# Optimal Portfolio Allocation with ESG Dynamics: A Stochastic Approach

by

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# Contents

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# Management Summary

Sustainable investing has become a critical topic in financial decision-making, with investors increasingly considering Environmental, Social, and Governance (ESG) factors in portfolio allocation (Global Sustainable Investment Alliance, 2022). This thesis develops a stochastic portfolio optimization model that integrates ESG preferences using a Hamilton-Jacobi-Bellman framework. The model seeks to determine an optimal asset allocation strategy that balances financial returns and ESG considerations while maintaining a mathematically rigorous foundation.

The ESG scores of assets are modeled as a Geometric Brownian Motion, reflecting their uncertain yet persistent nature in financial markets. The investment strategy is derived analytically, providing a closed-form solution for portfolio weights over time. The key finding is that the optimal allocation strategy remains constant over time, meaning that investors do not need to dynamically adjust their portfolio in response to ESG fluctuations. This stability is advantageous because it minimizes transaction costs and ensures a consistent approach to sustainable investing.

Empirical validation is performed using stock return and ESG score data, combined with Fama-French five-factor regressions to assess the financial impact of ESG factors. The results confirm that ESG preferences significantly influence portfolio decisions, leading to different allocations depending on an investor's risk aversion and sustainability preferences. This thesis contributes to the field of responsible investing by bridging financial optimization theory with ESG considerations in a novel way. The analytical framework provides a foundation for future research, which could explore alternative ESG dynamics, such as mean-reverting processes, or incorporate additional market frictions like transaction costs and liquidity constraints.

# 1 Introduction

The integration of Environmental, Social, and Governance (ESG) factors in financial decision-making has gained significant traction in recent years (Global Sustainable Investment Alliance, 2022). Investors are increasingly incorporating sustainability criteria into their investment strategies, reflecting both regulatory pressures and shifting societal preferences. The role of ESG in asset pricing, portfolio construction, and risk management has been widely studied (Albuquerque et al., 2019; Giese et al., 2019; Pástor et al., 2021; Pedersen et al., 2021), yet optimal allocation strategies that balance financial returns with ESG considerations remain an active area of research.

ESG investing has demonstrated its ability to influence financial performance, both positively and negatively. Empirical studies suggest that companies with strong ESG profiles exhibit lower risk and more stable returns (Giese et al., 2019), yet trade-offs exist between ESG considerations and financial objectives (Pedersen et al., 2021). The challenge for investors is to optimally allocate capital in a manner that aligns with both sustainability goals and expected financial returns.

While traditional portfolio optimization models focus on maximizing expected utility based on risk and return, they often neglect ESG preferences (Markowitz, 1952; Merton, 1969). This research aims to develop a stochastic optimization framework that explicitly incorporates ESG factors into portfolio allocation. A key challenge is modeling the dynamics of ESG scores and their interaction with asset prices, given their stochastic nature.

The primary objectives of this research are to model the dynamics of ESG scores using a stochastic process, specifically a Geometric Brownian Motion, to develop a theoretical framework for portfolio optimization that integrates ESG preferences, to derive an analytical solution using the Hamilton-Jacobi-Bellman equation, to conduct empirical analysis on historical ESG scores and stock returns to estimate key parameters, and to simulate and analyze the optimal allocation strategy under varying investor preferences.

Based on these objectives, the central research question of this thesis is formulated as follows.

*"How can a stochastic portfolio optimization framework be developed that integrates ESG factors to optimize asset allocation while balancing financial returns and sustainability preferences?"*

This research employs a combination of empirical and theoretical methods. The empirical component estimates the ESG dynamics and the characteristics of the stock return using regression and time series models. The theoretical component formulates the portfolio optimization problem as a stochastic control problem, solved using the Hamilton-Jacobi-Bellman equation.

This study contributes to the literature by developing a stochastic model for ESG scores and integrating it into a portfolio optimization framework, providing an analytical solution to the ESG-based portfolio allocation problem, offering empirical insights into the interaction between ESG scores and stock returns, and highlighting the implications of ESG investing for risk-return trade-offs.

The remainder of this thesis is structured as follows. Chapter 2 reviews the existing literature on ESG investing, portfolio optimization, and stochastic modeling. Chapter 3 describes the data set and data preparation and provides important insights in the data. Chapter 4 presents the empirical methods used to estimate key parameters. Chapter 5 presents the theoretical framework, including the stochastic modeling of ESG scores and the formulation of the optimization problem. Chapter 6 discusses the results of the empirical and theoretical analyses. Chapter 7 concludes the thesis with key takeaways and suggestions for future research.

## 2 Literature Review

The integration of ESG factors into financial decision-making has gained increasing attention in both academic research (Albuquerque et al., 2019; Giese et al., 2019; Pástor et al., 2021; Pedersen et al., 2021) and investment practice (Global Sustainable Investment Alliance, 2022). Investors are no longer solely focused on financial returns but also consider broader sustainability and risk mitigation aspects in their portfolios. As regulatory frameworks evolve and investor demand for ESG-aligned assets increases, the role of ESG factors in asset pricing, portfolio construction, and corporate financial performance is becoming more significant (Friede et al., 2015; Pástor et al., 2021). This section reviews the empirical evidence on ESG and financial performance, explores how ESG preferences can be incorporated into utility-based decision-making, and presents the theoretical advancements in ESG portfolio optimization.

Empirical research suggests that ESG factors influence both stock returns and firm risk. Friede et al. (2015) provide one of the most extensive meta-analyses in the field, aggregating findings from over 2200 individual studies. They conclude that approximately 90% of studies report either a neutral or positive relationship between ESG and corporate financial performance (CFP), with the majority showing a statistically significant positive correlation. Their findings indicate that ESG considerations are not necessarily a constraint on returns but may enhance long-term value creation, particularly in emerging markets and asset classes such as corporate bonds and sustainable real estate.

Pástor et al. (2021) extend this empirical evidence by developing a tractable equilibrium model that demonstrates how ESG preferences influence asset prices, portfolio holdings, and real economic outcomes. They show that green stocks have lower expected returns due to investors' preferences and their climate risk-hedging properties, but these stocks can outperform when ESG concerns unexpectedly increase. Additionally, sustainable investing leads to greener corporate behavior and a positive social impact by shifting investments from brown to green firms (Pástor et al., 2021).

Aswani et al. (2024) find that prior findings linking carbon emissions to returns are primarily driven by vendor-estimated emissions, which correlate with firm fundamentals rather than actual carbon performance. They argue that scaling emissions by firm size provides a more reliable measure, as unscaled emissions are mechanically correlated with productivity and size.

From a risk management perspective, ESG factors can enhance financial stability. Giese et al. (2019) analyze ESG scores within a discounted cash flow framework and identify three key transmission channels: (1) the cash-flow channel, where firms with strong ESG practices exhibit more stable earnings and operational efficiency; (2) the idiosyncratic risk channel, where ESG mitigates firm-specific risks such as legal disputes or reputational damage, leading to lower downside risk; and (3) the valuation channel, where ESG integration results in a lower cost of capital and higher firm valuations. Their findings support the argument that ESG considerations provide risk-mitigation benefits, particularly for long-term investors.

Engle et al. (2020) further explore how ESG factors interact with financial markets by constructing hedge portfolios that mitigate exposure to climate risk. Some studies highlight potential challenges and trade-offs in responsible investing. Kruger (2015) examines how stock markets react to corporate events and find that negative incidents, such as governance failures or environmental controversies, lead to significant stock price declines, whereas positive news does not always generate strong market reactions. This asymmetric pricing effect suggests that investors penalize poor performance more than they reward improvements. Furthermore, Bolton and Kacperczyk (2021) investigate the role of carbon emissions in asset pricing and find that firms with higher carbon dioxide emissions earn higher expected returns. Their results indicate that investors demand compensation for exposure to climate risk, creating a carbon premium that challenges the assumption that investing in ESG improves financial performance.

Albuquerque et al. (2019) develop an industry equilibrium model in which Corporate Social Responsibility (CSR) serves as a product differentiation strategy, leading to higher profit margins, lower systematic risk and increased firm value. Their empirical analysis supports this mechanism and shows that firms with higher CSR have a lower cost of equity, which can reduce overall portfolio risk.

Investor preferences for ESG can be integrated into portfolio models through utility functions. Epstein and Zin (1989) introduce recursive utility functions that separate risk aversion from intertemporal substitution, allowing investors to optimize their portfolios based on both financial returns and sustainabilit objectives. Abel (1990) further develops habit formation models, suggesting that investors' preferences evolve dynamically based on past investment experiences. These models provide a theoretical foundation for understanding the behavioral dynamics of ESG investing beyond traditional risk-return considerations.

A major contribution to ESG portfolio theory is the ESG-efficient frontier developed by Pedersen et al. (2021). Their model extends the traditional mean-variance framework by explicitly incorporating ESG preferences into portfolio selection. They demonstrate that ESG factors influence asset pricing through two mechanisms: (1) fundamental information, where ESG reflects firm quality and risk exposure, and (2) investor preferences, where demand for ESG-aligned assets alters equilibrium returns. Their key result is a four-fund separation theorem, where the optimal ESG portfolio is constructed as a combination of: 1. The risk-free asset (safe investments), 2. The traditional tangency portfolio (optimal risk-return trade-off), 3. The minimum-variance portfolio (low-risk assets), and 4. An

ESG-tilted tangency portfolio (a sustainability-adjusted efficient portfolio).

This framework allows investors to allocate capital based on both financial and sustainability considerations, showing that responsible investing can be optimal even within a risk-return framework. Their empirical findings confirm that ESG considerations impact equilibrium asset prices, affecting expected returns and risk premia (Pedersen et al., 2021).

Overall, the literature highlights the increasing role of ESG in financial decision-making, asset pricing, and portfolio optimization. Empirical studies support the notion that ESG factors enhance risk-adjusted returns, although challenges such as measurement inconsistencies, sector biases, and the carbon premium effect remain. Theoretical advancements in utility-based ESG investing and the ESG-efficient frontier provide a robust foundation for integrating sustainability preferences into investment decisions. As ESG investing continues to evolve, understanding these dynamics will be crucial for investors seeking to balance financial performance with sustainability goals.

# 3 Data

In this research, we utilize a comprehensive dataset to examine the relationship between ESG scores and stock returns, alongside traditional financial factors. This section provides details on the data sources, preprocessing steps, statistical properties and summary statistics.

## 3.1 Data Description

This research utilizes historical data from S&P 500 companies to analyze the relationship between ESG scores and stock returns. The stock price data, sourced from Refinitiv, spans monthly prices from 2011 to December 2023, while ESG scores, also obtained from Refinitiv, cover the period from 2010 to 2022. Additionally, the Fama-French 5 factors—market return, size (SMB), value (HML), profitability (RMW), and investment style (CMA)—were sourced from the Kenneth French online data library and are included in the model to explain stock returns alongside the ESG factor. The risk-free rate (Rf), which is used to calculate excess returns, was also sourced from the Kenneth French online data library. This ensures consistency in the financial data used for the analysis, aligning the risk-free rate with the Fama-French factors for a cohesive approach.

From the 500 companies in the S&P 500, 386 were included in the analysis. This selection was made based on the availability of complete data for both ESG scores and stock prices, ensuring that the dataset used for analysis was free from missing values.

In preparing the data for analysis, since ESG scores are typically reported annually, they were expanded to a monthly frequency to match the stock price data. This was done by repeating the annual ESG score for each month within the respective year, allowing for a consistent temporal alignment with the stock prices.

The analysis period was set from April 2011 to December 2023 for stock prices, while the ESG scores cover January 2010 to December 2022. This difference in the time period is due to the lagged use of ESG scores, as they are typically published annually in March and used from April onwards. For example, the ESG score for 2010 is published in March 2011 and used in investment decisions starting from April 2011. This results in a lag of approximately 15 months from the end of the year for which the ESG score applies to when it begins to impact investment decisions.

The Fama-French factors were synchronized with the stock price data by aligning the dates, ensuring that all datasets were consistent in terms of time frequency and coverage. These steps ensured that the data were thoroughly prepared and aligned, providing a solid foundation for the models and analyses presented in this research.

## 3.2 Summary Statistics

In this section, we present the summary statistics of some key variables used in our analysis: the ESG scores and the FF5 factors.

Table 1 shows the number of companies periods and the distribution of companies across industries.

Table 1: Number of Companies, Periods, and Companies per Industry

| **Industry** | **Number of Companies** |
|---|---|
| Communication Services | 12 |
| Consumer Discretionary | 43 |
| Consumer Staples | 28 |
| Energy | 17 |
| Financials | 55 |
| Health Care | 46 |
| Industrials | 54 |
| Information Technology | 53 |
| Materials | 22 |
| Real Estate | 26 |
| Utilities | 29 |
| **Total** | **386** |

Table 2 presents the yearly summary statistics of the ESG scores.

