonal matrix that contains the singular values of $A$ decreasing by absolute value. The matrices $U$ and $V$ have orthogonal columns which implies that

$$U^TU = 1, \quad (A.2)$$

and

$$V^TV = 1. \quad (A.3)$$

From A.1 it follows that the matrix $A$ can exactly be replicated by

$$\sum_{i=1}^{n} U(i) D(i,i) V^T(i), \quad (A.4)$$

where $U(i)$ denotes the $i$th column of $U$ and $V^T(i)$ the $i$th row of $V^T$. However, since the absolute values of the elements of $D$ are in decreasing order and often the first element has by far the greatest absolute value, $A$ can very well be fitted by just taking the first column of $U$, the first diagonal element of $D$, and the first row of $V^T$. 
APPENDIX A. THE SINGULAR VALUE DECOMPOSITION
Appendix B

Results for $\alpha, \beta$ and $\kappa$

<table>
<thead>
<tr>
<th>age</th>
<th>$\alpha$ (males)</th>
<th>$\beta$ (males)</th>
<th>$\alpha$ (females)</th>
<th>$\beta$ (females)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25–29</td>
<td>-7.07224</td>
<td>0.10316</td>
<td>-7.69887</td>
<td>0.08379</td>
</tr>
<tr>
<td>30–34</td>
<td>-6.92277</td>
<td>0.08297</td>
<td>-7.35081</td>
<td>0.08237</td>
</tr>
<tr>
<td>35–39</td>
<td>-5.97877</td>
<td>0.08844</td>
<td>-6.94344</td>
<td>0.07428</td>
</tr>
<tr>
<td>40–44</td>
<td>-6.12081</td>
<td>0.08929</td>
<td>-6.47495</td>
<td>0.05865</td>
</tr>
<tr>
<td>45–49</td>
<td>-5.90470</td>
<td>0.09848</td>
<td>-5.99236</td>
<td>0.05017</td>
</tr>
<tr>
<td>50–54</td>
<td>-5.05166</td>
<td>0.10778</td>
<td>-5.56748</td>
<td>0.05190</td>
</tr>
<tr>
<td>55–59</td>
<td>-4.52247</td>
<td>0.10076</td>
<td>-5.14195</td>
<td>0.05542</td>
</tr>
<tr>
<td>60–64</td>
<td>-4.01429</td>
<td>0.08746</td>
<td>-4.67328</td>
<td>0.06644</td>
</tr>
<tr>
<td>65–69</td>
<td>-3.53233</td>
<td>0.06618</td>
<td>-4.15449</td>
<td>0.08187</td>
</tr>
<tr>
<td>70–74</td>
<td>-3.04714</td>
<td>0.04590</td>
<td>-3.59503</td>
<td>0.09120</td>
</tr>
<tr>
<td>75–79</td>
<td>-2.56967</td>
<td>0.03420</td>
<td>-3.01031</td>
<td>0.09190</td>
</tr>
<tr>
<td>80–84</td>
<td>-2.11094</td>
<td>0.02530</td>
<td>-2.42980</td>
<td>0.07957</td>
</tr>
<tr>
<td>85–89</td>
<td>-1.66498</td>
<td>0.02422</td>
<td>-1.88849</td>
<td>0.06227</td>
</tr>
<tr>
<td>90–94</td>
<td>-1.25973</td>
<td>0.02190</td>
<td>-1.41712</td>
<td>0.04299</td>
</tr>
<tr>
<td>95+</td>
<td>-0.85120</td>
<td>0.02397</td>
<td>-0.97620</td>
<td>0.02719</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>males</th>
<th></th>
<th>females</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>sum</td>
<td>1.00000</td>
<td>1.00000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.1: Estimated age-dependent parameters of the Lee-Carter model
<table>
<thead>
<tr>
<th>year</th>
<th>males</th>
<th>females</th>
<th>year</th>
