



Limited Farsightedness and Pairwise Stability in Two-Sided Matching Problems

by

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1 Introduction

Two-sided matching problems can be used to represent a variety of interesting social and behavioral problems. Most famously introduced by Gale and Shapley (1962), these problems deal with finding efficient, stable, or "ideal" algorithms or solution concepts to the following problem: one set of agents (classically labeled Women) each have a series of ordered preferences over a second set of agents (classically Men), or else remaining single. Men, in turn, have a series of ordered preferences over women. A matching assigns the members of these sets to pairs with everyone matched to either a member of the opposite set of agents or themselves. The two groups must be paired using these preferences, and with a feasibility constraint that each agent can be assigned to at most one partner of the opposite set, or else remain single.

Finding "ideal" matchings is non-trivial. Gale and Shapley (1962) proposed, as we discuss later, the core as a solution concept (Roth and Sotomayor, 1990). As an alternative, some authors have utilized the von Neumann and Morgenstern stable set (von Neumann and Morgenstern, 1944), which requires that no matching within the set can be reached from (or dominated by) another matching within the set (internal stability), and that from any matching outside the set there exists an individually improving path leading to at least one matching within the set (external stability) (Ehlers, 2007). However, along these dominating paths, it is not necessary to assume that agents can see only actions which cause immediate improvements over their current state, as it is also possible to model problems where some or all agents can see further results of adopting a matching in a certain period (Mauleon, Vannetelbosch, and Vergote, 2011). In practice, heterogeneity of such farsightedness may be useful for applications: when modeling, for example, a one-to-one employer-employee market, every employer could be modeled as having more experience than every employee, resulting in farsightedness in the set of employers but not the set of employees. Or it could be that only some employers are experienced, or a minority of employees are also experienced, etc.

One way to formalize a model of such behavior is to view deviations from a given matching dynamically. Agents may move through a sequence of intermediate matchings generated by simple local operations: breaking an existing pair (so the two agents become single) and forming a new pair (which may simultaneously break prior partners). A *path* then represents a plausible adjustment process beginning from a given matching and ending at a new matching. Internal and external stability can be interpreted relative to this path-based notion of reachability, and a key modeling choice becomes how far ahead agents evaluate the consequences of initiating a change at a specific step of such a path.

However, the appropriate degree of this farsightedness along such paths is not an obvious modeling decision. Herings, Mauleon, and Vannetelbosch (2020), which serves as the primary inspiration for this thesis, study heterogeneous farsightedness where agents

are either fully farsighted (can see the final result of any possible path) or myopic (can only see one step ahead). These authors introduce the pairwise myopic-farsighted stable set and discuss its properties. One interesting property in the heterogeneous case is that farsighted agents may be able to create paths which dominate other matchings (even if impossible were all agents myopic), to the apparent disadvantage of myopic agents. Doğan and Ehlers (2023) show that this set exists for myopic-fully farsighted marriage problems. Following Herings, Mauleon, and Vannetelbosch (2025), an alternative and recent concept is limited farsightedness, where agents can be heterogeneously endowed with some natural number, or infinite, degree of farsightedness. We apply this concept of limited farsightedness to marriage problems, resulting in the pairwise horizon- k (PH- k) stable set.

Conceptually, the PH- k stable set is a von Neumann-Morgenstern stability requirement applied to our horizon-improving paths: we allow only pairwise deviations from a given element of a path, and we determine whether a deviation is admissible using an evaluation given by the endowed horizon- k farsightedness of the involved players. This retains the path-based perspective of farsighted stability, but avoids the extreme modeling choice of either complete myopia or full farsightedness.

We now summarize our main results. In *Section 2*, we formally define the marriage problem with horizon vector k , introduce the destroy and form operations that generate paths between matchings, and provide a formal definition for the PH- k stable set.

We also recall the man-optimal and woman-optimal stable matchings μ^M and μ^W resulting from the Deferred Acceptance algorithm of Gale and Shapley (1962). We then introduce a running example: a 2×2 market with diametrically opposed preferences. For the homogeneously myopic case, we show the unique PH- k stable set contains both woman and man-optimal matchings, and so coincides with the core. For the heterogeneous case with 3-farsighted women and myopic men, we show the unique PH- k stable set becomes the singleton $\{\mu^W\}$, which is a subset of the core. This demonstrates that farsightedness of just 3, rather than infinite farsightedness, suffices to provide a PH- k advantage to the more farsighted side.

Section 3 establishes some algorithmic properties of pairwise horizon- k improving paths. We prove that under homogeneously myopic horizons, reachability is transitive, and that paths can be reduced to eliminate repeated matchings.

In *Section 4*, we address individual rationality of the PH- k stable set. First we show some reachability properties of individual rational matchings. We then establish that individual rationality is preserved along pairwise horizon- k improving paths. We show that when all agents are myopic, every matching in any PH- k stable set is individually rational, a result previously shown in Herings, Mauleon, and Vannetelbosch (2020). We then extend this to show that in any marriage problem, every matching in a PH- k stable set is individually rational with respect to all myopic agents, regardless of the

farsightedness of other agents. We further show any singleton PH- k stable set must be individually rational, and that every PH- k stable set contains at least one individually rational matching.

Section 5 considers strategy-proofness. Strategy-proofness is traditionally defined in the context of centralized mechanisms, where a social planner who does not know agents' true preferences seeks to implement a matching rule that does not allow agents to improve by submitting false preferences. This is a deviation from previous material, as pairwise deviation is decentralized, but is natural to ask whether the PH- k stable set, which we define as a set-valued solution concept, satisfies strategy-proofness. We show by counterexample that PH- k stability fails strategy-proofness. This is a disappointing result with respect to a goal of using this set as a generalizable solution concept.

In *Section 6*, we make some effort to formalize cases where farsighted advantage observed can generally be shown. We show that two women who would prefer to swap their partners, and with farsightedness at least 3 can always reach their preferred matching, to the possible disadvantage of the matched men, provided the new matchings are individually rational. Our main existence result is *Proposition 7*, which establishes that when women are first-choice injective (FCI), men are partner-preferred, and $k_W \geq 3$ with $k_M = 1$, the unique PH- k stable set is the singleton $\{\mu^W\}$. In this case, we provide an explicit algorithm (the FCI Farsighted Algorithm) that constructs horizon- k improving paths from any matching to μ^W . We further show that μ^W is still within a PH- k stable set in this case, irregardless of the farsightedness of any M , which also provides a more general existence argument.

Finally, *Section 7* provides a complete classification of PH- k stable sets for the 2×2 diametric opposition case across some relevant horizon profiles, while *Section 8* concludes with final thoughts on future work in this field.

2 The k -Farsighted Marriage Problem

A (k -farsighted) marriage problem $(M, W, \succ, k)$ consists of a finite set of agents N , partitioned into a set of men M and set of women W , who have a *strict* ordering of transitive preferences $\{\succ_i\}$ over the set of possible partners of the opposite sex, together with the option of being single. We use m for a general man, w for a general woman, or i for a general agent irregardless of sex. If m prefers w_a to w_b , we write $w_a \succ_m w_b$. We may also represent m 's preferences as an ordered tuple; for example, $\succ_m = (w_1, w_2, w_3, m)$ would imply $w_1 \succ_m w_2 \succ_m w_3 \succ_m m$ (Herings, Mauleon, and Vannetelbosch, 2020). We also represent all preferences in matrices where the rows are these tuples. Let $k = (k_i) \in (\mathbb{N} \cup \{\infty\})^N$ be a *horizon vector*, where k_i is the number of steps ahead agent i looks (with $k_i = \infty$ *fully farsighted* and $k_i = 1$ *myopic*). We discuss implications of this horizon vector beginning following a discussion of the non-horizon elements of our marriage model.

A matching is a function $\mu : N \rightarrow N$ such that:

1. for every $m \in M, \mu(m) \in W \cup \{m\}$
2. for every $w \in W, \mu(w) \in M \cup \{w\}$
3. for every $i \in N, \mu(\mu(i)) = i$.

The set of all matchings μ is denoted by $\mathcal{M}$ (Herings, Mauleon, and Vannetelbosch, 2020). An agent i is single at μ if $\mu(i) = i$, implying, for example, that a for a single man m $\mu(m) \notin W$, and so $\mu(m) = m$. This setup is originally based upon the two-sided matching problem proposed by Gale and Shapley, (1962).

In the case that $t := |M| = |W|$, the number of matchings is given by:

$$\sum_{k=0}^t \binom{t}{k}^2 k!,$$

the number of partial bijections between M and W (Harris, Hirst, and Mossinghoff, 2008). This value increases very quickly in t , as can be seen by the following table.

t	$f(t)$
1	2
2	7
3	34
4	209
5	1,546
6	13,327
7	130,922
8	1,441,729
9	17,572,114
10	234,662,231

Table 1: Values of $f(t) = \sum_{k=0}^t \binom{t}{k}^2 k!$ for $t = 1, \dots, 10$.

We call a set of men (or, symmetrically, women) *partner-preferred* if every man prefers a matching to any $w \in W$ matching to being single. That is, if for all $m \in M$ we have $m \prec_m w$ for all $w \in W$. We call a matching problem *completely partner-preferred* if both M and W are partner-preferred. For ease of notation, when we state a set is partner-preferred, we will exclude each agent's preference for being single from our ordering of preferences.

Definition 1 (Individual Rationality). Let $(M, W, \succ, k)$ be a marriage problem. A matching $\mu \in \mathcal{M}$ is *individually rational* (IR) if

$$\mu(i) \succeq_i i \quad \text{for all } i \in N.$$

That is, no agent prefers being single to their assignment under the matching μ , (Roth and Sotomayor, 1990). Additionally, it is later useful to consider whether a matching is individual rationality for a given i instead of all $i \in N$. So we say a $\mu \in \mathcal{M}$ is *individually rational with respect to i* if $\mu(i) \succeq_i i$.

Example 1. Consider a matching market with two men $\{m_1, m_2\}$ and two women $\{w_1, w_2\}$. Suppose preferences are as follows:

$$m_1 : m_1 \succ w_1 \succ w_2, \quad m_2 : m_2 \succ w_2 \succ w_1,$$

$$w_1 : m_1 \succ m_2 \succ w_1, \quad w_2 : m_1 \succ m_2 \succ w_2.$$

Then the only possible individually rational matching would be

$$\mu = \{(m_1, m_1), (m_2, m_2), (w_1, w_1), (w_2, w_2)\}.$$

This is because no man prefers a woman to being single, and so an individually rational matching must have both men single, leaving no other matching options for the women. $\diamond$

For any coalition $C \subseteq N$, let $\mathcal{M}(C)$ denote the set of matchings defined on C , where agents in C are matched only among other agents in C or remain single.

Definition 2 (Core). Let $(M, W, \succ, k)$ be a marriage problem. For any coalition $C \subseteq N$, the coalition C *blocks* a matching $\mu \in \mathcal{M}$ if there exists a matching $\nu \in \mathcal{M}(C)$ such that

$$\nu(i) \succ_i \mu(i) \quad \text{for all } i \in C.$$

The *core* $\mathcal{C}$ is the set of matchings μ that are not blocked by any coalition.

For the marriage problem, Roth and Sotomayor (1990) show that the core coincides with the set of stable matchings, i.e. matchings that are individually rational and admit no blocking pairs. We therefore often refer to *core elements* as *stable*. When we later introduce the notion of Pairwise Horizon- k stability, the Pairwise Horizon- k stable set is, in our examples, a subset of the core.

Example 2. Consider a completely partner-preferred (so we need not list preferences for singleness) marriage problem, again with two men $\{m_1, m_2\}$ and two women $\{w_1, w_2\}$, and preferences as follows:

$$m_1 : w_1 \succ w_2, \quad m_2 : w_2 \succ w_1$$

$$w_1 : m_1 \succ m_2, \quad w_2 : m_1 \succ m_2.$$

Then, the matching $\mu = \{(m_1, w_2), (m_2, w_1)\}$ would not be a core element, that is $\mu \notin \mathcal{C}$, since (m_1, w_1) would block μ . (Clearly, this additionally violates our notion of pairwise stability.) However, it can be shown $\mu^* = \{(m_1, w_1), (m_2, w_2)\} \in \mathcal{C}$. $\diamond$

2.1 The Pairwise Horizon- k Stable Set

We now discuss our model of pairwise deviation. We begin with defining reachability with respect to farsightedness and the possible form and destroy operations. We then discuss paths resulting from such operations before formally defining the PH- k stable set and providing an example. Recall $k = (k_{i \in N}) \in (\mathbb{N} \cup \{\infty\})^N$, with k as a horizon vector.

Definition 3 (Lookahead function). Let $(M, W, \succ, k)$ be a marriage problem. Given a path length L and a step index $\ell \in \{0, \dots, L-1\}$, define the *lookahead function*

$$r_i(\ell) := \min\{L, \ell + k_i\}.$$

Intuitively, the role of $r_i(\ell)$ is to specify which future step along a possible path agent i uses for comparison when evaluating the step from μ_ℓ to $\mu_{\ell+1}$. If the planned path is short relative to k_i , then $r_i(\ell) = L$ and the agent evaluates the move using the terminal matching. (So having additional farsightedness would not provide any additional information in considering the path.) Otherwise, $r_i(\ell) = \ell + k_i$ and the agent evaluates using the matching at the edge of her horizon.

If all $k_m = k_n$ for all $m, n \in M$, then we assign k_M the value of all k_m , and symmetrically for k_W .

Given $\mu \in \mathcal{M}$ with $\mu(m) = w$, define the *destroy* operation $\mu - (m, w)$ as the matching that is identical to μ except that m and w are made single. If m and w are not matched to one another at μ , define the *form* operation $\mu + (m, w)$ as the matching that is identical to μ except that m and w are matched to one another and their former partners (if any) become single, (Herings, Mauleon, and Vannetelbosch, 2020).

Now, we discuss our definitions for horizon evaluations for each k_i .

Definition 4 (PH- k Improving Paths). Let $(M, W, \succ, k)$ be a marriage problem. A *pairwise horizon- k improving path* from $\mu \in \mathcal{M}$ to $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a finite sequence, or *path* P of matchings:

$$P := (\mu_0, \mu_1, \dots, \mu_L)$$

with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that, for each $\ell \in \{0, \dots, L-1\}$, either

(destroy) $\mu_{\ell+1} = \mu_\ell - (m, w)$ for some $(m, w) \in M \times W$, and at least one of the two agents strictly improves at his/her horizon:

$$\mu_{r_m(\ell)}(m) \succ_m \mu_\ell(m) \quad \text{or} \quad \mu_{r_w(\ell)}(w) \succ_w \mu_\ell(w).$$

(form) $\mu_{\ell+1} = \mu_\ell + (m, w)$ for some $(m, w) \in M \times W$, and both agents strictly prefer the move at their horizons:

$$\mu_{r_m(\ell)}(m) \succ_m \mu_\ell(m) \quad \text{and} \quad \mu_{r_w(\ell)}(w) \succ_w \mu_\ell(w).$$

Importantly, note that there is asymmetry in requirements for admissibility of either step: a destroy move can be initiated unilaterally, whereas a form move requires mutual agreement. We may say an agent m *initiates* a destroy move to indicate he strictly improves at his horizon at consideration of the destroy move. (Both players may improve at their horizon from such a move, but we use this to emphasize the improving player when only one improves.)