Table 2: Yearly Summary Statistics of the ESG Scores

| **Year** | **Mean** | **Median** | **StdDev** | **Min** | **Max** | **P25** | **P75** |
|---|---|---|---|---|---|---|---|
| 2010 | 49.17 | 48.38 | 20.04 | 2.81 | 95.16 | 32.89 | 65.85 |
| 2011 | 50.90 | 50.41 | 19.82 | 2.96 | 92.54 | 35.75 | 67.17 |
| 2012 | 51.45 | 51.16 | 19.19 | 1.90 | 91.31 | 36.60 | 67.40 |
| 2013 | 52.28 | 52.36 | 18.80 | 2.49 | 92.28 | 38.85 | 67.78 |
| 2014 | 53.17 | 53.94 | 18.13 | 2.46 | 92.39 | 39.07 | 67.67 |
| 2015 | 56.62 | 57.51 | 17.29 | 5.81 | 92.84 | 43.65 | 70.06 |
| 2016 | 59.32 | 60.62 | 16.33 | 15.45 | 91.29 | 46.72 | 71.31 |
| 2017 | 61.94 | 63.29 | 15.67 | 14.18 | 92.01 | 51.59 | 72.97 |
| 2018 | 63.44 | 65.53 | 15.24 | 20.87 | 93.03 | 52.49 | 74.54 |
| 2019 | 65.38 | 67.18 | 14.23 | 23.81 | 92.85 | 55.95 | 76.08 |
| 2020 | 66.89 | 69.49 | 13.94 | 22.39 | 93.24 | 59.15 | 76.90 |
| 2021 | 68.17 | 69.96 | 12.86 | 23.88 | 92.77 | 60.72 | 77.18 |
| 2022 | 68.81 | 70.62 | 11.77 | 24.80 | 91.71 | 61.90 | 77.13 |

Table 3 shows the drift ($\mu_\theta$) and volatility ($\sigma_\theta$) of raw ESG scores across industries and the average $\mu_\theta$ and $\sigma_\theta$ over all companies.

Table 3: ESG Drift and Volatility per Industry

| **Industry** | $\mu_\theta$ | $\sigma_\theta$ |
|---|---|---|
| Communication Services | 0.0050 | 0.0153 |
| Consumer Discretionary | 0.0040 | 0.0127 |
| Consumer Staples | 0.0019 | 0.0083 |
| Energy | 0.0028 | 0.0111 |
| Financials | 0.0037 | 0.0125 |
| Health Care | 0.0049 | 0.0161 |
| Industrials | 0.0050 | 0.0353 |
| Information Technology | 0.0038 | 0.0148 |
| Materials | 0.0032 | 0.0105 |
| Real Estate | 0.0074 | 0.0325 |
| Utilities | 0.0020 | 0.0098 |
| **Mean** | 0.0041 | 0.01947 |

Table 4 presents the summary statistics of the FF5 factors.

Table 4: Summary Statistics of FF5 Factors

| | **Mkt-RF** | **SMB** | **HML** | **RMW** | **CMA** | **RF** |
|---|---|---|---|---|---|---|
| Mean | 0.998 | -0.106 | -0.095 | 0.321 | 0.019 | 0.00077 |
| Median | 1.270 | -0.060 | -0.370 | 0.300 | -0.110 | 0.00010 |
| StdDev | 4.394 | 2.671 | 3.378 | 2.035 | 2.163 | 0.00120 |
| Min | -13.380 | -8.280 | -13.830 | -4.760 | -6.810 | 0.00000 |
| Max | 13.650 | 7.320 | 12.880 | 7.200 | 7.740 | 0.00470 |
| P25 | -1.600 | -1.900 | -1.845 | -1.183 | -1.375 | 0.00000 |
| P75 | 3.410 | 1.438 | 1.405 | 1.555 | 1.250 | 0.00130 |

These tables provide insights into the characteristics of the dataset, including trends in ESG scores, variation across industries, and the distribution of factor returns over time.

# 4 Empirical Analysis

This section presents the methodological framework used to investigate the relationship between ESG factors and asset returns. First, a regression analysis incorporating an ESG component is performed to estimate the drift of asset returns. Next, the volatility is derived from the historical data. Finally, the upward trend in ESG scores is investigated to determine the most appropriate dynamics, such as a Geometric Brownian Motion, for the analytical model. This empirical analysis estimates key parameters that are then incorporated into the analytical model.

## 4.1 Regression Analysis

The regression analysis in this study serves two primary purposes. Firstly, it investigates whether there exists a meaningful relationship between ESG factors and asset returns. By integrating ESG scores into an extended version of the Fama-French 5-factor model (Fama & French, 2015), the analysis provides insights into how sustainability practices influence financial performance while accounting for established risk factors. Subsequently, it is used to estimate the drift term ($\mu_S$) for each asset, which represents the expected return needed for simulations in the analytical model.

The extended Fama-French model is expressed as follows:

$$R_{i,t} - r_t^f = \alpha_i + \beta_{i,\text{Mkt}} \cdot \text{Mkt-Rf}_t + \beta_{i,\text{SMB}} \cdot \text{SMB}_t + \beta_{i,\text{HML}} \cdot \text{HML}_t +$$

$$\beta_{i,\text{RMW}} \cdot \text{RMW}_t + \beta_{i,\text{CMA}} \cdot \text{CMA}_t + \beta_{i,\text{ESG}} \cdot \text{ESG}_{t-15} + \epsilon_{i,t},$$

where the left-hand side, $R_{i,t} - r_t^f$, represents the historical excess return of asset $i$ at time $t$, defined as the return above the risk-free rate. The terms on the right-hand side include established risk factors as well as an ESG component. Specifically, the model accounts for market exposure ($\text{Mkt-Rf}_t$), size ($\text{SMB}_t$), value ($\text{HML}_t$), profitability ($\text{RMW}_t$), and investment style ($\text{CMA}_t$), which are the traditional components of the Fama-French 5-factor model. To extend this framework, a lagged ESG score ($\text{ESG}_{t-15}$) is added to capture the influence of sustainable practices on asset returns. This lagged ESG factor reflects the practical delay between the publication of ESG scores and their incorporation into investment decisions, ensuring that the model captures the financial impact of ESG practices in a realistic manner.

Each coefficient in the regression represents the sensitivity of the asset's return to a specific factor. The coefficients $\beta_{i,\text{Mkt}}, \beta_{i,\text{SMB}}, \beta_{i,\text{HML}}, \beta_{i,\text{RMW}}, \beta_{i,\text{CMA}}$ measure the exposure of asset $i$ to the traditional risk factors, while $\beta_{i,\text{ESG}}$ captures the sensitivity to ESG considerations. The intercept term $\alpha_i$ accounts for any return component that is not explained by the included factors, and the residual term $\epsilon_{i,t}$ captures the unexplained variation in returns. Estimating these coefficients allows us to decompose asset returns into components attributable to systematic risks, ESG considerations, and idiosyncratic effects.

The inclusion of the ESG component enables the analysis to explore whether ESG scores have a statistically significant relationship with asset returns. By examining the significance and magnitude of $\beta_{i,\text{ESG}}$, we can evaluate whether ESG considerations influence financial performance beyond what is captured by traditional risk factors. This is critical for understanding the potential role of sustainability in asset pricing.

### 4.2 Estimation of $\mu_S$

In addition to investigating the relationship between ESG factors and asset returns, the regression model is also used to estimate the drift term ($\mu_S$) for each asset. The drift term represents the expected growth rate of the asset's price over time and is a key input for the simulation of stock prices in the analytical model. Once the regression coefficients are estimated, $\mu$ is calculated by applying these coefficients to the historical averages of the corresponding factors. This process combines the intercept ($\alpha_i$) and the sensitivity to each factor ($\beta$) with their historical averages ($E[\cdot]$) to obtain the expected return for asset $i$:

$$\hat{\mu_i} = \alpha_i + \beta_{i,\text{Mkt}} \cdot E[\text{Mkt-Rf}] + \beta_{i,\text{SMB}} \cdot E[\text{SMB}] + \beta_{i,\text{HML}} \cdot E[\text{HML}] +$$

$$\beta_{i,\text{RMW}} \cdot E[\text{RMW}] + \beta_{i,\text{CMA}} \cdot E[\text{CMA}] + \beta_{i,\text{ESG}} \cdot E[\text{ESG}].$$

In this equation, $E[\cdot]$ represents the historical average of each factor over the period of analysis. The resulting $\hat{\mu_i}$ reflects the asset's expected return, incorporating both systematic risks and the specific influence of ESG factors. This estimated drift term is subsequently used in the analytical model to simulate stock price dynamics.

### 4.3 Regression Checks

To ensure the reliability and validity of the regression model, several diagnostic tests were performed. The White test was used to check for heteroscedasticity, ensuring that the variance of the residuals remains constant across all levels of the independent variables. A violation of this assumption would indicate heteroscedasticity, potentially leading to inefficient coefficient estimates. The Durbin-Watson test was conducted to detect the presence of autocorrelation in the residuals, which would violate the assumption of independence. This is particularly important when working with time series data, as residual autocorrelation can distort the standard errors of the coefficients. Additionally, the normality of the residuals was assessed using visual inspection through a histogram, ensuring that the residuals are approximately normally distributed—a key assumption for hypothesis testing in ordinary least squares regression. Finally, the linearity assumption was evaluated by examining scatter plots of residuals versus predicted values to confirm that the model adequately captures the linear relationship between the dependent and independent variables. These diagnostic tests collectively ensured that the regression results are robust and the estimated coefficients can be interpreted with confidence.

### 4.4 Estimation of $\sigma_S$

The volatility ($\sigma_S$) of each asset is estimated directly from the historical monthly log-returns ($\log(S_{t+1}/S_t)$) for each stock. The standard deviation of these returns provides a measure of the asset's total variability, capturing both systematic and idiosyncratic risks. This approach ensures that the estimated volatility reflects the real-world fluctuations in asset prices. Together with the estimated drift ($\mu_S$), these volatilities ($\sigma_S$) form the foundation for the stochastic modeling of asset prices, enabling the analytical solution to be tested under realistic conditions.

### 4.5 Time Series Analysis for the ESG Scores

Understanding the dynamics of ESG scores is crucial for modeling their impact on financial decision-making. To analyze the evolution of ESG scores over time, we investigate their historical trends and stationarity properties. This allows us to determine whether ESG scores exhibit predictable patterns or stochastic behavior that can be effectively captured. First, we examine whether ESG scores follow a systematic upward trend. To assess this, we analyze the historical ESG data. We observe a clear upward trend, representing continuous growth.

To determine whether ESG scores follow a stationary or non-stationary process, we conduct the Augmented Dickey-Fuller (ADF) test.

The results indicate that ESG scores are non-stationary, implying that they exhibit persistent long-term trends. This finding motivates the use of a model that incorporates a positive drift component to reflect the continuous growth in ESG performance.

Given these findings, we model ESG scores using a Geometric Brownian Motion, which allows for both systematic growth and stochastic fluctuations. A Geometric Brownian Motion includes a drift term ($\mu_\theta$), representing the expected growth rate of ESG scores, and a volatility term ($\sigma_\theta$), capturing the variability in ESG score changes. These parameters are estimated from historical data, where $\mu_\theta$ is computed as the average monthly return of ESG scores, and $\sigma_\theta$ is calculated as the standard deviation of these returns.

# 5 Theoretical Model

In this section, we present the theoretical framework that supports the portfolio optimization model. The objective is to derive an optimal investment strategy that balances financial returns with ESG performance. First, we introduce the general setup and utility specification. Next, we detail the stochastic processes governing wealth and the portfolio ESG score, and we derive the optimal portfolio allocation using the Hamilton-Jacobi-Bellman equation. Furthermore, we consider the possibility of modeling ESG scores as a mean-reverting process and discuss the simulation framework that will later be used to evaluate the model's performance.

## 5.1 Portfolio Optimization Model

The model seeks to determine the optimal investment strategy for an investor who aims to maximize overall utility, which depends on both wealth ($W_t$) and the portfolio ESG score ($\theta_{P,t}$).

The modeling approach follows the intertemporal portfolio optimization framework introduced by Merton (1969), who demonstrated how investors dynamically adjust their portfolios to maximize expected utility over time in a continuous-time setting.

To integrate ESG considerations into investment decisions, the portfolio ESG score ($\theta_{P,t}$) is modeled as a Geometric Brownian Motion. This modeling choice reflects the observed behavior of ESG scores, which exhibit long-term growth trends combined with stochastic fluctuations.

The initial focus is on a simplified portfolio comprising a single risky asset and a risk-free asset. This setup allows for the derivation of an analytical solution, which serves as the foundational case for understanding the dynamics of the model. The framework is then extended to include $N$ risky assets with varying ESG characteristics. This generalization introduces diversification into the model, allowing the investor to account for differences in the sustainability performance of individual assets.

The optimization process is guided by the investor's utility function, which explicitly balances preferences for wealth and ESG performance. This utility function enables the investor to make structured trade-offs between financial returns and sustainability objectives, ultimately determining the optimal allocation strategy for the portfolio.

The Hamilton-Jacobi-Bellman (HJB) equation is employed to solve this dynamic optimization problem, following the principles of continuous-time optimization as in Merton (1969).

By solving the Hamilton-Jacobi-Bellman equation, we derive the optimal allocation strategy for $\pi_t$, ensuring that the investor achieves the highest possible utility given their preferences, the stochastic nature of the market, and the dynamics of ESG scores. This analytical solution provides valuable insights into how investors balance traditional financial performance with sustainability objectives under varying market conditions.

#### 5.1.1 Value Function

The Value Function $J(W_t, \theta_{P,t}, t)$ represents the maximum expected utility that an investor can achieve from time $t$ onward, given their current wealth $W_t$ and the portfolio ESG score $\theta_{P,t}$. It reflects the investor's optimal dynamic decisions over time, balancing preferences for financial returns and ESG considerations.

The portfolio ESG score $\theta_{P,t}$ represents the weighted ESG performance of the investor's portfolio. For the single-asset case, this simplifies to:

$$\theta_{P,t} = \pi_t \theta_t,$$

where $\pi_t$ is the proportion of wealth allocated to the risky asset, and $\theta_t$ is the ESG score of the single asset. In the multi-asset case, $\theta_{P,t}$ generalizes to:

$$\theta_{P,t} = \sum_{i=1}^{N} \pi_{i,t}\theta_{i,t},$$

where $\pi_{i,t}$ is the proportion of wealth allocated to the $i$-th asset, and $\theta_{i,t}$ is its ESG score. This formulation ensures that the portfolio ESG score reflects the weighted contributions of all assets based on their ESG characteristics and allocation proportions.