<th>males</th>
<th>females</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>0.6944</td>
<td>3.5849</td>
<td>1982</td>
<td>0.5478</td>
<td>-0.9145</td>
</tr>
<tr>
<td>1961</td>
<td>0.5230</td>
<td>3.1616</td>
<td>1983</td>
<td>0.3621</td>
<td>-1.2290</td>
</tr>
<tr>
<td>1962</td>
<td>1.4631</td>
<td>3.3936</td>
<td>1984</td>
<td>0.2655</td>
<td>-1.2261</td>
</tr>
<tr>
<td>1963</td>
<td>1.6122</td>
<td>3.2745</td>
<td>1985</td>
<td>0.3513</td>
<td>-1.1647</td>
</tr>
<tr>
<td>1964</td>
<td>0.9054</td>
<td>2.5037</td>
<td>1986</td>
<td>0.3693</td>
<td>-1.1232</td>
</tr>
<tr>
<td>1965</td>
<td>1.4537</td>
<td>2.7571</td>
<td>1987</td>
<td>-0.3802</td>
<td>-1.8119</td>
</tr>
<tr>
<td>1966</td>
<td>1.4489</td>
<td>2.7867</td>
<td>1988</td>
<td>-0.5242</td>
<td>-1.8008</td>
</tr>
<tr>
<td>1967</td>
<td>1.2240</td>
<td>2.1666</td>
<td>1989</td>
<td>-0.2952</td>
<td>-1.4932</td>
</tr>
<tr>
<td>1968</td>
<td>1.8276</td>
<td>2.4582</td>
<td>1990</td>
<td>-0.6809</td>
<td>-1.6999</td>
</tr>
<tr>
<td>1969</td>
<td>1.8472</td>
<td>2.5541</td>
<td>1991</td>
<td>-0.9340</td>
<td>-1.7515</td>
</tr>
<tr>
<td>1970</td>
<td>2.0492</td>
<td>2.3626</td>
<td>1992</td>
<td>-1.3240</td>
<td>-1.9709</td>
</tr>
<tr>
<td>1971</td>
<td>1.8895</td>
<td>2.2068</td>
<td>1993</td>
<td>-0.6094</td>
<td>-1.3380</td>
</tr>
<tr>
<td>1972</td>
<td>2.2943</td>
<td>2.2130</td>
<td>1994</td>
<td>-1.6076</td>
<td>-1.8373</td>
</tr>
<tr>
<td>1973</td>
<td>1.6587</td>
<td>1.5733</td>
<td>1995</td>
<td>-1.5790</td>
<td>-1.8830</td>
</tr>
<tr>
<td>1974</td>
<td>1.1896</td>
<td>1.0930</td>
<td>1996</td>
<td>-1.6429</td>
<td>-1.8941</td>
</tr>
<tr>
<td>1975</td>
<td>1.8916</td>
<td>1.1592</td>
<td>1997</td>
<td>-2.4350</td>
<td>-2.1193</td>
</tr>
<tr>
<td>1976</td>
<td>1.8362</td>
<td>0.8329</td>
<td>1998</td>
<td>-2.5517</td>
<td>-2.1957</td>
</tr>
<tr>
<td>1977</td>
<td>0.8355</td>
<td>0.0483</td>
<td>1999</td>
<td>-2.6868</td>
<td>-1.9647</td>
</tr>
<tr>
<td>1978</td>
<td>1.2521</td>
<td>0.1146</td>
<td>2000</td>
<td>-3.0671</td>
<td>-2.1160</td>
</tr>
<tr>
<td>1979</td>
<td>0.6782</td>
<td>-0.3669</td>
<td>2001</td>
<td>-3.5454</td>
<td>-2.1897</td>
</tr>
<tr>
<td>1980</td>
<td>0.7368</td>
<td>-0.6414</td>
<td>2002</td>
<td>-3.6984</td>
<td>-2.0534</td>
</tr>
<tr>
<td>1981</td>
<td>0.5767</td>
<td>-0.7719</td>
<td>2003</td>
<td>-4.0552</td>
<td>-2.2575</td>
</tr>
</tbody>
</table>

Table B.2: Re-estimated values of the mortality index
Appendix C

Variance computation

In the text it was argued that the computation of the variance of the mortality index $\kappa$ requires the covariances of $\Delta_t$. The model that is used to describe changes in $\Delta_t$ reads

$$\Delta_t = \mu + \gamma_1 (\Delta_{t-1} - \mu) + \gamma_2 (\Delta_{t-2} - \mu) + \epsilon_t,$$