Definition 5 (Reachability). Let $(M, W, \succ, k)$ be a marriage problem. We write $\mu \rightarrow_L \mu'$ if there exists a pairwise horizon- k improving path of length L from μ to μ' . The

reachability set from μ is

$$h_k(\mu) := \{\mu' \in \mathcal{M} \mid \exists L \in \mathbb{N} \text{ with } \mu \rightarrow_L \mu'\}.$$

Definition 6 (PH- k Stability). Let $(M, W, \succ, k)$ be a marriage problem. A set $V \subseteq \mathcal{M}$ is a *pairwise horizon- k stable set* (PH- k stable set) if:

- (i) (k -internal stability) for all $\mu, \mu' \in V$, we have $\mu' \notin h_k(\mu)$;
- (ii) (k -external stability) for all $\mu \in \mathcal{M} \setminus V$, we have $h_k(\mu) \cap V \neq \emptyset$.

This follows the setup of von Neumann-Morgenstern stable sets, (von Neumann and Morgenstern, 1944). Additionally, for a given marriage problem we let $\mathcal{V}$ denote the superset collection of all PH- k stable sets.

(i) indicates that, for any $\mu \in V$ there exists no horizon- k improving path to another matching which is also in V . (ii) indicates that for any matching μ which is not within a pairwise horizon- k stable set, there exists a horizon- k improving path resulting in a matching μ' which is in the same pairwise horizon- k stable set.

Example 3 (An easy 3×3 myopic case). Consider a completely partner-preferred marriage problem with

$$M = \{m_1, m_2, m_3\}, \quad W = \{w_1, w_2, w_3\}, \quad k_i = 1 \text{ for all } i \in N.$$

Let preferences be such that each agent's top choice is completely symmetric, so the most ideal matching is obvious.

$$m_1 : w_1 \succ w_2 \succ w_3, \quad m_2 : w_2 \succ w_3 \succ w_1, \quad m_3 : w_3 \succ w_1 \succ w_2,$$

$$w_1 : m_1 \succ m_2 \succ m_3, \quad w_2 : m_2 \succ m_3 \succ m_1, \quad w_3 : m_3 \succ m_1 \succ m_2.$$

Clearly, the most ideal matching is to match each agent to their top preference, making everyone as happy as possible. It is easy to see that any other matching would be unstable, and any reasonable solution concept should return this matching. We write this “top matching” as:

$$\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}.$$

We claim the unique PH- k stable set is the singleton $\{\mu^*\}$.

Internal stability must clearly always be vacuously satisfied for any singleton set, such as $\{\mu^*\}$. (There exists no other element of a singleton set which may be in the horizon of μ^* , or vice versa).

To see external stability, fix any $\mu \in \mathcal{M}$ with $\mu \neq \mu^*$. Then there exists some $j \in \{1, 2, 3\}$ such that $\mu(m_j) \neq w_j$ (equivalently, $\mu(w_j) \neq m_j$). Consider the form move $\mu_1 = \mu + (m_j, w_j)$. Since $k_{m_j} = k_{w_j} = 1$, admissibility is evaluated at the next node. Under μ_1 we have $\mu_1(m_j) = w_j$ and $\mu_1(w_j) = m_j$, and by construction each is the other's first choice. Thus

$$\mu_1(m_j) = w_j \succ_{m_j} \mu(m_j) \quad \text{and} \quad \mu_1(w_j) = m_j \succ_{w_j} \mu(w_j),$$

so $\mu \rightarrow_1 \mu_1$ and μ_1 is a myopic improving step.

If $\mu_1 = \mu^*$ we are done. Otherwise, repeat the same argument for some remaining index $j' \in \{1, 2, 3\}$ with $\mu_1(m_{j'}) \neq w_{j'}$. Since each step forms one of the pairs

$$(m_1, w_1), (m_2, w_2), (m_3, w_3)$$

and never destroys an already formed pair (a man and woman in a formed pair are each other's top choice), after at most three such myopic form steps we reach μ^* . Hence $\mu^* \in h_k(\mu)$ for all $\mu \neq \mu^*$, so $\{\mu^*\}$ is k -externally stable.

Uniqueness follows immediately: $\mu^* \in h_k(\mu)$ for all $\mu \neq \mu^*$, so any possible PH- k stable set containing both μ^* and some other μ would violate internal stability. Furthermore, any possible PH- k stable set V' such that $\mu^* \notin V'$ would violate external stability, as $\mu^* \notin h_k(\mu^*)$ for any $\mu \in \mathcal{M}$. To see this, consider that no form or destroy move is ever admissible from μ^* , as it would be impossible for any agent to strictly improve. So we can also say that $\mathcal{V} = \{\{\mu^*\}\}$. $\diamond$

The above example shows some initial properties of the PH- k stable set in relation to matchings which are clearly, in some sense, optimal. For a more thorough discussion of true optimality we turn in the next subsection to optimal matchings as initially shown by Gale and Shapley, (1962).

2.2 Man-Optimal and Woman-Optimal Matchings

For any marriage problem, Gale and Shapley (1962) established the existence of two particular matchings which are core elements, computable via the Deferred Acceptance (DA) algorithm. The *man-optimal stable matching* μ^M is the unique core-element matching that every man weakly prefers to any other core-element matching; the *woman-optimal stable matching* μ^W is defined symmetrically. Roth and Sotomayor (1990) discuss in more detail DA and its properties.

We now illustrate our framework in a particularly tractable case $|M| = |W| = 2$, which we will refer to as the *diametric-opposition* case when men's and women's rankings are opposed, as defined in Example 4. When $n = 2$, by the rook polynomial values in Table 1, there are 7 possible matchings, which we enumerate in Figure 1.

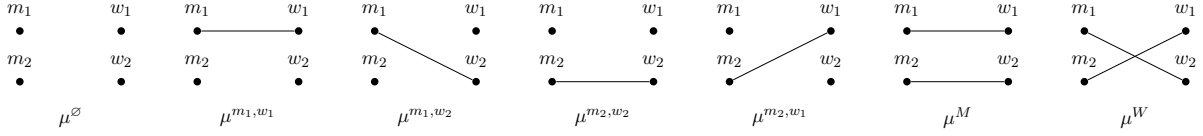


Figure 1: All matchings in the 2×2 marriage problem, including the empty matching, single-pair matchings, and the man- and woman-optimal stable matchings.

Example 4. Suppose we have $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$, and completely partner-preferred, diametrically opposed preferences:

$$m_1 : w_1 \succ w_2 \succ m_1, \quad m_2 : w_2 \succ w_1 \succ m_2,$$

$$w_1 : m_2 \succ m_1 \succ w_1, \quad w_2 : m_1 \succ m_2 \succ w_2.$$

For this example, the man-optimal and woman-optimal matchings are

$$\mu^W = \{(m_1, w_2), (m_2, w_1)\}, \quad \mu^M = \{(m_1, w_1), (m_2, w_2)\}.$$

We define each of the 7 possible matchings as follows, consistently with Figure 1:

$$\begin{aligned} \mu^\emptyset &= \{(m_1, m_1), (m_2, m_2), (w_1, w_1), (w_2, w_2)\} \quad (\text{all agents single}), \\ \mu^{m_1, w_1} &= \{(m_1, w_1), (m_2, m_2), (w_1, m_1), (w_2, w_2)\}, \\ \mu^{m_1, w_2} &= \{(m_1, w_2), (m_2, m_2), (w_1, w_1), (w_2, m_1)\}, \\ \mu^{m_2, w_2} &= \{(m_1, m_1), (m_2, w_2), (w_1, w_1), (w_2, m_2)\}, \\ \mu^{m_2, w_1} &= \{(m_1, m_1), (m_2, w_1), (w_1, m_2), (w_2, w_2)\}, \\ \mu^M &= \{(m_1, w_1), (m_2, w_2), (w_1, m_1), (w_2, m_2)\}, \\ \mu^W &= \{(m_1, w_2), (m_2, w_1), (w_1, m_2), (w_2, m_1)\}. \end{aligned}$$

Homogeneously Myopic Case

Consider now the case where $k_i = 1$ for all i . We argue $V = \{\mu^W, \mu^M\}$ is a PH- k stable set.

First, we show internal stability of $V = \{\mu^W, \mu^M\}$: First consider any path $\mu^W \rightarrow_L \mu^M$, consider any possible path. Consider the first step of this path $\mu^W \rightarrow \nu_1$. No form move is feasible as in μ^W both women are matched to their first preference, and so could not possibly improve at any future matching, and so would not agree to any new form operation. So now we consider only the possible destroy steps:

$$\nu_1 = \mu^W - (m_1, w_2) = \mu^{m_2, w_1} \quad \text{or} \quad \nu'_1 = \mu^W - (m_2, w_1) = \mu^{m_1, w_2}.$$

But, all i myopic. So, any such destroy step is clearly inadmissible under our destroy condition as both agents in the possible destroy function are myopic and partner-preferred. So no path exists, and $\mu^M \notin h_k(\mu^W)$. By symmetry of M and W , we can easily see similar reasoning would imply $\mu^W \notin h_k(\mu^M)$.

We now show external stability of $V = \{\mu^W, \mu^M\}$. Consider μ^{m_2, w_1} . We have $\mu^W \in h_k(\mu^{m_2, w_1})$ as there exists a path of length 1 such that $\mu^W = \mu^{m_2, w_1} + (m_1, w_2)$. This is admissible as $r_i(\ell) = \ell + 1$ for all myopic i , and both agents are partner-preferred. So, $w_2 = \mu_{r_{m_1}(0)}(m_1) \succ_{m_1} \mu^{m_2, w_1}(m_1) = m_1$, and $m_1 = \mu_{r_{w_2}(0)}(w_2) \succ_{w_2} \mu^{m_2, w_1}(w_2) = w_2$.

This case shows the admissibility of form operations between single men and women in this completely partner-preferred case. By the exact same reasoning, it follows that: $\mu^W \in h_k(\mu^{m_1, w_2})$, $\mu^M \in h_k(\mu^{m_2, w_2})$, $\mu^M \in h_k(\mu^{m_1, w_1})$. Additionally, once again by the admissibility of such form operations it is easy to see $\mu^{m_1, w_1}, \mu^{m_1, w_2}, \mu^{m_2, w_2}$, and $\mu^{m_2, w_1} \in h_k(\mu^\emptyset)$. Consider additionally that it is admissible to append to $\mu^{m_1, w_2} + (m_2, w_1) = \mu^W$, which shows an admissible path, and implies $\mu^W \in h_k(\mu^\emptyset)$. Similarly, we have $\mu^M \in h_k(\mu^\emptyset)$. So, for all 5 matchings not in V , clearly V is in the horizons of those matchings, and so external stability is shown.

Before turning to the case of heterogeneous horizons, note that in this 2×2 diametric-opposition case the set of stable matchings coincides with $\{\mu^M, \mu^W\}$. This mirrors the earlier 3×3 case in *Example 3*, where the women and man-preferred sets are the same. The difference here is that the market admits two stable extremes, and the role of horizons in PH- k analysis is precisely to determine whether these extremes remain both as elements of the PH- k stable set, or whether one becomes reachable from the other via an admissible sequence of destroy/form moves. We will now see this is the case when we introduce sufficiently heterogeneous farsightedness.

Heterogeneous Farsightedness Case

Now consider the case where $k_W = 3$, $k_M = 1$. That is, all men are myopic, and women have k -farsightedness of 3. Herings, Mauleon, and Vannetelbosch (2020) shows for this example that when $k_W = \infty$, $V = \{\mu^W\}$. We show this is also true when $k_W = 3$. Again, we show internal and then external stability of $\{\mu^W\}$.

First, observe $\{\mu^W\}$ is a singleton, so internal stability is vacuously true.

Now we show external stability. First and most interestingly, we exhibit a horizon- k improving path from μ^M to μ^W when $k_{m_1} = k_{m_2} = 1$ and $k_{w_1} = k_{w_2} = 3$. Consider the

path $P = (\nu_0, \nu_1, \nu_2, \nu_3, \nu_4)$ of length $L = 4$ defined by

$$\begin{aligned}\nu_0 &= \mu^M = \{(m_1, w_1), (m_2, w_2)\}, \\ \nu_1 &= \nu_0 - (m_1, w_1), \\ \nu_2 &= \nu_1 - (m_2, w_2), \\ \nu_3 &= \nu_2 + (m_2, w_1), \\ \nu_4 &= \nu_3 + (m_1, w_2) = \mu^W.\end{aligned}$$

We verify that each step is admissible as part of a pairwise horizon- k improving path. For men we have $k_m = 1$, so $r_m(\ell) = \ell + 1$, if $\ell \leq 3$. (Else, the horizon is the end of the path, which is the current step of the path.) For women $w \in W$ we have $k_w = 3$, so $r_w(\ell) = \min\{4, \ell + 3\}$, yielding

$$r_w(0) = 3, \quad r_w(1) = 4, \quad r_w(2) = 4, \quad r_w(3) = 4.$$

Step 0: $\nu_0 \rightarrow \nu_1$ (*destroy* (m_1, w_1)). Man m_1 becomes single, which is worse than w_1 , so he does not improve. However, woman w_1 looks ahead to $\nu_{r_{w_1}(0)} = \nu_3$, where she is matched with m_2 . Since $m_2 \succ_{w_1} m_1$, we have $\nu_3(w_1) \succ_{w_1} \nu_0(w_1)$, so the destroy step is admissible by our destroy rule.

Step 1: $\nu_1 \rightarrow \nu_2$ (*destroy* (m_2, w_2)). Similarly, m_2 becomes worse off immediately, but w_2 looks ahead to $\nu_{r_{w_2}(1)} = \nu_4$, where she is matched with m_1 , her top choice. Since $m_1 \succ_{w_2} m_2$, we have $\nu_4(w_2) \succ_{w_2} \nu_1(w_2)$, so this destroy step is also admissible.

Step 2: $\nu_2 \rightarrow \nu_3$ (*form* (m_2, w_1)). At $\ell = 2$ we have $r_{m_2}(2) = 3$ and $r_{w_1}(2) = 4$. Man m_2 compares $\nu_3(m_2) = w_1$ to $\nu_2(m_2) = m_2$ (single) and strictly prefers w_1 . Woman w_1 compares $\nu_4(w_1) = m_2$ to $\nu_2(w_1) = w_1$ (single) and strictly prefers m_2 . Thus both agents strictly improve at their horizons, and the form step is admissible.