The Utility Function $U(W_t, \theta_{P,t})$ combines wealth and the portfolio ESG score into a Constant Relative Risk Aversion (CRRA) Cobb-Douglas form (Cobb & Douglas, 1928):

$$U(W_t, \theta_{P,t}) = \frac{\left(W_t^{\alpha}\theta_{P,t}^{\beta}\right)^{1-\gamma}}{1-\gamma},$$

where $\alpha$ represents the investor's preference for wealth, $\beta$ captures the preference for higher portfolio ESG scores, and $\gamma$ is the coefficient of relative risk aversion. The parameters satisfy $\alpha + \beta = 1$, ensuring consistent weighting between wealth and ESG performance.

The Value Function $J(W_t, \theta_{P,t}, t)$ aggregates the expected utility over time, incorporating the investor's preferences and market dynamics. It is defined as:

$$J(W_t, \theta_{P,t}, t) = \sup_{\pi_t} E\left[\int_t^T U(W_s, \theta_{P,s})g(t)\, ds \mid W_t, \theta_{P,t}\right],$$

where $T$ is the terminal time, $g(t)$ is a general time-dependent function that discounts utility, reflecting time preferences, $\pi$ is the vector of allocations to the risky assets, and the expectation $E[\cdot]$ is taken over all possible future paths of $W_t$ and $\theta_{P,t}$.

For tractability, the model assumes a stationary solution of the form:

$$J(W_t, \theta_{P,t}, t) = g(t) \frac{\left(W_t^{\alpha} \theta_{P,t}^{\beta}\right)^{1-\gamma}}{1-\gamma},$$

where $g(t)$ captures the time-dependence and ensures consistency with the Hamilton-Jacobi-Bellman equation.

The key parameters of the Utility Function are central to the investor's decision-making process. The parameters $\alpha$ and $\beta$ define the trade-off between wealth and ESG considerations. A higher $\alpha$ indicates a focus on financial returns, while a higher $\beta$ reflects stronger preferences for sustainability. The parameter $\gamma$ governs the investor's risk preferences. For $\gamma > 1$, the investor is risk-averse, while $\gamma = 1$ corresponds to logarithmic utility, and $\gamma < 1$ indicates risk-seeking behavior.

By incorporating $\theta_{P,t}$ into the utility and value functions, this model allows for dynamic optimization that balances traditional financial goals with ESG objectives, providing a flexible framework suitable for both single- and multi-asset portfolios.

### 5.1.2 Wealth Dynamics

The investor's total wealth $X_t$ evolves over time based on their allocation to risky and risk-free assets. A proportion $\pi_t$ of the wealth is invested in the risky asset, while the remaining proportion $(1 - \pi_t)$ is allocated to the risk-free asset, which yields a constant return $r$. The dynamics of the wealth process depend on the evolution of the risky asset's price $S_t$, which follows a Geometric Brownian Motion.

For a single risky asset, the stock price dynamics are given by:

$$dS_t = S_t \left[\mu_S \, dt + \sigma_S \, dW_{1,t}\right],$$

where $\mu_S$ is the expected return (or drift) of the risky asset, $\sigma_S$ is its volatility, and $dW_{1,t}$ represents a standard Brownian motion capturing the uncertainty in price movements. The risky asset introduces stochasticity into the wealth dynamics.

The corresponding wealth dynamics are:

$$dX_t = X_t \left[r + \pi_t(\mu_S - r)\right] dt + X_t \pi_t \sigma_S \, dW_{1,t}.$$

This equation shows that the change in wealth arises from the risk-free return $r$, the excess return $(\mu_S - r)$ from the risky asset, and the volatility of the risky asset scaled by the proportion $\pi_t$.

For $N$ risky assets, the model generalizes to a portfolio, with each risky asset $S_{i,t}$ following its own Geometric Brownian Motion:

$$d\mathbf{S}_t = \mathbf{S}_t \circ \left(\boldsymbol{\mu}_S \, dt + \boldsymbol{\Sigma}_S \, d\mathbf{W}_{1,t}\right).$$

Here, $\mathbf{S}_t$ is the $N \times 1$ vector of risky asset prices, $\boldsymbol{\mu}_S$ is the $N \times 1$ vector of expected returns for these assets, and $\boldsymbol{\Sigma}_S$ is the $N \times N$ covariance matrix capturing the volatilities and correlations of the assets. The symbol $\circ$ denotes element-wise multiplication. The vector $d\mathbf{W}_{1,t}$ represents $N$ independent standard Brownian motions, one for each asset.

The wealth dynamics for $N$ risky assets are then expressed as:

$$dX_t = X_t \left[r + \boldsymbol{\pi}_t^T(\boldsymbol{\mu}_S - r\mathbf{1})\right] dt + X_t \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_S \, d\mathbf{W}_{1,t}.$$

In this case, $\boldsymbol{\pi}_t$ is the $N \times 1$ vector of proportions of wealth allocated to each risky asset, and $\mathbf{1}$ is an $N \times 1$ vector of ones used to distribute the risk-free rate $r$ across all assets. This equation highlights the aggregate impact of the portfolio's expected excess returns and its exposure to stochastic shocks, weighted by the allocation vector $\boldsymbol{\pi}_t$.

In the single-asset case, the dynamics focus on one risky asset with its drift $\mu_S$ and volatility $\sigma_S$. In the multi-asset case, these are replaced by the vector $\boldsymbol{\mu}_S$, which contains the expected returns of all $N$ risky assets, and the covariance matrix $\boldsymbol{\Sigma}_S$, which encodes the volatilities (on the diagonal) and covariances (off-diagonal entries) of the asset returns.

This generalization maintains the structure of the single-asset case while allowing for diversified portfolios and the additional complexities introduced by multiple assets.

### 5.1.3 Dynamics of ESG Scores ($\theta_t$ and $\theta_{P,t}$)

The ESG score ($\theta_t$) serves as a measure of the sustainability performance of assets and plays a critical role in this model. It captures the investor's preference for socially responsible investments and directly influences the utility function. Modeling $\theta_t$ appropriately is crucial for accurately representing its behavior over time. In this model, we assume that ESG scores follow a Geometric Brownian Motion. This choice is motivated by several key factors.

First, ESG scores exhibit proportional growth, where changes are relative to their current level. For instance, companies with lower initial ESG scores may experience larger proportional improvements compared to companies with already high scores, reflecting diminishing returns on sustainability investments. A Geometric Brownian Motion naturally captures this behavior, as its changes are scaled by the current value of $\theta_t$. Second, ESG scores are subject to both predictable trends and random shocks. Predictable trends, such as gradual improvements driven by corporate strategies or regulatory policies, are captured by the drift term of the Geometric Brownian Motion. Random shocks, which arise from events such as controversy or regulatory changes, are modeled through the stochastic term.

Third, the Geometric Brownian Motion aligns with empirical behaviors observed in historical ESG data, which often show stochastic variability and upward trends, particularly in industries with strong regulatory or market incentives for sustainability.

The Geometric Brownian Motion parameters, $\mu_\theta$ (drift) and $\sigma_\theta$ (volatility), offer flexibility to represent different scenarios. A higher $\mu_\theta$ reflects stronger trends in ESG improvement, such as in industries with robust sustainability commitments, while a lower or negative $\mu_\theta$ may represent stagnation or declines in ESG performance. Similarly, $\sigma_\theta$ captures the uncertainty in the evolution of ESG, with higher values indicating greater unpredictability.

In the context of portfolios with multiple assets, the dynamics of ESG scores must also account for the interrelationships between the scores of different assets. Inclusion of covariances between asset-specific ESG scores introduces the possibility of diversification effects in the portfolio ESG score.

The dynamics of the ESG scores, both for individual assets and the overall portfolio, are described as follows. For a single asset, the ESG score evolves according to:

$$d\theta_t = \theta_t \left[\mu_\theta \, dt + \sigma_\theta \, dW_{2,t}\right],$$

where $\mu_\theta$ is the expected growth rate of the ESG score, $\sigma_\theta$ represents the volatility, and $dW_{2,t}$ is a standard Brownian motion. This formulation ensures that $\theta_t$ evolves proportionally to its current level, with both deterministic and stochastic components.

For portfolios consisting of $N$ assets, the ESG scores of individual assets are represented by the vector $\boldsymbol{\theta}_t$, which follows:

$$d\boldsymbol{\theta}_t = \boldsymbol{\theta}_t \circ \left(\boldsymbol{\mu}_\theta \, dt + \boldsymbol{\Sigma}_\theta \, d\mathbf{W}_{2,t}\right),$$

where $\boldsymbol{\mu}_\theta$ is an $N \times 1$ vector of drift terms, $\boldsymbol{\Sigma}_\theta$ is an $N \times N$ covariance matrix capturing the relationships between ESG scores, and $d\mathbf{W}_{2,t}$ is an $N \times 1$ vector of Brownian motions.

The element-wise product $\circ$ ensures that the proportional dynamics are preserved for each individual asset.

The portfolio ESG score ($\theta_{P,t}$) represents the weighted sustainability performance of the portfolio and is a key component in the utility function. It is defined as:

$$\theta_{P,t} = \boldsymbol{\pi}_t^T \boldsymbol{\theta}_t + (1 - \mathbf{1}^T \boldsymbol{\pi}_t) \cdot \theta_{R_f},$$

where $\theta_{R_f}$ is the ESG score of the risk-free asset.

In this model, we assume that the ESG score of the risk-free asset is $\theta_{R_f} = 0$. This reflects the idea that the risk-free asset does not contribute to the portfolio's sustainability performance. Consequently, the portfolio ESG score simplifies to:

$$\theta_{P,t} = \boldsymbol{\pi}_t^T \boldsymbol{\theta}_t.$$

where $\boldsymbol{\theta}_t$ represents the vector of ESG scores for the individual assets in the portfolio and weights determined by the allocation vector $\boldsymbol{\pi}_t$.

This assumption implies that the portfolio ESG score depends entirely on the allocations to risky assets and their respective ESG scores. If a significant portion of the portfolio is allocated to the risk-free asset ($1 - \mathbf{1}^T \pi_t \gg 0$), the portfolio ESG score $\theta_{P,t}$ will naturally decrease.

While the choice of $\theta_{R_f} = 0$ simplifies the model, it introduces an important consideration: investments in the risk-free asset effectively dilute the portfolio's ESG score. This assumption aligns with the idea that sustainability improvements are driven by investments in risky assets with measurable ESG characteristics, rather than by allocations to risk-free assets, which are treated as neutral in this framework.

In practice, this assumption incentivizes higher allocations to risky assets with high ESG scores for investors with strong sustainability preferences. The optimal allocation to risky assets ($\pi_t$) balances the trade-offs between financial risk, returns, and the portfolio's ESG performance, as captured by the utility function.

The dynamics of $\theta_{P,t}$ are given by:

$$d\theta_{P,t} = \theta_{P,t} \left[ \boldsymbol{\pi}_t^T \boldsymbol{\mu}_\theta \, dt + \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_\theta \, d\mathbf{W}_{2,t} \right].$$

By adopting this framework, the model captures both the individual behavior of asset-specific ESG scores and their aggregate impact on the portfolio. The use of Geometric Brownian Motion dynamics ensures mathematical tractability while aligning with empirical characteristics of ESG performance, such as non-negativity, proportional changes, and exposure to both predictable trends and random shocks.

## 5.2 Hamilton-Jacobi-Bellman Equation

To determine the optimal portfolio allocation $\boldsymbol{\pi}_t^*$, the Hamilton-Jacobi-Bellman equation is formulated as:

$$ZJ = \sup_{\boldsymbol{\pi}_t} \left\{ \frac{\partial J}{\partial t} + J_X X_t \left[ \boldsymbol{\pi}_t^\top (\boldsymbol{\mu}_S - r\mathbf{1}) + r \right] + J_{\theta_P} \theta_{P,t} \boldsymbol{\pi}_t^\top \boldsymbol{\mu}_\theta \right.$$

$$\left. + \frac{1}{2} J_{XX} X_t^2 \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S \boldsymbol{\pi}_t + \frac{1}{2} J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t + J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t \right\}.$$

Here, $Z$ is a constant scaling the value function $J(X_t, \theta_{P,t}, t)$. The terminal condition for the value function is:

$$J(X_T, \theta_{P,T}, T) = U(X_T, \theta_{P,T}) = \frac{(X_T^\alpha \theta_{P,T}^\beta)^{1-\gamma}}{1-\gamma},$$

where $\alpha + \beta = 1$, ensuring a balance between the investor's preferences for wealth ($X_t$) and ESG performance ($\theta_{P,t}$) and $\gamma > 0, \gamma \neq 1$, representing the coefficient of relative risk aversion.

The terms $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ capture the correlation structures between risky asset returns and ESG scores:

- $\boldsymbol{P}$ is the $N \times N$ correlation matrix for risky asset returns, with elements $P_{ij}$ representing the correlation between the $i$-th and $j$-th risky assets.
- $\boldsymbol{Q}$ is the $N \times N$ correlation matrix for ESG scores of assets, where $Q_{ij}$ represents the correlation between the $i$-th and $j$-th asset's ESG scores.
- $\boldsymbol{R}$ is the $N \times N$ matrix capturing correlations between the Brownian motions driving the risky asset returns and the ESG scores. Specifically, $R_{ij}$ represents the correlation between the $i$-th risky asset Brownian motion and the $j$-th ESG score Brownian motion.

