A Spectral Measure of Fragility in Financial Networks

by
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B.Sc. Tilburg University 2023

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in
Quantitative Finance and Actuarial Science

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Date: 29 April 2024

Abstract

This thesis introduces an experimental spectral measure of fragility within financial networks, designed to capture the non-linearities in liability payments that can exacerbate systemic risk during large systematic shocks. The study is motivated by the need for more precise tools to analyze and predict the propagation of financial distress within interconnected banking systems. This spectral method is compared to standard linear measures of network centrality.

Using a combination of theoretical modeling based on the seminal work of Eisenberg and Noe (2001) and recent enhancements, this thesis develops a framework that incorporates various recent theoretical contributions to literature. The research focuses on the role of dependency cycles and their impact on the robust-yet-fragile nature of highly interconnected financial systems. Through theoretical special cases of financial networks and strong components, the study explores how different network configurations affect the spectral fragility measure.

The theoretical building blocks that the spectral measure is built upon are controlled for using a naive simulation method, demonstrating the practical applications of the spectral fragility measure and its ability to capture the desired non-linearity. The results highlight the potential of this measure to inform regulatory strategies and risk management practices by identifying fragile components within the network and links within financial networks that are most susceptible to cascading failures.

The thesis concludes with a discussion of the implications of these findings for policymakers and financial institutions, emphasizing the need for enhanced monitoring tools that can dynamically adjust to changing shape of financial networks.

Acknowledgments

I would like to express my gratitude to my supervisor, Prof. Dr. Jean-Jacques Herings, for his invaluable guidance and support throughout my research process. His vast experience as a prominent researcher in the field and ability to tailor advice to the specific challenges I faced during my previous thesis was particularly beneficial.

I am also deeply thankful to my family and friends for their unwavering support throughout my academic pursuits. I extend a special gratitude to my mother, whose dedication to understanding my needs during a challenging period led to significant personal and academic growth.

I would also like to extend my heartfelt appreciation to my girlfriend, whose patience and understanding were invaluable as I dedicated many long hours to my studies.

Contents

1 Introduction

"The robust-yet-fragile property of networks helps make sense of (...) non-linear financial dynamics. Though they looked and felt like chaos, these dynamics were in fact manifestations of a new network order" (Haldane, 2009, p. 5).

The increased levels of globalization over the past few decades are reflected in the increased interconnectedness of financial institutions on a global scale. Bilateral contractual obligations link financial institutions and expose these institutions to counterparty risk (Bernard et al., 2022).

The risks are compounded by the network structure of the financial system, which can be robust-yet-fragile (Haldane, 2009; Acemoglu et al., 2015; Jackson and Pernoud, 2020).

While interconnectedness can diversify some risks, it also makes the system more fragile to widespread losses that can cascade through the network (Haldane, 2009; Jackson and Pernoud, 2020).

Contagion refers to the process by which financial distress in one institution spreads to other institutions (Bernard et al., 2022). Contagion is just one of the dimensions of systemic risk. Bardoscia et al. (2021) consider different financial networks and review models of contagion.

An illuminating summary of the different dimensions of systemic risk is provided in the online appendix of Jackson and Pernoud (2020). These dimensions are explained by Duffie (2019) in the context of the 2008 financial crisis.

In one respect, financial intermediation plays a crucial role in economies by leveraging economies of scope and scale to pool risks, providing trading opportunities, and matching funds with investments (Jackson and Pernoud, 2020).

In another respect, the potential for disaster is exacerbated if sufficiently many institutions are invested in similarly distressed portfolios, as was the case in 2008 (Jackson and Pernoud, 2019). This crisis underscored the financial system's capacity to absorb small shocks while highlighting its vulnerability to large shocks (Jackson and Pernoud, 2019).

To illustrate the level of interconnectedness post-crisis, Duarte and Jones (2017) estimate that 23% of all assets of bank holding companies and 48% of liabilities still come from other financial intermediaries.

The seminal work by Eisenberg and Noe (2001) introduced a model where financial entities are linked through nominal liabilities. This model is the first to consider the fundamental aspect of financial structures that the inability of a bank to pay the nominal liabilities in full, affects the solvency of other banks which, in turn, are unable to pay the initially insolvent bank, ultimately furthering its insolvency.

Clearing payments are payments following a default that must satisfy several rules.

Eisenberg and Noe (2001) highlights in an example that the presence of a directed cycle without cash injections can imply the existence of inefficient equilibria of clearing payments. Rogers and Veraart (2013) find the inclusion of default costs allows for inefficient clearing payment even with cash injections. This thesis considers similar default costs to capture the consequences of inefficient clearing payments.

The presence of these directed cycles forms a fundamental building block in the spectral fragility measure as defined in this thesis.

Eisenberg and Noe (2001) provide mild regularity conditions for the uniqueness of clearing payments. Glasserman and Young (2015) and Kusnetsov and Veraart (2019) provide alternative uniqueness conditions. These results are generalized to weaker sufficiency conditions and more general bankruptcy rules by Csóka and Herings (2024). Another generalization exists for a reasonable subset of default costs (Jackson and Pernoud, 2023). The latter two papers reinforce the importance of directed cycles for inefficiencies in financial networks. Both these papers indicate that it is impossible to have inefficient clearing payments if there are no directed liability cycles in the network.

Acemoglu et al. (2015) consider banks with identical roles and find that if a network is just a directed cycle, this always produces the largest number of defaults because of the spillover effect.

Acemoglu et al. (2015) argue that the standard spectral measures to find contagious banks fail to capture the non-linear jumps in liability payments in case of default.

In response, this thesis aims to define a spectral measure that captures these non-linearities with a particular focus on the fragility of financial networks to large systematic shocks.

Eisenberg and Noe (2001) and much of the work that is based on this framework assumes a central clearing agency performs all bankruptcy proceedings.

As Elsinger et al. (2006) and Gai and Kapadia (2010) note, the governing bodies possess only partial information on the true connections between intermediaries. Furthermore, Franken (2012) explains how in international insolvency proceedings, courts typically operate in parallel.

In response, the work by Csóka and Herings (2018) on decentralized clearing algorithms captures the consequences of the non-cooperative nature of bankruptcy proceedings and sequential bankruptcy filings prevalent in real-world financial systems. Csóka and Herings (2018) find that decentralized clearing algorithms typically lead to the least clearing payments and provide an upper bound on the differences with the optimal clearing payments. As a consequence, their work concludes that the costs could outweigh the benefits of organizing a central clearing mechanism.

The goal of this spectral measure is to predict the difference in aggregate costs of financial networks for the least and greatest clearing payments.

The spectral measure proposed in this thesis is a weighted average of the fragility of subcomponents in the network. These subcomponents, which are explained in later sections as strong components, can contain many directed cycles. The larger the number of dependency cycles within a strong component, the more interconnected this component is. The more interconnected this component is, the more fragile it is to large systematic shocks.

These weights are based on the fraction of total liabilities exposed to the credit risk of these fragile components. Different parameters allow for network topology-based adjustments to the weights and fragility measures for different strong components.

The spectral measure as defined in this thesis could allow regulators to assign risk capital to strong components based on their fragility.

Risk capital allocation methods could then decide the allocation to the individual banks in a strong component. See, e.g., Bauer and Zanjani (2013) and Baione et al. (2018), among others, for a description of desirable properties of risk capital allocations.

In essence, the spectral fragility measure may pave the way toward pricing a part of systemic risk.

Note that a measure of fragility in terms of the degree of interconnectedness of banks in a financial network is automatically a measure of robustness. These properties of financial networks are two sides of the same coin. After all, the fragility of large systematic shocks is inherently linked to the robustness against small idiosyncratic shocks.

The focus of this thesis lies on the fragility side of this coin. The context is a financial crisis, following a large systematic shock. The motivation is in part to compare the empirical performance of standard network measures in literature to simulated results in the Eisenberg and Noe (2001) model as a control for the spectral fragility measure.

Groote Schaarsberg et al. (2013); Csóka and Herings (2018, 2023, 2024) consider different rules, such as constraint equal loss and the pairwise netting pro-rata rule.

Gai et al. (2011) highlight that methods from statistical physics for complex networks could significantly improve the understanding of financial network structures based on partial information.

Mezić et al. (2019) provides an efficient algorithm that calculates an eigenvalue-based complexity measure that mainly highlights the presence of cyclical structures in complex networks. In particular, this method approximates a dominant cycle in directed networks by utilizing the ability of complex eigenvalues to describe cyclical patterns. This dominant cycle partitions the banks in a financial network into clusters. This method supplements the standard spectral measures, e.g., the spectral gap (Fiedler, 1973; Chung, 2005; Montenegro and Tetali, 2006), and the borrower and lender Bonacich (Bonacich and Lloyd, 2001) and Katz (Katz, 1953) centrality measures.

Craig et al. (2013) find that the Bonacich lender centrality measure for German banks had a significant positive effect on sealed bids for liquidity in refinancing operations of the ECB during the 2008 financial crisis. However, this effect is relatively very small. Craig et al. (2013) find that the direct links in the network much more consistently explain the willingness to pay for liquidity. Intuitively, during a financial crisis, when exposed to a high degree of credit risk, financial institutions are willing to pay more for liquid assets on average relative to institutions that are less exposed to credit risk (Craig et al., 2013).

Acemoglu et al. (2015) argue that this difference in effect can be explained by the failure of the Bonacich centrality to capture non-linearities in financial contracts.

Thus, the presence of directed cycles in financial networks, measured through Bonacich lender centrality, was much less important for German banks during the 2008 crisis than the direct exposure to credit risk for liquidity demand.

Periphery banks that directly lend to highly Bonacich lender central banks bid much more aggressively to cover their liquidity needs during the 2008 financial crisis (Craig et al., 2013).

This indicates that the Katz lender centrality measure (Katz, 1953; Bonacich and Lloyd, 2001; Puhr et al., 2012; Glasserman and Young, 2016) with a high discount parameter is preferable to the borrower or lender Bonacich measure and the borrower Katz measure. That is, the Katz centrality scales down the indirect relationships exponentially, such that the indirect liability links between banks are still considered with significantly less weight relative to the direct links. In contrast, the presence of the directed cycles had an estimated dominating impact on failures in the *U.S.* treasury market.

However, Puhr et al. (2012) finds that the Katz borrower centrality measure is useful for detecting the potential for contagion through a particular bank. These results follow from simulations based on data from the Austrian interbank markets and it is not clear whether their conclusions stem from the simulations or the underlying data (Glasserman and Young, 2016).

This thesis considers a measure based on a version of the Katz-Bonacich borrower centrality measure to control for the predictive power regarding the aggregate costs in the network due to inefficient clearing payments.

The goal is to compare the effectiveness of assessing complex tools based on simple assumptions.

This measure holds high predictive power which is likely due to the oversimplified simulation methods. In addition, the Katz-Bonacich discount parameter that optimizes the predictive power of the measure is incredibly small. This could provide some motivation for the poor empirical performance of Katz-Bonacich centrality measures as predictors of contagion. In addition, Fleming and Keane (2021) estimate that approximately three-quarters of failures in the *U.S.* treasury market in March 2020 were potentially caused by directed cycles in the network that would have been prevented if claims were centrally cleared.

Glasserman and Young (2016) argue that the lack of consensus in the empirical literature on systemic risk measures could indicate that standard network measures from other fields are ill-equipped to capture systemic risk in financial networks.

However, experimental systemic risk measures from, e.g., statistical physics and engineering should not be disregarded.

The paper is organized as follows. In Section 2, the network model is defined. Section 3 establishes the proportional payment rule. Section 4 generalizes the framework to include default costs. Section 5 links the network structure to inefficient clearing payments. Section 6 establishes the theoretical foundation of the spectral fragility measure. Section 7 defines the spectral fragility measure. Section 8 establishes the simulation methods and results. Section 9 summarizes the key findings of this thesis and discusses directions for future research.

2 Model Description

2.1 Financial Network

To start, the most basic version of the model is described based on the seminal work by Eisenberg and Noe (2001).

The standard framework is the following. There are n financial entities, referred to as banks for simplicity. These banks are part of a complex system of mutual liabilities and earn exogenous cash from regular operations. Sometimes these banks are referred to as nodes, which indicates their presence in the graph of a financial network.

This complex system allows for many interpretations which are rarely consistent across different articles. Other interpretations of the operational cash flow pertain, e.g., net asset value from an investment portfolio (Jackson and Pernoud, 2023) or general initial endowments (Csóka and Herings, 2023). Groote Schaarsberg et al. (2013) and Csóka and Herings (2023) consider the structure of nominal liabilities payments into an upper triangular matrix of , one part of this focuses on transformation of liability

Definition 1 (Financial Network)
A *financial network* is a tuple (N, L, c), where $N = \{1, ..., n\}$ represents the finite set of banks, $L \in \mathbb{R}_+^{n \times n}$ is the non-negative *liability matrix*, and $c \in \mathbb{R}_+^n$ is the non-negative *cash vector*.