Step 3: $\nu_3 \rightarrow \nu_4$ (*form* (m_1, w_2)). At $\ell = 3$ we have $r_{m_1}(3) = 4$ and $r_{w_2}(3) = 4$. Both m_1 and w_2 are single at ν_3 and matched to each other at ν_4 . Similar to Step 2, since the market is completely partner-preferred, each strictly prefers being matched to the other to being single, so this form step is admissible.

Hence every step of P is horizon- k improving, and we conclude $\mu^W \in h_k(\mu^M)$.

For the remaining matchings, paths to μ^W are easily verified. For instance, from $\mu^{m_1, w_2} = \{(m_1, w_2)\}$ the single form step (m_2, w_1) produces μ^W , and from $\mu^{m_2, w_1} = \{(m_2, w_1)\}$ the form step (m_1, w_2) produces μ^W . From μ^{m_1, w_1} and μ^{m_2, w_2} we can first destroy the existing pair to reach $\mu^\emptyset$ and then form (m_2, w_1) and (m_1, w_2) in succession to obtain μ^W . Finally, from the empty matching $\mu^\emptyset$ we reach μ^W by forming first one and then the other pair. (Which will clearly be myopically admissible for partner-preferred men.)

Thus, for every $\mu \in \mathcal{M} \setminus \{\mu^W\}$ we have $\mu^W \in h_k(\mu)$, so the singleton $V = \{\mu^W\}$

satisfies k -external stability as well. Combined with the (vacuous) internal stability of a singleton, this shows that $\{\mu^W\}$ is a PH- k stable set in the heterogeneous farsightedness case $k_{m_1} = k_{m_2} = 1$, $k_{w_1} = k_{w_2} = 3$.

It remains to be argued that this is the unique PH- k stable set. Consider any other possible matching. Recall our argument in the myopic case that no admissible path exists from any μ^W to any other matching. This argument required only the myopia of the men, which remains, so it is easy to see $\mu \notin h_k(\mu^W)$ for any $\mu \neq \mu^W$. We have already shown an admissible path exists from each of the other 6 matchings to μ^W , and have now shown no path exists from μ^W to another matching, so by external stability $\mu^W \in V$ for every V . But, having another matching ν in any V would violate k -internal stability, as we have established $\mu^W \in h_k(\nu)$. So $\{\mu^W\}$ is the only PH- k stable set.

We add that our reasoning for the matchings applies for a path of this length with a more general $k_W \geq 3$, as increasing each woman's horizon in the path P will only affect w_1 's action at ν_0 as her look-ahead value could now be 4. However, at $\nu_4(w_1) \succ_{w_1} \nu_0(w_1)$ the destroy action would still be admissible, and none of the other look-ahead values would change as they would be bounded by $L = 4$. So we can say more generally for the 2×2 diametrically opposed case, that if $k_W \geq 3$, and $k_M = 1$, $\{\mu^W\} = V$, while $\mathcal{V} = \{\{\mu^W\}\}$. $\diamond$

This example indicates some sort of "advantage" in the heterogeneous case favoring, for example W in the set of PH- k stable matchings over M in some cases if W has higher farsightedness than M .

3 Path Characteristics

We now discuss some characteristics of Pairwise Horizon- k improving paths under myopic horizons which will prove helpful for our later results, and are independently interesting. First, we show transitivity of horizons, and then that paths with repeating elements can be reduced.

Definition 7. Let $(M, W, \succ, k)$ be a marriage problem. We say that the horizon reachability h_k is *transitive* if for any $\mu_a, \mu_b, \mu_c \in \mathcal{M}$,

$$\mu_a \in h_k(\mu_b) \quad \text{and} \quad \mu_b \in h_k(\mu_c) \quad \implies \quad \mu_a \in h_k(\mu_c).$$

Lemma 1 (Transitivity of reachability for myopic horizons). Let $(M, W, \succ, k)$ be a marriage problem with $k_i = 1$ for all $i \in N$. The horizon reachability h_k is transitive.

Proof. By assumption $\mu_a \in h_k(\mu_b)$, so there exists $L_1 \in \mathbb{N}$ and a horizon- k improving path

$$P^1 := (\mu_0^1, \mu_1^1, \dots, \mu_{L_1}^1)$$

With $\mu_b = \mu_0^1$, and $\mu_{L_1}^1 = \mu_a$. Similarly, $\mu_b \in h_k(\mu_c)$ implies the existence of $L_2 \in \mathbb{N}$ and a horizon- k improving path

$$P^2 := (\mu_0^2, \mu_1^2, \dots, \mu_{L_2}^2)$$

With $\mu_c = \mu_0^2$, and $\mu_{L_2}^2 = \mu_b$.

We can construct an adjoined, or *concatenated* sequence P of length $L_1 + L_2$ from μ_c to μ_a by

$$\mu_0 = \mu_c, \mu_1 = \mu_1^2, \dots, \mu_{L_2} = \mu_{L_2}^2 = \mu_b, \quad \mu_{L_2+1} = \mu_1^1, \dots, \mu_{L_2+L_1} = \mu_{L_1}^1 = \mu_a.$$

We claim that P is a horizon- k improving path.

Since $k_i = 1$ for all i , for any path of length $L = L_1 + L_2$ we have

$$r_i(\ell) = \min\{L, \ell + 1\} = \ell + 1$$

for all $\ell = 0, \dots, L - 1$. This implies the admissibility of a step at index ℓ depends only on the current matching μ_ℓ and the next matching $\mu_{\ell+1}$: each moving agent compares her/his partner at μ_ℓ to her/his partner at $\mu_{\ell+1}$.

In the concatenated path P , every step $\mu_\ell \rightarrow \mu_{\ell+1}$ is exactly one of the steps from P^2 or from P^1 , with the same ordered pair of matchings. Since each such step is horizon- k improving in its original path and the comparison is always between μ_ℓ and $\mu_{\ell+1}$, the same inequalities hold in P . Hence every step of P is horizon- k improving, so P is a valid horizon- k improving path from μ_c to μ_a .

Therefore $\mu_a \in h_k(\mu_c)$, and reachability h_k is transitive under myopic horizons. $\square$

For additional clarity, we add the following point regarding the path from μ_c to μ_a . In particular, and for example, consider $\mu_{L_2}^2 = \mu_b \rightarrow \mu_1^1$. If it is admissible as part of path P^1 , then it must also be admissible as part of path P , because in P^1 at μ_b agents make decisions for μ_{b+1} at their highest possible horizon 1, which result in μ_1^1 . In P , the exact same decision can be made because μ_1^1 is the same and the horizon of both agents involved in the pairwise change are the same.

This argument does not follow for general k . The concatenation method may fail, but this does not imply transitivity fails in individual cases. We show this by example.

Example 5. Concatenating Paths Fails with Farsightedness

Consider the following case:

$$m_1 : w_1 \succ w_2, \quad m_2 : w_2 \succ w_1$$

$$w_1 : m_2 \succ m_1, \quad w_2 : m_1 \succ m_2.$$

Suppose $k_M = 2$, $k_W = 3$.

Consider the matchings $\mu_a = \{(m_2, w_1)\}$, $\mu_b = \{(m_1, w_1)\}$, and

$$\mu_c = \{(m_1, m_1), (m_2, m_2), (w_1, w_1), (w_2, w_2)\} := \mu^\emptyset.$$

Clearly $\mu_b \in h_k(\mu_c)$, by the path: $P^1 = (\mu_c, \mu_c + (m_1, w_1) = \mu_b)$, and $\mu_a \in h_k(\mu_b)$, by the path $P^2 = (\mu_b, \mu_b - (m_1, w_1), \mu_b - (m_1, w_1) + (m_2, w_1) = \mu_a)$. Observe however that adjoining P^1 and P^2 would not be a valid path, because at $\ell = 0$, m_1 would not take the *form* action, as he does not strictly improve in two steps, which is his horizon. However, it is also easy to see that $\mu_a \in h_k(\mu_c)$ by a 1-step path. $\diamond$

We now show an interesting result regarding the reducibility of paths when all agents are myopic. This is also an illustrative exercise in the appearance myopic pairwise improving paths can take.

Lemma 2 (Reducibility of paths under myopic horizons). Let $(M, W, \succ, k)$ be a marriage problem with $k_i = 1$ for all agents i . Suppose $\mu, \mu' \in \mathcal{M}$ and $\mu' \in h_k(\mu)$. Then there exists a pairwise horizon- k improving path $P' = (\nu_0, \nu_1, \dots, \nu_T)$ from $\mu = \nu_0$ to $\mu' = \nu_T$ such that:

1. all matchings along the path are distinct, i.e. $\nu_s \neq \nu_t$ whenever $0 \leq s < t \leq T$;
2. $T \leq |\mathcal{M}| - 1$.

Proof. By assumption, $\mu' \in h_k(\mu)$, so there exists at least one pairwise horizon- k improving path

$$P = (\mu_0, \mu_1, \dots, \mu_L)$$

with $\mu_0 = \mu$ and $\mu_L = \mu'$. Because $k_i = 1$ for all agents, the lookahead function satisfies

$$r_i(\ell) = \min\{L, \ell + 1\} = \ell + 1 \quad \text{for all } i, \ell \in \{0, \dots, L - 1\}.$$

In particular, the admissibility of the step from μ_ℓ to $\mu_{\ell+1}$ depends only on the current state μ_ℓ and future state $\mu_{\ell+1}$, and not on orderings of previous states or states past $\ell + 1$.

If all matchings on P are distinct, we are done. Otherwise, there exist indices $0 \leq s < t \leq L$ such that $\mu_s = \mu_t$; if multiple such pairs exist, then among all such pairs choose one with $t - s$ maximal. Since $\mu_s = \mu_t$, we can form a shorter path $P^{(1)}$ by deleting the loop between indices s and t :

$$P^{(1)} = (\mu_0^{(1)}, \mu_1^{(1)}, \dots, \mu_{L-(t-s)}^{(1)}) := (\mu_0, \dots, \mu_s, \mu_{t+1}, \dots, \mu_L).$$

Clearly $\mu_0^{(1)} = \mu_0 = \mu$ and $\mu_{L-(t-s)}^{(1)} = \mu_L = \mu'$, so $P^{(1)}$ still connects μ and μ' .

We now check that $P^{(1)}$ is still a pairwise horizon- k improving path. For steps that do not cross the point where the loop was removed, nothing changes: for any index ℓ with either $\ell < s$ or $\ell \geq t + 1$, the adjacent pair in $P^{(1)}$ coincides with the corresponding adjacent pair in P , so admissibility is preserved by myopic horizons.

The only potentially problematic step is the new transition from $\mu_s^{(1)}$ to $\mu_{s+1}^{(1)}$. By construction,

$$\mu_s^{(1)} = \mu_s = \mu_t, \quad \mu_{s+1}^{(1)} = \mu_{t+1}.$$

Thus the new step $\mu_s^{(1)} \rightarrow \mu_{s+1}^{(1)}$ in $P^{(1)}$ is identical to the original step $\mu_t \rightarrow \mu_{t+1}$ in P . Since the original step was a valid form or destroy move according to the pairwise horizon- k improving definition, and myopic horizons implies that admissibility depends only on this adjacent pair, the new step is also admissible.

Consequently $P^{(1)}$ is again a pairwise horizon- k improving path from μ to μ' , but with strictly fewer elements than P . If $P^{(1)}$ still contains a repeated matching, we can repeat the same loop-removal procedure. Each iteration strictly reduces the length of the path, so the process terminates after finitely many steps with a path

$$P' = (\nu_0, \nu_1, \dots, \nu_T)$$

from μ to μ' in which all matchings are distinct.

Finally, since $\mathcal{M}$ is finite, a path with distinct matchings between μ and μ' can include at most $|\mathcal{M}|$ different matchings, so the number of transitions satisfies $T \leq |\mathcal{M}| - 1$. This shows the claim. $\square$

4 Individual Rationality of the PH- k stable set

In this section we first show an interesting result regarding reachability of individually rational matchings from matchings in which no agents are matched. We then begin a series of arguments culminating in propositions which partially characterize the individual rationality of the PH- k stable set.

One interesting problem concerns the reachability of certain matchings via pairwise horizon- k improving paths. In particular, consider the matching where all agents are single, denoted $\mu^\emptyset$. We now show that for every marriage problem and every individually rational matching $\mu \in \mathcal{M}$, there exists a pairwise horizon- k improving path from $\mu^\emptyset$ to μ .

Proposition 1. Consider a marriage problem $(M, W, \succ, k)$ and let $\mu \in \mathcal{M}$ be any individually rational matching. Then there exists a pairwise horizon- k improving path $P = (\mu_0, \mu_1, \dots, \mu_L)$ such that $\mu_0 = \mu^\emptyset$, $\mu_L = \mu$, and

$$L = |\{w \in W \mid \mu(w) \neq w\}|.$$

Proof. Let $\{w \in W : \mu(w) \neq w\}$ be the set of women who are matched in μ . Write

$$L = |\{w \in W : \mu(w) \neq w\}|$$

and index these women arbitrarily as $\{w_1, \dots, w_L\}$, where $\mu(w_\ell) = m_\ell$ for each $\ell = 1, \dots, L$.

Define a sequence of matchings by the path $P = (\mu_0, \mu_1, \dots, \mu_L)$ recursively by

$$\begin{aligned} \mu_0 &= \mu^\emptyset, \\ \mu_\ell &= \mu_{\ell-1} + (m_\ell, w_\ell), \quad \ell = 1, \dots, L. \end{aligned}$$

Since every man and every woman appears in at most one pair of μ , and we start from the empty matching, each (m_ℓ, w_ℓ) consists of two agents who are single at $\mu_{\ell-1}$. Moreover, after step ℓ the pair (m_ℓ, w_ℓ) is never broken: for all $j \geq \ell$ we have

$$\mu_j(m_\ell) = w_\ell \quad \text{and} \quad \mu_j(w_\ell) = m_\ell$$

by construction. Hence μ_L coincides with μ .

We now show that each step $\mu_{\ell-1} \rightarrow \mu_\ell$ is a valid form move in the sense of a pairwise horizon- k improving path. Fix $\ell \in \{1, \dots, L\}$ and consider the pair (m_ℓ, w_ℓ) . At $\mu_{\ell-1}$ both are single:

$$\mu_{\ell-1}(m_\ell) = m_\ell, \quad \mu_{\ell-1}(w_\ell) = w_\ell.$$

At μ_ℓ we have

$$\mu_\ell(m_\ell) = w_\ell, \quad \mu_\ell(w_\ell) = m_\ell,$$

and this assignment coincides with their partner in the final matching μ .