For example, $\boldsymbol{P}$ could take the form, with $\rho_{i,j}$ the correlation between risky asset $i$ and $j$:

$$\boldsymbol{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{2N} & \cdots & 1 \end{bmatrix}.$$

If the Brownian motions driving the risky asset returns and the ESG scores are uncorrelated, $\boldsymbol{R}$ becomes the zero matrix. This simplifies the cross-term $J_{X\theta_P}$ and decouples the dynamics of $X_t$ and $\theta_{P,t}$.

### 5.3 Optimal Investment Strategy

The objective is to derive the optimal portfolio allocation $\boldsymbol{\pi}_t^*$ that maximizes the value function $J(X_t, \theta_{P,t}, t)$.

To find $\boldsymbol{\pi}_t^*$, we extract the terms in the Hamilton-Jacobi-Bellman equation that depend on $\boldsymbol{\pi}_t$ and take the derivative:

$$J_X X_t \boldsymbol{\mu}_S - r J_X X_t \mathbf{1} + J_{\theta_P} \theta_{P,t} \boldsymbol{\mu}_\theta + J_{XX} X_t^2 \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S \boldsymbol{\pi}_t + J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t$$

$$+ 2 J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t = 0.$$

This results in:

$$\boldsymbol{A} \boldsymbol{\pi}_t = \boldsymbol{b},$$

where:

$$\boldsymbol{A} = J_{XX} X_t^2 \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta + 2 J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta,$$

$$\boldsymbol{b} = -\left( J_X X_t \boldsymbol{\mu}_S - r J_X X_t \mathbf{1} + J_{\theta_P} \theta_{P,t} \boldsymbol{\mu}_\theta \right).$$

Then the partial derivatives of $J$ are substituted in this expression:

$$\boldsymbol{A} = g(t) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \left[ (\tilde{\alpha} - 1)\alpha \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + (\tilde{\beta} - 1)\beta \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta + 2\alpha\beta(1-\gamma) \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \right].$$

$$\boldsymbol{b} = -g(t) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \left[ \alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta \boldsymbol{\mu}_\theta \right].$$

where $\tilde{\alpha} = \alpha(1-\gamma), \quad \tilde{\beta} = \beta(1-\gamma)$. The optimal $\boldsymbol{\pi}_t^*$ is then:

$$\boldsymbol{\pi}_t^* = \frac{\alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta \boldsymbol{\mu}_\theta}{(1-\tilde{\alpha})\alpha \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + (1-\tilde{\beta})\beta \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta - 2\alpha\beta(1-\gamma) \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta}.$$

From the structure of the Hamilton-Jacobi-Bellman equation and the derivation of $\boldsymbol{\pi}_t^*$, we observe that $\boldsymbol{\pi}_t^*$ depends solely on the model parameters. Since these partial derivatives do not explicitly depend on time $t$, $\boldsymbol{\pi}_t^*$ is constant over time. That means we can write it as $\boldsymbol{\pi}^*$. This is a direct consequence of the separable structure of the value function $J(X_t, \theta_{P,t}, t)$, which assumes a specific functional form.

By substituting $\boldsymbol{\pi}^*$ back into the Hamilton-Jacobi-Bellman equation, we validate the consistency of the solution and show that the value function $J$ evolves as expected under the given dynamics. We rewrite the Hamilton-Jacobi-Bellman equation as:

$$ZJ = (A + B + C + D + E + F + G) \cdot J,$$

where $Z$ is a constant and the terms $A$ to $G$ correspond to the constant contributions from the partial derivatives of $J$ and their interactions with $\boldsymbol{\pi}^*$. The solution for $\boldsymbol{\pi}^*$ provides a constant optimal allocation vector for the portfolio, balancing risk, return, and ESG preferences. The result highlights the role of correlations between asset returns ($\boldsymbol{P}$), ESG scores ($\boldsymbol{Q}$), and cross-effects ($\boldsymbol{R}$) in determining the optimal weights.

## 5.4 Limitations of Geometric Brownian Motion for $\theta_t$

The use of a Geometric Brownian Motion to model ESG scores ($\theta_t$) introduces certain limitations. While a Geometric Brownian Motion captures proportional growth and stochastic fluctuations effectively, it allows ESG scores to grow unboundedly over time. This can result in unrealistic outcomes, as most ESG scoring frameworks cap scores at a maximum value, such as 100. Furthermore, a Geometric Brownian Motion does not exhibit mean-reverting behavior, which may better reflect the natural tendency of ESG scores to stabilize over time due to diminishing returns on sustainability improvements or regulatory constraints. To address these limitations, the dynamics of $\theta_t$ are redefined by applying an exponential transformation to a mean-reverting process inspired by Vasicek (1977):

$$\theta_t = e^{Z_t}, \quad dZ_t = \kappa(Z^* - Z_t)\,dt + \sigma_Z\,dW_{2,t}.$$

The Exponential Vasicek process ensures positivity and introduces mean-reversion, preventing $\theta_t$ from exceeding practical bounds. Here, $\kappa > 0$ represents the speed of mean reversion, $Z^*$ is the long-term mean of $Z_t = \ln(\theta_t)$, and $\sigma_Z$ is the volatility.

Using Itô's Lemma for $\theta_t = e^{Z_t}$, the dynamics of $\theta_t$ are derived as:

$$d\theta_t = \theta_t\kappa\left(\ln(\theta^*) - \ln(\theta_t)\right)\,dt + \frac{\sigma_\theta^2}{2\theta_t}\,dt + \sigma_\theta\,dW_{2,t},$$

where $\sigma_\theta = \theta_t\sigma_Z$. This formulation incorporates mean-reversion and adjusts for stochastic fluctuations in $\theta_t$.

By Itô's Lemma we find that the dynamics of $\theta_P$ become:

$$d\theta_{P,t} = \pi_t\left[\theta_t\kappa\left(\ln(\theta^*) - \ln(\theta_t)\right)\,dt + \frac{\sigma_\theta^2}{2\theta_t}\,dt + \sigma_\theta\,dW_{2,t}\right].$$

The wealth dynamics ($X_t$) remain unchanged and follow:

$$dX_t = X_t\left(r + \pi_t(\mu_S - r)\right)\,dt + X_t\pi_t\sigma_S\,dW_{1,t}.$$

The Hamilton-Jacobi-Bellman equation for the value function $J(X_t, \theta_t, t)$ is formulated as:

$$0 = \sup_{\pi_t}\left\{\frac{\partial J}{\partial t} + J_X X_t\Big(r + \pi_t(\mu_S - r)\Big) + J_\theta\Big(\theta_t\kappa\left(\ln(\theta^*) - \ln(\theta_t)\right) + \frac{\sigma_\theta^2}{2\theta_t}\Big)\right.$$

$$+\frac{1}{2}J_{XX}X_t^2\pi_t^2\sigma_S^2+\frac{1}{2}J_{\theta\theta}\sigma_\theta^2+J_{X\theta}X_t\pi_t\sigma_S\sigma_\theta\rho\Bigg\}.$$

The terminal condition is defined as:

$$J(X_T,\theta_T,T)=U(X_T,\theta_T),$$

where $U(X_T,\theta_T)$ represents the utility derived from wealth $X_T$ and the ESG score $\theta_T$ at the terminal time $T$. While this terminal condition reflects the investor's preferences, it does not impose a specific functional form for $J(X_t,\theta_t,t)$. Consequently, a closed-form solution for $J$ cannot be derived, but the optimization process and first-order conditions provide a practical framework for finding the optimal controls $\pi_t^*$.

By rewriting the model to use $\theta_t$ as a state variable, the ESG dynamics become independent of portfolio weights, enabling a more direct analysis of ESG trends and their impact on utility. This reformulation simplifies the Hamilton-Jacobi-Bellman equation and aligns the dynamics with the Exponential Vasicek process, ensuring positivity and realistic ESG score behavior. The absence of a specific form for $J(X_t,\theta_t,t)$ highlights the complexity of the problem, but the terminal condition $J(X_T,\theta_T,T)=U(X_T,\theta_T)$ ensures alignment with investor preferences.

## 5.5 Simulation Framework

This subsection presents the methodology for simulating the dynamics of ESG scores and portfolio wealth over time. The simulations are designed to evaluate the behavior of the theoretical model under varying conditions and to analyze the impact of different parameter configurations and investment strategies on optimal allocation and investor utility.

The stochastic evolution of ESG scores and portfolio wealth is modeled using a Geometric Brownian Motion for ESG scores and a wealth dynamic equation driven by optimal portfolio allocation. Parameter values for these simulations are derived from empirical analyses to ensure alignment with realistic market behavior and historical trends. This approach captures the interplay between ESG preferences and wealth growth, reflecting the inherent uncertainty in both dimensions.

By bridging the empirical analysis with the theoretical model, this simulation framework enables a comprehensive exploration of the model's implications, contributing to a better understanding of the relationship between ESG dynamics and investment decisions.

### 5.5.1 Discrete Implementation of Model Dynamics

The stochastic dynamics of ESG scores and portfolio wealth are defined in continuous time. To implement these dynamics numerically, they are transformed into discrete-time equations, allowing for simulation over a finite number of time steps. This section outlines the discrete implementations of the ESG score dynamics and portfolio wealth evolution, as well as the incorporation of correlation between the two processes in a multi-asset setting.

**Discrete ESG Dynamics ($\boldsymbol{\theta_t}$)** The ESG scores of individual assets, denoted as $\boldsymbol{\theta_t} = (\theta_{1,t}, \theta_{2,t}, ..., \theta_{N,t})^\top$, follow a Geometric Brownian Motion, given by:

$$d\boldsymbol{\theta_t} = \boldsymbol{\theta_t} \circ (\boldsymbol{\mu_\theta} dt + \boldsymbol{\sigma_\theta} d\boldsymbol{W_t}),$$

where $\boldsymbol{\mu_\theta}$ is the vector of expected ESG drifts, $\boldsymbol{\sigma_\theta}$ is the vector of ESG volatilities, and $d\boldsymbol{W_t}$ is a vector of standard Brownian motions. The operator $\circ$ denotes element-wise multiplication. The discretized form used in simulations is:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta_t} \circ \exp\left((\boldsymbol{\mu_\theta} - \frac{1}{2}\boldsymbol{\sigma_\theta}^2)\Delta t + \boldsymbol{\sigma_\theta}\sqrt{\Delta t}\boldsymbol{Z_t}\right),$$

where $\boldsymbol{Z_t}$ is a vector of independent standard normal random variables. The ESG scores evolve stochastically while maintaining their positive growth properties. ESG scores are only updated once a year, remaining constant for 12 consecutive months before being adjusted.

**Discrete Portfolio ESG Score Dynamics ($\theta_{P,t}$)** The portfolio-level ESG score, $\theta_{P,t}$, is defined as the weighted sum of individual asset ESG scores:

$$\theta_{P,t} = \boldsymbol{\pi}^{*\top}\boldsymbol{\theta_t}.$$

Using the discretized ESG score dynamics, $\theta_{P,t}$ is updated at each time step based on the changing asset-level ESG scores and portfolio weights. Given that portfolio allocations $\boldsymbol{\pi}^*$ are constant in the analytical solution, the portfolio ESG score evolves as a function of the ESG drifts, volatilities, and Brownian shocks applied to the asset-level ESG scores.

**Discrete Wealth Dynamics ($W_t$)** The evolution of portfolio wealth follows:

$$dW_t = W_t\left[r + \boldsymbol{\pi}^\top(\boldsymbol{\mu_S} - r\mathbf{1})\right]dt + W_t\boldsymbol{\pi}^\top\boldsymbol{\sigma_S}d\boldsymbol{W_t},$$

where $r$ is the risk-free rate, $\boldsymbol{\mu_S}$ is the expected return vector of risky assets, $\boldsymbol{\sigma_S}$ is the volatility vector, and $\boldsymbol{\pi}$ represents the portfolio allocation. The discrete version is:

$$W_{t+1} = W_t\left(1 + \boldsymbol{\pi}^\top(\boldsymbol{\mu_S} - r\mathbf{1})\Delta t + \boldsymbol{\pi}^\top\boldsymbol{\sigma_S}\sqrt{\Delta t}\boldsymbol{Z_t}\right),$$

where $\boldsymbol{Z_t}$ is a vector of standard normal random variables.

**Correlation Between ESG Scores and Portfolio Returns** Correlations between ESG scores ($\boldsymbol{\theta_t}$) and portfolio returns ($W_t$) are incorporated by generating correlated Brownian motions. Given two independent standard normal vectors, $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$, the correlated shocks are obtained as:

$$\begin{bmatrix} \boldsymbol{Z}_{\text{correlated},\mathbf{1}} \\ \boldsymbol{Z}_{\text{correlated},\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \boldsymbol{R} \\ \boldsymbol{R}^\top & \mathbf{I} \end{bmatrix}^{1/2} \cdot \begin{bmatrix} \boldsymbol{Z_1} \\ \boldsymbol{Z_2} \end{bmatrix},$$

where $\boldsymbol{R}$ is the correlation matrix between stock returns and ESG scores. These correlated shocks ensure that ESG scores and asset returns reflect realistic dependencies.

### 5.5.2 Simulation Setup

To numerically simulate the dynamics of ESG scores and portfolio wealth, a structured simulation setup is implemented. This section describes the time horizon, discretization scheme, number of simulations, and initialization of parameters.

**Time Horizon and Discretization** The simulations are conducted over a finite time horizon of $T = 10$ years, using monthly time steps. Since the ESG scores and portfolio returns are already available as monthly observations, no further discretization is applied. The discrete equations for ESG scores ($\boldsymbol{\theta_t}$) and portfolio wealth ($W_t$) are iteratively updated at each time step, aligned with the available data frequency.