The *ij-th* entry of the liability matrix, $\forall i, j \in N : L_{ij}$ represents the nominal liability of bank i to j. The nominal liabilities of any node to itself are always zero, i.e., $\forall i \in N : L_{ii} = 0$. The total nominal liabilities of a bank $i \in N$ are denoted $l_i = \sum_{j \in N} L_{ij}$. These total nominal liabilities are captured in the *liability vector* $L\iota = l$, where $\iota \in \mathbb{R}^n$ is the vector of ones.

The *operational cash flow* of bank $i \in N$ is a result of regular business operations and is denoted $c_i \geq 0$. These operational cash flows are captured in the cash vector $c = (c_i)_{i \in N} \in \mathbb{R}_+^n$.

The *nominal asset value* of bank $i \in N$, is the amount of cash available to pay off liabilities in case all the debtors of this bank do not default on their loans. Formally, the nominal asset value of node $i \in N$ is

$$\eta_i = c_i + \sum_{k \in N} L_{ki}. \tag{1}$$

These nominal asset values are captured in the vector $\eta = c + L^T \iota$.

Consider a financial network (N, L, c). The primary variable of a financial network is the payment. A payment occurs between banks to pay off the liabilities. $\forall i, j \in N : P_{ij} \in [0, L_{ij}]$ denotes the payment from node $i \in N$ to $j \in N$. These payments are collected in a non-negative *payment matrix* $P \in \mathbb{R}_+^{n \times n}$, where the ij-th entry is a P_{ij} for all banks $i, j \in N$. The total payments by bank $i \in N$ are denoted $p_i = \sum_{j \in N} P_{ij}$ and are collected in the payment vector $P\iota = p$.

Given a payment matrix P, the *asset value* of node $i \in N$ and the nominal asset value η_i are the total amount of cash available for their liability payments. Formally, the asset value of node $i \in N$ is a function $a_i : [\mathbf{0},\ L] \mapsto [0, \eta_i]$, defined by

$$a_i(P) = c_i + \sum_{k \in N} P_{ki}, \tag{2}$$

where $\mathbf{0} \in \mathbb{R}^{n \times n}$ is the zero matrix. The asset values are collected in the vector $a(P) = c + P^T \iota$. Under regular circumstances, the liabilities in a financial network are paid in full, i.e., $\forall i, j \in N : P_{ij} = L_{ij}$. In contrast, a problem occurs when any node $i \in N$ is unable to pay the nominal liabilities in full due to, e.g., an idiosyncratic (Glasserman and Young, 2015) or systematic (Rogers and Veraart, 2013) drop in the operational cash flow c_i. When at least one bank is unable to pay all its liabilities, there must be a set of rules that dictate the resulting payments following default.

Given a payment matrix, the *book value* of a bank is the capital shortfall where negative values represent insolvency. That is, the book value of bank $i \in N$ is a function $v_i : [\mathbf{0}, L] \mapsto \mathbb{R}$, defined by

$$v_i(P) = a_i(P) - l_i. \tag{3}$$

The book values are collected in the book vector $v(P) = a(P) - l$.

A bank $i \in N$ defaults in case the book value is negative following payments, i.e., $v_i(P) < 0$. The interpretation of solvency for $v_i(P) = 0$ lies in the idea that at least some banks in the network are willing to roll over on their loans because the bank was able to pay off all its liabilities. In the short term, these banks are not able to verify the financial health of the barely solvent bank. In turn, this allows the regular operations of banks to continue which facilitates their new debt and operational cash flows.

The tighter regulations in the banking sector since the 2008 crisis greatly reduced such scenarios (Duffie, 2019). However, regional and sectoral regulatory arbitrage is designed to game regulations and a large part of the risk is potentially unaccounted for (Nouy, 2017).

In contrast, Rogers and Veraart (2013) implicitly assume that zero book values represent insolvency because they require the total equity improvement to be strictly positive for banks to merge and save the initial set of defaulting banks. That is, even the banks that would be able to pay all their liabilities if and only if they agreed to a merger had no incentive because the equity improvement would be zero.[1]

The *equity* of bank $i \in N$ is the positive part of its book value and thus a function $e_i : [\mathbf{0}, L] \mapsto \mathbb{R}_+$, defined by

$$e_i(P) = (v_i(P))^+. \tag{4}$$

The equity values are collected in the equity vector $e(P) = (v(P))^+$.

Clearing payments are payments between banks that settle claims of creditors on a bank that defaults on their loans by establishing payment rules (Eisenberg and Noe, 2001; Csóka and Herings, 2018, 2023).

The first condition, limited liability, requires that no bank $i \in N$ can pay more than the available assets $a_i(P)$ allow, i.e., $\sum_{i \in N} P_{ik} \leq a_i(P)$. The second property, the absolute priority of creditors, requires equity holders to be paid last for a bank that defaults, i.e., $\forall i \in N : a_i(P) < l_i \implies e_i(P) = 0$.

For a financial network (N, L, c), a clearing matrix must thus be part of the set of feasible payment matrices, given by

$$\mathcal{Q} = \left\{ P \in \mathbb{R}_+^{n \times n} : P \leq L, p \leq a(P) \right\}, \tag{5}$$

where the first inequality ensures the clearing matrix is a payment matrix and the second inequality imposes limited liability.

Let $K \wedge M = \min\{K, M\}$, denote the coordinate-wise minimum operator for any two matrices $K, M \in \mathbb{R}^{r \times z}$ for general dimensions $r, z \in \mathbb{N}$. If a payment matrix in $\mathcal{Q}$ satisfies the absolute priority of creditors, it is a clearing matrix which is formalized in the following definition.

Definition 2 (Clearing Matrix and Vector)
A payment matrix, $\tilde{P} \in \mathcal{Q}$, is a *clearing matrix* if it complies with the absolute priority of creditors, i.e., $\tilde{p} = l \wedge a(\tilde{P})$, for $\tilde{p} = \tilde{P}\iota$.

[1]This is easily prevented by positive merger costs. This ensures that there is no incentive as both merging and not merging lead to insolvency. Otherwise, the conclusion that there is no incentive for insolvent banks to become solvent is problematic.

A *clearing vector* is a payment vector corresponding to clearing payments.

Let the set of clearing matrices be denoted $\mathcal{M}$ and the clearing vectors be denoted CV.

2.2 Objective Function

It is important to quantify the economic damages suffered for different clearing payments in the same, or a different financial network.

For an appropriate objective function, it is possible to obtain the largest clearing payments as the solution to an optimization problem.

Assume $f : \mathcal{Q} \mapsto \mathbb{R}$ is any decreasing function in the sense that $P' \lneq P''$ implies $f(P') > f(P'')$. Csóka and Herings (2018) show that

$$P^* = \arg\min_{P \in \mathcal{Q}} \{f(P)\}, \tag{6}$$

satisfies absolute priority of creditors, i.e., $P^* \in \mathcal{M}$.

The proof is based on the principle that in a financial network (N, L, c), the problem is essentially a problem that assigns the total value of the network to claimants with mutual liabilities that have unique final allocations (Groote Schaarsberg et al., 2013; Csóka and Herings, 2024).

In practice, there are additional constraints on the clearing payments, such as the pro-rata rule as modeled by Eisenberg and Noe (2001).[2] Before the inclusion of an additional bankruptcy rule, the greatest clearing matrix of the financial network (N, L, c) provides a lower bound for the optimal objective function after the inclusion of the additional bankruptcy rule, see, e.g., Calafiore et al. (2022a). After all, additional constraints can only yield weakly less optimal solutions to any optimization problem.

The next section establishes the clearing payment rule applied throughout this thesis.

3 Pro-Rata

This section establishes the pro-rata rule which is a common principle for clearing payments in case of default. The pro-rata rule states that if a default occurs, the defaulting bank pays all

[2]For alternative rules to the pro-rata rule, see, e.g., Groote Schaarsberg et al. (2013); Csóka and Herings (2023).

claimant banks in proportion to the size of their nominal claims on the assets of the defaulting bank (Rogers and Veraart, 2013). After some definitions, the first result establishes a relationship between the structure of a network graph and optimal and sub-optimal clearing payments.

The following definitions are provided by Ross (2019). A non-negative square matrix $A \in \mathbb{R}^{n \times n}$ is said to be *row-stochastic* if all its rows sum to 1: $\forall i \in N : \sum_{j \in N} A_{ij} = 1$, and *row sub-stochastic* if $\forall i \in N : \sum_{j \in N} a_{ij} \leq 1$. Matrix $A \geq 0$ is stochastic if $A\iota = \iota$ and sub-stochastic if $A\iota \leq \iota$. A matrix A is strictly sub-stochastic if at least one row does not sum up to one. That is $A\iota \lneq \iota$.

For their analysis, Eisenberg and Noe (2001) define a proportionality matrix which is by definition *strictly sub-stochastic* if at least one of the nodes does not have any liabilities. It is possible to define the matrix that it is always stochastic which may be useful for specific applications.[3] However, the sub-stochastic property is useful in later sections of this thesis. Define the proportionality matrix $A \in [0,1]^{n \times n}$ by

$$A_{ij} = \begin{cases} \frac{L_{ij}}{l_i} & \text{if } l_i > 0, \\ 0 & \text{else.} \end{cases} \tag{7}$$

The pro-rata rule implies $\forall i, j \in N : P_{ij} = A_{ij} p_i$ where p_i is the total payment by node $i \in N$. Because of the pro-rata rule it is thus possible to identify all clearing payments by just the total amount paid by each node and the fraction of total liabilities owed to each other node provided in A. In addition, the nominal liabilities are identified in the same sense that $L_{ij} = A_{ij} l_i$ are the nominal liabilities from node i to j.

It is now possible to reformulate the financial system (N, L, c) using the proportionality matrix A, the vector of total liabilities l, and the cash vector c.

Definition 3 (Pro-Rata Network)
A *pro-rata network* is a tuple (N, A, l, c), where N represents the set of banks, A is the proportionality matrix, l is the liability vector, and c is the cash vector.

Thus, the financial network (N, L, c) under the pro-rata rule is equivalent to the pro-rata network (N, A, l, c).

[3] Eisenberg and Noe (2001) do mention the possibility of a node without liabilities to serve as the receiver of operating costs if one seeks to model negative operational cash flow. However, in their model, Eisenberg and Noe (2001) treat their proportionality matrix as row-stochastic and thus implicitly assume that there is no explicitly modeled bank without liabilities. This is not problematic because none of their results rely on row-stochasticity.

Consider (N, A, l, c). All functions of payment matrices are reduced to functions of payment vectors. In particular, the asset value, book value, and equity value are functions $a_i, v_i, e_i : [0, l] \mapsto \mathbb{R}$, defined by

$$a_i(p) = c_i + (A^T p)_i, \tag{8}$$
$$v_i(p) = a_i(p) - l_i, \tag{9}$$
$$e_i(p) = (v_i(p))^+. \tag{10}$$

These values are collected, respectively, in the vectors $a(p) = c + (A^T p)$, $v(p) = a(p) - l$, and $e(p) = (v(p))^+$. Clearly, $a(\cdot)$ is weakly increasing, i.e., $\forall p' \leq p'' : a(p') \leq a(p'')$. In addition, if $p' \lneq p''$ then $a(p') \lneq a(p'')$. That is, if at least one coordinate of p increases then all coordinates weakly increase and at least one coordinate strictly increases of $a(p)$.

Given a pro-rata network, the set of defaulting banks is a function of the payment vector. Let $\mathcal{P}(N)$ be the power set of N. Formally, the default set is a function $D : [0, l] \mapsto \mathcal{P}(N)$, defined by,

$$D(p) = \{i \in N : v_i(p) < 0\}. \tag{11}$$

The set of payment vectors that satisfy limited liability is referred to as the set of feasible payment vectors and is given by (Eisenberg and Noe, 2001),

$$Q = \left\{p \in \mathbb{R}^n_+ : p \leq l, p \leq a(p)\right\}. \tag{12}$$

This facilitates a reformulation of clearing payments in terms of total payments by each node, collected in a clearing vector.

Definition 4 (Clearing Vector in a Pro-Rata Network)
A vector $p^* \in Q$ is a *clearing vector*, i.e., $p^* \in CV$, if it complies with absolute priority of creditors. That is,

$$p^* = l \wedge a(p^*). \tag{13}$$

It is possible and useful to reformulate the objective function in terms of the set of feasible vectors Q.