Because μ is individually rational, and we have strict preferences, we have

$$\mu(m_\ell) = w_\ell \succ_{m_\ell} m_\ell, \quad \mu(w_\ell) = m_\ell \succ_{w_\ell} w_\ell.$$

Moreover, since (m_ℓ, w_ℓ) is never destroyed along the path, for any horizon lengths $k_{m_\ell}, k_{w_\ell} \geq 1$ we have

$$\mu_{r_{m_\ell}(\ell-1)}(m_\ell) = w_\ell \quad \text{and} \quad \mu_{r_{w_\ell}(\ell-1)}(w_\ell) = m_\ell.$$

Therefore,

$$\mu_{r_{m_\ell}(\ell-1)}(m_\ell) = w_\ell \succ_{m_\ell} \mu_{\ell-1}(m_\ell) = m_\ell,$$

and

$$\mu_{r_{w_\ell}(\ell-1)}(w_\ell) = m_\ell \succ_{w_\ell} \mu_{\ell-1}(w_\ell) = w_\ell.$$

Thus both agents strictly prefer their horizon partner to being single, and the form step $\mu_{\ell-1} \rightarrow \mu_\ell$ is admissible.

Since this holds for every $\ell = 1, \dots, L$, the sequence

$$P = (\mu_0, \mu_1, \dots, \mu_L)$$

is a pairwise horizon- k improving path from the empty matching to μ , of length L . □

A very useful characteristic of our model to show is that it is not possible to form a path from a matching which is individually rational with respect to some agent to the final element of the path is not individually rational with respect to that agent. We will later use this Lemma in our discussion of general individual rationality of PH- k stable sets.

Lemma 3. Suppose a marriage problem $(M, W, \succ, k)$, with $\mu, \mu' \in \mathcal{M}$ such that $\mu' \in h_k(\mu)$. If for some agent m , $\mu(m) \succeq_m m$, then $\mu'(m) \succeq_m m$.

Proof. Let $P = (\nu_0, \nu_1, \dots, \nu_L)$ be a pairwise horizon- k improving path with $\nu_0 = \mu$ and $\nu_L = \mu'$. Suppose $\nu_0(m) \succeq_m m$.

If agent m 's matching is not affected by any move within P , then $\nu_L(m) = \nu_0(m) \succeq_m m$, and we are done.

If m 's matching is affected only by a *destroy* move along P , or the last move which effects m 's matching *destroys* his matching to a partner, then recall at any step $\nu_\ell \in P$

where m 's matching is destroyed we have $\nu_{\ell+1}(m) = m \succeq_m m$, so again $\nu_L(m) \succeq_m m$. Note that this covers the case where m 's matching is destroyed by a *destroy* move by his partner or a *form* move by his partner with another man, destroying his current matching.

Now suppose that m participates in at least one *form* move along P . Let ℓ_1 be the smallest index such that the transition $\nu_{\ell_1-1} \rightarrow \nu_{\ell_1}$ is a form move involving m . Because previous moves either destroyed m 's original matching or m 's matching unchanged, $\nu_{\ell_1-1}(m) \succeq_m m$.

By admissibility of this form move with respect to k_m , there exists a path element $\nu_{r_m(\ell_1-1)} \in P$, with the index $r_m(\ell_1-1) \geq \ell_1$ such that

$$\nu_{r_m(\ell_1-1)}(m) \succ_m \nu_{\ell_1-1}(m).$$

Since $\nu_{\ell_1-1}(m) \succeq_m m$ (either m is single or was in an individually rational matching), transitivity of preferences implies $\nu_{r_m(\ell_1-1)}(m) \succ_m m$, hence $\nu_{r_1}(m) \succeq_m m$.

If there is no form move involving m after the step $\nu_{r_m(\ell_1-1)} \in P$, then from $\nu_{r_m(\ell_1-1)} \in P$ to $\nu_L \in P$ agent m is either unaffected or only involved in a move which destroys his current matching, both of which preserve that each subsequent matching be individually rational. Hence, in this case, $\nu_L(m) \succeq_m m$.

Otherwise, let $\ell_2 > r_m(\ell_1-1)$ be the smallest index such that $\nu_{\ell_2} = \nu_{\ell_2-1} + (m, w')$ is a form move involving m and some $w' \in W$. By admissibility, there exists a matching we index as $\mu_{r_m(\ell_2-1)}$, with $r_m(\ell_2-1) \geq \ell_2$ such that

$$\nu_{r_m(\ell_2-1)}(m) \succ_m \nu_{\ell_2-1}(m).$$

Since $\nu_{\ell_2-1}(m) \succeq_m m$ and so eventually m is better off at $\nu_{r_m(\ell_2-1)}$, we again obtain $\nu_{r_m(\ell_2-1)}(m) \succeq_m m$.

Iterating this argument for j possible *form* moves yields a strictly increasing finite sequence of indices $r_m(\ell_1-1) < r_m(\ell_2-1) < \dots < r_m(\ell_j-1) \leq L$ such that $\nu_{r_j}(m) \succeq_m m$. Because the path has finite length L , the iteration must terminate, and after the final such index no further *form* move affects m . Observe finally that after r_j , no *form* move occurs, and any possible *destroy* move keeps individual rationality for subsequent matchings.

Hence, $\mu'(m) = \nu_L(m) \succeq_m m$, as we wished to show. $\square$

However, it is certainly not the case that any agent i who participates in moves along some path P from $\mu \rightarrow_L \mu'$ that $\mu'(i) \succeq_i \mu(i)$. We show this by the following example.

Example 6. Consider a partner-preferred marriage problem $(M, W, \succ, k)$, with $k_W = 3$,

$k_M = 1$, which is completely partner-preferred, and the following remaining preferences:

$$\begin{array}{l|l} m_1 : & w_1 \succ w_2 \succ w_3 \succ w_4 \\ m_2 : & w_2 \succ w_3 \succ w_4 \succ w_1 \\ m_3 : & w_3 \succ w_2 \succ w_1 \succ w_4 \end{array} \quad \begin{array}{l|l} w_1 : & m_1 \succ m_2 \succ m_3 \\ w_2 : & m_2 \succ m_1 \succ m_3 \\ w_3 : & m_2 \succ m_1 \succ m_3 \\ w_4 : & m_1 \succ m_3 \succ m_2 \end{array}$$

Consider the matchings $\mu = \{(m_1, w_3), (m_2, w_4), (m_3, w_2)\}$, and

$\mu' = \{(m_1, w_4), (m_2, w_1), (m_3, w_3)\}$. We have $\mu' \in h_k(\mu)$ by the following path: $P = (\mu = \nu_0, \dots, \nu_4 = \mu')$, such that:

$$\begin{aligned} \nu_0 &= \mu, \\ \nu_1 &= \nu_0 + (m_1, w_2), \\ \nu_2 &= \nu_1 + (m_2, w_2), \\ \nu_3 &= \nu_2 + (m_3, w_3), \\ \nu_4 &= \nu_3 + (m_1, w_4) = \mu' \end{aligned}$$

For conciseness, and as we demonstrate similar reasoning in other examples, we claim this path as admissible by repeated application of our definition of the *form* move.

We care in particular about m_1 . Note that m_1 begins with a *form* move which is admissible on his myopic horizon as $\nu_1(m_1) \succ_{m_1} \nu_0(m_1)$, and it can be similarly verified that w_2 improves at her horizon. However, at ν_2 , m_1 is involuntarily made single by w_2 *forming* with her most preferred matching. As he is now single, the last move is admissible as $\nu_4(m_1) = w_4 \succ_{m_1} m_1 = \nu_3(m_1)$, by the assumption of complete partner-preference. We have therefore demonstrated that there are horizon- k improving paths which make participating agents strictly worse. Note however that, following Lemma 3, m_1 remains in an individually rational matching at μ' , because μ was also individually rational with respect to m_1 . $\diamond$

One important question regarding our PH- k stable set is whether its elements are always individually rational (IR). This is a key result if we are to consider the PH- k stable set as a solution concept to general marriage problems: (Roth, Sotomayor, 1990) If we were to attempt to assign a non-individually rational matching to some agent i , she could always unilaterally reject this matching by becoming single, as we soon show. Our proof structure here is formatted following Herings, Mauleon, and Vannetelbosch (2020), which show general individual rationality for cases where farsightedness is, by fiat, either myopic or fully farsighted. Using this structure we re-prove their myopic result, but expand to include more general results for unrestricted k .

We now show two helpful Lemmas for our later, more broad individual rationality arguments.

Lemma 4. Consider a marriage problem $(M, W, \succ, k)$, and a matching $\mu \in \mathcal{M}$ such that for some $m \in M$ we have $\mu(m) = w \prec_m m$. Then $\mu - (m, w) \in h_k(\mu)$.

Proof. Consider a path of length $L = 1$. Then, irregardless of k , $r_m(0) = 1$. Call $\mu = \mu_0$. Consider $\mu_1 = \mu_0 - (m, w)$. For this to be admissible as a path of length 1, we will show m is always willing to initiate a destroy action at μ_0 . We have $\mu_1(m) = \mu_{r_m(0)}(m) = m \succ_m \mu_0(m) = w$. This shows $\mu - (m, w) = \mu_1 \in h_k(\mu)$. $\square$

To see that this generalizes for any i consider the symmetry of the problem with respect to an arbitrary m or w .

This implies that there is always an admissible path from a non-individually rational matching to a matching which is individually rational. This also implies that if there was some μ such that multiple i were matched irrationally, an admissible path exists such that all i are matched individually rationally. We see a formalized application of this argument later, in *Proposition 4*.

Lemma 5. Consider a marriage problem $(M, W, \succ, k)$ and a matching $\mu \in \mathcal{M}$ such that for some m and his match w , $\mu(m) = w \prec_m m$. Then for $\mu' := \mu - (m, w)$, we have $\mu \notin h_k(\mu')$.

Proof. The matching μ' is clearly individually rational with respect to m , as m is single. By assumption μ is not individually rational with respect to m . Therefore, by *Lemma 3*, $\mu \notin h_k(\mu')$. $\square$

Once again, to see that this generalizes for any $i \in N$ consider the symmetry of the problem with respect to an arbitrary m or w .

Combined the two previous Lemmas provide us with sufficient background for our first result in individual rationality. Our first result, *Proposition 2*, has already been shown by similar reasoning in Herings, Mauleon, and Vannetelbosch (2020).

Proposition 2. Consider a marriage problem $(M, W, \succ, k)$ such that $k_i = 1$ for all $i \in N$, and a corresponding PH- k stable set V . Then for any $\mu \in V$, μ is individually rational.

Proof. Suppose for contradiction a matching $\mu \in V$ such that μ is not individually rational for some $m \in M$. We have already shown for a general k that for $\mu' = \mu - (m, w)$, $\mu' \in h_k(\mu)$. Additionally, we have shown $\mu \notin h_k(\mu')$. Under our assumption $\mu \in V$, the internal stability of the PH- k stable set this implies $\mu' \notin V$, which in turn implies by the external stability of V , there exists a $\mu'' \in V$ (with $\mu'' \neq \mu$) such that $\mu'' \in h_k(\mu')$.

As we have earlier shown, for a completely myopic k , horizons are transitive. So, $\mu' \in h_k(\mu)$, and $\mu'' \in h_k(\mu')$ implies $\mu'' \in h_k(\mu)$, but we have assumed $\mu \in V$ and shown this implies $\mu'' \in V$, but this contradicts the internal stability of V . So any $\mu \in V$ is individually rational in the myopic case. $\square$

Observe that the above argument holds, by symmetry, without loss of generality for any $i \in N$. Additionally, we can further expand upon this argument by showing individual rationality holds for all myopic agents in any marriage problem, without restricting all agents to myopia. We show this as follows.

Proposition 3. Consider a marriage problem $(M, W, \succ, k)$, and a corresponding PH- k stable set V . Then for any $\mu \in V$, μ is individually rational with respect to all myopic agents.

Proof. Suppose for contradiction there exists a matching $\mu \in V$ such that μ is not individually rational for some myopic m . We have already shown in *Lemma 4* that for $\mu' = \mu - (m, w)$, $\mu' \in h_k(\mu)$. Additionally, we have shown in *Lemma 5* $\mu \notin h_k(\mu')$. Under our assumption $\mu \in V$, the internal stability of the PH- k stable set this implies $\mu' \notin V$, which in turn implies, by the external stability of V , there exists exists a $\mu'' \in V$ (with $\mu'' \neq \mu$) such that $\mu'' \in h_k(\mu')$.

Consider the path $P = (\mu' = \mu_0, \mu_1, \mu_2, \dots, \mu_L = \mu'')$. If m is myopic, as assumed, then at any step $\ell \in \{0, 1, \dots, L-1\}$ of the path m will only form a new matching precisely if $\mu_{\ell+1}(m) \succ_m \mu_\ell(m)$. Because at μ_0 m is unmatched, this implies that for every ℓ m will only be single or matched to some $\hat{w}$ which he prefers to being single. Therefore, every matching in P is individually rational for m , as is μ'' . Now consider the concatenated path: $P^* = (\mu = \mu_0, \mu' = \mu_1, \mu_2, \mu_3, \dots, \mu_L = \mu'')$. The step from $\mu_0 \rightarrow \mu_1$ is permissible because a myopic m will still prefer being unmatched to μ_0 , and the remaining steps are permissible because their permissibility is invariant to previous steps by our definitions of form and destroy actions. But this shows that $\mu'' \in h_k(\mu)$, which contradicts the internal stability of the PH- k stable set. Thus μ is individually rational with respect to all myopic agents. $\square$

We can use similar reasoning to easily show that, irrespective of k , any singleton PH- k stable set must also be individually rational. This follows by a simple proof by contradiction.

Proposition 4. Consider a marriage problem $(M, W, \succ, k)$. Any corresponding PH- k stable set which is a singleton is individually rational.

Proof. Consider for a contradiction a singleton PH- k stable set V , with $V = \{\mu\}$ such that μ is not individually rational with respect to, without loss of generality, some $m \in M$. The external stability of V would imply that $\mu \in h_k(\bar{\mu})$ for all $\bar{\mu} \neq \mu$. But, we know there exists $\mu' = \mu - (m, \mu(m))$ such that $\mu \notin h_k(\mu')$. This contradicts the external stability of V . So, any singleton V is individually rational. $\square$

It is also possible to show, that there is at least one element of any PH- k stable set which must be individually rational with respect to every agent. Once again, this

is a further generalization of our previous individual rationality arguments, as it holds without restriction on k .

Proposition 5. Consider a marriage problem $(M, W, \succ, k)$. In every corresponding PH- k stable set V there is at least one matching $\mu \in V$ such that μ is individually rational.

Proof. Suppose for a contradiction there exists no matching $\mu \in V$ which is individually rational. Consider the set $\mu^\emptyset$ in which every agent is single. There are two possible cases, either $\mu^\emptyset \in V$ or $\mu^\emptyset \notin V$.

If $\mu^\emptyset \in V$, then we have a contradiction, as $\mu^\emptyset$ is individually rational.

If $\mu^\emptyset \notin V$, then by external stability of V there exists $\mu' \in V$ such that $\mu' \in h_k(\mu^\emptyset)$. But, by *Lemma 3*, $\mu' \in V$ must also be individually rational, which also contradicts our assumption. Therefore there exists at least one $\mu \in V$ such that μ is individually rational. $\square$

The last Proposition, in particular, is an important result with respect to possible practical applications of the PH- k stable set. Suppose all agents in a marriage problem agreed to resolve upon a final matching by a given solution concept, such as the PH- k stable set. It would be a reasonable expectation of any agent agreeing to use the PH- k stable set that they are guaranteed to least weakly prefer the PH- k stable result to being single. Any agent matched so as not to be individually rational by the PH- k stable set could simply unilaterally choose not to cooperate. So, in practice, if designing a centralized mechanism, it could be agreed to use the subset, (perhaps $\hat{V}$) of any PH- k stable set which is at least individually rational for every agent.