**Number of Simulations** To ensure robust statistical results, $N_{\text{sim}} = 10{,}000$ independent simulations are performed. Each simulation generates a trajectory for both ESG scores and portfolio wealth, allowing for the evaluation of average behavior and distributional properties.

**Initial Values** The initial values for ESG scores and portfolio wealth are set as:

$$\boldsymbol{\theta_0} = \tilde{\boldsymbol{X}}, \quad \theta_{P,0} = \boldsymbol{\pi}^{*\top} \tilde{\boldsymbol{X}}, \quad W_0 = 100,$$

where $\tilde{\boldsymbol{X}}$ represents the last observed ESG score vector from historical data. The initial portfolio ESG score, $\theta_{P,0}$, is determined by applying the optimal portfolio weights $\boldsymbol{\pi}^*$ to the historical ESG scores.

**Parameter Inputs** The parameters used in the simulations, including $\boldsymbol{\mu_\theta}$, $\boldsymbol{\sigma_\theta}$, $\boldsymbol{\mu_S}$, $\boldsymbol{\sigma_S}$, $r$, and $\boldsymbol{R}$, are derived from the empirical analysis described earlier. These values remain fixed throughout the simulation process.

The simulation setup provides a structured and reproducible framework for numerically evaluating the model. By integrating empirically derived parameters, a large number of simulations, and appropriate initial values, the framework enables a detailed exploration of the stochastic dynamics of ESG scores and portfolio wealth over time.

# 6 Results

This section presents the main findings of the study, including both empirical and theoretical results.

## 6.1 Empirical Results

To assess the influence of ESG scores on stock returns, we conduct an empirical analysis based on historical data. This subsection presents the key findings from the regression and time series analysis.

### 6.1.1 Regression Analysis

In this section, we present the results of the regression analysis using an extended version of the Fama-French 5-factor model, which includes an additional ESG factor. The primary objective of this analysis is to isolate and quantify the impact of ESG scores on stock returns while controlling for the well-established Fama-French factors: market return (Mkt - Rf), size (SMB), value (HML), profitability (RMW), and investment style (CMA). By including the ESG factor alongside these traditional factors, we can assess whether and to what extent ESG scores contribute to the overall expected return of a portfolio. This analysis is crucial for understanding the role of sustainability metrics in financial performance and for guiding investment strategies that incorporate ESG considerations.

Table 5 presents the estimated beta coefficients and their associated p-values for the intercept and each factor in the model. The beta coefficients represent the sensitivity of the stock returns to the respective factors, while the p-values indicate the statistical significance of these coefficients.

The regression results indicate that all Fama-French factors, including market (**Mkt - Rf**),

Table 5: Regression Results: Betas and p-values for FF5 and ESG Factors

| | **Intercept** | **Mkt - Rf** | **SMB** | **HML** | **RMW** | **CMA** | **ESG** |
|---|---|---|---|---|---|---|---|
| $\beta$ | -0.0028 | 0.0100 | 0.0012 | 0.0029 | 0.0009 | -0.0012 | -0.0009 |
| **p-value** | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 |

size (**SMB**), value (**HML**), profitability (**RMW**), and investment style (**CMA**), have statistically significant impacts on stock returns. The market, size, value, and profitability factors all exhibit positive beta coefficients, indicating that these factors contribute positively to returns. In contrast, the investment factor (**CMA**) has a negative beta, suggesting a negative relationship with returns. The statistical significance of all factors strengthens the robustness of the model and implies that these factors are meaningful in explaining stock returns.

To visualize the variability across different factors, Figure 1 provides a boxplot of the beta coefficients for each factor, illustrating the spread and central tendency. The boxplot helps demonstrate the consistency of the factor impacts across different companies.

Table 6: Statistics for ESG Betas

| | **Mean** | **StdDev** | **Min** | **Max** | **Median** |
|---|---|---|---|---|---|
| **ESG Betas** | -0.0009 | 0.0055 | -0.0166 | 0.0357 | -0.0011 |

Table 6 represents the statistics for the beta of the ESG Score. The ESG factor has a negative average beta of -0.0009, which indicates that, on average, stocks with higher ESG scores may slightly underperform those with lower scores when all other factors are held constant. However, this relationship is statistically significant, suggesting that ESG considerations do influence stock returns in this context.

The descriptive statistics in Table 6 show that the ESG betas range from -0.0166 to 0.0357, indicating a considerable variation in the influence of ESG scores on stock returns across different companies. The standard deviation of 0.0055 also suggests a notable spread in the impact, which means that while the average influence is negative, there are cases where ESG scores positively impact stock returns. This allows our model to incorporate both positive and negative effects of ESG on portfolio returns, reflecting the diverse impact that ESG factors can have on different companies. The spread in ESG betas may also be attributed to industry-specific dynamics or varying investor perceptions of ESG performance. This variation is also evident in Figure 1, which provides a boxplot of the beta coefficients for each factor. The boxplot for ESG shows that the distribution contains both negative and positive values, reinforcing the conclusion that ESG scores can have mixed effects on returns.

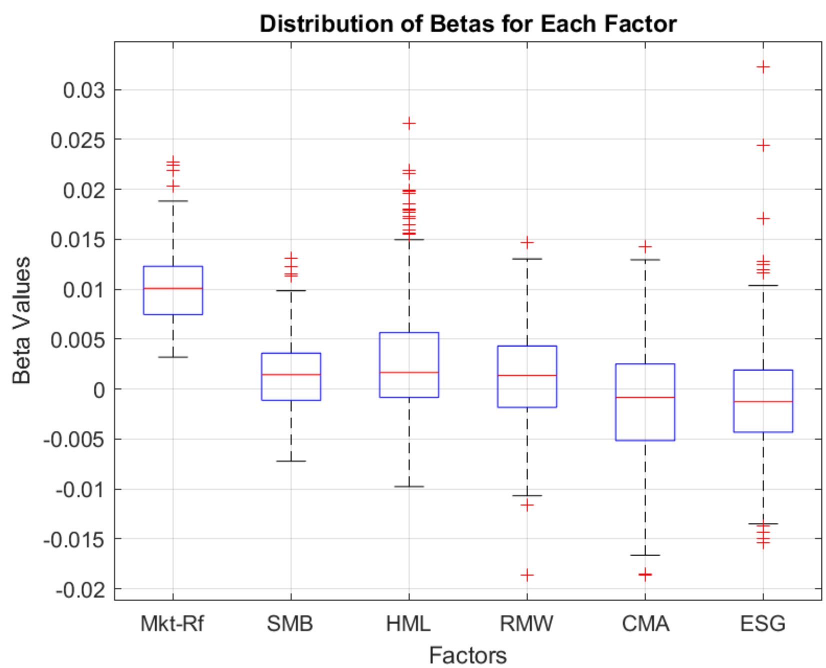


Figure 1: Distribution of Betas for Each Factor

We present the diagnostic checks performed to ensure the validity of the regression model.

These checks are crucial for confirming that the assumptions of linear regression are met. To assess whether the residuals have constant variance, we performed the White test for homoscedasticity. The average p-value for the White test is 0.263, which indicates that there is no evidence of heteroscedasticity, and the assumption of constant variance is satisfied.
To check for the presence of autocorrelation in the residuals, we used the Durbin-Watson statistic. The average Durbin-Watson statistic across all companies is 2.1653, which is close to the ideal value of 2, indicating that there is no significant autocorrelation in the residuals.
Figure 2 presents a scatter plot of residuals against predicted values to check the assumption of linearity. The scatter plot shows no discernible pattern, which suggests that the relationship between predictors and response is linear.

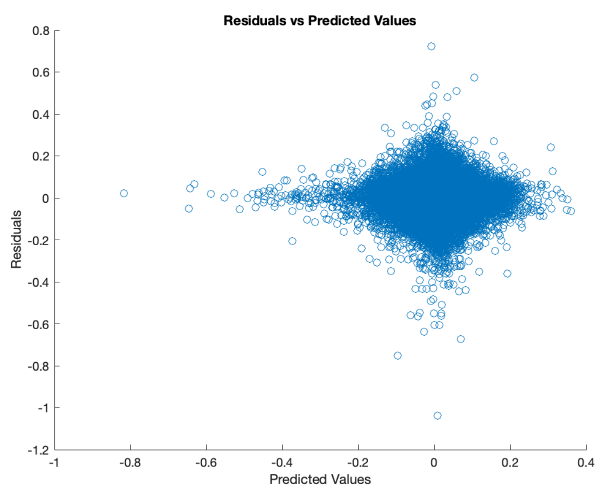


Figure 2: Scatterplot of Residuals vs Predicted Values

Figure 3 provides a histogram of residuals, indicating that the residuals are approximately normally distributed, which supports the assumption of normality.

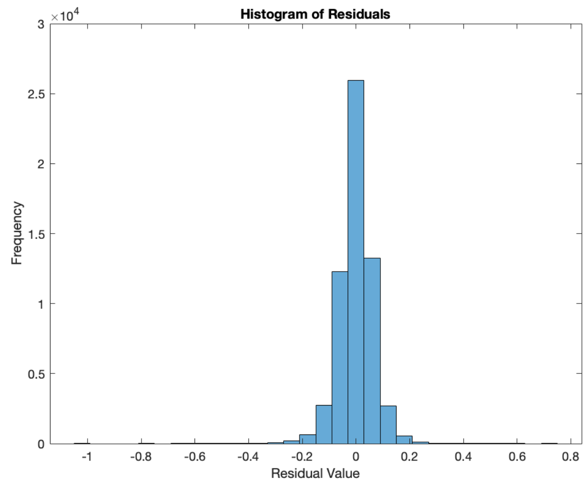


Figure 3: Histogram of Residuals

The results of these diagnostic checks confirm that the regression model meets the key assumptions for linear regression, including no significant heteroscedasticity, no autocorrelation, linearity, and normally distributed residuals. This supports the reliability of the regression results presented in this research.

#### 6.1.2 Time Series Analysis

The historical ESG scores exhibit a clear upward trend over time, which is illustrated in Figure 4. This upward trend indicates that companies are, on average, continuously improving their sustainability practices. This trend can be attributed to increased regulatory pressure, investor expectations, and societal focus on sustainability. The plotted ESG scores represent the average ESG score across all companies in the dataset.

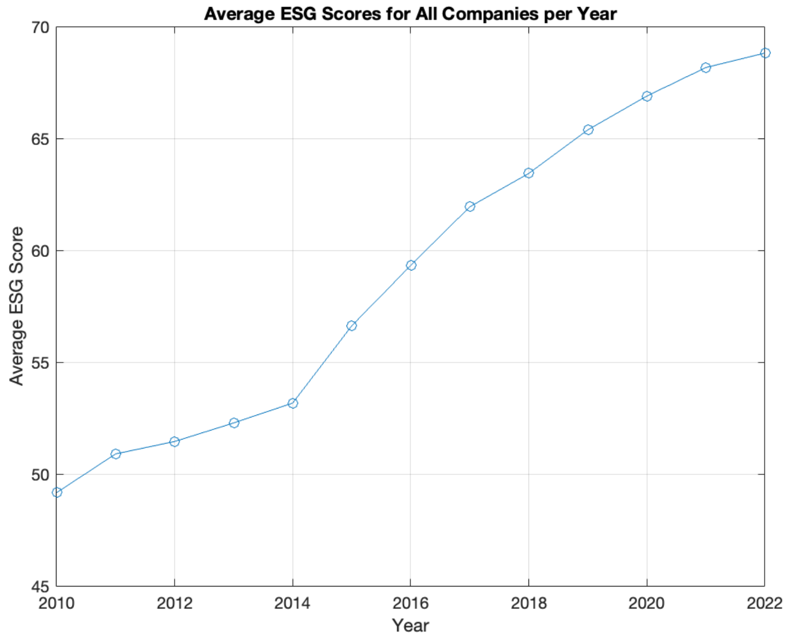


Figure 4: Average ESG scores over time, showing a clear upward trend.

The observed upward trend supports the use of a Geometric Brownian Motion for modeling ESG scores, as it allows for a positive drift, representing continuous growth. This characteristic is consistent with the trend seen in the historical data, where ESG scores tend to increase over time.

We applied the Augmented Dickey-Fuller (ADF) test to each time series to assess its stationarity. If a series was found to be non-stationary, we applied differencing ($d = 1$) to achieve stationarity. Table 7 summarizes the ADF test results before and after differencing.

Table 7: Summary of ADF Test Results

| **Metric** | **Value** |
|---|---|
| Total Companies | 386 |
| Stationary After Differencing | 363 |
| Non-Stationary After Differencing | 23 |
| Mean $p$-Value (Original Series) | 0.79141 |
| Mean $p$-Value (First Difference) | 0.012381 |
| Min $p$-Value (First Difference) | 0.001 |
| Max $p$-Value (First Difference) | 0.18613 |

The ADF test results indicate that, out of 386 companies, 363 became stationary after first differencing ($d = 1$). This means that the majority of ESG score time series could be converted into a stationary form, which is necessary for fitting ARIMA models. The mean $p$-value for the original series was 0.79141, indicating non-stationarity, while after differencing, it dropped to 0.012381, highlighting the improved stationarity. This supports the use of a Geometric Brownian Motion, as the original non-stationary behavior is consistent with a process that includes both a stochastic trend and a growth component, which a Geometric Brownian Motion can effectively model.

## 6.2 Theoretical Results

This subsection presents the theoretical findings derived from the Hamilton-Jacobi-Bellman framework. We analyze the optimal portfolio allocation under ESG preferences and evaluate how different model parameters influence investment decisions.