Assume that $f : Q \to \mathbb{R}$ is any decreasing function in the sense that $p', p'' \in \mathbb{R}^n : p' \lneq p''$ implies $f(p') > f(p'')$. Any solution, $p^* \in Q$, to the following problem

$$\min_{p \in Q} \{f(p)\}. \tag{14}$$

is a clearing vector (Eisenberg and Noe, 2001), i.e., $p^* \in CV$. Let the set of clearing vectors under the pro-rata rule be denoted $CV = CV \cap Q$.

Calafiore et al. (2022a) define the system-level cost as the total deviations between the nominal asset value and asset value values of all nodes. This choice is formalized in the following definition.

Definition 5 (System-Level Costs)
The *system-level costs* is a function $f : Q \mapsto \mathbb{R}$, defined by

$$f(p) = \iota^T(\eta - a(p)). \tag{15}$$

Clearly, this objective function is decreasing in $p \geq 0$ through $a(p)$ in the appropriate sense and thus satisfies (13).

As mentioned at the start of this section, any such optimization function requires a central clearing agency that possesses all information. For simplicity, the central clearing agency is assumed to do all payments simultaneously.

The existence of an all-knowing central clearing agency does not reflect reality (Gai and Kapadia, 2010; Elsinger et al., 2006).

3.1 Equivalent Weighted Digraphs

The following definitions are standard in graph theory and are useful tools to intuitively understand and theoretically derive results that reveal the structure of complex networks.

Every non-negative square matrix $A = (A_{ij})_{i,j \in N}$ corresponds to a *weighted digraph* $G[A] = (N, E[A], A)$ whose nodes are indexed by N and whose set of arcs is defined as $E[A] = \{(i,j) \in N \times N : A_{ij} > 0\}$. The value $A_{ij} > 0$ represents the weight of arc $(i,j) \in E[A]$. A sequence of arcs $(h_0, h_1),\ (h_1, h_2),\ \ldots,$
$(h_{s-1},\ h_s) \in E[A]$ form a directed path between nodes $h_0 \in N$ and $h_s \in N$ in graph $G[A]$. The set of nodes $J \subseteq N$ is *reachable* from node i if $u \in J$ or a path from i to some element $j \in N$ exists; J is called *globally reachable* in the graph if it is reachable from every node $u \notin J$.

In a directed graph $G[A]$, a component is a subset of nodes $S \subseteq N$ that can hold certain properties, in terms of reachability. An important example is, that a set of nodes $S \subseteq N$ forms a strong component if it is a maximal subgraph where every pair of nodes $i, j \in S$ are mutually reachable. This implies that, starting from any node in S it is possible to reach every other node. Strong refers to the strong relationship between any two nodes in such a component. Furthermore, the maximal part of the definition implies that it is impossible to add another node to the component without violating the mutual reachability for all nodes in the component.

A strong component is *non-trivial* in case it contains at least two nodes.

This intuitively implies that there are multiple strong components in a digraph as long as not all nodes are mutually reachable.

In contrast, if any pair of nodes in a digraph is mutually reachable, then the digraph is connected If there is a direct path between any pair of nodes in both directions, the digraph is *fully connected*.

A *sink component* has no arcs leaving it, a *source component* has no arcs entering it.

Additional definitions for digraphs, $G[A] = (N, E[A], A)$ are relevant for the study of inefficient equilibria outcomes in case of default costs (Jackson and Pernoud, 2023) or bare minimum sufficiency conditions for uniqueness of the clearing vector with no default costs (Csóka and Herings, 2024). In particular, a dependency cycle is a directed path (i_0, i_1), ..., $(i_{s-1},\ i_s)$ such that for $0 \leq j \leq s-1 : (i_j, i_{j+1}) \in E[A],\ s \geq 2, i_s = i_0$, and $\forall j \notin \{0, s\} \forall h \neq j : i_j \neq i_h$. Note that the last requirement differs from similar definitions by, e.g., Csóka and Herings (2024) and Jackson and Pernoud (2023) that allow for cycles or dependency cycles to pass the same node twice. Thus, a dependency cycle is a directed path of positive nominal liabilities that starts and ends at the same node, passes at least one other node, and never passes the same node twice.

Note, that any dependency cycle must be part of a strong component. Either the dependency cycle is the strong component or multiple dependency cycles form the strong component. Thus, roughly speaking, a non-trivial strong component starting from node $i \in N$ is the largest collection of dependency cycles that contains $i \in N$. For this reason, Csóka and Herings (2024) and Jackson and Pernoud (2023) define the cycle to be the strong component. Both the presence of a dependency cycle and a non-trivial strong component provide the same necessary condition for the existence of inefficient clearing vectors. These results underline Proposition 1.

3.2 Uniqueness Condition for the Clearing Vector

Eisenberg and Noe (2001) add a bankruptcy rule and highlight that the equity after clearing payments is constant and is essentially an allocation of total operational cash flows. This result is generalized by Groote Schaarsberg et al. (2013) to the entire set of bankruptcy rules. That is, choosing a different bankruptcy rule does not change the invariance of equity for different clearing vectors.

The works by Eisenberg and Noe (2001); Glasserman and Young (2015); Kusnetsov and Veraart (2019), describe conditions for unique clearing vectors. In addition, Csóka and Herings (2024) provide a uniqueness condition that combines these conditions and holds Eisenberg and Noe (2001); Glasserman and Young (2015) as special cases.

Proposition 1 Consider a pro-rata network (N, A, l, c).

- (i) There is a unique least and greatest clearing vector, i.e., $\exists p_*, p^* \in CV \ \forall p \in CV : p_* \leq p \leq p^*$ (Eisenberg and Noe, 2001).
- (ii) The solution to (14) is unique, the largest clearing vector, and independent of f provided f is decreasing (Eisenberg and Noe, 2001).
- (iii) Let $\tilde{p}, \hat{p} \in CV$. Then $e(\tilde{p}) = e(\hat{p})$ (Eisenberg and Noe, 2001; Groote Schaarsberg et al., 2013).
- (iv) If each dependency cycle has at least one bank with positive operational cash flow or a path of liabilities toward a sink node, then the clearing vector is unique and thus $CV = \{p^*\}$, where p^* is the unique solution to (14) (Csóka and Herings, 2024).

Proof. The proof of (i) is provided in the next section in a more general setting. The proof of (ii) follows from (i) and that each solution must be a clearing vector in combination with the decreasing property of f in (14). The proof of (iii) is provided explicitly below because of the relevant distinction with the general setting of this thesis in Section 4. This proof follows the seminal work by Eisenberg and Noe (2001).

The following proof of (iv) utilizes the proof provided by Eisenberg and Noe (2001) and is based in large part on property (iii). Alternative proofs for these properties are more technical and available in Glasserman and Young (2016); Calafiore et al. (2022b); Csóka and Herings (2024).

- (iii) Note, $\forall p \in CV : e(p) = (v(p))^+ = (A^T p - l)^+ = A^T p - p$. In addition, $\iota^T A^T = \iota^T$ because A^T is column stochastic and thus total equity satisfies $\iota^T e(p^*) = \iota^T(A^T p^* + c - p^*) = \iota^T(p^*(1-1)+c) = \iota^T c = \iota^T(p(1-1)+c) = \iota^T(A^T p + c - p) = \iota^T e(p)$ for the largest clearing vector $p^* \in CV$ and another clearing vector $p \in CV \setminus \{p^*\}$. Thus, total equity is equal to total operational cash flow irrespective of the clearing vector.

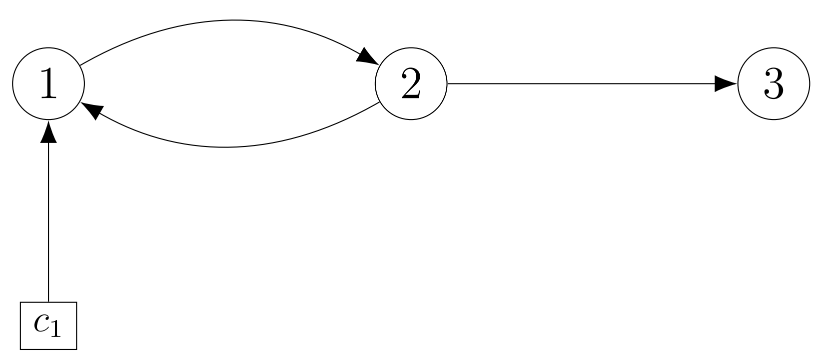
Suppose now, $\exists \hat{p} \in CV \setminus \{p^*\} : e(\hat{p}) \neq e(p^*)$. Because of (i), it holds that $\forall p \leq p^* : e(p) \leq e(p^*)$ such that $e(\hat{p}) \lneq e(p^*)$. Then, $0 = \iota^T(c - c) = \iota^T(e(p) - e(p^*)) < 0$ which provides a contradiction.

(iv) Eisenberg and Noe (2001) show the uniqueness of the clearing vector if each node reaches at least one bank with positive operational cash flow. In addition, Glasserman and Young (2015) show that another uniqueness condition is provided if each bank reaches an outside bank with no nominal liabilities.

Both these results follow from (iii) in Proposition 1 because there is at least one insolvent bank that pays different amounts for the least and greatest clearing vector to a bank that is solvent such that equity changes of the solvent bank which is a contradiction.

The core idea is that only non-trivial strong components can create non-unique clearing vectors. This property is established by (i) of Theorem 2 in Section 5.

If there is a strong component that is not a sink, there is a directed path from that strong component toward a sink component because the path either ends at a sink node or a non-trivial strong sink component. The path cannot reenter the strong component because all nodes on the path leaving the strong component would be inside the strong component. This implies that all the banks with potentially non-unique payments and not part of a strong sink component have a directed path toward a sink component with at least one solvent bank. If the strong component is a sink there is at least one solvent bank as well because total equity must equal total operational cash flow and thus be strictly positive in the sink. By (iii) and the arguments above, multiple clearing vectors would contradict constant equity. After all, total equity must equal total positive operational cash flow. The positive cash flow cannot escape the sink component. This is indicated in the sketch of a digraph below.



Assume $c_1 > 0$ is the only positive operational cash flow. The total equity in this network must equal $c_1 > 0$. Banks 1 and 2 are part of a non-trivial strong component and bank 3 is a sink node. As a consequence, the positive constant equity of bank 3 forces banks 1 and 2 to have unique clearing payments.

The set of all nodes reachable from node $i \in N$ is denoted $R(i)$. $\forall i \in N$: $R(i)$ is a sink component. Suppose not, then there is a bank $j \in N \setminus R(i)$ on an outwards pointing path from $R(i)$. Thus, $j \in R(i)$ provides the necessary contradiction.

Suppose there is another clearing vector $p \in CV : p \lneq p^*$. The equity for the largest clearing vector can only be weakly larger than the equity for a sub-optimal clearing vector because $e(p) = (a(p) - l)^+$ is weakly increasing in p. Then, $\exists j \in N \setminus D(p^*) : e_j(p^*) > e_j(p)$. That is, the payments along a directed path from an insolvent bank toward a solvent bank increase which results in larger equity for the bank solvent for both clearing vectors. This contradicts constant equity in (iii).

□

In summary, there always is a clearing vector due to (i). In addition, the largest clearing vector uniquely optimizes the system-level costs by (ii). Furthermore, the constant equity property (iii) ensures that a clearing vector is unique as long as each insolvent bank always reaches at least one solvent bank. This last property facilitates the uniqueness condition in (iv). Separating all banks into a zero or positive equity subset and the existence of a directed path from the first to the latter are both essential to proving the uniqueness condition. The existence of a path toward a sink node or the existence of positive operational cash flow among the nodes reachable from dependency cycles ensures that multiple equilibria are contradictory of constant equity.

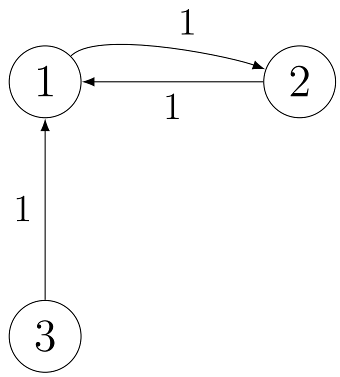
Eisenberg and Noe (2001) establish that a sufficient condition for uniqueness of (13) is provided by $\forall i \in N : c_i > 0$. Eisenberg and Noe (2001) suggest that any negative operational cash flow can be modeled separately by the inclusion of a bank without nominal liabilities where all banks $i \in N$ that receive negative operational cash flow instead hold positive nominal liabilities equal to the size of the negative cash flow. That is, $N = \{0, 1, ..., n\}$ and $\exists i \in N : L_{u0} > 0$. This inclusion implies from any strong component there is positive operational cash flow or there is a directed path towards a sink component. By (iv) the clearing vector is unique.