Define

$$V = \text{var} (\Delta_t),$$
$$C = \text{cov} (\Delta_t, \Delta_{t-1}),$$
$$\sigma^2 = \text{var} (\epsilon_t).$$

(C.1)

Since $\Delta_t$ is stationary over time and $\epsilon_t$ is white noise, it follows that

$$V = \gamma_1^2 V + \gamma_2^2 V + 2 \gamma_1 \gamma_2 C + \sigma^2,$$
$$C = \gamma_1 V + \gamma_2 C.$$  

(C.2)

This is a system of two equations and two unknowns. Straightforward computations lead to these solutions for $V$ and $C$:

$$V = \frac{1}{1-\gamma_1^2-\gamma_2^2} \left( \frac{2\gamma_1 \gamma_2 \sigma^2}{1-\gamma_2-2\gamma_1 \gamma_2} + \sigma^2 \right),$$
$$C = \frac{\gamma_1 \sigma^2}{(1-\gamma_2-2\gamma_1 \gamma_2)}.$$  

(C.3)

Next, a formulation for the covariance between $\Delta_t$ and $\Delta_{t-n}$ for $n \geq 2$ is needed:

$$\text{cov} (\Delta_t, \Delta_{t-n}) = \text{cov} (\mu + \gamma_1 (\Delta_{t-1} - \mu) + \gamma_2 (\Delta_{t-2} - \mu) + \epsilon_t, \Delta_{t-n})$$
$$= \text{cov} (\gamma_1 \Delta_{t-1} + \gamma_2 \Delta_{t-2}, \Delta_{t-n})$$
$$= \text{cov} (\gamma_1 \Delta_{t-1}, \Delta_{t-n}) + \text{cov} (\gamma_2 \Delta_{t-2}, \Delta_{t-n}) \gamma_2 \Delta_{t-2}, \Delta_{t-n}$$
$$= \gamma_1 \text{cov} (\Delta_t, \Delta_{t-n+1}) + \gamma_2 \text{cov} (\Delta_t, \Delta_{t-n+2}).$$  

(C.4)
where the last equality follows from the fact that \( \Delta_t \) is stationary. Hence, all covariances can be computed recursively using C.2 and C.4.
Appendix D

Dutch pension terms and their English translation

In this thesis English pension terminology is used. Many of the used pension terms are originally Dutch and are translated into English. The bulk of the translations contained in this thesis are based upon the consultation document “Financial assessment framework consultation document” ([7] DNB, 2004). However, sometimes readers that are used to the Dutch terms might not quite understand the English versions in this thesis. For instance, it could be that they are used to different translations. Therefore, this appendix gives an overview of the used terminology as well as the Dutch counterpart.
## Table D.1: English pension term and corresponding Dutch translations