### 6.2.1 Closed-Form Solution

The optimal portfolio allocation $\boldsymbol{\pi}^*$ represents the balance between financial returns, ESG preferences, and risk, as dictated by the model parameters and the interactions between asset returns and ESG scores.

$$\boldsymbol{\pi}^* = \frac{\alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta\boldsymbol{\mu}_\theta}{(1-\tilde{\alpha})\alpha\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S + (1-\tilde{\beta})\beta\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta - 2\alpha\beta(1-\gamma)\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta}.$$

The constancy of $\boldsymbol{\pi}^*$ greatly simplifies the problem, as the portfolio weights remain fixed throughout the investment horizon, and no dynamic rebalancing is required.

The numerator of the expression for $\boldsymbol{\pi}^*$, given by $\alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta\boldsymbol{\mu}_\theta$, captures the investor's trade-off between expected excess returns and ESG-driven benefits. The term $\boldsymbol{\mu}_S - r\mathbf{1}$ represents the excess return of an asset over the risk-free rate. Higher values of $\mu_S$ imply greater expected returns, making an asset more attractive from a traditional financial standpoint. The investor's preference for wealth, captured by $\alpha$, determines the extent to which excess returns influence the allocation decision.

The second component, $\boldsymbol{\mu}_\theta$, represents the expected change in the ESG score of an asset over time. When $\boldsymbol{\mu}_\theta$ is high, it indicates that the asset is improving in terms of sustainability, making it more attractive for ESG-conscious investors. The strength of this ESG preference is determined by $\beta$, implying that investors with a strong sustainability focus will allocate more capital to assets with higher ESG drifts, even if their financial returns are not the highest.

The denominator incorporates the risk structure of both financial returns and ESG dynamics. The first term, $(1-\tilde{\alpha})\alpha\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S$, reflects the sensitivity of the allocation to the variance of asset returns. When asset returns exhibit high volatility, the optimal allocation tends to be more conservative, particularly for risk-averse investors with high $\gamma$. The second term, $(1-\tilde{\beta})\beta\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta$, accounts for the risk associated with ESG scores, recognizing that ESG dynamics can introduce uncertainty into the portfolio's sustainability profile.

A key interaction term, $-2\alpha\beta(1-\gamma)\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta$, captures the correlation between financial returns and ESG scores. When asset returns and ESG scores are positively correlated

($\boldsymbol{R} > 0$), an investor with ESG preferences benefits from allocating capital to assets with high sustainability scores, as these assets tend to offer both financial and ESG advantages. In this scenario, sustainable investments not only align with ethical considerations but also contribute positively to financial performance, reinforcing the investor's objectives.

Conversely, when returns and ESG scores are negatively correlated ($\boldsymbol{R} < 0$), a trade-off emerges between financial returns and sustainability. In this case, assets with higher ESG scores tend to underperform financially, making it more costly for an investor to prioritize sustainability. The extent to which an investor is willing to hold such assets depends on their ESG preference parameter $\beta$ relative to their preference for financial returns $\alpha$. If $\beta$ is high, the investor may still allocate a significant portion of their wealth to ESG-friendly assets despite lower financial returns. However, if $\alpha$ dominates, the investor will allocate less to these assets, as their underperformance outweighs the sustainability benefits.

### 6.3 Scenario Analysis

Investors exhibit varying preferences when balancing financial returns and ESG considerations, leading to different optimal allocation strategies. To explore these trade-offs, we analyze how utility evolves under different configurations of $\alpha$ and $\beta$. By examining multiple scenarios, we gain insights into how investor choices influence long-term portfolio performance and the implications of prioritizing sustainability over pure financial gains.

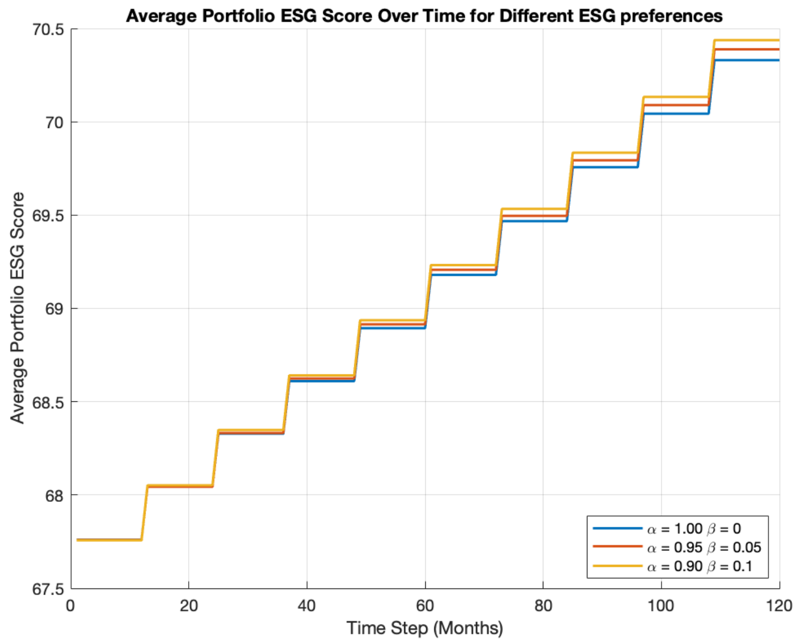


Figure 5: Average Portfolio ESG Score Over Time for Different ESG Preferences.

Figure 5 shows the average portfolio ESG score for different values of $\alpha$ and $\beta$. The results illustrate how increasing $\beta$ (higher ESG preference) leads to a gradual increase in the portfolio's average ESG score, while $\alpha = 1$ ($\beta = 0$) corresponds to a purely wealth-driven strategy with no ESG considerations. The stepwise increases reflect the annual update of ESG scores in the model, as defined by its dynamics.

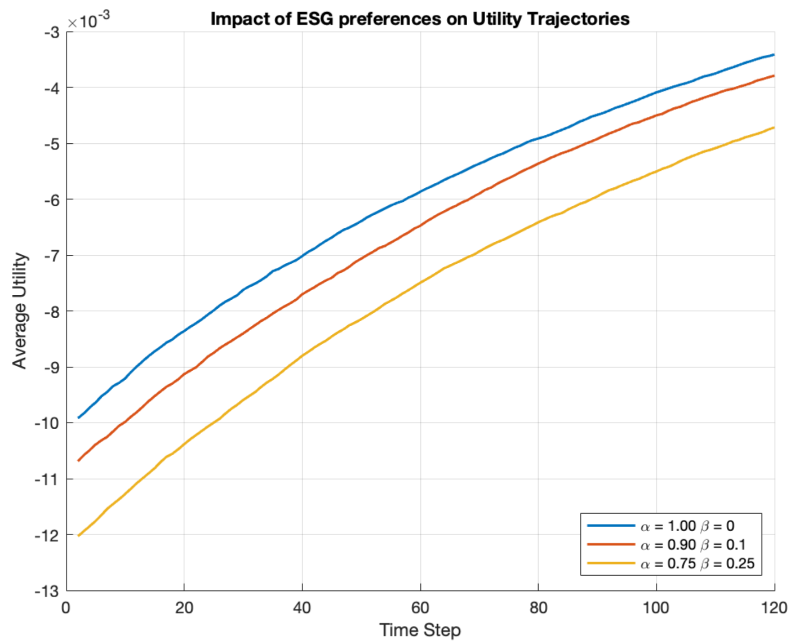


Figure 6: Impact of ESG Preferences on Utility Trajectories

Figure 6 illustrates the evolution of average utility over time for different configurations of $\alpha$ and $\beta$, which represent preferences for wealth and ESG scores, respectively. The blue line corresponds to $\alpha = 1.0$ and $\beta = 0$, representing an investor focused solely on wealth without consideration for ESG preferences. This configuration results in the highest utility as it allows optimization purely for financial returns. The red line, associated with $\alpha = 0.9$ and $\beta = 0.1$, introduces a small preference for ESG, leading to a slight reduction in utility compared to the wealth-only scenario. This reflects the trade-off between financial returns and incorporating ESG objectives. The yellow line, corresponding to $\alpha = 0.75$ and $\beta = 0.25$, represents a stronger emphasis on ESG preferences, which results in the lowest utility among the three configurations. The utility reduction demonstrates the increasing cost of prioritizing ESG factors over financial returns. Despite the differences in utility levels, all configurations exhibit steady growth over time, reflecting the cumulative benefits of portfolio diversification. As $\beta$ increases, indicating a stronger preference for ESG, the trade-off becomes more pronounced, with utility decreasing as the focus shifts from financial returns to sustainability considerations.

The interaction between financial returns and ESG scores thus plays a crucial role in shaping the optimal allocation. Investors must carefully assess the trade-offs involved, considering not only their personal preferences but also the broader market dynamics that influence the relationship between ESG performance and financial returns. Investors with strong ESG preferences (high $\beta$) will allocate more to assets with high $\boldsymbol{\mu}_\theta$, even if their financial returns are moderate. Risk-averse investors (high $\gamma$) will be more cautious in their allocations, particularly when the volatility of returns or ESG scores is high. Moreover, the correlation between ESG and financial returns plays a crucial role: when ESG improvements align with strong financial performance, sustainable investing becomes an even more compelling strategy.

This formulation of $\boldsymbol{\pi}^*$ demonstrates that optimal portfolio selection is not merely a function of maximizing returns but involves a nuanced interplay between return expectations, sustainability trends, risk aversion, and market dynamics. The investor's preferences, as captured by $\alpha$ and $\beta$, interact with the statistical properties of the assets to determine the final allocation, making the model well-suited to describe the behavior of investors incorporating ESG factors into their decision-making.

## 7 Conclusion

The findings of this research highlight the practicality of a constant optimal portfolio strategy $\pi^*$ that remains independent of time. Through the integration of ESG scores into portfolio allocation, this study has demonstrated that investors can balance financial performance with sustainability preferences without requiring dynamic rebalancing. The closed-form analytical solution derived using the Hamilton-Jacobi-Bellman equation confirms that $\pi^*$ is determined solely by the underlying parameters of the model, making it a stable and predictable strategy over the investment horizon. The empirical analysis of historical ESG scores and stock returns has provided robust estimates for key model parameters, supporting the choice of a Geometric Brownian Motion for ESG dynamics. The simulation results further reinforce the notion that a fixed optimal allocation can achieve long-term sustainability and return objectives. By maintaining a time-invariant strategy, investors avoid the complexity and transaction costs associated with frequent reallocation while still aligning their portfolios with ESG considerations.

From a theoretical perspective, this research contributes to the growing field of ESG-integrated asset pricing by demonstrating that ESG scores can be systematically incorporated into optimal portfolio decisions. The relationship between ESG scores and stock returns, as captured through the empirical estimation of drift and volatility parameters, suggests that ESG factors can play a meaningful role in investment decision-making. The findings align with previous studies that highlight the impact of ESG on financial performance (Giese et al., 2019; Pedersen et al., 2021) while providing a structured optimization framework that is mathematically tractable.

For investors, the practical implication of a constant $\pi^*$ strategy is the ability to adopt a long-term perspective without continuously adjusting allocations. This is particularly relevant for institutional investors and ESG-focused funds, which seek stable investment policies that integrate sustainability without excessive trading costs. Policymakers and regulators may also find these insights valuable in developing guidelines that promote ESG integration in financial markets without imposing burdensome reallocation requirements.

Despite these contributions, certain limitations should be acknowledged. The assumption of a Geometric Brownian Motion for ESG scores, while empirically justified, may not capture all aspects of ESG evolution, particularly in response to abrupt policy changes or market shocks. Additionally, while the model accounts for ESG preferences, it does not explicitly consider liquidity constraints, transaction costs, or investor heterogeneity, which could further refine the applicability of the results. Future research can extend this work by exploring alternative ESG modeling approaches, such as mean-reverting processes, to capture potential stationarity in ESG scores. Additionally, incorporating dynamic investor preferences or multi-period optimization could offer further insights into how ESG considerations evolve over time. Empirical studies analyzing the impact of different ESG policies on portfolio performance could also enhance the practical applicability of the model.

In conclusion, this research establishes a rigorous framework for ESG-integrated portfolio optimization, showing that a constant $\pi^*$ strategy provides a stable, efficient, and theoretically grounded approach to sustainable investing. The findings support the integration of ESG factors into investment decision-making while ensuring that optimal allocations remain independent of time, thereby offering a robust and practical solution for long-term investors.