Consider the following example to demonstrate the implications of propostion 1 in an intuitive example.

Example 1 Suppose the pro-rata network (N, A, l, c), where A has the equivalent weighted digraph
$G[A]$:

Table 1: Pro-rata network parameters for example 1

N	A	l	c	$G[A]$
$\{1,2,3\}$	$\begin{bmatrix}0&1&0\\1&0&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\c_3\end{bmatrix}$	$(N,\{(1,2),(2,1),(3,1)\},A)$



First, consider $c_3 = 0$. (ii) implies the largest clearing vector exists which must be the unique solution to (14). Furthermore, there are infinitely many clearing vectors. In particular, the set of clearing vectors is all convex combinations of the least and largest clearing vectors. That is,
$CV = conv\{[0,0,0]^T, [1,1,0]^T\}$. Plug any vector $\lambda \times [1,1,0]^T$ for $\lambda \in [0,1]$ into the definition of a clearing vector (13) to see this. This example provides a similar context to the example by Eisenberg and Noe (2001) to demonstrate the possibility for multiple clearing vectors. The presence of the dependency cycle provides some intuition behind (iv) in Proposition 1 imposing restrictions on dependency cycles. Note that a lack of dependency cycles trivially satisfies the requirement of (iv). In that sense, if there are no dependency cycles, it is a certainty that there is just one clearing vector. The fundamental importance of the dependency cycle for the model by Eisenberg and Noe (2001)is obvious as this seminal paper already mentions that their contribution to the literature is the first to consider the dependency cycle. This importance generalizes well to more complex assumptions, such as default costs, as will be formally established in Section 5.

For $p^* = [1,1,0]^T$, both 1 and 2 are solvent, whereas, for $p \in CV \setminus \{p^*\}$, both 1 and 2 are insolvent. Note, that the equity of nodes 1 and 2 must be zero by Proposition 1 because these banks are insolvent for at least one clearing vector. As demonstrated in the proof for (iii) of Proposition 1, total equity must equal total operational cash flow, i.e., $\forall p \in CV : e_1(p) + e_2(p) + e_3(p) = c_1 + c_2 + c_3 = 0$.

Note, if $c > 0$ then (iv) and the unique clearing vector $p^* \in CV$ optimizes system-level costs by (ii).

In the following, the definition of a clearing vector is used to demonstrate how cash injections affect dependency cycles.

For sure, $[1, 1, c_3 \wedge l_3]^T \in CV$ as $c_3 > 0$ is not required to have $p_1 = p_2$. This could seem odd. However, there is no order in the clearing payments. The central clearing agency wires all payments simultaneously, by assumption. The implicit assumption is that the agency observes that banks 1 and 2 owe each 1 in mutual liabilities. Thus, bank 1 pays bank 2 because bank 2 pays bank 1 via the central clearing agency.

Suppose there is a $\bar{p} \in CV \setminus \{p^*\}$. For $c_3 > 0$, if either 1 or 2 is insolvent, then at least 2 is insolvent because $\bar{p}_1 = l_1 \wedge c_3 + \bar{p}_2$. Thus, $\bar{p}_1 > \bar{p}_2$ as $\bar{p}_2 = l_2$ implies $\bar{p}_1 \geq c_3 + \bar{p}_2 > \bar{p}_2 = l_1$ and thus both banks would be solvent. Because of absolute priority of creditors, $\bar{p}_2 = a_2(\bar{p}) = \bar{p}_1 = c_3 + \bar{p}_2 \wedge l_1 = c_3 + \bar{p}_2 \wedge l_2 > \bar{p}_2 \implies 0 < 0$. This only covers the case where $\bar{p}_1 = l_1$. If $\bar{p}_1 < l_1$, the same argument implies $\bar{p}_2 < \bar{p}_2 + c_3 = \bar{p}_1 = a_2(\bar{p})$ which again is a contradiction. $\square$

Another derivation for $c_3 \in (0, 1)$ is the following. Any digraph that is not part of a larger digraph is a sink. Any sink with at least some positive operational cash flow has at least one solvent bank Eisenberg and Noe (2001). If $0 < c_3 < 1$, 3 is not solvent. Thus $p^* = [1, 1, c_3]^T$ is the unique clearing vector following the same arguments in the last derivation where the absolute priority of creditors ensures node 2 is solvent if and only if 1 is. This is precisely the reason that positive operational cash flow for any dependency cycle that does not have a directed path towards a sink node ensures there is always a path towards a solvent node which facilitates the uniqueness by (iv) in Proposition 1. Node 3 is a source. Of course, if $c_3 = 1$, each bank is solvent.

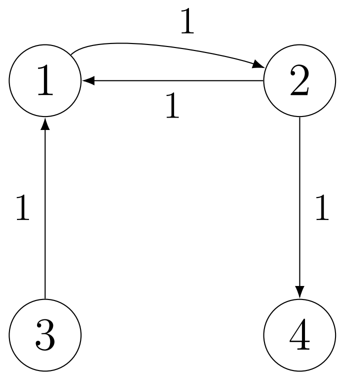
Table 2: Pro-rata network parameters for example 2

N	A	l	c	$G[A]$
$\{1, 2, 3, 4\}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$(N, \{(1, 2), (2, 1), (3, 1), (2, 4)\}, A)$

Example 2 Consider the same network now with an added glob-

ally reachable sink node (N, A, l, c), where A has the equivalent weighted digraph

$G[A]$:



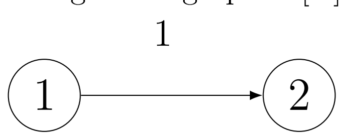
The solution is now unique due (iv) in Proposition 1. Without proof, the unique solution is now $[0, 0, 0, 0]^T$.[4]

Consider the following example to illustrate how the system-level costs objective function in (15) behaves under specific conditions.

Table 3: Pro-rata network parameters for example 3

N	A	l	c	$G[A]$
$\{1, 2\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$	$(N, \{(1, 2)\}, A)$

Example 3 Consider a simple pro-rata (N, A, l, c) with equivalent weighted digraph $G[A]$:



Among all exogenous quantities in the above pro-rata network, only the operational cash flow of node 1 varies across the next three scenarios.

Sink node 2 is globally reachable and the clearing vector is thus unique. First, suppose $c_1 = 1$. This implies 0 system-level costs. Thus, all nominal liabilities are paid and there are no defaulting banks. Second, suppose $c_1 = \frac{1}{2}$ which results in an objective function value of $\frac{1}{2}$ and bank 1 defaults. Finally, suppose $c_1 = 0$, then the objective function value is 1, and bank 1 defaults. An important question now arises about the difference in objective

[4] For a proof, use the least algorithm in Section 4.2 for zero default costs, i.e., $\alpha, \beta = 0$.

values between the last two scenarios. Is the third situation twice as bad as the second situation? Should the situation with an extra default and $\frac{1}{2}$ less equity decrease the objective function as much as losing another $\frac{1}{2}$ in equity without another default? This observed problem in the interpretation of the objective function is solved by the inclusion of default costs.

4 Default Costs

In a pro-rata network (N, A, l, c), the incoming clearing payments from a defaulting bank are proportional to the entire asset value plus the operational cash flow. In reality, there are typically consequences for when and how much the claimants are paid when a bank defaults (Battiston et al., 2016). It is not uncommon for bankruptcy proceedings to reduce the available cash to around half of nominal liabilities. More precisely, Jackson and Pernoud (2020) highlight in a survey based on several studies that the bankruptcy recovery rates are between $56-57\%$. This statistic motivates the inclusion of default costs in the model, in addition to the interpretation problem of the system-level costs at the end of the previous section. Furthermore, Bennett and Unal (2015) estimate the costs of bank resolutions using FDIC data over the period 1986–2007 for the 25 largest banks in their sample, these costs ranged from $0.33-13.19\%$. Similar to Rogers and Veraart (2013), Jackson and Pernoud (2019) propose default costs linear in the asset value $a_i(p)$ of the defaulting bank i. The default costs of bank $i \in N$ are defined as a function of the payment vector and are generally considered heterogeneous across different banks.

It is useful to distinguish between default costs in case a bank defaults and actual default costs.

Definition 6 (Pro-Rata Cost Network)
A pro-rata cost network is a tuple $(N, A, l, c, \alpha, \beta)$ where N represents the set of banks, A is the proportional liability matrix, l is the liability vector, c is the cash vector, and the vectors α and β contain the default cost coefficients in the potential default costs. The potential default costs of bank $i \in N$ is a function $\gamma_i : [0, l] \mapsto [0, l_i]$, defined by,

$$\gamma_i(p) = (\alpha_i + \beta_i \times a_i(p)) \wedge a_i(p), \tag{16}$$

where $0 \leq \alpha_i$ and $0 \leq \beta_i \leq 1$ ensure that $\gamma_i \in [0, l_i]$.[5]

[5]Jackson and Pernoud (2023) choose to allow for default costs to supersede the asset

Notably, these default costs do not depend on how insolvent a bank is at default. In particular, bank $i \in N$ defaults whether $l_i - p_i =$ €0.01 or $l_i - p_i =$€1000000.00. If a bank operates under regular market conditions, liquidity is available via short-term repurchase agreements (Jackson and Pernoud, 2023). In contrast, the bank run scenario in the 2008 financial crisis demonstrates that losses in confidence can deplete the liquid capital available for short-term borrowing (Duffie, 2019; Jackson and Pernoud, 2023).

Thus, the independence of the potential default costs of how insolvent a bank is lies in the fact that no short-term liquidity is available, and the moment a bank becomes insolvent the tremendously costly legal proceedings initiate.

The marginal costs, i.e., β_i, are estimated to be around 20% to 30% of the bank's assets and even worse in financial crises (Davydenko et al., 2012; Jackson and Pernoud, 2023). The fixed costs α_i are harder to estimate and presumably positive due to the costly nature of legal proceedings that vastly exceeds the marginal costs (Jackson and Pernoud, 2023).

The pro-rata network (N, A, l, c) is equivalent to the pro-rata cost network $(N, A, l, c, 0, 0)$.

The default costs of bank $i \in N$ is a function $\delta_i : [0, l] \mapsto [0, l_i]$, defined by

$$\delta_i(p) = \begin{cases} \gamma_i(p) & \text{if } a_i(p) < l_i, \\ 0 & \text{else.} \end{cases} \tag{17}$$

The possibility of $\alpha = 0$ and $\beta = 0$ is mainly included to be able to write the pro-rata network as a special case of a pro-rata network with default costs. Any conclusions for general vectors $0 \leq \alpha$ and $0 \leq \beta \leq 1$ must hold for general pro-rata networks.

The inclusion of positive default costs of this type has two important effects. The inclusion of $\alpha_i > 0$ ensures that for each bank $i \in N$ that receives asset value $a_i(p) > 0$ there is at least some consequence to collapsing in terms of the system-level costs. In addition, the coefficient $0 < \beta_i \leq 1$ ensures that the cash asset value of node $i \in N$ is reflected in the size of the system-level costs.

Consider a pro-rata cost network $(N, A, l, c, \alpha, \beta)$. The set of

value of a bank. As Jackson and Pernoud (2023) argue, the costs that exceed the assets still represent real costs, e.g., debts or legal costs that are never paid, capital or labor that are idled, etc., which can be incurred by the bank itself if it does not act under *(ii)* limited liability, or by the government or agents outside of the network.

feasible payment vectors is given by

$$\mathcal{Q} = \left\{ p \in \mathbb{R}^n_+ : p \leq l, p \leq a(p) - \delta(p) \right\}. \tag{18}$$

The clearing vector can be defined similarly to the pro-rata network counterpart.

Definition 7 (Clearing Vector)
Consider the pro-rata cost network $(N, A, l, c, \alpha, \beta)$. A vector $p^* \in \mathcal{Q}$ is a clearing vector, i.e., $p^* \in CV$,if it complies with absolute priority of creditors. That is,

$$p^* = l \wedge (a(p^*) - \delta(p^*)). \tag{19}$$

Similar to Rogers and Veraart (2013), the clearing vector for the pro-rata payment rule with default costs can be expressed as a fixed point of a function $\Phi : [0, l] \mapsto [0, l]$, defined by

$$\Phi_i(p) = \begin{cases} l_i \text{ if } l_i \leq a_i(p), \\ a_i(p) - \delta_i(p) \text{ else.} \end{cases} \tag{20}$$

Importantly, the first case prevents the scenario where banks have sufficiently many funds available to pay all liabilities and still default based on default costs alone, instead of the consequences. This point is elaborated on in Jackson and Pernoud (2023). Thus, even though default costs are incurred when $p_i < l_i$, the definition of the clearing vector ensures this only occurs for clearing payments where the asset value is insufficient to cover liabilities.