<table>
<thead>
<tr>
<th>English term</th>
<th>Dutch term</th>
</tr>
</thead>
<tbody>
<tr>
<td>accrual percentage</td>
<td>opbouwpercentage</td>
</tr>
<tr>
<td>average pay plan</td>
<td>middelloonregeling</td>
</tr>
<tr>
<td>basic provisions</td>
<td>basisvoorzieningen</td>
</tr>
<tr>
<td>continuity analysis</td>
<td>continuïteitsanalyse</td>
</tr>
<tr>
<td>deferred</td>
<td>slaper, gewezen deelnemer</td>
</tr>
<tr>
<td>Defined Benefit scheme</td>
<td>toegezegdpensioenregeling</td>
</tr>
<tr>
<td>Defined Contribution scheme</td>
<td>beschikbare premie regeling</td>
</tr>
<tr>
<td>disability pension</td>
<td>arbeidsongeschiktheidspensioen</td>
</tr>
<tr>
<td>equity capital</td>
<td>eigen vermogen</td>
</tr>
<tr>
<td>experienced death</td>
<td>ervaringssterfte</td>
</tr>
<tr>
<td>final pay plan</td>
<td>eindloonregeling</td>
</tr>
<tr>
<td>financial assessment framework</td>
<td>financieel toetsingskader (FTK)</td>
</tr>
<tr>
<td>freely disposable assets</td>
<td>vrij beschikbare activa</td>
</tr>
<tr>
<td>funding ratio</td>
<td>dekkingsgraad</td>
</tr>
<tr>
<td>loan capital</td>
<td>vreemd vermogen</td>
</tr>
<tr>
<td>offset</td>
<td>franchise</td>
</tr>
<tr>
<td>old age pension</td>
<td>ouderdomspensioen</td>
</tr>
<tr>
<td>paid up pension commitments</td>
<td>premievrije aanspraken</td>
</tr>
<tr>
<td>pension agreement</td>
<td>pensioenovereenkomst</td>
</tr>
<tr>
<td>pension benefit</td>
<td>pensioenmijtkening</td>
</tr>
<tr>
<td>pension commitment</td>
<td>pensioentoezegging</td>
</tr>
<tr>
<td>pension commitment</td>
<td>pensioen aanspraak</td>
</tr>
<tr>
<td>pension scheme</td>
<td>pensioenregeling</td>
</tr>
<tr>
<td>pensionable salary</td>
<td>pensioengrondslag</td>
</tr>
<tr>
<td>pillar</td>
<td>pijler</td>
</tr>
<tr>
<td>present value</td>
<td>contante waarde</td>
</tr>
<tr>
<td>provision for pension commitments</td>
<td>voorziening pensioenverplichtingen (VPV)</td>
</tr>
<tr>
<td>realistic value</td>
<td>actuele waarde</td>
</tr>
<tr>
<td>retirement age</td>
<td>pensioenleeftijd</td>
</tr>
<tr>
<td>society of actuaries</td>
<td>Actuarieel Genootschap (AG)</td>
</tr>
<tr>
<td>solvency test</td>
<td>solvabiliteitstoets</td>
</tr>
<tr>
<td>survivor’s pension</td>
<td>nabestaandenpensioen</td>
</tr>
<tr>
<td>total foreseeable liabilities</td>
<td>totaal aan voorzienbare verplichtingen</td>
</tr>
<tr>
<td>transfer of value</td>
<td>waardeoverdracht</td>
</tr>
<tr>
<td>(un)conditional indexation</td>
<td>(on)voorwaardelijke indexatie</td>
</tr>
<tr>
<td>years of service</td>
<td>dienstjaren</td>
</tr>
</tbody>
</table>
Appendix E

M-files

E.1 fundsandkappa.m

% This m-file simulates N trajectories for the mortality index kappa by using
% the parameters of the stochastic mortality model.
% These trajectories are for years 2004 to 2120.
% With these trajectories pability distributions are derived.
% Four different pability distributions are computed:
% For males and females;
% And for the total population as well as for an insured population
% For years 2006 and further also the cumulative survival distributions are
computed.
% These cumulative distributions are used to determine the payo¤s of the
mortality bonds.
% Prices of these bonds are computed using the mean and variance of the
payoffs of the bonds.
% Also within each trajectory, for each pension fund a fund simulations is
executed.
% This fund simulations is used to determine the liabilities
% for each fund for every year starting 2006.

N=1000;
variables; % this m-file contains the parameters of the stochastic
mortality model
	simerrom=norminv(rand(N,117),0,sigmam); % random values to sim-
ulate trajectories
	 simerrorf=norminv(rand(N,117),0,sigmaf);