5 Strategy-proofness of PH- k Stability as a Solution Concept

A natural question is whether PH- k -stability can serve as a *solution concept* for two-sided matching in general. Following Roth and Sotomayor (1990), two ideal characteristics of a solution concept are individual rationality and strategy-proofness. The former was addressed in the previous section; we now turn to strategy-proofness. Fundamentally, strategy-proofness requires that no agent or coalition can benefit from misreporting their true preferences. Roth (1982) has shown as a seminal result that no solution concept is both strategy-proof and stable.

However, in general, much previous work in this field has only defined strategy-proofness for a solution concept which is a singleton. But as we have observed, as in previous examples, and can later be seen in *Section 7*, the PH- k stable set need not be a singleton. Hence, we first establish a definition of strategy-proofness to work with set-valued solution concepts. An example of a more generalized definition of strategy-proofness can be found in Kelly (1977). In order to define strategy-proofness here, and for a different case than Kelly, we must first describe a more formal definition for what precisely is a solution concept for a marriage problem. Let $\mathcal{S}$ be the set of all possible preference profiles $\succ$. We define a solution concept, following Roth and Sotomayor (1990) as a mapping between the set of preferences $\mathcal{S}$, given M and W , of a marriage problem, and the set of matchings $\mathcal{M}$.

Definition 8 (Solution Concept). Consider the preference profiles of M and W . The set containing all possible such preference profiles is $\mathcal{S}$. We call $\Phi : \mathcal{S} \rightarrow \mathcal{M}$ a *solution concept* that maps the preferences $\succ$ of M and W to a set of matchings within $\mathcal{M}$.

Definition 9 (Coalitional Domination). Let $(M, W, \succ, k)$ be a marriage problem and let $C \subseteq N$ be a coalition and let nonempty $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$. We say that $\mathcal{A}$ *weakly dominates* $\mathcal{B}$ for coalition C , written $\mathcal{A} \succeq_C \mathcal{B}$, if for every $\mu' \in \mathcal{B}$ there exists $\mu \in \mathcal{A}$ such that

$$\mu(i) \succeq_i \mu'(i) \quad \text{for all } i \in C.$$

Now, we define precisely what a misreport is, relative to the coalition which may choose to misreport.

Definition 10 (Misreport). Let $(M, W, \succ, k)$ be a marriage problem. A *misreport by coalition* $C \subseteq N$ is a preference profile $\succ' \in \mathcal{S}$ such that

$$\succ'_i = \succ_i \quad \text{for all } i \notin C$$

Definition 11 (Strategy-proofness). Let $\Phi : \mathcal{S} \rightarrow \mathcal{M}$ be a solution concept. We say that Φ is *strategy-proof* if for every $\succ \in \mathcal{S}$, for every coalition $C \subseteq N$, and every misreport $\succ' \in \mathcal{S}$ by C , we have

$$\Phi(\succ) \succeq_C \Phi(\succ').$$

Define the solution concept $\Phi' : \mathcal{S} \rightarrow \mathcal{M}$ such that $\Phi'(\succ) = V$ for a marriage problem $(M, W, \succ, k)$, of which V is a PH- k stable set. Note that we have not shown if the PH- k stable set always exists. If there exists no PH- k stable set under $\succ$ or $\succ'$, we do not attempt to define strategy-proofness, as in *Definition 9* we have only defined coalitional domination with respect to nonempty sets.

We show by counterexample that PH- k stability fails strategy-proofness.

Example 7 (A counterexample to strategy-proofness of PH- k). Consider again the completely partner-preferred 2×2 diametric-opposition market with

$$m_1 : w_1 \succ w_2 \succ m_1, \quad m_2 : w_2 \succ w_1 \succ m_2,$$

$$w_1 : m_2 \succ m_1 \succ w_1, \quad w_2 : m_1 \succ m_2 \succ w_2,$$

and the following heterogeneous horizons (men myopic, women 3-farsighted),

$$k_{m_1} = k_{m_2} = 1, \quad k_{w_1} = k_{w_2} = 3.$$

With *true* preferences $\succ$, we established in *Section 2* that in this case, the unique PH- k stable set is the singleton

$$\Phi'(\succ) = V = \{\mu^W\}, \quad \mu^W = \{(m_1, w_2), (m_2, w_1)\}.$$

Now let man m_1 *misreport* by declaring w_2 unacceptable relative to being single:

$$\succ'_{m_1} : w_1 \succ m_1 \succ w_2,$$

keeping all other agents' preferences truthful (so $\succ'_i = \succ_i$ for all $i \neq m_1$). Denote the resulting profile by $\succ'$.

μ^W is no longer reachable from μ^M under $\succ'$.

Recall $\mu^M = \{(m_1, w_1), (m_2, w_2)\}$. Under $\succ'$, any horizon- k path from μ^M to μ^W must eventually include a *form* step that matches m_1 with w_2 (since $(m_1, w_2) \in \mu^W$). At the index ℓ of that *form* step, myopic m_1 evaluates at $r_{m_1}(\ell) = \ell + 1$ and must satisfy

$$\mu_{\ell+1}(m_1) \succ'_{m_1} \mu_{\ell}(m_1)$$

for admissibility of *form*. But $\mu_{\ell+1}(m_1) = w_2$, while in the predecessor state μ_{ℓ} , m_1 is

either single or matched to w_1 . Under $\succ'_{m_1}$ we have

$$w_1 \succ'_{m_1} m_1 \succ'_{m_1} w_2,$$

so m_1 does *not* strictly prefer being matched with w_2 at the next node. Hence no *form* step creating (m_1, w_2) can be admissible, and therefore

$$\mu^W \notin h_k(\mu^M) \quad \text{under } \succ'.$$

In particular, $\{\mu^W\}$ cannot satisfy k -external stability under $\succ'$.

Under $\succ'$, the unique PH- k stable set becomes $\{\mu^M\}$.

First, note that under $\succ'$ the matching μ^W is no longer individually rational for m_1 , since $m_1 \succ'_{m_1} w_2$. Using this, we can show that from μ^W there is a horizon- k improving path to μ^M :

$$\begin{aligned} \nu_0 &= \mu^W = \{(m_1, w_2), (m_2, w_1)\}, \\ \nu_1 &= \nu_0 - (m_1, w_2) = \mu^{m_2, w_1}, \\ \nu_2 &= \nu_1 + (m_2, w_2) = \mu^{m_2, w_2}, \\ \nu_3 &= \nu_2 + (m_1, w_1) = \mu^M. \end{aligned}$$

At $\nu_0 \rightarrow \nu_1$, agent m_1 initiates the *destroy* move, since $m_1 \succ'_{m_1} w_2$. At $\nu_1 \rightarrow \nu_2$, the form (m_2, w_2) is admissible since $w_2 \succ_{m_2} w_1$ and w_2 prefers m_2 to being single. At $\nu_2 \rightarrow \nu_3$, both m_1 and w_1 are single and prefer matching to each other. Hence each step is admissible under $\succ'$.

Now observe that, $h_k(\mu^M) = \emptyset$ under $\succ'$. At μ^M , man m_1 is matched to w_1 , his top choice under $\succ'_{m_1}$, and m_2 is matched to w_2 , his top choice. Neither man will initiate a destroy. Any destroy initiated by a woman would require her to obtain a better partner within her horizon. Women are farsighted, and so under $\succ$, as previously shown, such a path would exist. However, under $\succ'$, w_2 would see that no *form* move with m_1 will ever be admissible, (By *Lemma 1* such a path cannot exist) and so cannot improve over m_2 , and so will not initiate any move which *destroys* her matching.

Finally, μ^M is reachable from every other matching under $\succ'$. From $\mu^\emptyset$, form (m_1, w_1) then (m_2, w_2) . From μ^{m_1, w_1} or μ^{m_2, w_2} , a single form step reaches μ^M . From μ^{m_1, w_2} , agent m_1 initiates destroy (since $m_1 \succ'_{m_1} w_2$), reaching $\mu^\emptyset$. From μ^{m_2, w_1} , agent m_2 can form with w_2 (since $w_2 \succ_{m_2} w_1$), reaching μ^{m_2, w_2} . Combined (because this step is myopically admissible and so can be adjoined) with the path from μ^W shown above, we have $\mu^M \in h_k(\mu)$ for all $\mu \neq \mu^M$. Hence

$$\Phi'(\succ') = \{\mu^M\}.$$

Conclusion.

Under the true profile $\succ$ we have $\Phi'(\succ) = \{\mu^W\}$, while under the misreport $\succ'$ we have $\Phi'(\succ') = \{\mu^M\}$. Since

$$\mu^M(m_1) = w_1 \succ_{m_1} w_2 = \mu^W(m_1),$$

it follows that $\Phi'(\succ) \not\preceq_{\{m_1\}} \Phi'(\succ')$, violating Definition 11. Therefore PH- k stability (as a set-valued solution concept) is not strategy-proof.

◇

6 Farsighted Advantage in Heterogeneous Cases

As a main result, we show general several proofs regarding cases where high farsightedness of a given set allows that set to create paths towards their more-preferred matchings by strategic combinations of destroy and form moves. Our first result shows that two swap-preferred women (or, symmetrically, men) with sufficient finite farsightedness can obtain their more preferred matching if their initially matched men are myopic.

Definition 12 (Swap Preference). Let $(M, W, \succ, k)$ be a marriage problem. Two distinct women w_a, w_b are said to be *swap-preferred* with respect to some matching $\mu \in \mathcal{M}$, if for some distinct men $m_c, m_d \in M$,

$$\mu(w_a) = m_c, \quad \mu(w_b) = m_d,$$

while preferences satisfy:

$$m_d \succ_{w_a} m_c, \quad m_c \succ_{w_b} m_d.$$

Proposition 6. Consider a completely partner-preferred marriage problem $(M, W, \succ, k)$ such that there exists $\mu \in \mathcal{M}$, w_a, w_b swap-preferred with respect to μ . Suppose also $k_{w_a} \geq 3$, $k_{w_b} \geq 3$ and $k_{m_c} = k_{m_d} = 1$. Consider an μ' such that

$$\mu'(w_a) = m_d, \quad \mu'(w_b) = m_c,$$

while $\mu'(i) = \mu(i)$ for all $i \notin \{w_a, w_b, m_c, m_d\}$. Then $\mu' \in h_k(\mu)$.

Proof. Consider the path $P = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ defined by

$$\begin{aligned} \mu_0 &= \mu, \\ \mu_1 &= \mu_0 - (m_c, w_a), \\ \mu_2 &= \mu_1 - (m_d, w_b), \\ \mu_3 &= \mu_2 + (m_d, w_a), \\ \mu_4 &= \mu_3 + (m_c, w_b) =: \mu'. \end{aligned}$$

We verify admissibility step by step.

Step 0 (destroy (m_c, w_a)). Man m_c is myopic and does not improve by becoming single, but woman w_a has $k_{w_a} \geq 3$ and looks ahead to $\mu_{r_{w_a}(0)}$. Since $r_{w_a}(0) \geq 3$ and $\mu_3(w_a) = m_d \succ_{w_a} m_c = \mu_0(w_a)$ by swap-preference, the destroy step is admissible.

Step 1 (destroy (m_d, w_b)). Similarly, w_b looks ahead to $\mu_{r_{w_b}(1)}$ and, since $r_{w_b}(1) \geq 4$ and $\mu_4(w_b) = m_c \succ_{w_b} m_d = \mu_1(w_b)$, the destroy step is admissible.

Steps 2 and 3 (form moves). At μ_2 , both (m_d, w_a) and (m_c, w_b) are formed with myopic men who strictly prefer being matched to being single, and with women who obtain their preferred partner in the swap. Hence each form step is admissible.

Hence $\mu' = \mu_4 \in h_k(\mu)$, completing the argument $\square$

6.1 The FCI Farsighted Algorithm: A Limited Existence Argument

Proposition 6 showed some degree of advantage in path construction favoring agents with higher farsightedness-especially farsightedness greater than 2. We now show that with such farsightedness a set of women or men could guarantee their optimal matching under partner-preference of the opposite sex and another property of first choice injectiveness, which we now define.

Definition 13 (First Choices). Consider the preferences $\succ_w$ of some $w \in W$. We denote the *first choice* $f(w) \in M \cup \{w\}$ of w as the first entry in $\succ_w$.

Definition 14 (FCI). The set W is said to be *first-choice injective* (FCI) if for all distinct $w, w' \in W$ we have $f(w) \neq f(w')$.

That is, W is *first-choice injective* if each woman in it has a first preference which is different from every other woman. We do not restrict that this preference must be for the opposite set; it could also be to be single. But, if $\succ_{w_1} = (m_1, \dots)$ so that $f(w_1) = m_1$, then no other woman $w \neq w_1$ can have first choice m_1 .

Remark 1: It is easy to see, by Gale and Shapley (1962) that for any marriage problem $(M, W, \succ, k)$, which is partner-preferred with respect to men, and such that W is FCI, $\mu^W(w) = f(w)$ for all $w \in W$. The Deferred Acceptance algorithm would terminate in one round, in which each woman proposes to her preferred partner and is accepted.

Proposition 7. Consider any marriage problem $(M, W, \succ, k)$ such that W is FCI. Suppose M is partner-preferred. Suppose further $k_w \geq 3$ for all $w \in W$ and $k_M = 1$. Then the unique PH- k stable set is the singleton $\{\mu^W\}$.

To show this Proposition, we first establish the following algorithm.

FCI Farsighted Algorithm. For ease, if $\mu^W(w) \neq w$, index these women so that $\mu^W(w^f) = m^f$ for each such $w^f \in W$.

Input: a matching $\mu \in \mathcal{M} \setminus \mu^W$.

Output: a pairwise horizon- k improving path P from μ to μ^W .

Consider every $w \in W$ such that $\mu(w) \neq \mu^W(w)$. Call this subset $\bar{W} \subseteq W$. Construct a path P of matchings μ_ℓ for $\ell \in \{0, 1, \dots, L\}$ with $\mu_0 = \mu$ as follows.

Part 1.

Let $\alpha := |\{w^a \in \bar{W} : f(w^a) = w^a\}|$. For $\ell = 0, 1, \dots, \alpha$, consider a $w^a \in \bar{W}$ in some order. If

$$\mu_\ell(w^a) \prec_{w^a} w^a,$$

while

$$f(w^a) = \mu^W(w^a) = w^a,$$

then destroy in $\mu_{\ell+1}$ so that

$$\mu_{\ell+1} = \mu_\ell - (\mu_\ell(w^a), w^a).$$

This destroys matches that are not individually rational for those women whose first preference is to be single. These women never become matched again, as they are already in their most optimal matching. This part concludes at μ_α .