The interpretation of the clearing vector p^* does not change. The clearing vector is the amount each node has available to pay their liabilities. If the asset value of cash to node $i \in N$ is at least l_i, i.e., $a_i(p^*) \geq l_i$, then node i meets their financial obligations. If node i fails to do so, the node must liquidate its assets, and due to, e.g., legal costs, payment delays (Battiston et al., 2016), liquidity (Strömberg, 2000; Cifuentes et al., 2005), and all other costly inefficiencies of the liquidation process, there is a fixed penalty $\alpha_i > 0$ and proportional penalty $\beta_i \times a_i(p^*)$. Of course, it is not possible to reduce more than all of the asset value of the asset value, thus, these penalties are summed up as no larger than the total asset value. This implies the total recovered asset value to pay off the liabilities at default is equal to $(a_i(p^*) - \gamma_i)^+$.

The definitions of the book and equity values change due to the inclusion of default costs. In particular, for the pro-rata cost network, the book value, and equity value of bank $i \in N$ are functions $v_i, e_i : [0, l] \mapsto \mathbb{R}$, defined by

$$v_i(p) = a_i(p) - l_i - \delta_i(p), \tag{21}$$

$$e_i(p) = (v_i(p))^+. \tag{22}$$

These values are collected, respectively, in the vectors $v(p) = a(p) - l - \delta(p)$, and $e(p) = (v(p))^+$. This implies that $a_i(p) < l_i \iff v_i(p) < 0$. Thus, bank $i \in N$ defaults if and only if it does not hold sufficient funds to cover all liabilities.

Note, Proposition 1 is unlikely to hold because total equity reduces whenever default costs are incurred.

Furthermore, the default set is a function $D : [0, l] \mapsto \mathcal{P}(N)$, defined by

$$D(p) = \{i \in N : v_i(p) < 0\}. \tag{23}$$

Following Rogers and Veraart (2013), the following simple and important properties hold for the mapping Φ.

Lemma 1 The mapping Φ satisfies:

(i) Φ is bounded from below by 0 and above by l. For any p we have $\Phi(p) \leq l$.

(ii) Φ is monotone. If $\tilde{p} \leq p$ then $\Phi(\tilde{p}) \leq \Phi(p)$.

Proof. (i) The first property follows from the definition.

(ii) To prove the second property, note if $\tilde{p} \leq p$ then $D(p) \subseteq D(\tilde{p})$. Thus, if $i \in D(p)$, then $\Phi_i(\tilde{p}) = ((1 - \beta_i)a_i(\tilde{p}) - \alpha_i)^+ \leq ((1 - \beta_i)a_i(p) - \alpha_i)^+ = \Phi_i(p)$ because $a(p)$ weakly increases in p. If $i \in D(\tilde{p}) \setminus D(p)$, then $\Phi(\tilde{p}) < l_i = \Phi_i(p)$. If $u \notin D(\tilde{p})$ then $\Phi_i(\tilde{p}) = l_i = \Phi_i(p)$.

□

The following existence Theorem and proof are based on a similar result by Rogers and Veraart (2013).

Theorem 1 (Existence of the Clearing Vector)
For every pro-rata cost network $(N, A, l, c, \alpha, \beta)$, there exists a least clearing vector p_* and greatest clearing vector p^*. That is, $\exists p_*, p^* \in CV$ such that $p_* \leq \tilde{p} \leq p^*$ for each clearing vector $\tilde{p} \in CV$.

Proof. The following proof is more complex than it needs to be, to demonstrate the structure of an algorithm to arrive at the least and greatest clearing vector. Furthermore, this version of the proof uses common analytical principles taught in real analysis courses. In contrast, an alternative proof that requires additional knowledge is available in Appendix A.1.

Define a sequence, $\hat{p}_0 = l$, define $\forall k \in \mathbb{N} : \hat{p}_k = \Phi\left(\hat{p}_{k-1}\right)$. By lemma 1, the sequence $\{\hat{p}_i\}_{u=0}^{\infty}$, is decreasing and bounded from below. Thus, the sequence is convergent. The following holds because of convergence combined with a finite number of nodes, namely n. The set $F_k = \{u \in N : a_i(\hat{p}_k) = l_i\}$, becomes constant after k has become sufficiently large. This is evident as there are at most n jumps due to default costs, and, as $k \to \infty$, eventually, the last jump must occur. Note, that this does not prove that the last jump occurs in finitely many iterations. Because the transformation $\Phi(.)$ is continuous from above, where jumps occur at $l_i = a_i(p)$, the limit of the sequence belongs to the same n-dimensional line segment as the last tail of the sequence. The last tail of the sequence occurs at the value k such that F_k no longer changes. Thus, $\lim\limits_{k\to\infty} \hat{p}_k = p^* = \Phi(p^*)$.

Now, consider the sequence $\hat{p}_0 = 0$ and $\forall k \in \mathbb{N} : \hat{p}_k = \Phi(\hat{p}_{k-1})$. Using the same arguments, now for an increasing sequence bounded from above, the sequence will converge. However, the limit may not be a clearing vector, as the limit may lay outside of the set of sequence elements, such that $\forall k \in \mathbb{N} : \hat{p}_k < v$ and $v = \lim\limits_{k\to\infty} \hat{p}_k$. If $\Phi(v) \gneq v$, the jump occurs in v, and thus, as basic analysis dictates, $\lim\limits_{k\to\infty} \hat{p}_k \lneq \Phi(\lim\limits_{k\to\infty} \hat{p}_k)$. That is, the supremum of an increasing sequence is the limit, which need not be inside of the set which is not closed. This is intuitively clear when you consider that $\hat{p}_k$ never needs to reach the limit exactly, just approach it arbitrarily close. The value after the jump is not close to any element in the sequence in a limiting sense. This requires the sequence to start again starting at $\hat{p}_0 = v$. Then, the first limit which is a clearing vector, is the least clearing vector.

Finally, although not explicitly demonstrated, it is not possible to surpass the least and greatest clearing vector. Although lemma 1 indicates $\Phi(\cdot)$ is increasing in a weak sense, it is not addressed whether it is possible to surpass, e.g., the least clearing vector, in the following way. Suppose $\hat{p}_k \leq p_*$ and $\hat{p}_{k+1}$ is larger in some and smaller in other elements than p_*. This would not contradict the weak monotonicity in lemma 1, though is still impossible. The proof of this property is based on the principles underlying Tarski's fixed point (Zeidler, 1986) which will be used to demonstrate an important second proof of this model. The importance of this second proof lies in the general applicability to many similar models. □

An additional, faster proof requires a specific characterization of sets and element-wise ordering. This proof is based on Tarski's fixed point and is available and provided in appendix A.1.

4.1 Importance of the Least Clearing Vector

The inclusion of default costs has some important implications when considering the difference between a centralized and decentralized clearing mechanism. Csóka and Herings (2018) find that a decentralized algorithm based in only partial information will lead to the least clearing matrix. This decentralized algorithm terminates in finitely many steps because each payment must be a multiple of the smallest unit of account. To illustrate the importance of having at least some minimal level of payment, consider the following simplified example of a clearing mechanism without default costs. Example 2: Assume the following nominal liability matrix

$$L = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \tag{24}$$

where first node 1 owns net assets $c_1 = 1$ and $c_2 = c_3 = 0$. Then no default occurs after infinitely many payments. This happens in the following steps.

- Step 1: Node 1 pays $P_{12} = P_{13} = \frac{1}{2}$.
- Step 2: Node 2 and 3 pay both separately in total $\frac{1}{2}$ via $P_{21} = P_{23} = \frac{1}{4} = P_{31} = P_{32}$.
- Step 3: Node 2 and 3 pay both separately in total $\frac{1}{4}$ via $P_{21} = P_{23} = \frac{1}{8} = P_{31} = P_{32}$.
- step 3: ...

Then, after infinitely many steps, node 2 and 3 have paid off their nominal liabilities as well. That is, both 2 and 3 have paid $\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + ... = \frac{1}{1-\frac{1}{2}} - 1 = 1$. Of course, in real life, there is a smallest unit of account. It is not necessary to assume all payments are a multiple of the smallest unit of account to fix this particular problem. It suffices to assume that there is some $\epsilon > 0$ such that any remaining liabilities smaller than this value are forgiven. Csóka and Herings (2018) consider what happens if there are no default costs and the unit of account approaches zero. Their findings indicate that the decentralized algorithm approaches the maximum clearing matrix and provides bounds that indicate the equity costs of decentralized clearing are relatively small and do not necessitate central bankruptcy proceedings.

One interesting aspect to note is that, if a decentralized model is designed such that all clearing payments occur at the first possibility to do so, the analysis for decentralized and centralized versions of the model coincide. The order of payments, which distinguishes the decentralized from the centralized models, is no longer relevant. This should be kept in mind if one aims to compare simple examples from both types of model. For example, the hierarchical financial network (Csóka and Herings, 2024). A hierarchical contains no dependency cycles.

4.2 Least and Greatest Clearing Algorithm

The following algorithms to find the least or greatest clearing vector for any pro-rata cost network $(N, A, l, c, \alpha, \beta)$ closely follow the algorithms by Eisenberg and Noe (2001); Rogers and Veraart (2013) to find the greatest clearing vector. Rogers and Veraart (2013) adjusts the fictitious default algorithm by Eisenberg and Noe (2001) to account for default costs. These algorithms take at most n iterations and arrive at the least and greatest clearing vectors respectively.

The idea of the greatest algorithm is that you identify which banks default if all banks meet their nominal liabilities. These banks will always default because it is not possible to receive more. Set the corresponding clearing payments to variables while keeping the remaining banks at nominal. Solve a set of linear equations where only the defaulting banks thus have variable payments. The solution must be unique if either $\beta > 0$ or $\alpha = \beta = 0$. Start the next iteration with the solution of the last iteration. Select the banks that will additionally default for this payment vector. Repeat this process until the set of defaulting banks remains constant for an iteration and terminate the algorithm. The process of payment vectors results in the greatest clearing vector at termination.

The least algorithm utilizes the same principles in the opposite direction (Rogers and Veraart, 2013). Start with the zero payment vector and identify which banks are solvent. Set those clearing payments to nominal liabilities and set the payments of the remaining banks to variable and the resulting vector is the payment vector for the next iteration. At each iteration check whether new banks become solvent until the set of defaulting banks does not change. This results in the least clearing vector.

Consider $(N, A, l, c, \alpha, \beta)$.

1. $\mu = 0$
   Least Algorithm: Set $p_{*0} = 0$, $U_{-1} = \emptyset$,
   Greatest Algorithm: Set $p_0^* = l$, $Y_{-1} = \emptyset$.
2. Default sets: $U_\mu = D(p_{*\mu})$,$\mathrm{Y}_\mu = D(p_\mu^*)$,
   Solvent bank sets: $S_\mu = N \setminus U_\mu$, $Z_\mu = N \setminus Y_\mu$;
3. If $U_\mu = U_{\mu-1}$, terminate the Least Algorithm and set $p_* = p_{*\mu}$.
   Else, set $p_{*\mu i} = l_i$ for $i \in S_\mu$. Then solve,
   $x_i = a_i(I_{S_\mu} l + I_{U_\mu} x) - \gamma_i(I_{S_\mu} l + I_{U_\mu} x)$ for $i \in U_\mu$, where I_{S_μ} is the identity matrix where the *j-th* diagonal element is replaced with zero in case $j \notin S_\mu$, and I_{U_μ} is defined analogously. Set $p_{*\mu i} = x_i$ for $i \in U_\mu$.
   If $Y_\mu = Y_{\mu-1}$, terminate the Greatest Algorithm and set $p^* = p_\mu^*$.
   Else, set $p_{\mu i}^* = l_i$ for $i \in Z_\mu$. Then solve,
   $x_i = a_i(I_{Z_\mu} l + I_{Y_\mu} x) - \gamma_i(I_{Z_\mu} l + I_{Y_\mu} x)$ for $i \in Y_\mu$, where I_{Z_μ} is the identity matrix where the *j-th* diagonal element is replaced with zero in case $j \notin Z_\mu$, and I_{Y_μ} is defined analogously. Set $p_{\mu i}^* = x_i$ for $i \in Y_\mu$.;
4. $\mu \to \mu + 1$ and go back to step 2.;

The idea behind such an algorithm is intuitive. By starting from below the least clearing payments, if banks $i, j \in N$ become solvent in an iteration, then bank $i \in N$ can only be better off in the next iteration when the solvency of $j \in N$ is processed in the clearing vector in the next iteration. Analogously, when the algorithm starts from above the greatest clearing payments and banks $i, j \in N$ become insolvent in an iteration, then $i \in N$ can only be worse off in the next iteration when the insolvency of $j \in N$ is processed in the clearing vector.