cumulsurvivalm=zeros(61,N);
cumulsurvivalf=zeros(61,N);
ptotm(116:250,1:N)=1;
pinsm(116:250,1:N)=1; %140 has death probability 1
ptotf(116:250,1:N)=1;
pinsf(116:250,1:N)=1;
productpayoff=zeros(12,115,N);
liabilities=zeros(N,115,8);
for j=1:N
fundnumbers; % this m-file contains the fundpopulations
delta1m(j)=-0.3568; %value of 2003
delta2m(j)=-0.1530; %value of 2002
delta1f(j)=-0.2040; %value of 2003
delta2f(j)=0.1362; %value of 2002
for i=1:15
  lnmum1(j,i)=logmum(i);
  lnmuf1(j,i)=logmuf(i);
end
for k=2004:2120
deltam(j)=mum+gamma1m*(delta1m(j)-mum)+gamma2m*(delta2m(j)-mum)+simerrorm(j,k-2003);
deltaf(j)=muf+gamma1f*(delta1f(j)-muf)+gamma2f*(delta2f(j)-muf)+simerrorf(j,k-2003);
for i=1:15
  lnmum(j,i)=lnmum1(j,i)+betam(i)*deltam(j);
  forcem(j,i)=exp(lnmum(j,i));
  lnmuf(j,i)=lnmuf1(j,i)+betaf(i)*deltaf(j);
  forcef(j,i)=exp(lnmuf(j,i));
end
  lnmum1=lnmum;
delta2m(j)=delta1m(j);
delta1m(j)=deltam(j);
lnmuf1=lnmuf;
delta2f(j)=delta1f(j);
delta1f(j)=deltaf(j);
for i=27:5:87 %interpolation to derive probabilities for every age
  qtotm(i-24,j)=1-exp(-forcem(j,(i-2)/5-4)); %27 to 87
  qtotm(i-23,j)=1-exp(-0.8*forcem(j,(i-2)/5-4)-0.2*forcem(j,(i-2)/5-3)); %28 to 88
  qtotm(i-22,j)=1-exp(-0.6*forcem(j,(i-2)/5-4)-0.4*forcem(j,(i-2)/5-3)); %29 to 89
  qtotm(i-21,j)=1-exp(-0.4*forcem(j,(i-2)/5-4)-0.6*forcem(j,(i-2)/5-3)); %30 to 90
  qtotm(i-20,j)=1-exp(-0.2*forcem(j,(i-2)/5-4)-0.8*forcem(j,(i-2)/5-3)); %31 to 91
  qtotf(i-24,j)=1-exp(-forcef(j,(i-2)/5-4)); %27 to 87
  qtotf(i-23,j)=1-exp(-0.8*forcef(j,(i-2)/5-4)-0.2*forcef(j,(i-2)/5-3)); %28 to 88
  qtotf(i-22,j)=1-exp(-0.6*forcef(j,(i-2)/5-4)-0.4*forcef(j,(i-2)/5-3)); %29 to 89
  qtotf(i-21,j)=1-exp(-0.4*forcef(j,(i-2)/5-4)-0.6*forcef(j,(i-2)/5-3)); %30 to 90
E.1. FUNDSANDKAPPA.M

qtotf(i-20,j)=1-exp(-0.2*forcef(j,(i-2)/5-4)-0.8*forcef(j,(i-2)/5-3)); %31 to 91
end
for i=25:26 %interpolation to derive probabilities for every age
qtotm(i-24,j)=1-exp(-(1-(i-27)/5)*forcem(j,1)+((i-27)/5)*forcem(j,2));
qtotf(i-24,j)=1-exp(-(1-(i-27)/5)*forcef(j,1)+((i-27)/5)*forcef(j,2));
end
for i=92:139 %interpolation to derive probabilities for every age
qtotm(i-24,j)=1-exp(-(1-(i-92)/8)*forcem(j,14)-((i-92)/8)*forcem(j,15));
qtotf(i-24,j)=1-exp(-(1-(i-92)/8)*forcef(j,14)-((i-92)/8)*forcef(j,15));
end
qinsm(:,j)=cm.*qtotm(:,j);
qinsf(:,j)=cf.*qtotf(:,j);
for i=25:10:75
if k==2006
cumulsurvivalm(i-24,j)=1-qtotm(i-24,j);
cumulsurvivalf(i-24,j)=1-qtotf(i-24,j);
elseif k>2006
cumulsurvivalm(i-24,j)=cumsurvivalm1(i-24,j)*(1-qtotm(i-24+k-2006,j));
cumulsurvivalf(i-24,j)=cumsurvivalf1(i-24,j)*(1-qtotf(i-24+k-2006,j));
end
end
for i=1:6
if k>2005
if 10*i+k-1991>64 %payments if age>= 65
productpayoff(i,k-2005,j)=1000*cumulsurvivalm(10*(i-1)+1,j)*discount(k-2005);
productpayoff(i+6,k-2005,j)=1000*cumulsurvivalf(10*(i-1)+1,j)*discount(k-2005);
end
end
end
cumulsurvivalm1=cumulsurvivalm;
cumulsurvivalf1=cumulsurvivalf;
if k>2005
payments=zeros(61,1);
if k>=2006
payments=bene…t;
else
payments(2047-k:61)=bene…t(2047-k:61);
end
end
end
numberm=mnumbers;
numberf=fnumbers;
for fund=1:8
qm(1:61,fund)=[qinsm(1+k-2006:61+k-2006,j)];
qf(1:61,fund)=[qinsf(1+k-2006:61+k-2006,j)];
end
end
numberm=numberm-binornd(numberm,qm);
numb
m=numberm;
numberf=numberf-binornd(numberf,qf);
numb
f=numberf
for fund=1:8
liabilities(j,k-2005,fund)=discount(k-2005)*payments*(numberm(1:61,fund)+numberf(1:61,fund));
end
end
end
end
for i=1:12;
for j=1:50
pricematrix(i,j)=mean(sum(productpayoff(i,:,:))+(j-1)*0.1*std(productpayoff(i,:,:)));
end
end