Part 2.

If $\alpha = |\bar{W}|$ then $\mu_\alpha = \mu^W$ and we are done. Otherwise, our path must have $L > \alpha$, so consider $\ell > \alpha$. Alternatively, if for $\ell > \alpha$ we have

$$\mu_\ell(m^f) \prec_{m^f} w^f$$

for some $w^f \in \bar{W}$, then form $\mu_{\ell+1}$ such that

$$\mu_{\ell+1} = \mu_\ell + (m^f, w^f),$$

i.e. we form the pair (m^f, w^f) , breaking any existing match of m^f and of w^f if necessary.

Do this for all $w^f \in \bar{W}$ such that $\mu_\ell(m^f) \prec_{m^f} w^f$. Let W^* be the set of such women which form a new matching in this part or Part 1; write $|W^*| = \beta$. By construction, by μ_β we have matched all $w \in \bar{W}$ such that their first-preferred partner strictly preferred them to his current match.

Part 3.

If $|\bar{W}| - |W^*| = 0$, then $\mu_\beta = \mu^W$ and we are done. Otherwise our path must have $L > \beta$, so consider $\ell > \beta$.

For any remaining $w^f \in \bar{W} \setminus W^*$ we have $\mu_\beta(w^f) \neq m^f$, and by construction we are in the case where

$$\mu_\beta(m^f) \succ_{m^f} w^f.$$

Since W is FCI, and men are partner-preferred, whenever $\mu_\beta(m^f) \neq m^f$, the current partner $\mu_\beta(m^f)$ must itself belong to $\bar{W} \setminus W^*$ (otherwise m^f would have formed with w^f in Part 2). Consider a directed bipartite graph with each node consisting of either all women $w^f \in \bar{W} \setminus W^*$ or their corresponding target men $\{m^f : w^f \in \bar{W} \setminus W^*\}$. Construct directed edges as follows: each woman w^f points to her preferred partner m^f , and each man m^f points to his current partner $\mu_\beta(m^f) \in \bar{W} \setminus W^*$. In this graph, every woman has out-degree 1 (points to her unique preferred partner) and in-degree 1 (exactly one man has her as his current partner); similarly, every man has out-degree 1 (points to his unique current partner) and in-degree 1 (exactly one woman has him as her preferred

partner, by FCI property). Since every node has in-degree 1 and out-degree 1, the graph decomposes into one or more disjoint directed cycles (Harris, Hirst, and Mossinghoff, 2008).

For a given cycle, re-index its elements such that

$$\begin{aligned} w^{f_1} &\rightarrow m^{f_1} \quad \text{but} \quad \mu_\beta(m^{f_1}) = w^{f_2}, \\ w^{f_2} &\rightarrow m^{f_2} \quad \text{but} \quad \mu_\beta(m^{f_2}) = w^{f_3}, \\ &\vdots \\ w^{f_r} &\rightarrow m^{f_r} \quad \text{but} \quad \mu_\beta(m^{f_r}) = w^{f_1}. \end{aligned}$$

So under μ_β we have the matched pairs

$$(m^{f_1}, w^{f_2}), (m^{f_2}, w^{f_3}), \dots, (m^{f_r}, w^{f_1}).$$

For a given cycle $(w^{f_1}, \dots, w^{f_r})$ with corresponding men $(m^{f_1}, \dots, m^{f_r})$, construct the following sequence starting from the current node μ_ℓ :

$$\begin{aligned} \nu_0 &= \mu_\ell, \\ \nu_1 &= \nu_0 - (m^{f_1}, w^{f_2}), \\ \nu_2 &= \nu_1 - (m^{f_2}, w^{f_3}), \\ \nu_3 &= \nu_2 + (m^{f_2}, w^{f_2}), \\ \nu_4 &= \nu_3 - (m^{f_3}, w^{f_4}), \\ \nu_5 &= \nu_4 + (m^{f_3}, w^{f_3}), \\ &\vdots \\ \nu_{p-2} &= \nu_{p-3} - (m^{f_r}, w^{f_1}), \\ \nu_{p-1} &= \nu_{p-2} + (m^{f_r}, w^{f_r}), \\ \nu_p &= \nu_{p-1} + (m^{f_1}, w^{f_1}). \end{aligned}$$

Here each destroy step destroys the *non-ideal* current match of the involved woman (that is, the match $(m^{f_j}, w^{f_{j+1}})$ as it exists in ν_t), and each “+” step forms the *woman-ideal* match (m^{f_j}, w^{f_j}) (indices taken cyclically). By the time we reach ν_p , every woman in the cycle is matched to her preferred man m^{f_j} . Perform this procedure cycle-by-cycle until no women remain in $\bar{W} \setminus W^*$; the resulting matching is μ^W .

Recognizing the above algorithm is complicated, we re-summarize. The FCI Far-sighted Algorithm performs the following: First, in parts 1 and 2, we move from some $\mu \neq \mu^W$ towards μ^W by matching all of the elements of μ^W which are myopically obtainable from the given μ by an individual one-step path. It is important to note that in part

2 a new form move may be inadmissible at μ_α but admissible before μ_β . For example, suppose w^f such that $\mu_\alpha(w^f) = m^f$, but that $\mu_\alpha(m^f) = w^{f'} \succ_{m^f} w^f$. However, it could be that $w^{f'}$ becomes matched to $m^{f'}$ at $\mu_{\alpha+1}$, implying that the form move would now be admissible for m^f at the next step in the path. Only if m^f never becomes single in the algorithm are we guaranteed a cycle as described in part 3. By part 3, the remaining women who are not matched to their preferred μ^W matching must form one or more disjoint cycles in preferences, so we resolve the cycles in a way similar to our more simple swap-preferred case in *Proposition 6*. We now use this algorithm to show *Proposition 7*.

Proof. We show that for every matching $\mu \neq \mu^W$ with $\mu \in \mathcal{M}$, we have $\mu^W \in h_k(\mu)$.

Consider any such matching μ . The **FCI Farsighted Algorithm** constructs an admissible path P from μ to μ^W , i.e. a path $\mu \rightarrow_L \mu^W$.

Admissibility of Part 1. Part 1 only uses our destroy move. Within the algorithm, once some $w^a \in \bar{W}$ with $f(w^a) = w^a$ becomes single, which is her first preference, she never *forms* again. Moreover, she improves over μ by design, so each step in this part is admissible.

Admissibility of Part 2. Part 2 forms pairs (m^f, w^f) only when $\mu_\ell(m^f) \prec_{m^f} w^f$. Since $k_M = 1$, each such man evaluates myopically and strictly improves at the next node. Each such woman forms with her top preference $\mu^W(w^f) = m^f$ and never becomes matched to a different agent later in the algorithm. Hence every form step in Part 2 is admissible.

Admissibility of Part 3. Fix any cycle constructed in Part 3 and the associated sequence $\nu_0, \dots, \nu_p$. Each destroy step $\nu_t \rightarrow \nu_{t+1}$ breaks the *current* non-ideal match of the involved woman. Since $k_w \geq 3$ for all women, each such woman evaluates the destroy within her horizon and, by construction of the cycle-update, is matched to her preferred partner (her μ^W partner) within three subsequent steps. Thus at each destroy step, the destroying woman strictly improves at her horizon.

Each form step $\nu_t \rightarrow \nu_{t+1}$ is admissible for the man, who is myopic ($k_M = 1$) and strictly prefers the woman he forms with to being single, and admissible for the woman since she forms with her most-preferred partner and remains matched to him for the rest of the cycle update. Therefore every step of the cycle sequence is pairwise horizon- k admissible, and its endpoint matches all women in the cycle to their partners in μ^W . Applying this cycle-by-cycle completes Part 3 and yields μ^W .

Thus, the algorithm produces an admissible horizon- k improving path from μ to μ^W , and we conclude $\mu^W \in h_k(\mu)$ for every $\mu \neq \mu^W$. Therefore $V = \{\mu^W\}$ satisfies external stability, and internal stability trivially since it is a singleton.

To see uniqueness, consider some V such that $\mu^W \notin V$. This violates external stability, since $\mu \notin h_k(\mu^W)$ for any $\mu \neq \mu^W$: no form or destroy step is admissible because women will never initiate, and men cannot myopically improve by *destroying* their current matching. Now suppose V contains some $\mu \neq \mu^W$. Then $\mu^W \in h_k(\mu)$, as

shown above, contradicting k -internal stability. Therefore $V = \{\mu^W\}$, and in particular $\mathcal{V} = \{\{\mu^W\}\}$. $\square$

If we do not wish to show uniqueness, this can easily be extended without restricting the farsightedness of men.

Proposition 8. Let $(M, W, \succ, k)$ be a marriage problem such that W is FCI, and M is partner-preferred. Suppose $k_w \geq 3$ for all $w \in W$. Then a PH- k stable set is the singleton $\{\mu^W\}$.

Proof. In the proof for *Proposition 7*, observe that each form step l involving any $m \in M$ is myopically admissible, so $\nu_l(m) \succ_m \nu_{l-1}(m)$. But, after any form involving m , at no $\bar{l} > l$ is m matched to a different partner. So for any k_m , $r_{k_m}(\nu_{l-1})(m) = \nu_l(m)$, so if the step of the path was myopically admissible, it is also admissible for any $k_m > 1$. So, irrespective of k_m for any m , a path still exists from any $\mu \neq \mu^W$, with $\mu \in \mathcal{M}$, such that $\mu^W \in h_k(\mu)$. So μ^W must be a singleton PH- k stable set. $\square$

Note that we no longer argue for uniqueness, for generally farsighted men. For an example, see the following corollary.

Proposition 9. Consider any completely partner-preferred marriage problem $(M, W, \succ, k)$, such that W and M are FCI. Suppose $k_i \geq 3$ for all $i \in N$. Then the for superset $\mathcal{V}$ of PH- k stable sets, we have: $\{\mu^W\}, \{\mu^M\} \in \mathcal{V}$.

Proof. From *Proposition 8*, we know that if women are partner-preferred with $k_w \geq 3$ for all $w \in W$, then $\{\mu^W\}$ is a singleton PH- k stable set, irrespective of the farsightedness of men. But, because we needn't condition that argument on men, and by symmetry, it must also be the case that if men are partner-preferred with $k_m \geq 3$ for all $m \in M$, then $\{\mu^M\}$ is a singleton PH- k stable set. $\square$

These Propositions constitute a significant result, as it is the first subcase of the general marriage problem for which he have shown an existence argument and description of the PH- k stable set.

We show the functionality of this algorithm by example.

Example 8. Consider a marriage problem $(M, W, \succ, k)$, with $k_W = 3$, $k_M = 1$, which is completely partner-preferred, and the following remaining preferences:

$$\begin{array}{l|l} m_1 : & w_1 \succ w_2 \succ w_3 \succ w_4 \\ m_2 : & w_4 \succ w_1 \succ w_3 \succ w_2 \\ m_3 : & w_3 \succ w_2 \succ w_4 \succ w_1 \\ m_4 : & w_1 \succ w_3 \succ w_2 \succ w_4 \\ m_5 : & w_2 \succ w_4 \succ w_3 \succ w_1 \end{array} \quad \begin{array}{l|l} w_1 : & m_1 \succ m_2 \succ m_3 \succ m_4 \succ m_5 \\ w_2 : & m_3 \succ m_4 \succ m_1 \succ m_2 \succ m_5 \\ w_3 : & m_2 \succ m_1 \succ m_4 \succ m_3 \succ m_5 \\ w_4 : & m_5 \succ m_3 \succ m_2 \succ m_1 \succ m_4 \end{array}$$

Consider $\mu = \{(m_2, w_4), (m_3, w_3), (m_4, w_1), (m_5, w_2)\}$. Note that all 4 matched men are matched to their top-ranked woman. It is clear to see that if women were myopic, there would be no admissible path from μ to $\mu^W = \{(m_1, w_1), (m_2, w_3), (m_3, w_2), (m_5, w_4)\}$. But consider an application of our FCI Farsighted algorithm to see that such a path exists with women of farsightedness 3. The first step our path is to form $+(m_1, w_1)$, because this is myopically admissible as an element of Part 2. We then recognize the preferences and current matches of w_2, w_3 , and w_4 form a cycle. (see *Figure 2*) So we form the following path:

$$\begin{aligned}
\nu_0 &= \mu, \\
\nu_1 &= \nu_0 + (m_1, w_1), (\text{end of phase 1 and 2}) \\
\nu_2 &= \nu_1 - (m_5, w_2), \\
\nu_3 &= \nu_2 - (m_3, w_3), \\
\nu_4 &= \nu_3 + (m_3, w_2), \\
\nu_5 &= \nu_4 - (m_2, w_4), \\
\nu_6 &= \nu_5 + (m_2, w_3), \\
\nu_7 &= \nu_6 + (m_5, w_4) = \mu^W
\end{aligned}$$

It is easy to see, by reasoning similar to our previous examples, that this is an admissible path, following our algorithm. Also, observe that this path would be admissible if $k_{w_2} = 2$ and/or $k_{w_4} = 2$, as their myopically inadmissible destroy steps become admissible in only 2 steps. We therefore conjecture that in general there are cases where our Proposition holds when some members of the more farsighted group have farsightedness less than 3. $\diamond$

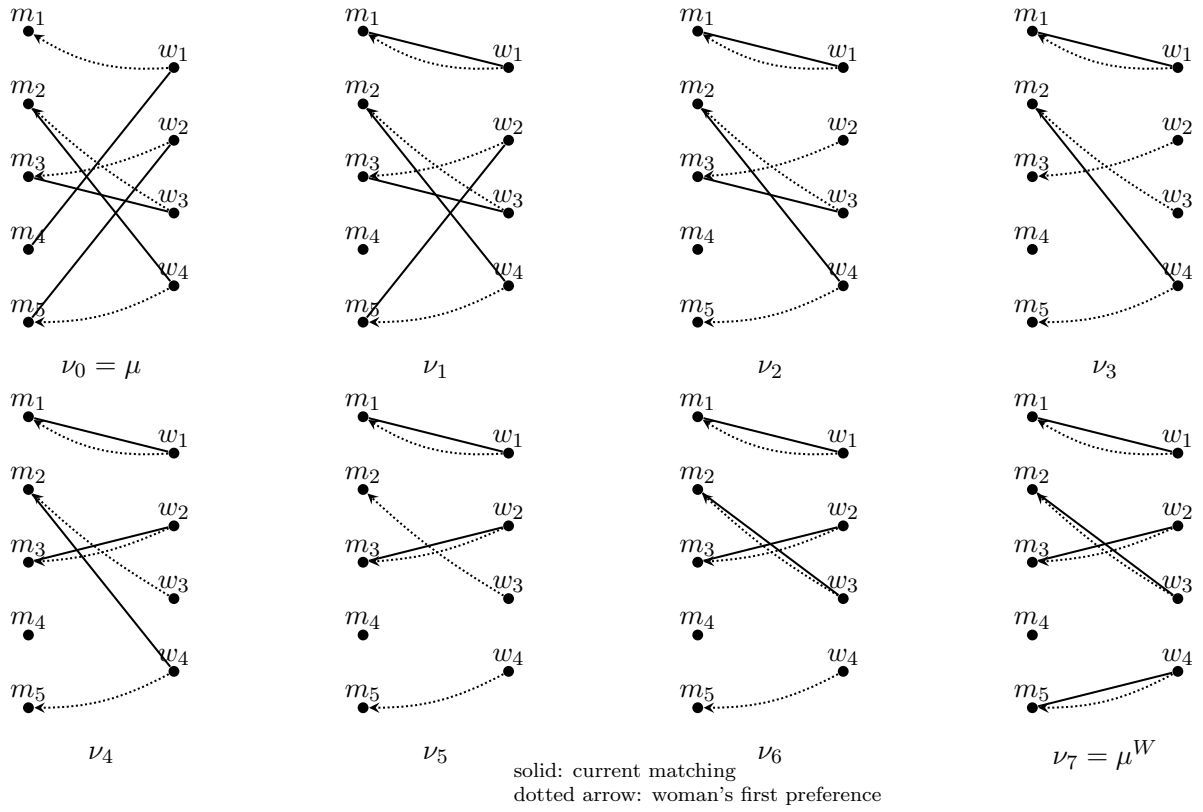


Figure 2: Each panel shows the current matching (solid) and women's first-choice pointers in μ^W (dotted).