The solutions x_i, for $i \in U_\mu \cup Y_\mu$, in step 3 are unique if $\beta > 0$ (Rogers and Veraart, 2013) because it is possible to write the right-hand sides in vector form as contractions of $I_{U_\mu} x$. The proof is provided in appendix A.2.

For $\alpha = \beta = 0$, the uniqueness of the x_i values are slightly different, and sufficient conditions are available Eisenberg and Noe (2001). In particular, if $c > 0$, or more generally, if (*iii*) in Proposition 1 holds, the clearing vector is unique. Moreover, (*iii*) ensures solutions x_i in step 3 are unique.

Note, for $b = 0$, $a = 0$ is essential to guarantee the uniqueness of the solution in step 3 if (*iii*) holds, in contrast to the case where $\beta > 0$. $a = 0$ ensures constant equity which is essential for uniqueness in step 3. A detailed explanation is provided in appendix A.2.

The algorithm is applied to the pro-rata network in example 1, now with default costs.

Table 4: Pro-rata and pro-rata cost network parameters for example 4

N	A	l	c	α	β	$G[A]$
$\{1,2,3\}$	$\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\\frac{1}{4}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$	$(N, \{(1,2),(2,1),(1,3)\}, A)$

Example 4 Consider $(N, A, l, c, \alpha, \beta)$. Thus, for each $i \in N$ the potential default costs are $\gamma_i(p) = \frac{1}{4}(1 + a_i(p)) \wedge a_i(p)$.

1. $\mu = 0$
   Least Algorithm: Set $p_{*0} = [0,0,0]^T$, $U_{-1} = \emptyset$,
   Greatest Algorithm: Set $p_0^* = [1,1,1]^T$, $Y_{-1} = \emptyset$.
2. $\forall i \in \{1,2\} : v_i(p_{*0}) = a_i(p_{*0}) - l_i = 0 - 1 = -1$, $v_3(p_{*0}) = a_3(p_{*0}) - l_3 = \frac{1}{4} - 1 = -\frac{3}{4}$,
   $\forall i \in \{1,2\} : v_i(p_0^*) = a_i(p_0^*) - l_i = 1 - 1 = 0$, $v_3(p_0^*) a_3(p_0^*) - l_3 = \frac{1}{4} - 1 = -\frac{3}{4}$;
   $U_0 = \{1,2,3\}$, $Y_0 = \{3\}$,
   $S_0 = \emptyset$, $Z_0 = \{1,2\}$;
3. $U_0 \neq U_{-1}$.
   $\forall i \in U_0 : x_i = 0 = a_i([0,0,0]^T) - \beta_i([0,0,0]^T)$. Set $p_{*0i} = x_i = 0$ for $i \in U_0$.
   $Y_0 \neq Y_{-1}$.
   Set $p_{0i}^* = l_i = 1$ for $i \in Z_0$.

$x_3 = 0 = r_3([1,1,0]^T) - \beta_3([1,1,0]^T)$. Set $p_{0i}^* = x_i = 0$ for $i \in Y_0$.;

1. $\mu = 1$;
2. Note, $p_{*1} = [0,0,0]^T = p_{*0}$. Terminate the least algorithm as no change in the clearing vector implies the set of insolvent banks will not change. Thus, $p_* = [0,0,0]^T$.
   $p_1^* = [1,1,0]^T \neq p_0^*$,
   $\forall i \in Z_0 : v_i(p_1^*) = a_i(p_1^*) - l_i = 1 - 1 = 0$, $v_3(p_1^*) = a_3(p_1^*) - l_3 = \frac{1}{4} - 1 = -\frac{3}{4}$;
   $Y_1 = \{3\}$, $Z_1 = \{1,2\}$;
3. $Y_1 = Y_0$, terminate the Greatest Algorithm. Thus, $p^* = p_1^* = [1,1,0]^T$.

5 The Structure of Inefficient Clearing Payments

5.1 Dependency Cycles

A large part of literature, e.g., Eisenberg and Noe (2001); Rogers and Veraart (2013); Acemoglu et al. (2015); Glasserman and Young (2016); Csóka and Herings (2018), hints at the prominence of dependency cycles for inefficient clearing payments. Jackson and Pernoud (2023) provide a precise formulation of necessary and sufficient conditions for inefficient clearing payments for a large collection of default costs. These results require mild regularity conditions on the default costs and are significant for the network topology approach to systemic risk. These results formalize and reinforce the importance of directed cycles of nominal liabilities. Notable is their choice of the book value as the primary variable. Importantly, this choice does not change the underlying structure of the model. In particular, there exists a bijection from the set of clearing vectors to the vector of book values.

This equivalence motivates the derivation and study of their results under the equivalent clearing vector formulation that is both intuitive and consistent with most work based on the seminal work by Eisenberg and Noe (2001).

Eisenberg and Noe (2001); Csóka and Herings (2018); Jackson and Pernoud (2023) argue that if payments occur sequentially, as in the real world, self-fulfilling default cycles can occur in the following sense. If bank A must pay bank B which must pay bank C which in turn, must pay A. This represents a dependency cycle. If none of these banks have the funds for the payment at the current time, each will default, even though a small cash injection into one of these banks would be able to start a cycle of payments that makes all banks solvent again (Eisenberg and Noe, 2001; Jackson and Pernoud, 2023).

This is precisely what occurs in Example 1. Banks 1 and 2 form a dependency cycle with no operational cash flows and bank 3 acts as a potential cash injection. If the operational cash flow of node 3 is zero, there is the potential for 1 or 2 to be solvent, or not, with arbitrary gaps in clearing vector values and nominal liabilities. The choice of any arbitrarily small positive operational cash flow of bank 3 forces a unique clearing vector where both banks 1 and 2 are solvent. Importantly, a dependency cycle need not lead to inefficient clearing vectors. This is demonstrated in

Example 1 as well. Only $c = 0$ leads to non-unique clearing vectors.

In contrast, Proposition 1 no longer holds, as the inclusion of positive default costs reduces the equity of solvent banks that receive clearing payments from defaulting banks. As a consequence, (*iii*) in Proposition 1 no longer holds because this result is fundamentally built upon constant equity.

Thus, $c > 0$, or, equivalently, the presence of a globally reachable sink node is no longer sufficient to ensure $p_* = p^*$.

The most important result of this section is that uniqueness is guaranteed in case there are no dependency cycles. If one of the two arcs between bank 1 and 2 was removed in Example 1, there would not have been multiple clearing vectors.

In reality, these self-fulfilling default cycles are commonly prevented through short-term contracts, such as repo contracts (Cifuentes et al., 2005; Jackson and Pernoud, 2020). However, the decreasing confidence in short-term contracts as occurred in 2008, greatly affected the liquidity of such contracts leading to the collapse of Lehman Brothers (Jackson and Pernoud, 2020). Without explicitly modeling the state of the economy, the default costs vary in interpretation. In case of an economic crisis, the default costs are default costs, or if the economy is not in crisis, the default costs can be interpreted as the loss due to costs associated with delayed payments (Jackson and Pernoud, 2023).

The following result describes the relationship between financial network structures and inefficient clearing payments in a pro-rata cost network.[6]

Theorem 2 (Jackson and Pernoud, 2023)
Consider a pro-rata cost network $(N, A, l, c, \alpha, \beta)$. Let $p_*, p^* \in CV$ denote the least and greatest clearing vectors, respectively.

(*i*) If there is no dependency cycle, then all clearing vectors coincide, i.e., $p_* = p^*$.

(*ii*) Conversely, if there is a dependency cycle, then there exist potential default costs $\gamma(\cdot)$ as described in (16), and operational cash flows $c \geq 0$ such that the least and greatest clearing vector are distinct, i.e., $\exists p_*, p^* \in CV : p_* \lneq p^*$.

[6] Apart from the formulation in terms of clearing payments, the corresponding proof deviates slightly from the approach by Jackson and Pernoud (2023) to utilize a common method of reformulating one problem as a special case of another. This approach indicates general properties of the latter problem must hold for the first problem. Thus, an 'efficient' algorithm for the latter problem can solve the first problem efficiently, provided the reformulation itself is efficient. This application is an important example. For details, see Cook (1971).

(*iii*) Any clearing vector that differs from the greatest clearing vector, i.e., $p \in CV \setminus \{p^*\}$, corresponds to a default set that contains the default set for the greatest clearing vector, i.e., $D(p^*) \subseteq D(p)$, plus all banks of at least one entire dependency cycle. Any other banks defaulting in this equilibrium but not in the best equilibrium lie on outpointing paths from the original defaulting banks and the newly defaulting dependency cycles.

The networks without dependency cycles are referred to as hierarchical networks.

Before the proof is provided, consider the following nuances regarding property (*iii*).

The distinction between a dependency cycle and a strong component is important here. There can be many dependency cycles in a strong component. (*iii*) implies that inefficient clearing payments result in at least one strong component that contains a defaulting dependency cycle that did not default for the best clearing payments. In contrast, it is possible that all the dependency cycles in the component default as well. However, this is not a requirement. It is even possible that there are many strong components with each a dependency cycle that defaults only for inefficient clearing payments.

and only one of these cycles must default in addition to the banks that default for the sub-optimal clearing vector. That is, This motivates the distinction between the term dependency cycle and strong component. An entire dependency cycle must certainly default additionally, for if not, then the dependency cycles can have at least one bank in each dependency cycle that is solvent for multiple clearing vectors such that it can be replaced by adjusting the parameters such that there no longer are dependency cycles. In turn, there cannot be non-unique clearing vectors and thus a contradiction is provided. This fundamental aspect is also used in the proof.

Proof. (*i*) Let $\forall i \in N : R^{-1}(i)$ denote the set of banks that hold nominal liabilities towards bank i, i.e., $R^{-1}(i) = \{j \in N : L_{ji} > 0\}$.
By Theorem 1, there exists a clearing vector $p \in CV$.
There are no dependency cycles, thus there is a non-empty set $X_0 \subseteq N$ such that for each $i \in X_0$ it holds that $R^{-1}(i) = \emptyset$[7]. Thus, for each $i \in X_0$, it holds that $\forall p \in [0, l] : \gamma_i(p) = \alpha_i + \beta_i \times a_i(p) = \alpha_i + \beta_i \times c_i$. Thus,

$$\forall i \in X_0 : p_i = (c_i - (\alpha_i + \beta_i \times c_i)\mathbb{1}_{\{c_i < l_i\}})^+ \wedge l_i, \tag{25}$$

is constant, where $\mathbb{1}$ denotes an indicator function.

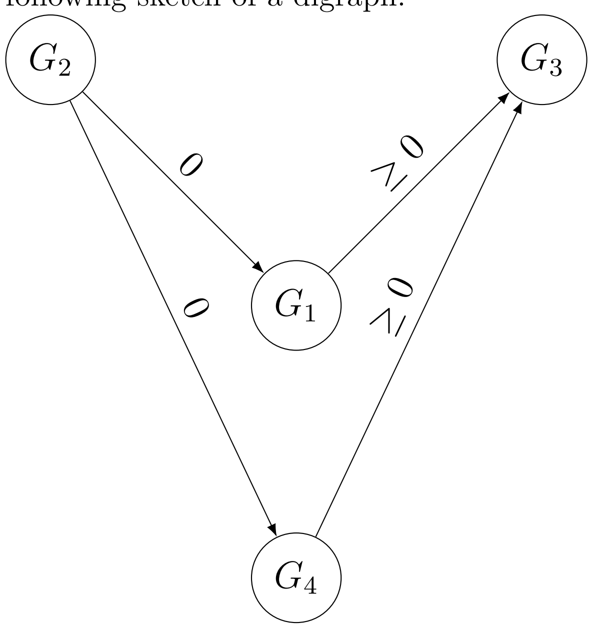
[7]This property is explained in Appendix A.3

Define by $X_1 \subseteq N$ the set that receives nominal liabilities only from X_0. Then, $\forall i \in X_1 : a_i(p) = c_i + \sum_{u \in X_0} A_{ui} p_i$ is constant. Thus,

$$\forall i \in X_1 : p_i = \left(a_i(p) - [\alpha_i + \beta_i \times a_i(p)] \, \mathbb{1}_{\{a_i(p) < l_i\}} \right)^+ \wedge l_i, \tag{26}$$

is constant. Creating a sequence of sets X_k which only receive liabilities form $\bigcup_{s=1}^{k-1} X_s$ in this way, there are at most $n = |N|$ iterations of this sequence until $X_l = N$ is reached for some $l \leq n$ and $\forall i \in N : p_i$ is constant. Thus, $p_* = p^*$. □

(ii) To start of this proof, note that there are at most 4 possible exhaustive and mutually disjoint sets of nodes. Let G_0 contain the nodes of a particular dependency cycle. Let $G_1 \subseteq N$ be the strong component that contains G_0. Let $G_2 \subseteq N$ be the set of banks that have nominal debt flow towards G_1 and are not in G_1. Let $G_3 \subseteq N \setminus G_1$ be the set of nodes that receive nominal debt flow from G_1 and are not part of G_1 and thus $G_2 \cap G_3 = \emptyset$.[8] Let $G_4 \equiv N \setminus (G_1 \cup G_2 \cup G_3)$ be the set that neither receives nor pays nominal liabilities towards any element in the specified strong component. Note, there is no nominal debt flow from G_3 and G_4 to G_2 and thus G_1. Set $c_i = 0$ for $i \in G_2$. This implies that all cash flow to G_1 elements stems from c_i for $i \in G_1$. The possible clearing payments are indicated in the following sketch of a digraph.