E.2 Costfunction.m

% This function computes the costs for fund f, in case the pricing principle
% is principle p, and in case the coefficient vector is omega.
% N denotes the number of kappa trajectories (1000)
% First the portfolio payoff is computed.
% Next the needed capital is computed
% After this the costs of buying omega assets is calculated, as well
% as the capital needed to satisfy the 75% probability level.
% Finally the total costs are determined.

function [costs] = costfunction(omega)
load simresults
portfoliopayooff=zeros(N,115);
fun
liabilities=liabilities(:,:,f);
largevalue=10000000000000;
for j=1:12
if omega(j)<0
  costs=largevalue; % no short selling
end
end
if costs<largevalue
  for i=1:N
    portfoliopayooff(i,:)=omega*productpayoff(i,:,:);
  end
  capitalneed=fundliabilities-assets;
  prices=pricematrix(:,p);
  productcost=omega*prices;
  capital=max(sum(capitalneed,2);0); % no negative capital
end
end
sortedcapital=sort(capital);
capitalcost=sortedcapital(ceil(0.75*N));
costs=productcost+capitalcost;
end
Appendix F

Optimal bond allocations

In this appendix the optimal bond allocations for the fictitious pension funds are displayed for three different values of the risk parameter $\xi$. For a correct understanding of these results, first the characteristics of the twelve mortality bonds and the eight funds are given in table F.1.

<table>
<thead>
<tr>
<th>bond</th>
<th>gender</th>
<th>age in 2006</th>
<th>fund</th>
<th>average age</th>
<th>number of participants</th>
<th>males/ females</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>male</td>
<td>25</td>
<td>1</td>
<td>36.8</td>
<td>75,000</td>
<td>40/60</td>
</tr>
<tr>
<td>2</td>
<td>male</td>
<td>35</td>
<td>2</td>
<td>36.8</td>
<td>75,000</td>
<td>60/40</td>
</tr>
<tr>
<td>3</td>
<td>male</td>
<td>45</td>
<td>3</td>
<td>33.7</td>
<td>500</td>
<td>30/70</td>
</tr>
<tr>
<td>4</td>
<td>male</td>
<td>55</td>
<td>4</td>
<td>33.7</td>
<td>500</td>
<td>70/30</td>
</tr>
<tr>
<td>5</td>
<td>male</td>
<td>65</td>
<td>5</td>
<td>66.6</td>
<td>75,000</td>
<td>33/67</td>
</tr>
<tr>
<td>6</td>
<td>male</td>
<td>75</td>
<td>6</td>
<td>66.6</td>
<td>75,000</td>
<td>67/33</td>
</tr>
<tr>
<td>7</td>
<td>female</td>
<td>25</td>
<td>7</td>
<td>68.9</td>
<td>500</td>
<td>40/60</td>
</tr>
<tr>
<td>8</td>
<td>female</td>
<td>35</td>
<td>8</td>
<td>68.9</td>
<td>500</td>
<td>60/40</td>
</tr>
<tr>
<td>9</td>
<td>female</td>
<td>45</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>female</td>
<td>55</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>female</td>
<td>65</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>female</td>
<td>75</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table F.1: Characteristics of the mortality bonds and the fund populations
APPENDIX F. OPTIMAL BOND ALLOCATIONS