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# 8 Appendix

## 1. Derivation Optimal $\pi^*$ for Portfolio with a Single Asset, $\theta$ follows a Geometric Brownian Motion

### Dynamics

$$dS_t = S_t\,[\mu_S\,dt + \sigma_S\,dW_{1,t}]$$

$$d\theta_{P,t} = \theta_{P,t}\,[\mu_\theta\,dt + \sigma_\theta\,dW_{2,t}]$$

$$dX_t = X_t\,[r + \pi_t(\mu_S - r)]\;dt + X_t\pi_t\sigma_S\,dW_{1,t}$$

$$d\theta_{P,t} = \theta_{P,t}\,[\pi_t\mu_\theta\,dt + \pi_t\sigma_\theta\,dW_{2,t}]$$

### Hamilton-Jacobi-Bellman Equation

$$ZJ = \sup_{\pi_t}\Bigg[\frac{\partial J}{\partial t} + J_X X_t\pi_t(\mu_S - r) + J_{\theta_P}\theta_{P,t}\pi_t\mu_\theta + J_X X_t r + \frac{1}{2}J_{XX}X_t^2\pi_t^2\sigma_S^2$$
$$+\frac{1}{2}J_{\theta_P\theta_P}\theta_{P,t}^2\pi_t^2\sigma_\theta^2 + J_{X\theta_P}X_t\theta_{P,t}\pi_t^2\sigma_S\sigma_\theta\rho\Bigg].$$

*Terminal condition:*

$$J(X_T, \theta_{P,T}, T) = U(X_T, \theta_{P,T}) = \frac{(X_T^\alpha\theta_{P,T}^\beta)^{1-\gamma}}{1-\gamma},$$

### All terms involving $\pi_t$

$$J_X X_t\pi_t(\mu_S - r) + J_{\theta_P}\theta_{P,t}\pi_t\mu_\theta + \frac{1}{2}J_{XX}X_t^2\pi_t^2\sigma_S^2 +$$

$$\frac{1}{2}J_{\theta_P\theta_P}\theta_{P,t}^2\pi_t^2\sigma_\theta^2 + J_{X\theta_P}X_t\theta_{P,t}\pi_t^2\sigma_S\sigma_\theta\rho.$$

**First Order Conditions**

$$J_X X_t(\mu_S - r) + J_{\theta_P}\theta_{P,t}\mu_\theta + J_{XX}X_t^2\pi_t\sigma_S^2 + J_{\theta_P\theta_P}\theta_{P,t}^2\pi_t\sigma_\theta^2 + 2J_{X\theta_P}X_t\theta_{P,t}\pi_t\sigma_S\sigma_\theta\rho = 0.$$

$$\pi_t\left(J_{XX}X_t^2\sigma_S^2 + J_{\theta_P\theta_P}\theta_{P,t}^2\sigma_\theta^2 + 2J_{X\theta_P}X_t\theta_{P,t}\sigma_S\sigma_\theta\rho\right) = -\left(J_X X_t(\mu_S - r) + J_{\theta_P}\theta_{P,t}\mu_\theta\right).$$

$$\pi_t = -\frac{J_X X_t(\mu_S - r) + J_{\theta_P}\theta_{P,t}\mu_\theta}{J_{XX}X_t^2\sigma_S^2 + J_{\theta_P\theta_P}\theta_{P,t}^2\sigma_\theta^2 + 2J_{X\theta_P}X_t\theta_{P,t}\sigma_S\sigma_\theta\rho}.$$

**Partial Derivatives of $J$**

$$\tilde{\alpha} = \alpha(1-\gamma), \quad \tilde{\beta} = \beta(1-\gamma).$$

**First-order derivatives:**

$$J_X = g(t)\cdot\alpha X_t^{\tilde{\alpha}-1}\theta_{P,t}^{\tilde{\beta}}.$$

$$J_{\theta_P} = g(t)\cdot\beta X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}-1}.$$

**Second-order derivatives:**

$$J_{XX} = g(t)\cdot\alpha(\tilde{\alpha}-1)X_t^{\tilde{\alpha}-2}\theta_{P,t}^{\tilde{\beta}}.$$

$$J_{\theta_P\theta_P} = g(t)\cdot\beta(\tilde{\beta}-1)X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}-2}.$$

**Cross-term derivative:**

$$J_{X\theta_P} = g(t) \cdot \alpha\beta(1-\gamma)X_t^{\tilde{\alpha}-1}\theta_{P,t}^{\tilde{\beta}-1}.$$

**Time derivative:**

$$\frac{\partial J}{\partial t} = g'(t) \cdot \frac{(X_t^{\alpha}\theta_{P,t}^{\beta})^{1-\gamma}}{1-\gamma}.$$

**Substituting the Partial Derivatives of $V$**

$$\pi_t\Big[g(t)\Big((\tilde{\alpha}-1)\alpha X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\sigma_S^2 + (\tilde{\beta}-1)\beta X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\sigma_\theta^2 + 2\alpha\beta(1-\gamma)X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\sigma_S\sigma_\theta\rho\Big)\Big] =$$

$$\Big[-g(t)\Big(\alpha X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}(\mu_S - r) + \beta X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\mu_\theta\Big)\Big].$$

**Optimal $\pi^*$**

$$\pi^* = -\frac{\alpha(\mu_S - r) + \beta\mu_\theta}{(\tilde{\alpha}-1)\alpha\sigma_S^2 + (\tilde{\beta}-1)\beta\sigma_\theta^2 + 2\alpha\beta(1-\gamma)\sigma_S\sigma_\theta\rho}.$$

**Hamilton-Jacobi-Bellman Equation with $\pi^*$**

$$ZJ = \frac{\partial J}{\partial t} + J_X X_t\pi^*(\mu_S - r) + J_{\theta_P}\theta_{P,t}\pi^*\mu_\theta + J_X X_t r + \frac{1}{2}J_{XX}X_t^2\pi^{*2}\sigma_S^2 + \frac{1}{2}J_{\theta_P\theta_P}\theta_{P,t}^2\pi^{*2}\sigma_\theta^2 + J_{X\theta_P}X_t\theta_{P,t}\pi^{*2}\sigma_S\sigma_\theta\rho.$$

**Substituting the partial derivatives:**

**First term: $\frac{\partial J}{\partial t}$**

$$\frac{\partial J}{\partial t} = g'(t) \cdot \frac{X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}}{1-\gamma} = \frac{g'(t)(1-\gamma)}{g(t)} \cdot J = A \cdot J.$$

**Second term:** $J_X X_t \pi^*(\mu_S - r)$

$$J_X X_t \pi^*(\mu_S - r) = g(t) \cdot \alpha X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \pi^*(\mu_S - r) = J \cdot \alpha(1-\gamma)\pi^*(\mu_S - r) = B \cdot J.$$

**Third term:** $J_{\theta_P} \theta_{P,t} \pi^* \mu_\theta$

$$J_{\theta_P} \theta_{P,t} \pi^* \mu_\theta = g(t) \cdot \beta X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \pi^* \mu_\theta = J \cdot \beta(1-\gamma)\pi^* \mu_\theta = C \cdot J.$$

**Fourth term:** $J_X X_t r$

$$J_X X_t r = g(t) \cdot \alpha X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} r = J \cdot \alpha(1-\gamma) r = D \cdot J.$$

**Fifth term:** $\frac{1}{2} J_{XX} X_t^2 \pi_t^{*2} \sigma_S^2$

$$\frac{1}{2} J_{XX} X_t^2 \pi_t^{*2} \sigma_S^2 = g(t) \cdot \frac{1}{2}\alpha(\tilde{\alpha}-1) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \pi_t^{*2} \sigma_S^2 = J \cdot \frac{1}{2}\alpha(1-\gamma)(\tilde{\alpha}-1)\pi_t^{*2}\sigma_S^2 = E \cdot J.$$

**Sixth term:** $\frac{1}{2} J_{\theta_P \theta_P} \theta_{P,t}^2 \pi_t^{*2} \sigma_\theta^2$

$$\frac{1}{2} J_{\theta_P \theta_P} \theta_{P,t}^2 \pi_t^{*2} \sigma_\theta^2 = g(t) \cdot \frac{1}{2}\beta(\tilde{\beta}-1) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \pi_t^{*2} \sigma_\theta^2 = J \cdot \frac{1}{2}\beta(1-\gamma)(\tilde{\beta}-1)\pi_t^{*2}\sigma_\theta^2 = F \cdot J.$$

**Seventh term:** $J_{X\theta_P} X_t \theta_{P,t} \pi_t^{*2} \sigma_S \sigma_\theta \rho$

$$J_{X\theta_P} X_t \theta_{P,t} \pi_t^{*2} \sigma_S \sigma_\theta \rho = g(t) \cdot \alpha\beta(1-\gamma) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \pi_t^{*2} \sigma_S \sigma_\theta \rho = J \cdot \alpha(1-\gamma)\beta(1-\gamma)\pi_t^{*2}\sigma_S\sigma_\theta\rho = G \cdot J.$$

**Conclusion**

That means that we can rewrite the Hamilton-Jacobi-Bellman equation as:

$$ZJ = (A + B + C + D + E + F + G) \cdot J$$

## 2. Derivation Optimal $\pi^*$ for Portfolio with N Assets, $\theta$ follows a Geometric Brownian Motion

### Dynamics

$$d\mathbf{S}_t = \mathbf{S}_t \circ (\boldsymbol{\mu}_S \, dt + \boldsymbol{\Sigma}_S \, d\mathbf{W}_{1,t})$$

- $\mathbf{S}_t$: $N \times 1$ vector of asset prices.
- $\boldsymbol{\mu}_S$: $N \times 1$ vector of drift rates of $\mathbf{S}$.
- $\boldsymbol{\Sigma}_S$: $N \times N$ matrix of volatilities and covariances.
- $d\mathbf{W}_{1,t}$: $N \times 1$ vector of Brownian motions.

$$d\boldsymbol{\theta}_t = \boldsymbol{\theta}_t \circ (\boldsymbol{\mu}_\theta \, dt + \boldsymbol{\Sigma}_\theta \, d\mathbf{W}_{2,t})$$

- $\boldsymbol{\theta}_t$: $N \times 1$ vector of ESG scores.
- $\boldsymbol{\mu}_\theta$: $N \times 1$ vector of drift rates of $\boldsymbol{\theta}$.
- $\boldsymbol{\Sigma}_\theta$: $N \times N$ matrix of volatilities and covariances for $\boldsymbol{\theta}$.
- $d\mathbf{W}_{2,t}$: $N \times 1$ vector of Brownian motions.

$$dX_t = X_t \left[ r + \boldsymbol{\pi}_t^T (\boldsymbol{\mu}_S - r\mathbf{1}) \right] \, dt + X_t \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_S \, d\mathbf{W}_{1,t}$$

- $X_t$: Scalar representing the total wealth at time $t$.
- $r$: Risk-free rate.

- $\boldsymbol{\pi}_t$: $N \times 1$ vector of proportions of wealth invested in each risky asset at time $t$.
- $\boldsymbol{\mu}_S$: $N \times 1$ vector of expected returns of the risky assets.
- $\mathbf{1}$: $N \times 1$ vector of ones, used to multiply $r$ across all assets.
- $\boldsymbol{\Sigma}_S$: $N \times N$ matrix of volatilities and covariances of the risky assets.
- $d\mathbf{W}_{1,t}$: $N \times 1$ vector of Brownian motions.

$$d\theta_{P,t} = \theta_{P,t} \left[ \boldsymbol{\pi}_t^T \boldsymbol{\mu}_\theta \, dt + \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_\theta \, d\mathbf{W}_{2,t} \right]$$

- $\theta_{P,t}$: Scalar representing the portfolio ESG score at time $t$.
- $\boldsymbol{\pi}_t$: $N \times 1$ vector of proportions of wealth invested in each asset at time $t$.
- $\boldsymbol{\mu}_\theta$: $N \times 1$ vector of drift rates for the ESG scores.
- $\boldsymbol{\Sigma}_\theta$: $N \times N$ matrix of volatilities and covariances for the ESG scores.
- $d\mathbf{W}_{2,t}$: $N \times 1$ vector of Brownian motions, representing the uncertainty in the ESG score dynamics.

**Hamilton-Jacobi-Bellman Equation**

$$ZJ = \sup_{\boldsymbol{\pi}_t} \left\{ \frac{\partial J}{\partial t} + J_X X_t \left[ \boldsymbol{\pi}_t^\top (\boldsymbol{\mu}_S - r\mathbf{1}) + r \right] + J_{\theta_P} \theta_{P,t} \boldsymbol{\pi}_t^\top \boldsymbol{\mu}_\theta \right.$$

$$\left. + \frac{1}{2} J_{XX} X_t^2 \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S \boldsymbol{\pi}_t + \frac{1}{2} J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t + J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}_t \right\}.$$

*Terminal condition:*

$$J(X_T, \theta_{P,T}, T) = U(X_T, \theta_{P,T}) = \frac{(X_T^{\alpha}\theta_{P,T}^{\beta})^{1-\gamma}}{1-\gamma},$$

- $\boldsymbol{P}$ is the $N \times N$ correlation matrix for risky asset returns, where $P_{ij}$ represents the correlation between the $i$-th and $j$-th risky assets.
- $\boldsymbol{Q}$ is the $N \times N$ correlation matrix for ESG scores of assets, where $Q_{ij}$ represents the correlation between the $i$-th and $j$-th asset's ESG score.
- $\boldsymbol{R}$ is the $N \times N$ correlation matrix representing the correlation between risky asset Brownian motions and ESG score Brownian motions. Specifically, $R_{ij}$ represents the correlation between the $i$-th Brownian motion of the risky asset return and the $j$-th ESG Brownian motion.