Assume that $\alpha_i = 0$ and $\beta_i = 1$. Note, $\forall p \in [0, l] \forall i \in G_1 : a_i(p) = c_i + \sum_{u \in G_1} A_{ui} p_i$. For all $i \in G_1$ take c_i as the smallest element ensuring that

[8] Jackson and Pernoud (2023) do not state that G_1 must be the nodes of a strong component containing a particular cycle, but any node in a dependency cycle, and still treat $G_2 \cap G_3 = \emptyset$ as given. However, there can be two dependency cycles such that receiving nominal liabilities from one and paying to another is possible without being part of a dependency cycle.

each $a_i(l) \geq l_i$. Thus, take $c_i = (l_i - \sum_{j \in G_1} A_{ji} l_j)^+$. Under the assumption that $\gamma_i(p) = a_i(p)$, as each element of a dependency cycle receives and pays both positive nominal liabilities, it holds that, if $a_i(v) = c_i$ for each $i \in G_1$, then, $p_i = c_i(1-1) = 0 < l_i$.

Thus, $p_* \lneq p^*$ because the clearing payments change from positive to zero for at least some banks.

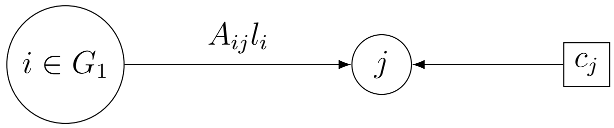
To show it is possible for the more reasonable default parameters of this thesis, i.e., $\alpha_i > 0$ and $0 < \beta < 1$, suppose the following. Let $\alpha_i > 0$ and $\beta_i < 1$, yet sufficiently high such that it does not bridge the gap between $p_i < l_i$.

(iii) Because the set of clearing vectors is a complete lattice, those nodes that default in the best equilibrium must default in any other equilibrium because $\forall p \in CV : p \leq p^* \iff D(p^*) \subseteq D(p)$ as demonstrated in the proof of lemma 1. In particular, $\forall i \in D(p^*) \forall p \in CV : p_i \leq p_i^* < l_i$. Furthermore, the set of all banks that only default in an inefficient equilibrium contains all nodes of at least one entire cycle. Suppose not, by (i) there must be a dependency cycle where inefficient equilibria occur and there is at least one sub-optimal clearing vector for which each cycle contains one solvent bank.

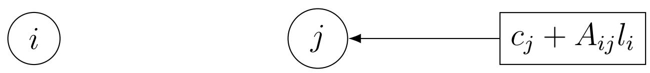
Let G_1 be as in the proof of (ii). Thus, there is an $i \in G_1$ for each cycle $G_0 \subseteq G_1$ such that $p_i = l_i$ for an $p \in CV \setminus \{p^*\}$. Let all these solvent banks in G_1 be collected in the set G_{-1}.

Define a new pro-rata cost network, in the following way. If $A_{ij} > 0$, set $c_j \to c_j + A_{ij} l_i$ and $l_i \to 0$. This implies $a_j(p) \to a_j(p) + A_{ij} l_i - A_{ij} l_i$ for the cash flows of nodes $j \in N$ which received liabilities from i. Thus, each clearing payment that depends on payments from $i \in G_{-1}$, remains the same. Only $\forall i \in G_{-1} : p_i$ changes and does so by a constant, namely $p_i \to 0$. That is, p solves the original problem if and only if p after subtracting l_i from the *i-th* element solves the second problem. Let G[A] be the digraph corresponding to the original problem and $G'[A]$ be the digraph concerning the newly defined problem.

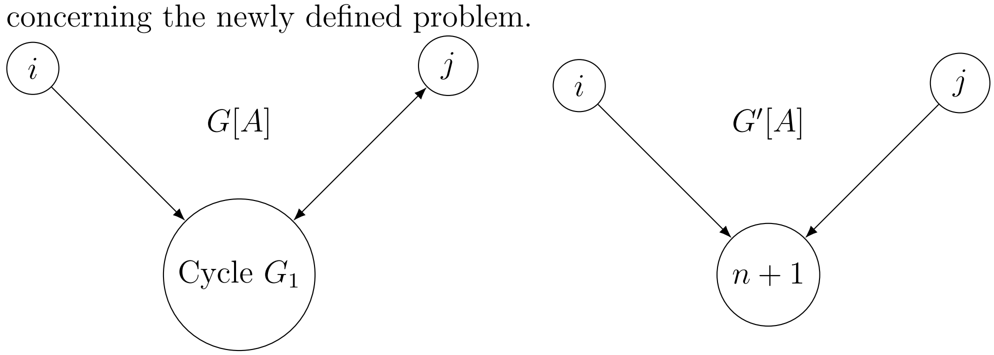
$G[A]$



$G'[A]$



Because of (i), the second problem has a unique solution. This is equivalent to the first problem having a unique solution. This contradicts the assumption of at least one sub-optimal clearing vector. The only property left to prove is that other nodes that only default for sub-optimal clearing vectors must lie on outwards pointing paths of nodes in $D(p^*)$ and the dependency cycles that default for sub-optimal clearing vectors. Suppose not, then those nodes are not affected by any defaulting cycle, removing said cycle and taking all values that are owed to the cycle are now owed to a newly defined node, $n + 1$, does not change the clearing payments of the newly defaulting $i \in N$ for any clearing vector. Let G[A] be the digraph corresponding to the original problem and $G'[A]$ be the digraph concerning the newly defined problem.



Thus, the clearing payments of node $i \in N$ have been made unique without changing them. This provides the necessary contradiction.

□

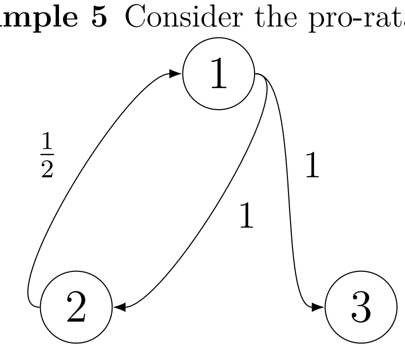
This Theorem establishes there is a dependency cycle if and only if it is possible to find a cash vector $c \geq 0$ and appropriate potential default costs default costs $\gamma(p)$ such that there are multiple clearing vectors, i.e., $p_* \neq p^*$.

One important distinction between a pro-rata network (N, A, l, c) and a pro-rata cost network $(N, A, l, c, \alpha, \beta)$ is the following. (i) implies that if there were no dependency cycle in the pro-rata network in Example 1, it would not have been possible to have multiple clearing vectors. (i) tells us that there is no combination of reasonable default costs, including 0, and operational cash flows such that it is possible to have multiple clearing vectors. However, (ii) does not prove that there are operational cash flows such that the Example 1 has multiple clearing vectors. Thus, if there are no dependency cycles in a pro-rata network, there must be a unique clearing vector. If there is a dependency cycle, there could be multiple clearing vectors. However, there need not be multiple clearing vectors. That there need not be multiple equilibria for no default costs is not surprising. The proof of (ii) utilizes strictly positive default costs. There can be multiple clearing vectors as already proven in Example 1. The fact that there need not be multiple clearing vectors if there are no default costs is apparent in the following example.

Table 5: Pro-rata and pro-rata cost network parameters for Example 5

N	A	l	c	α	β	$G[A]$
$\{1,2,3\}$	$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ \frac{1}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1\frac{1}{2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ \frac{3}{4} \\ \alpha_3 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$	$(N, \{(1,2),(2,1),(1,3)\}, A)$

Example 5 Consider the pro-rata network (N, A, l, c):



It is immediately clear that the solution is unique because there is a globally reachable sink node by (iv) in Proposition 1. Apply the least algorithm in Section 4.2 to find the unique clearing vector. Because no node has positive operational cash flow, $x_1 = x_2$ and $x_2 = \frac{1}{2}x_1$ imply $x_1 = x_2 = 0$ in step 3. Thus,

$p_* = p^* = [0,0,0]^T$.

To ensure there are multiple clearing vectors, set the operational cash flows for banks 1 and 2 to $c_i = (l_i - (A^T l)_i)^T$ as in the proof of (*ii*). Afterward, include sufficiently large default costs for all nodes in the cycle. An obvious choice for the required positive default costs is to set $\alpha_i = (A^T l)_i + c_i$. Thus, consider the pro-rata cost network $(N, A, l, c, \alpha, \beta)$.

If 1 or 2 defaults, their clearing payments are 0 because their default costs are larger than their asset value. If 1 pays nominal, then 2 pays nominal, and vice versa. It is thus that $p^* = [2, \frac{3}{4}, \ 0] \gneqq p_* = [0,0,0]^T$.

6 Spectral Analysis for Financial Networks

The previous section highlights the importance of dependency cycles for the existence of inefficient clearing vectors and Section 4.1 highlights the relevance and decentralized interpretation of the least clearing vector. Furthermore, Haldane and May (2011) and Acemoglu et al. (2015) highlight that a fully connected network is both robust against small idiosyncratic shocks and fragile against large systematic shocks. This fragility is the focus of this thesis. Throughout the remainder of this thesis, fragility refers to the level of interconnectedness between banks that potentially expose these banks to large systematic shocks. This fragility concerns both individual components and entire networks.

These core theoretical and empirical results regarding the structures of financial networks and the complexities of intersectoral bankruptcy proceedings motivate a measure based on the network structure that indicates the presence of a large gap between the best and worst clearing payments.

Furthermore, Cerutti and Zhou (2017) highlight the lack and biases of data on intersectoral liability relations. The last two arguments support a frugal systemic risk measure that can be computed efficiently and only depends on the parameters of the general framework provided by Eisenberg and Noe (2001).

From a regulatory perspective, it is important to identify the most problematic components of the financial network.

The costs of organizing a central clearing mechanism potentially outweigh the benefits of moving from the worst to the best clearing payments (Csóka and Herings, 2018).

Even if this generally holds, assigning values to strong components that indicate the fragility of these components could pave the way toward pricing some dimensions of systemic risk.[9]

Consider, e.g., various insurers that belong to different strong components with different levels of fragility to large systematic shocks.

Based on the points mentioned above, this section defines a systemic risk measure that efficiently calculates the contribution of individual network components to systemic risk. The individual components can then be represented in a hierarchical graph that highlights and neatly summarizes the linkages between the different components. This representation of linkages in the network highlights both the role of individual nodes as inducers and receivers of systemic risk.

The individual components, in particular, the non-trivial strong components, serve as building blocks of the systemic risk of the entire network. These components themselves can be characterized in relationship to two fundamental building blocks of non-trivial strong sink components, adjusted for the proportion of liabilities that leave the non-sink components. The adjustment for proportions of liabilities leaving these blocks behaves in a way that is consistent with the systemic risk interpretation.

In the following, non-trivial strong components are referred to as strong components because the role of non-trivial strong components is represented by a node with certain characteristics in a hierarchical network. In a hierarchical network, all nodes are trivial strong components thus such a distinction is superfluous.

For special cases, it is possible to derive exact properties that describe the approximate properties of more complex strong components.

Before the intuition is provided behind this quantification of systemic risk, it is necessary to formally define the values that provide this quantification.

6.1 The Spectrum of Proportionality Matrices

In linear algebra, an eigenvector ς remains directionally unchanged under a linear transformation via a matrix-vector product $A\varsigma$, being scaled by a factor $\lambda \in \mathbb{R}$, i.e., $A\varsigma = \lambda\varsigma$ (Burden and Faires,

[9] In particular, contagion and potentially amplification to the extent this results in a large shock to the network.

1993). By definition $\varsigma \neq 0$ because this describes no properties of the matrix. The scalar λ is the eigenvalue associated with ς (Burden and Faires, 1993). The spectrum of a matrix refers to the set of eigenvalues of a matrix. From hereon out i can refer to imaginary units. It is clear from the context when $i \in N$ or when $i \in \mathbb{C}$.