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>13,022</td>
<td>978</td>
<td>79</td>
<td>209</td>
<td>431</td>
<td>1,112</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>64,562</td>
<td>8,781</td>
<td>510</td>
<td>771</td>
<td>6,749</td>
<td>9,340</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>37,021</td>
<td>30,260</td>
<td>295</td>
<td>627</td>
<td>27,764</td>
<td>32,050</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>19,264</td>
<td>158,977</td>
<td>0</td>
<td>0</td>
<td>76,503</td>
<td>171,798</td>
<td>584</td>
<td>996</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>5,218</td>
<td>319,821</td>
<td>0</td>
<td>0</td>
<td>156,110</td>
<td>331,074</td>
<td>2,423</td>
<td>2,869</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1,933</td>
<td>414,256</td>
<td>0</td>
<td>0</td>
<td>214,067</td>
<td>422,911</td>
<td>1,541</td>
<td>2,385</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>15,287</td>
<td>339</td>
<td>155</td>
<td>47</td>
<td>699</td>
<td>380</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_8$</td>
<td>86,011</td>
<td>3,389</td>
<td>535</td>
<td>436</td>
<td>9,052</td>
<td>3661</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_9$</td>
<td>90,363</td>
<td>20,457</td>
<td>453</td>
<td>275</td>
<td>25,314</td>
<td>21,216</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_{10}$</td>
<td>25,710</td>
<td>59,189</td>
<td>0</td>
<td>0</td>
<td>122,830</td>
<td>63,973</td>
<td>796</td>
<td>548</td>
</tr>
<tr>
<td>$\omega_{11}$</td>
<td>10,980</td>
<td>159,284</td>
<td>0</td>
<td>0</td>
<td>295,483</td>
<td>165,646</td>
<td>2,497</td>
<td>1,800</td>
</tr>
<tr>
<td>$\omega_{12}$</td>
<td>4,589</td>
<td>219,537</td>
<td>0</td>
<td>0</td>
<td>517,830</td>
<td>225,025</td>
<td>2,749</td>
<td>2,006</td>
</tr>
</tbody>
</table>

Table F.2: Optimal bond allocation in case the risk parameter equals 0.1

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>9,774</td>
<td>711</td>
<td>0</td>
<td>136</td>
<td>12,849</td>
<td>277</td>
<td>0</td>
<td>55</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>52,359</td>
<td>6,955</td>
<td>0</td>
<td>589</td>
<td>113,411</td>
<td>5,097</td>
<td>0</td>
<td>364</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>31,863</td>
<td>26,733</td>
<td>0</td>
<td>513</td>
<td>13,412</td>
<td>22,479</td>
<td>0</td>
<td>158</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>16,732</td>
<td>142,898</td>
<td>475</td>
<td>0</td>
<td>38,112</td>
<td>63,321</td>
<td>812</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>4,799</td>
<td>294,728</td>
<td>2,064</td>
<td>0</td>
<td>8,937</td>
<td>138,976</td>
<td>2,590</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>1,763</td>
<td>376,284</td>
<td>1,371</td>
<td>0</td>
<td>6,129</td>
<td>190,482</td>
<td>2,287</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_7$</td>
<td>11,264</td>
<td>259</td>
<td>0</td>
<td>32</td>
<td>5,812</td>
<td>482</td>
<td>0</td>
<td>107</td>
</tr>
<tr>
<td>$\omega_8$</td>
<td>71,832</td>
<td>2,811</td>
<td>0</td>
<td>314</td>
<td>40,910</td>
<td>6,956</td>
<td>0</td>
<td>405</td>
</tr>
<tr>
<td>$\omega_9$</td>
<td>65,782</td>
<td>14,751</td>
<td>0</td>
<td>201</td>
<td>49,116</td>
<td>17,591</td>
<td>0</td>
<td>311</td>
</tr>
<tr>
<td>$\omega_{10}$</td>
<td>24,690</td>
<td>58,191</td>
<td>716</td>
<td>0</td>
<td>15,349</td>
<td>108,452</td>
<td>482</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_{11}$</td>
<td>9,978</td>
<td>146,764</td>
<td>2,221</td>
<td>0</td>
<td>2,457</td>
<td>260,003</td>
<td>1,537</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_{12}$</td>
<td>4,231</td>
<td>202,116</td>
<td>2,389</td>
<td>0</td>
<td>1,434</td>
<td>465,893</td>
<td>1,784</td>
<td>0</td>
</tr>
</tbody>
</table>