7 Classification of the 2×2 Diametric Opposition Case

In this section we derive the PH- k stable sets corresponding to the previously introduced 2×2 diametric example for some interesting horizon vectors. Throughout we keep the completely partner-preferred, diametrically opposed preferences

$$m_1 : w_1 \succ w_2 \succ m_1, \quad m_2 : w_2 \succ w_1 \succ m_2,$$

$$w_1 : m_2 \succ m_1 \succ w_1, \quad w_2 : m_1 \succ m_2 \succ w_2,$$

and we continue with the notation $k_{m_1} = k_{m_2} = k_M$ and $k_{w_1} = k_{w_2} = k_W$ only when horizons are homogeneous within a set.

We use the previously established notation for the seven possible matchings and then analyze homogeneous and heterogeneous horizon profiles. The analysis is uniform with respect to each case: for each horizon combination we (1) describe the set of horizon- k improving paths, (2) summarize the resulting reachability relation, and (3) identify the PH- k stable sets.

There are two matchings in which both men and both women are matched (namely μ^M and μ^W), four matchings with exactly one pair, and the all-single matching $\mu^\emptyset$. In particular, recall that μ^M is the man-optimal stable matching and μ^W is the woman-optimal stable matching.

First we show several easy and useful arguments for horizons which always hold.

Lemma 6. In the 2×2 diametric opposition case, for any k , $\mu^M \in h_k(\mu^{m_2, w_2})$.

Proof. This follows from the assumed partner preference of all agents. Consider the path

$$P = (\nu_0, \nu_1) \text{ with } \nu_0 = \mu^{m_2, w_2}, \nu_1 = \nu_0 + (m_1, w_1) = \mu^M.$$

As a one-step path k is irrelevant, and by assumption both agents m_1 and w_1 prefer this matching to being single, so this is admissible. $\square$

Lemma 7. In the 2×2 diametric opposition case, for any k , $\mu^W \in h_k(\mu^{m_2, w_1})$.

Proof. This follows from the assumed partner preference of all agents. Consider the path

$$P = (\nu_0, \nu_1) \text{ with } \nu_0 = \mu^{m_2, w_1}, \nu_1 = \nu_0 + (m_1, w_2) = \mu^W.$$

As a one-step path k is irrelevant, and by assumption both agents m_1 and w_2 prefer this matching to being single, so this is admissible. $\square$

One implication of Lemmas 6–7 is that, by the internal stability of a PH- k stable set, if μ^M is in a PH- k stable set, then μ^{m_2, w_2} cannot be. Similarly, if μ^W is in a PH- k stable set,

then μ^{m_2, w_1} cannot be. Observe also that symmetry implies the analogous statements for μ^{m_1, w_1} and μ^{m_1, w_2} , etc. We use this type of symmetry argument to condense our arguments to show paths only for interesting cases.

7.1 Homogeneous horizons

We start with the symmetric case $k_M = k_W = k$ for all agents and vary k .

Case $k_i = 1$ for all i (all agents myopic).

We have already addressed this case in *Example 4*, but here we summarize horizons for more detail. Under partner-preference and $k_i = 1$ for all i , no destroy move is admissible: a destroy step makes both agents single at $\ell + 1$, and with myopic evaluation ($r_i(\ell) = \ell + 1$) neither of the two destroying agents can (weakly) improve relative to being currently matched. Hence every admissible step is a single *form* move $\mu \mapsto \mu + (m, w)$ in which both m and w strictly prefer their partners at the next node.

From $\mu^\emptyset$ (all single) any acceptable pair can form in one step, giving $\mu^{m_1, w_1}, \mu^{m_1, w_2}, \mu^{m_2, w_2}, \mu^{m_2, w_1}$. A second *form* then yields μ^M (add (m_2, w_2) to μ^{m_1, w_1} or (m_1, w_1) to μ^{m_2, w_2}) or μ^W (add (m_2, w_1) to μ^{m_1, w_2} or (m_1, w_2) to μ^{m_2, w_1}). (Recall myopic horizons are transitive). Thus

$$h_k(\mu^\emptyset) = \{\mu^{m_1, w_1}, \mu^{m_1, w_2}, \mu^{m_2, w_2}, \mu^{m_2, w_1}, \mu^M, \mu^W\}.$$

Summarizing horizons. Consider the following examples of horizons; these also imply other cases which are clearly symmetric.

- $\mu^M \in h_k(\mu^{m_1, w_1})$ via one-step *form* (m_2, w_2) (both are single).
- $\mu^{m_2, w_1} \in h_k(\mu^{m_1, w_1})$ via one-step *form* (m_2, w_1) since $w_1 \succ_{m_2} m_2$ and $m_2 \succ_{w_1} m_1$.
- $\mu^W \in h_k(\mu^{m_1, w_1})$ via the two-step path that first forms (m_2, w_1) and then forms (m_1, w_2) .
- $\mu^{m_2, w_2} \in h_k(\mu^{m_1, w_1})$ by concatenating the admissible one-step moves $\mu^{m_1, w_1} \rightarrow \mu^{m_2, w_1}$ (form (m_2, w_1)) and $\mu^{m_2, w_1} \rightarrow \mu^{m_2, w_2}$ (destroy (m_2, w_1) and form (m_2, w_2) is not admissible when myopic; instead use the direct one-step *form* from μ^{m_1, w_1} : form (m_2, w_2) yields μ^M and then destroy is not allowed. Hence, for clarity, we omit this chain and rely on Table 2 below for the complete relation).
- $\mu^{m_1, w_2} \in h_k(\mu^{m_1, w_1})$ (by symmetry and the one-step *form* move available from $\mu^\emptyset$, as summarized in Table 2).

In this myopic case it is simplest to read the full reachability relation directly from Table 2. From μ^M and μ^W no *form* step is admissible (each candidate pair would make one participant strictly worse at the next node), and destroy is never admissible when $k = 1$. Therefore $h_k(\mu^M) = h_k(\mu^W) = \emptyset$.

Table 2: Myopic reachability $h_k(\mu)$ in the 2×2 diametric case

μ	$\mu^\emptyset$	μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
$\mu^\emptyset$		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_1}			μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_2}		μ^{m_1, w_1}		μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_2, w_2}		μ^{m_1, w_1}	μ^{m_1, w_2}		μ^{m_2, w_1}	μ^M	μ^W
μ^{m_2, w_1}		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}		μ^M	μ^W
μ^M							
μ^W							

PH-k stability. Internal stability: $h_k(\mu^M) = h_k(\mu^W) = \emptyset$, so any set containing only μ^M and/or μ^W is internally stable, and no set containing one such element and another element can be internally stable because every $\mu \in \{\mu^\emptyset, \mu^{m_1, w_1}, \mu^{m_1, w_2}, \mu^{m_2, w_2}, \mu^{m_2, w_1}\}$ reaches μ^M or μ^W . Additionally, note $\{\mu^M\}$ individually cannot be PH- k stable, as $\mu^M \notin h_k(\mu^W)$, so this singleton set would violate external stability. By symmetry the same is true for $\{\mu^W\}$.

External stability: by Table 2, for each $\mu \notin \{\mu^M, \mu^W\}$ we have $\mu^M \in h_k(\mu)$ and $\mu^W \in h_k(\mu)$.

Proposition 10. In the 2×2 diametric opposition example with $k_M = k_W = 1$, the unique PH- k stable set is

$$V = \{\mu^M, \mu^W\}.$$

Which coincides with the core in this instance.

Case $k_i = 2$ for all i .

Here $k_i = 2$ for all i , so each agent evaluates a deviation at node ℓ by the assignment at

$$r_i(\ell) = \min\{L, \ell + 2\}.$$

Any one-step move that is admissible when $k_i = 1$ remains admissible when $k_i = 2$, since k is irrelevant to one-step paths. The question is whether any additional multi-step reachability becomes admissible.

In this example, any genuinely new reachability relative to $k_i = 1$ would have to come from a path whose first step is a destroy move (since all one-step *form* moves were

already admissible under $k_i = 1$). In particular, the only candidates that could change reachability are paths that attempt to start from a fully matched outcome and use destroy steps supported by agents who can now evaluate two steps ahead. However, as we explain below, such deviations are still blocked when $k_i = 2$.

Paths no longer admissible. Some multi-step paths that were admissible under $k_i = 1$ cease to be admissible when agents evaluate two steps ahead. For example, consider

$$\mu_0 = \mu^{m_1, w_1} \rightarrow \mu_1 = \mu^{m_2, w_1} \rightarrow \mu_2 = \mu^{m_2, w_2}.$$

This path is not horizon- k admissible when $k_i = 2$ because at μ_0 woman w_1 evaluates at $r_{w_1}(0) = 2$ and sees that she ends up single at μ_2 , so she weakly prefers her current partner at μ_0 to the evaluated outcome at μ_2 and will not support the initial move. Similar reasoning applies to the symmetric chains that required two or three steps in the myopic case.

The resulting reachability table is thus:

Table 3: Reachability $h_k(\mu)$ in the 2×2 diametric case with $k_i = 2$ for all i

μ	$\mu^\emptyset$	μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
$\mu^\emptyset$		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_1}					μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_2}		μ^{m_1, w_1}				μ^M	μ^W
μ^{m_2, w_2}			μ^{m_1, w_2}			μ^M	μ^W
μ^{m_2, w_1}				μ^{m_2, w_2}		μ^M	μ^W
μ^M							
μ^W							

PH- k stability. Although the reachability relation is smaller than in the myopic case, the stability conclusion is unchanged. Internal stability: $h_k(\mu^M) = h_k(\mu^W) = \emptyset$, so any set containing only μ^M and/or μ^W is internally stable, and no set containing one such element and another element can be internally stable because every $\mu \notin \{\mu^M, \mu^W\}$ reaches μ^M and μ^W . External stability: by Table 3, for each $\mu \notin \{\mu^M, \mu^W\}$ we have $\mu^M \in h_k(\mu)$ and $\mu^W \in h_k(\mu)$. Therefore the PH- k stable set remains unchanged.

Proposition 11. In the 2×2 diametric opposition example with $k_M = k_W = 2$, the unique PH- k stable set is

$$V = \{\mu^M, \mu^W\}.$$

Case $k_i = 3$ for all i .

We now consider the case in which all agents have horizon $k_i = 3$. Most of the reachability structure from the homogeneous $k_i = 2$ case carries over unchanged. In particular, as before, any one-step deviation that was admissible when $k_i = 2$ remains admissible when $k_i = 3$. The only new phenomenon is that deviations between the two fully matched outcomes μ^M and μ^W have now become admissible, since destroying agents can evaluate farther ahead.

New paths between μ^M and μ^W .

Lemma 8. In the 2×2 diametric opposition case with $k_i = 3$ for all i , we have

$$\mu^M \in h_k(\mu^W) \quad \text{and} \quad \mu^W \in h_k(\mu^M).$$

Proof. We give the path $\mu^W \rightarrow \mu^M$; by symmetry the reverse direction is analogous.

Recall $\mu^W = \{(m_1, w_2), (m_2, w_1)\}$ and $\mu^M = \{(m_1, w_1), (m_2, w_2)\}$. Consider the path

$$\begin{aligned} \nu_0 &= \mu^W, \\ \nu_1 &= \nu_0 - (m_1, w_2), \\ \nu_2 &= \nu_1 - (m_2, w_1), \\ \nu_3 &= \nu_2 + (m_1, w_1), \\ \nu_4 &= \nu_3 + (m_2, w_2) = \mu^M. \end{aligned}$$

The two *form* moves are admissible by partner preference (each participant is single before the move and matched at the relevant evaluation node along the path). For the first destroy move, m_1 evaluates at $r_{m_1}(0) = \min\{0+3, 4\} = 3$ and sees $\nu_3(m_1) = w_1 \succ_{m_1} w_2 = \nu_0(m_1)$, so destroying (m_1, w_2) is horizon-admissible for m_1 . The second destroy move is admissible for m_2 by the analogous argument. Hence the entire path is admissible and $\mu^M \in h_k(\mu^W)$, and by symmetry $\mu^W \in h_k(\mu^M)$. $\square$

Reachability. Putting this together, the reachability relation $h_k(\mu)$ for $k_i = 3$ is exactly as in the $k_i = 2$ case, except that μ^M and μ^W now appear in each other's horizon:

Table 4: Reachability $h_k(\mu)$ in the 2×2 diametric case with $k_i = 3$ for all i

μ	$\mu^\emptyset$	μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
$\mu^\emptyset$		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_1}					μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_2}		μ^{m_1, w_1}				μ^M	μ^W
μ^{m_2, w_2}			μ^{m_1, w_2}			μ^M	μ^W
μ^{m_2, w_1}				μ^{m_2, w_2}		μ^M	μ^W
μ^M							μ^W
μ^W						μ^M	

PH- k stability.

Proposition 12. In the 2×2 diametric opposition example with $k_M = k_W = 3$, there are exactly two PH- k stable sets:

$$V_1 = \{\mu^M\} \quad \text{and} \quad V_2 = \{\mu^W\}.$$

So we say $\mathcal{V} = \{\{\mu^M\}, \{\mu^W\}\}$.

7.2 Heterogeneous horizons

We now consider heterogeneous horizon profiles in which men and women have different (but within-group homogeneous) horizons, that is, $k_{m_1} = k_{m_2} = k_M$ and $k_{w_1} = k_{w_2} = k_W$. Note that the case $k_W = 3$, $k_M = 1$ was already analyzed in detail earlier, where we showed the unique PH- k stable set is $\{\mu^W\}$. We include additional heterogeneous cases here to complete the classification.