### All terms involving $\boldsymbol{\pi}$

$$J_X X_t \boldsymbol{\pi}_t^\top (\boldsymbol{\mu}_S - r\mathbf{1}) + J_{\theta_P}\theta_{P,t}\boldsymbol{\pi}_t^\top \boldsymbol{\mu}_\theta + \frac{1}{2}J_{XX}X_t^2\boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S\boldsymbol{\pi}_t$$
$$+\frac{1}{2}J_{\theta_P\theta_P}\theta_{P,t}^2\boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}_t + J_{X\theta_P}X_t\theta_{P,t}\boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}_t.$$

### First Order Conditions

$$J_X X_t\boldsymbol{\mu}_S - rJ_X X_t\mathbf{1} + J_{\theta_P}\theta_{P,t}\boldsymbol{\mu}_\theta + J_{XX}X_t^2\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S\boldsymbol{\pi}_t + J_{\theta_P\theta_P}\theta_{P,t}^2\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}_t$$
$$+2J_{X\theta_P}X_t\theta_{P,t}\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}_t = 0.$$

We can write it as

$$\boldsymbol{A}\boldsymbol{\pi}_t = \boldsymbol{b},$$

where:

$$\boldsymbol{A} = J_{XX} X_t^2 \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta + 2 J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta,$$

$$\boldsymbol{b} = -\left( J_X X_t \boldsymbol{\mu}_S - r J_X X_t \mathbf{1} + J_{\theta_P} \theta_{P,t} \boldsymbol{\mu}_\theta \right).$$

**Substituting the Partial Derivatives of $V$**

$$\boldsymbol{A} = g(t) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \left[ (\tilde{\alpha} - 1)\alpha \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + (\tilde{\beta} - 1)\beta \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta + 2\alpha\beta(1-\gamma) \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \right].$$

$$\boldsymbol{b} = -g(t) X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \left[ \alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta \boldsymbol{\mu}_\theta \right].$$

## Optimal $\pi^*$

$$\boldsymbol{\pi}_t^* = \left[ (\tilde{\alpha} - 1)\alpha \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S + (\tilde{\beta} - 1)\beta \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta + 2\alpha\beta(1-\gamma) \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \right]^{-1}$$

$$\cdot - \left[ \alpha(\boldsymbol{\mu}_S - r\mathbf{1}) + \beta \boldsymbol{\mu}_\theta \right].$$

## Hamilton-Jacobi-Bellman Equation with $\pi^*$

$$\delta J = \frac{\partial J}{\partial t} + J_X X_t \left[ \boldsymbol{\pi}^{*\top} (\boldsymbol{\mu}_S - r\mathbf{1}) + r \right] + J_{\theta_P} \theta_{P,t} \boldsymbol{\pi}^{*\top} \boldsymbol{\mu}_\theta$$

$$+ \frac{1}{2} J_{XX} X_t^2 \boldsymbol{\pi}^{*\top} \boldsymbol{\Sigma}_S \boldsymbol{P} \boldsymbol{\Sigma}_S \boldsymbol{\pi}^* + \frac{1}{2} J_{\theta_P \theta_P} \theta_{P,t}^2 \boldsymbol{\pi}^{*\top} \boldsymbol{\Sigma}_\theta \boldsymbol{Q} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}^* + J_{X\theta_P} X_t \theta_{P,t} \boldsymbol{\pi}^{*\top} \boldsymbol{\Sigma}_S \boldsymbol{R} \boldsymbol{\Sigma}_\theta \boldsymbol{\pi}^*.$$

## Substitute partial derivatives

**First term:**

$$\frac{\partial J}{\partial t} = g'(t) \cdot \frac{X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}}}{1-\gamma} = \frac{g'(t)(1-\gamma)}{g(t)} \cdot J = A \cdot J.$$

**Second term:**

$$J_X X_t \boldsymbol{\pi}^{*\top}(\boldsymbol{\mu}_S - r\mathbf{1}) = g(t)\cdot \alpha X_t^{\tilde{\alpha}} \theta_{P,t}^{\tilde{\beta}} \boldsymbol{\pi}^{*\top}(\boldsymbol{\mu}_S - r\mathbf{1}) = J\cdot \alpha(1-\gamma)\boldsymbol{\pi}^{*\top}(\boldsymbol{\mu}_S - r\mathbf{1}) = B\cdot J.$$

**Third term:**

$$J_{\theta_P}\theta_{P,t}\boldsymbol{\pi}^{*\top}\boldsymbol{\mu}_\theta = g(t)\cdot \beta X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\boldsymbol{\pi}^{*\top}\boldsymbol{\mu}_\theta = J\cdot\beta(1-\gamma)\boldsymbol{\pi}^{*\top}\boldsymbol{\mu}_\theta = C\cdot J.$$

**Fourth term:**

$$J_X X_t r = g(t)\cdot\alpha X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}} r = J\cdot\alpha(1-\gamma)r = D\cdot J.$$

**Fifth term:**

$$\frac{1}{2}J_{XX}X_t^2\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S\boldsymbol{\pi}^* = g(t)\cdot\frac{1}{2}\alpha(\tilde{\alpha}-1)X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S\boldsymbol{\pi}^*$$

$$= J\cdot\frac{1}{2}\alpha(1-\gamma)(\tilde{\alpha}-1)\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{P}\boldsymbol{\Sigma}_S\boldsymbol{\pi}^* = E\cdot J.$$

**Sixth term:**

$$\frac{1}{2}J_{\theta_P\theta_P}\theta_{P,t}^2\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^* = g(t)\cdot\frac{1}{2}\beta(\tilde{\beta}-1)X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^*$$

$$= J\cdot\frac{1}{2}\beta(1-\gamma)(\tilde{\beta}-1)\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_\theta\boldsymbol{Q}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^* = F\cdot J.$$

**Seventh term:**

$$J_{X\theta_P}X_t\theta_{P,t}\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^* = g(t)\cdot\alpha\beta(1-\gamma)X_t^{\tilde{\alpha}}\theta_{P,t}^{\tilde{\beta}}\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^*$$

$$= J\cdot\alpha(1-\gamma)\beta(1-\gamma)\boldsymbol{\pi}^{*\top}\boldsymbol{\Sigma}_S\boldsymbol{R}\boldsymbol{\Sigma}_\theta\boldsymbol{\pi}^* = G\cdot J.$$

## Conclusion

That means that we can rewrite the Hamilton-Jacobi-Bellman equation as:

$$\delta J = (A + B + C + D + E + F + G) \cdot J$$

**3. Derivation Optimal $\pi^*$ for Portfolio with one asset, $\theta$ follows an Exponential Vasicek process**

**Dynamics**

$$dS_t = S_t \left( \mu_S \, dt + \sigma_S \, dW_{1,t} \right)$$

$$dX_t = X_t \left( r + \pi_t(\mu_S - r) \right) \, dt + X_t \pi_t \sigma_S \, dW_{1,t}$$

$$d\theta_t = \theta_t \kappa \left( \ln(\theta^*) - \ln(\theta_t) \right) \, dt + \frac{\sigma_\theta^2}{2\theta_t} \, dt + \sigma_\theta \, dW_{2,t}$$

$$d\theta_{P,t} = \pi_t \left[ \theta_t \kappa \left( \ln(\theta^*) - \ln(\theta_t) \right) \, dt + \frac{\sigma_\theta^2}{2\theta_t} \, dt + \sigma_\theta \, dW_{2,t} \right].$$

To find the dynamics of $\theta_t$, we use Itô's Lemma for the function $\theta_t = e^{Z_t}$.

By Itô's Lemma:

$$df(Z_t) = f'(Z_t) \, dZ_t + \frac{1}{2} f''(Z_t) \, (dZ_t)^2$$

Filling in $f(Z_t) = e^{Z_t}$, we find:

$$f'(Z_t) = e^{Z_t}, \quad f''(Z_t) = e^{Z_t}$$

Using the dynamics of $Z_t$:

$$dZ_t = \kappa (Z^* - Z_t) \, dt + \sigma_Z \, dW_{2,t}$$

We substitute $f'(Z_t)$, $f''(Z_t)$, and $dZ_t$:

$$d\theta_t = e^{Z_t}\, dZ_t + \frac{1}{2} e^{Z_t}\, \sigma_Z^2\, dt = \theta_t\, dZ_t + \frac{1}{2}\theta_t\, \sigma_Z^2\, dt$$

Replacing $dZ_t$:

$$d\theta_t = \theta_t \left( \kappa(Z^* - Z_t) + \frac{1}{2}\sigma_Z^2 \right) dt + \theta_t \sigma_Z\, dW_{2,t}$$

The relationship between $Z_t$ and $\theta_t$, and between $\sigma_Z$ and $\sigma_\theta$ is:

$$e^{Z_t} = \theta_t, \quad Z_t = \ln(\theta_t), \quad \sigma_\theta = \theta_t \sigma_Z = e^{Z_t}\sigma_Z, \quad \sigma_Z = \frac{\sigma_\theta}{\theta_t} = \frac{\sigma_\theta}{e^{Z_t}}$$

Finally, express all $Z_t$ in terms of $\theta_t$:

$$d\theta_t = \theta_t \kappa \left(\ln(\theta^*) - \ln(\theta_t)\right)\, dt + \frac{\sigma_\theta^2}{2\theta_t}\, dt + \sigma_\theta\, dW_{2,t}$$

To find the dynamics of the portfolio ESG score $\theta_{P,t}$, we use the definition:

$$\theta_{P,t} = \pi_t \theta_t,$$

where $\pi_t$ is assumed to be constant (i.e., $d\pi_t = 0$). Applying the product rule:

$$d\theta_{P,t} = \pi_t\, d\theta_t.$$

Substitute the dynamics of $\theta_t$:

$$d\theta_{P,t} = \pi_t \left[ \theta_t \kappa \left(\ln(\theta^*) - \ln(\theta_t)\right)\, dt + \frac{\sigma_\theta^2}{2\theta_t}\, dt + \sigma_\theta\, dW_{2,t} \right].$$

Finally, rewrite the dynamics in terms of $\theta_{P,t}$:

$$d\theta_{P,t} = \theta_{P,t}\kappa \left( \ln(\theta^*) - \ln\left(\frac{\theta_{P,t}}{\pi_t}\right)\right) dt + \frac{\pi_t \sigma_\theta^2}{2\theta_{P,t}}\, dt + \pi_t \sigma_\theta \, dW_{2,t}.$$

**Derivative with respect to individual ESG Score:**

$$J_\theta = g(t)\beta X_t^{\tilde{\alpha}} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-1}$$

$$J_{\theta\theta} = g(t)\beta(\tilde{\beta}-1) X_t^{\tilde{\alpha}} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-2}$$

$$J_{X\theta} = g(t)\alpha\beta(1-\gamma) X_t^{\tilde{\alpha}-1} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-1}$$

- **First-order derivative:**

  $$\frac{\partial J}{\partial \theta_{P,t}} = \frac{\partial J}{\partial \theta_t} \frac{\partial \theta_t}{\partial \theta_{P,t}}.$$

  Since $\theta_{P,t} = \pi_t \theta_t$, we have:

  $$\frac{\partial \theta_t}{\partial \theta_{P,t}} = \frac{1}{\pi_t}.$$

  Therefore:

  $$\frac{\partial J}{\partial \theta_{P,t}} = \frac{1}{\pi_t} \frac{\partial J}{\partial \theta_t}.$$

- **Second-order derivative:**

  $$\frac{\partial^2 J}{\partial \theta_{P,t}^2} = \frac{\partial}{\partial \theta_{P,t}} \left( \frac{1}{\pi_t} \frac{\partial J}{\partial \theta_t} \right).$$

  Since $\pi_t$ is constant:

  $$\frac{\partial^2 J}{\partial \theta_{P,t}^2} = \frac{1}{\pi_t^2} \frac{\partial^2 J}{\partial \theta_t^2}.$$

- **Cross-term between $X_t$ and $\theta_{P,t}$:**

$$\frac{\partial^2 J}{\partial X_t \partial \theta_{P,t}} = \frac{\partial}{\partial X_t}\left(\frac{1}{\pi_t}\frac{\partial J}{\partial \theta_t}\right).$$

  Again, since $\pi_t$ is constant:

$$\frac{\partial^2 J}{\partial X_t \partial \theta_{P,t}} = \frac{1}{\pi_t}\frac{\partial^2 J}{\partial X_t \partial \theta_t}.$$

Concluding:

$$J_{\theta_P} = \frac{1}{\pi_t} \cdot J_\theta = \frac{1}{\pi_t} \cdot g(t)\beta X_t^{\tilde{\alpha}} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-1}$$

$$J_{\theta_P\theta_P} = \frac{1}{\pi_t^2} \cdot J_{\theta\theta} = \frac{1}{\pi_t^2} \cdot g(t)\beta(\tilde{\beta}-1) X_t^{\tilde{\alpha}} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-2}$$

$$J_{X\theta_P} = \frac{1}{\pi_t} \cdot J_{X\theta} = \frac{1}{\pi_t} \cdot g(t)\alpha\beta(1-\gamma) X_t^{\tilde{\alpha}-1} \pi^{\tilde{\beta}} \theta_t^{\tilde{\beta}-1}$$

**Hamilton-Jacobi-Bellman Equation**

$$0 = \sup_{\pi_t}\Bigg\{\frac{\partial J}{\partial t} + J_X X_t\Big(r + \pi_t(\mu_S - r)\Big) + J_\theta\Big(\theta_t\kappa\left(\ln(\theta^*) - \ln(\theta_t)\right) + \frac{\sigma_\theta^2}{2\theta_t}\Big)$$
$$+\frac{1}{2}J_{XX}X_t^2\pi_t^2\sigma_S^2 + \frac{1}{2}J_{\theta\theta}\sigma_\theta^2 + J_{X\theta}X_t\pi_t\sigma_S\sigma_\theta\rho\Bigg\}$$

*Terminal condition:*

$$J(X_T, \theta_{P,T}, T) = U(X_T, \theta_{P,T}) = \frac{(X_T^\alpha \theta_{P,T}^\beta)^{1-\gamma}}{1-\gamma},$$

**All terms involving $\pi$**

$$-J_X X_t \pi_t (\mu_S - r) + \frac{1}{2} J_{XX} X_t^2 \pi_t^2 \sigma_S^2 + J_{X\theta} X_t \pi_t \sigma_S \sigma_\theta \rho$$

**First Order Conditions**

$$J_X X_t (\mu_S - r) + J_{XX} X_t^2 \pi_t \sigma_S^2 + J_{X\theta} X_t \sigma_S \sigma_\theta \rho = 0$$

**Optimal $\pi^*$**

$$\pi_t^* = -\frac{J_X(\mu_S - r) + J_{X\theta} \sigma_S \sigma_\theta \rho}{J_{XX} X_t \sigma_S^2}$$