Table F.3: Optimal bond allocation in case the risk parameter equals 0.5
Table F.4: Optimal bond allocation in case the risk parameter equals one

<table>
<thead>
<tr>
<th>fund</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_1)</td>
<td>7,129</td>
<td>540</td>
<td>0</td>
<td>101</td>
<td>9,461</td>
<td>199</td>
<td>0</td>
<td>38</td>
</tr>
<tr>
<td>(\omega_2)</td>
<td>37,624</td>
<td>4,590</td>
<td>0</td>
<td>389</td>
<td>78,356</td>
<td>3,579</td>
<td>0</td>
<td>265</td>
</tr>
<tr>
<td>(\omega_3)</td>
<td>28,722</td>
<td>23,347</td>
<td>0</td>
<td>468</td>
<td>12,261</td>
<td>20,414</td>
<td>0</td>
<td>139</td>
</tr>
<tr>
<td>(\omega_4)</td>
<td>15,139</td>
<td>134,232</td>
<td>456</td>
<td>0</td>
<td>35,234</td>
<td>59,983</td>
<td>757</td>
<td>0</td>
</tr>
<tr>
<td>(\omega_5)</td>
<td>4,431</td>
<td>274,861</td>
<td>2,007</td>
<td>0</td>
<td>8,119</td>
<td>129,210</td>
<td>2,421</td>
<td>0</td>
</tr>
<tr>
<td>(\omega_6)</td>
<td>1,617</td>
<td>311,287</td>
<td>1,282</td>
<td>0</td>
<td>5,736</td>
<td>175,297</td>
<td>1,996</td>
<td>0</td>
</tr>
<tr>
<td>(\omega_7)</td>
<td>7,589</td>
<td>179</td>
<td>0</td>
<td>21</td>
<td>3,779</td>
<td>302</td>
<td>0</td>
<td>69</td>
</tr>
<tr>
<td>(\omega_8)</td>
<td>52,511</td>
<td>2,193</td>
<td>0</td>
<td>243</td>
<td>31,102</td>
<td>5,276</td>
<td>0</td>
<td>302</td>
</tr>
<tr>
<td>(\omega_9)</td>
<td>65,982</td>
<td>14,739</td>
<td>0</td>
<td>188</td>
<td>48,967</td>
<td>17,311</td>
<td>0</td>
<td>379</td>
</tr>
<tr>
<td>(\omega_{10})</td>
<td>19,363</td>
<td>45,489</td>
<td>566</td>
<td>0</td>
<td>12,553</td>
<td>87,110</td>
<td>396</td>
<td>0</td>
</tr>
<tr>
<td>(\omega_{11})</td>
<td>8,823</td>
<td>129,674</td>
<td>1,913</td>
<td>0</td>
<td>2,223</td>
<td>231,299</td>
<td>1,411</td>
<td>0</td>
</tr>
<tr>
<td>(\omega_{12})</td>
<td>4,002</td>
<td>192,171</td>
<td>2,377</td>
<td>0</td>
<td>1,322</td>
<td>438,762</td>
<td>1,624</td>
<td>0</td>
</tr>
</tbody>
</table>
References