Case $k_W = 2$, $k_M = 1$.

We now consider a heterogeneous horizon profile in which women are 2-farsighted and men are myopic:

$$k_{w_1} = k_{w_2} = 2, \quad k_{m_1} = k_{m_2} = 1.$$

Relative to the homogeneous $k_i = 2$ case, the only difference is that men have a shorter horizon. In this particular example there are no new admissible paths relative to the homogeneous $k_i = 2$ case.

Admissible one-step moves. Every one-step move that was admissible for $k_i = 2$ remains admissible for $(k_W, k_M) = (2, 1)$, since k is irrelevant to one-step paths.

Multi-step paths and potential new reachability. Any new reachability compared to the homogeneous $k_i = 2$ case under $(k_W, k_M) = (2, 1)$ would have to come from a multi-step path that was blocked when $k_M = 2$ because a man anticipated being worse off at his horizon, but that becomes admissible when he only evaluates the next step. In this example, the multi-step paths that are inadmissible in the homogeneous $k_i = 2$ case are blocked because some *woman* compares her partner at her evaluation node to her current partner and refuses the move. Since k_W remains 2, the same comparisons apply here, and these paths continue to be blocked. On the other hand, any path that was admissible under $k_i = 2$ clearly remains admissible. Therefore, the reachability relation under $(k_W, k_M) = (2, 1)$ coincides exactly with the homogeneous $k_i = 2$ case.

Reachability and PH- k stability. The resulting reachability $h_k(\mu)$ is therefore given by the same table as in the homogeneous $k_i = 2$ case. Consequently, the PH- k stable set is characterized exactly as in Proposition 11.

Table 5: Reachability $h_k(\mu)$ in the 2×2 diametric case with $k_W = 2$, $k_M = 1$

μ	$\mu^\emptyset$	μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
$\mu^\emptyset$		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_1}					μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_2}		μ^{m_1, w_1}				μ^M	μ^W
μ^{m_2, w_2}			μ^{m_1, w_2}			μ^M	μ^W
μ^{m_2, w_1}				μ^{m_2, w_2}		μ^M	μ^W
μ^M							
μ^W							

Proposition 13. In the 2×2 diametric opposition example with $k_W = 2$ and $k_M = 1$, the unique PH- k stable set is

$$V = \{\mu^M, \mu^W\}.$$

Case $k_W = 3$, $k_M = 2$.

We now consider a heterogeneous horizon profile in which women are 3-farsighted and men are 2-farsighted:

$$k_{w_1} = k_{w_2} = 3, \quad k_{m_1} = k_{m_2} = 2.$$

A pairwise horizon- k improving path from μ^M to μ^W . Relative to the homogeneous $k_i = 2$ case, there is now a horizon- k improving path, which is the same given by *Lemma 8*. Observe that the argument given in that lemma holds for any k_{m_1} or k_{m_2} . Hence $\mu^W \in h_k(\mu^M)$ when $(k_W, k_M) = (3, 2)$.

No admissible path from μ^W to μ^M . We show that $\mu^M \notin h_k(\mu^W)$ under $(k_W, k_M) = (3, 2)$.

Any path from μ^W to μ^M must at some point destroy the pair (m_1, w_2) (otherwise m_1 remains matched to w_2 throughout and can never be matched to w_1 as required in μ^M). At any node where (m_1, w_2) is present, the destroy move $-(m_1, w_2)$ cannot be supported by w_2 , since w_2 is matched to her top choice m_1 and therefore cannot improve along any continuation in which she becomes single at the next node. Hence the only possible supporter of destroying (m_1, w_2) is m_1 .

Consider any candidate horizon- k improving path from μ^W to μ^M and let ℓ be the first index at which (m_1, w_2) is destroyed. Immediately after this step, m_1 is single. Since $k_{m_1} = 2$, he evaluates no later than $r_{m_1}(\ell) = \ell + 2$. But to reach μ^M one must first destroy (m_2, w_1) and then form (m_1, w_1) , so m_1 cannot be guaranteed to be matched with w_1 within two steps of destroying (m_1, w_2) . Therefore, within m_1 's two-step evaluation window, he evaluates himself as single and strictly prefers keeping w_2 to participating in

the destroy move. Hence the initial destroy step is not horizon-admissible for m_1 , and no horizon- k improving path from μ^W to μ^M exists. Thus $\mu^M \notin h_k(\mu^W)$.

Reachability and PH- k stability. Putting this together, the reachability relation $h_k(\mu)$ for $(k_W, k_M) = (3, 2)$ is exactly as in the homogeneous $k_i = 2$ case, except that μ^W now appears in the horizon of μ^M :

Table 6: Reachability $h_k(\mu)$ in the 2×2 diametric case with $k_W = 3$, $k_M = 2$

μ	$\mu^\emptyset$	μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
$\mu^\emptyset$		μ^{m_1, w_1}	μ^{m_1, w_2}	μ^{m_2, w_2}	μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_1}					μ^{m_2, w_1}	μ^M	μ^W
μ^{m_1, w_2}		μ^{m_1, w_1}				μ^M	μ^W
μ^{m_2, w_2}			μ^{m_1, w_2}			μ^M	μ^W
μ^{m_2, w_1}				μ^{m_2, w_2}		μ^M	μ^W
μ^M							μ^W
μ^W							

Proposition 14. In the 2×2 diametric opposition example with $k_W = 3$ and $k_M = 2$, the unique PH- k stable set is

$$V = \{\mu^W\}.$$

Case $k_{w_1} = 3$ and $k_{m_1} = k_{m_2} = k_{w_2} = 1$.

We now consider the heterogeneous horizon profile in which only woman w_1 is 3-farsighted and all other agents are myopic:

$$k_{w_1} = 3, \quad k_{w_2} = 1, \quad k_{m_1} = k_{m_2} = 1.$$

The key difference relative to the homogeneously myopic case is that w_1 can now support an initial destroy move from μ^M , generating a horizon- k improving path to μ^W . As before, this implies μ^W becomes the unique PH- k stable set.

A horizon- k improving path from μ^M to μ^W . We exhibit a horizon- k improving path from μ^M to μ^W in this horizon profile. Consider the path $P = (\nu_0, \nu_1, \nu_2, \nu_3)$ of length $L = 3$ defined by

$$\begin{aligned} \nu_0 &= \mu^M, \\ \nu_1 &= \nu_0 - (m_1, w_1), \\ \nu_2 &= \nu_1 + (m_1, w_2) = \mu^{m_1, w_2}, \\ \nu_3 &= \nu_2 + (m_2, w_1) = \mu^W. \end{aligned}$$

For myopic agents $i \in \{m_1, m_2, w_2\}$ we have $r_i(\ell) = \ell + 1$, and for w_1 we have $r_{w_1}(\ell) = \min\{L, \ell + 3\} = 3$ for $\ell = 0, 1, 2$.

Step $\nu_0 \rightarrow \nu_1$ (destroy (m_1, w_1)). Man m_1 is worse off at the next node, but woman w_1 evaluates at $\nu_{r_{w_1}(0)} = \nu_3$, where she is matched with m_2 . Since $m_2 \succ_{w_1} m_1$, we have $\nu_3(w_1) \succ_{w_1} \nu_0(w_1)$, so the destroy step is admissible.

Step $\nu_1 \rightarrow \nu_2$ (form (m_1, w_2)). At ν_1 , m_1 and w_2 are single. Since both are myopic, they compare at $\ell + 1$. At ν_2 , each is matched and strictly prefers being matched to being single, so this *form* step is admissible.

Step $\nu_2 \rightarrow \nu_3$ (form (m_2, w_1)). At ν_2 , both m_2 and w_1 are single. With partner preference, each strictly prefers being matched to the other to being single, so this *form* step is admissible (for m_2 myopically and for w_1 also at her horizon).

Therefore, $\mu^W \in h_k(\mu^M)$ in this horizon profile.

No horizon- k path from μ^W to μ^M . Any path from μ^W to μ^M must destroy, for example, (m_1, w_2) . Woman w_2 will not support destroying (m_1, w_2) since she is matched to her top choice m_1 , and myopic m_1 will not support it since he becomes single at the next node. The same is symmetrically true for (m_2, w_1) . Hence no admissible first step exists, so $\mu^M \notin h_k(\mu^W)$.

Proposition 15. In the 2×2 diametric opposition example with $k_{w_1} = 3$ and $k_{m_1} = k_{m_2} = k_{w_2} = 1$, the unique PH- k stable set is

$$V = \{\mu^W\}.$$

7.3 Summary

We summarize the findings of this section in Table 7.

Table 7: Classification of PH- k stable sets in the 2×2 diametric opposition case

Horizon	Unique PH- k stable set(s)	$ \mathcal{V} $
$k_M = k_W = 1$ (all myopic)	$\{\mu^M, \mu^W\}$	1
$k_M = k_W = 2$	$\{\mu^M, \mu^W\}$	1
$k_M = k_W = 3$	$\{\mu^M\}$ and $\{\mu^W\}$	2
$k_M = 1, k_W = 2$	$\{\mu^M, \mu^W\}$	1
$k_M = 1, k_W = 3$	$\{\mu^W\}$	1
$k_M = 2, k_W = 3$	$\{\mu^W\}$	1
$k_{w_1} = 3$, all others myopic	$\{\mu^W\}$	1

Several patterns are clear from this classification. First, when all agents share the same horizon, the threshold $k_i = 3$ for some i marks a transition in this 2×2 example: when $k_i \leq 2$ for all i , both set-optimal matchings coexist in a single PH- k stable set, whereas when at least one agent has farsightedness 3 we tend to see variance in the set in

this case. This reflects the fact that farsightedness of at least 3 is required to support the destroy–destroy–form–form path structure that allows transitions between μ^M and μ^W .

Second, in the heterogeneous cases, the side with greater farsightedness may obtain an advantage. When $k_W = 3$ and $k_M = 2$, the unique PH- k stable set is $\{\mu^W\}$; by symmetry, $k_M = 3$ and $k_W = 2$ would yield $\{\mu^M\}$. The case $k_{w_1} = 3$ with all other agents myopic demonstrates that a single sufficiently farsighted agent can shift the stable set in favor of her preferred outcome. This asymmetry arises because farsighted agents can support destroy moves that myopic agents would never support, thereby enabling paths, even involving myopic agents within the same set, toward their preferred matching.

These findings complement the general results established elsewhere, where we show in this partner-preferred case that farsightedness of at least 3 combined with first-choice injectivity guarantees the set-optimal matching as a PH- k stable set. The 2×2 diametric opposition case, while simple, captures the essential mechanism by which heterogeneous farsightedness confers strategic advantage in two-sided matching, and is also a direct example of our previous swap preference result in *Proposition 6*.

8 Conclusion and Further Questions

Building on the von Neumann–Morgenstern approach of Herings, Mauleon, and Vannetelbosch (2020) and the more recent literature on limited horizons (Herings, Mauleon, and Vannetelbosch, 2025), we introduced the PH- k stable set for marriage problems. Within this framework, we characterized pairwise horizon- k improving paths, established basic algorithmic properties in the myopic case, and ultimately obtained individual rationality results for PH- k stable sets when all agents are myopic and, more generally, for all myopic agents in matching problems with general horizons, as well as additional special cases.

We also showed by example that if PH- k stable sets are applied as a set-valued solution concept, this solution is not in general strategy-proof: an agent can change favorably the PH- k stable set by misreporting preferences.

Perhaps our most conceptually interesting insight is that even modest degrees of finite farsightedness can generate a persistent advantage for sufficiently farsighted agents when horizons are heterogeneous. In the 2×2 diametric example we showed that, with myopic men and women who can look only three steps ahead, the woman-optimal deferred-acceptance matching becomes a PH- k stable set in a way that parallels the fully farsighted case in Herings, Mauleon, and Vannetelbosch (2020). We also formalize this structure by showing that, under first-preference injectivity and sufficient farsightedness of, for example women, and partner-preferred men, that the woman-preferred matching is guaranteed to be a PH- k stable set. This also shows some possible underlying relationship between the PH- k stable set and woman-optimal and man-optimal core elements guaranteed by Gale and Shapley (1962). However, we have left important questions unaddressed.

A first open question concerns the existence of PH- k stable sets. Throughout the thesis we have focused on explicit constructions of PH- k stable sets in small markets and under particular structural assumptions with respect to preferences and horizons. In all examples considered so far, at least one PH- k stable set exists. Additionally, we have shown that whenever a side is first-choice injective and sufficiently farsighted (with the opposite side partner-preferred), the PH- k stable set exists and contains at least one of the woman-optimal or man-optimal stable matchings generated by Deferred Acceptance. However, these results do not yet constitute a general existence theorem: it remains open whether a PH- k stable set exists for arbitrary horizon vectors k and arbitrary preference profiles.

A second natural question concerns the relationship between PH- k stability and individual rationality. Both core stability and previous work on farsighted–myopic stable sets typically imply individual rationality, since agents can always unilaterally deviate to being single (Herings, Mauleon, and Vannetelbosch, 2017). For the PH- k framework, we established individual rationality in several important cases (including the fully myopic case and related extensions), but we have not been able to show that *every* element of a

PH- k stable set must be individually rational, nor have we found a counterexample.

Our main difficulty is that the standard contradiction arguments used previously by Herings, Mauleon, and Vannetelbosch, for myopic or fully farsighted settings do not automatically extend to limited horizons: when admissibility is evaluated at a finite horizon node, concatenating a path to reproduce the relevant contradiction may fail if a key step is only admissible relative to a particular foreseeable horizon (see *section 4*). It therefore remains open whether a PH- k stable set can contain a matching in which some agent is matched to a not individually rational partner.

Finally, an important direction for future work would be whether an algorithm to a PH- k stable set can be guaranteed or shown in general. Doğan and Ehlers (2023) establish a close connection between the EADA algorithm (an adaptation of Deferred Acceptance) and the farsighted-myopic stable set, a previously mentioned special case of our framework in which each agent is either fully farsighted or myopic. This raises the question of whether PH- k stable sets can be characterized algorithmically in a comparable way.

Our results provide some intuitive evidence in this direction. In particular, we show that when one side is sufficiently farsighted and first-choice injective (and the other side partner-preferred), the corresponding man-optimal or woman-optimal stable matching (an outcome of Deferred Acceptance, hence a core element) emerges through an algorithm as a singleton PH- k stable set. This suggests that PH- k stability may capture DA outcomes or outcomes of some other algorithm.

However, it remains unclear whether such a relationship can be established more generally. In non-FCI settings, the structure of admissible paths becomes significantly more complex. The development of an algorithmic procedure that relaxes the FCI restrictions while still yielding outcomes that coincide with PH- k stable sets would be a significant and interesting result.

In summary, our analysis suggests that modeling heterogeneity in constrained farsighted behavior may be informative for applications where the ability to anticipate and coordinate along rationally optimizing paths is possibly unevenly distributed across agents.

9 References

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