Additional mathematical properties are described at a later point after the following intuition. The eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$ are determined by solving for the roots of the characteristic polynomial, i.e., $\det(\lambda I - A) = 0$, which has n roots in the complex numbers. The determinant must be zero because $A\varsigma = \lambda\varsigma = I\lambda\varsigma \iff \lambda I\varsigma - A\varsigma = 0$. Thus, the eigenvector must be in the null space of $\lambda I - A$ which contains non-zero vectors if and only if the determinant is zero.

The equation, $\det(\lambda I - A) = 0$, is referred to as the characteristic polynomial of A because it characterizes the multiplicative properties of A corresponding to eigenvectors.

Note that, if the order of banks is changed in N, this does not change the underlying liabilities between the banks. Such a reordering is referred to as a permutation. This implies, in particular, that the proportionality matrix of a network is unique up to a permutation. That is, by reordering the banks in N, the relationship between the banks does not change, only the position of elements in A.

Such a permutation of the proportionality matrix A does not change the eigenvalues of A. This further validates the use of eigenvalues of the proportionality matrix without consideration of the order of banks in N. This invariance to a permutation furthermore holds if only a subset of banks is considered.

One useful purpose of eigenvalues is to efficiently calculate a matrix-vector product where the matrix is a power of another matrix, i.e., $A^k\varsigma = \lambda^k\varsigma$. This can additionally describe the notion of payment by a bank that returns to a bank through a dependency cycle in the network. In particular, if the payment vector is an eigenvector, this can be represented by $(A^T)^d p = \lambda^d p$ if there is a dependency cycle of length d. At most 1 of the payment can return to a bank, and it is no coincidence that the size of eigenvalues for the proportionality matrix is at most 1.

This does not mean payment vectors have to be eigenvectors. If only part of the payments returns to banks it is unlikely to result in a clearing eigenvector. Even if individual strong components are considered, the corresponding payment vector entries need not represent an eigenvector of the submatrix.

6.2 Eigenvalues Versus Dependency Cycles

Consider the financial network (N, L, c).

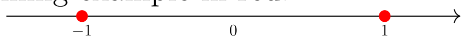
Our goal is to use the average eigenvalues of the matrices corresponding to individual strong components in a weighted average to describe the contributions of these components to overall systemic risk. Roughly speaking, the closer these eigenvalues are to zero, the more fragile a strong component and the banks that receive liability payments from these components are to large systematic shocks. In particular, these eigenvalues move closer to zero if there are many nominal liabilities toward other banks in or outside the strong component. The fragility of a strong component thus refers to the interconnectedness within and outside of a strong component as measured by these eigenvalues. The term fragility is deliberately chosen over contagion because a large number of nominal liability relations within a strong component can act as a buffer against small idiosyncratic shocks (Haldane and May, 2011; Acemoglu et al., 2015).

For special cases, the eigenvalues can be determined exactly. For more complex configurations, the eigenvalues behave approximately in line with these special cases. This section establishes these results, starting with the most simple example.

The running example to explain this interpretation pertains to a dependency cycle that is a strong component by itself. From here on out, such a strong component is referred to as a circular component.

Consider a circular sink component of length 2, described by $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Suppose there is a payment of $p_1 = 1$ by bank 1. If bank 2 pays back this value, this can be represented by the matrix-vector product $A^T \times p = 1 \times p$, where $p = [1, 1]^T$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The intuition here is that all payments move one step in the cycle. The payment of 1 by bank 1 is done by bank 1 to bank 2 and vice versa. The other eigenvalue of A is -1, with corresponding eigenvector $[1, -1]^T$. This is not a payment vector. However, there is still intuition behind the eigenvalue, $\lambda = -1$. The core idea is that, for a cycle of length 2, for all eigenvalues of the submatrix, it is required that λ^2 is positive. This requirement generalizes to the notion that for a circular strong component of length d, the *d-th* power of all eigenvalues must be positive and will be formalized in Theorem 3.

The eigenvalues are represented in the graph below for the running example in red.

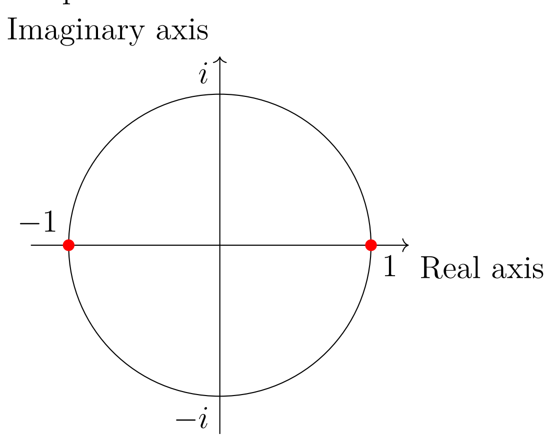


It is thus possible to describe this pattern using $[-1, 1]$. This is a special case of describing the eigenvalues on the complex unit circle.

The core idea behind complex numbers is that there is a real term, $y \in \mathbb{R}$, and an *imaginary* term $z * i$, where $i^2 = -1$ and $z \in \mathbb{R}$. There is no literal interpretation of i, it is merely a definition. In a sense, the space of all complex numbers denoted $\mathbb{C}$, is a generalization of the real line where even roots of negative numbers exist.

Complex numbers have special properties. The most important property for this thesis is the bijective relationship with any vector in $\mathbb{R}^2$. This is immediately clear because the complex number $y + zi$ describes $[y, z]^T$ and vice versa. This allows us to describe the unit circle in $\mathbb{R}^2$ by using the complex unit circle, i.e., a number in $[-1, 1]$ plus a number in $[-i, i]$ that jointly describes two dimensions.

Consider the eigenvalues of the submatrix corresponding to the running example of a dependency cycle of length 2, now on the complex unit circle.



Starting from 1, a rotation over π radians counterclockwise, arrives at -1. Doing another rotation over the same angle, one arrives again at 1. It is possible to describe this pattern using $\forall j \in \{1, 2\} : \lambda_j = \cos\left(\frac{2j\pi}{2}\right) + \sin\left(\frac{2j\pi}{2}\right) i$. The sinus term is always zero because this function is always zero if the input is an integer multiple of π, i.e., $\forall j \in \mathbb{Z} : \sin(j\pi) = 0$.

In general, for a circular sink component of length d, it is possible to describe the eigenvalues of the corresponding submatrix as $\forall j \in \{1, ..., d\} : \lambda_j = \cos\left(\frac{2j\pi}{d}\right) + \sin\left(\frac{2j\pi}{d}\right) i$.

From this representation of the eigenvalues, it is clear that the eigenvalues are on the complex unit circle, such that λ_{j+1} is obtained via a counterclockwise rotation from λ_j, over an angle $\frac{2\pi}{d}$.

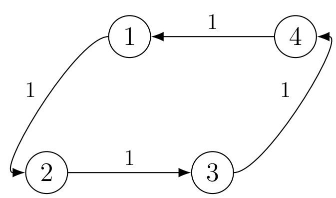
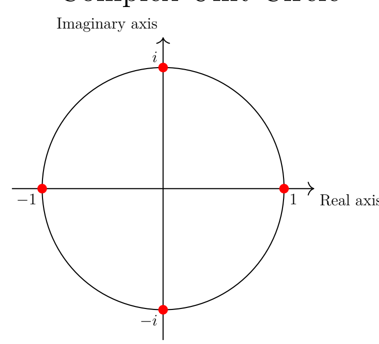
For any dependency cycle of length $d > 2$, at least two of the eigenvalues must be complex. This is represented in Table 6 for $d = 4$. Furthermore, using Euler's identity, it is possible to describe these eigenvalues as $\forall j \in \{1, ..., d\} : \lambda_j = \exp\left(\frac{2j\pi}{d} i\right)$.

Euler's identity intuitively demonstrates that the *d-th* power if the d eigenvalues all equal 1. In particular, $\exp\left(\frac{2j\pi}{d} i\right)^d = \exp\left(\frac{2j\pi}{d} di\right) = \exp\left(2j\pi i\right) = \cos\left(2j\pi\right) + \sin\left(2j\pi\right) i = 1$, for each $j \in \{1, ..., d\}$.

Without going into any details, the size of $y + zi \in \mathbb{C}$ is given $|y + zi| = \sqrt{y^2 + z^2} \in \mathbb{R}$ which coincides with the l_2 norm in $\mathbb{R}^2$.[10] The size of the largest eigenvalue corresponding to a matrix A is referred to as the *Spectral Radius* and is denoted $\rho\left(A\right)$.

The eigenvalues for the circular strong component are represented on the complex unit circle in Table 6 for $d = 4$ in red.

Table 6: Circular Strong Sink Component of Length 4

A	G[A]	Complex Unit Circle
$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$		

Thus, the presence of dependency cycles is equivalent to the possibility of describing the properties of these circular strong components in terms of rotation. This pattern can be generalized to more complex strong components and is known as the Perron-Frobenius Theorem. Before this theorem is provided, more context is necessary.

A node $i \in N$ has periodicity $d(i) \in \mathbb{N}$ if i can reach itself in only multiples of $d(i)$ steps. That is, for all paths from i to i, the

[10]See Appendix A.2 for details on the l_1 and l_2 norms.

number of arcs is always a multiple of $d(i)$. A strong component only has nodes with the same periodicity. For this reason, the period of all nodes in a strong component is denoted in shorthand d. If the period of a node or non-trivial strong component is 1, this node or component is a-periodic.

The period of the strong component is precisely what is reflected in the rotation along the complex unit circle at the start of this section.

The goal of this thesis is to decompose the financial network into a hierarchy of non-trivial strong components and use the eigenvalues corresponding combined with the position within this hierarchy to measure their contribution to overall systemic risk. For this reason, the remainder of this section will be under the assumption that it is possible to find at least one non-trivial strong component.

From hereon, assume there always are $\kappa \geq 1$ non-trivial strong components in the financial network $(N, E[A], A)$ and let these be collected in the set $\mathcal{SC} = \{SC_1, ..., SC_\kappa\} \subseteq \mathcal{P}(N)$. A principal submatrix is a submatrix where only rows and columns with the same index can be removed. In a similar sense to a permutation, no relationship between nodes is changed. A principal submatrix of the proportionality matrix describes a subgraph of the network. Let A_{SC} be the principal submatrix corresponding to a strong component $SC \in \mathcal{SC}$. Thus, $G[A_{SC}]$ is a subgraph of $G[A]$ corresponding to the strong component $SC \in \mathcal{SC}$.

The fundamental link between strong components and proportionality matrices is the following. A matrix is referred to as irreducible if it is not possible to permute it into a triangular matrix. This implies that it is not possible to rearrange the elements of the matrix to reflect a hierarchical structure without changing the underlying structure. In Table 6, A reflects this for the circular component.[11].

There are many proofs of the Perron-Frobenius Theorem, see, e.g., Smyth (2002). The proof will only be provided for a special case regarding a circular component that need not be a sink, in

[11]To provide some intuition, consider the proportionality matrix that describes a circular network, without loss of generality, it is possible to permute this matrix into the same structure as that in 6. It is necessary to have in order a path from the first node to the final node and back. That is, there must be n positive elements not on the diagonal such that there is precisely one per row and column. Such a value can only occur at most $n-1$ above or below the diagonal. Thus, the final arc that creates the cycle cannot be added without violating the triangular requirement. Note, this is a sketch of another proof that if there is no dependency cycle there is at least one bank that does not receive nominal liabilities. A graphical approach is described in A.2

Proposition 2.

Theorem 3 (Perron-Frobenius)
Let $SC \in \mathcal{SC}$ with period d and let A_{SC} have spectral radius $\rho(A_{SC})$. There are d eigenvalues with size $\rho(A_{SC})$, given by $\forall j \in \{1, ..., d\} : \exp\left(\frac{2j\pi}{d} i\right) \rho(A_{SC})$.

Before the proposition concerning the eigenvalues of a circular component is provided, consider the following nuances of the Perron-Frobenius Theorem regarding the spectral radius.

First of all, a non-trivial strong component is a sink if and only if $\rho(A_{SC}) = 1$. A non-trivial strong component is not a sink if and only if $0 < \rho(A_{SC}) < 1$. These last two properties follow from the substochastic nature of A_{SC}. In this sense, the d eigenvalues corresponding to the period of such a component are closer to 0 if there are nominal liabilities that leave the component. This already hints at the general property of strong components that the larger the fraction of liabilities that leave the strong components ceteris paribus, the closer the eigenvalues are to zero. [12]

Now suppose the two-bank circular component in the running example is no longer a sink. What if, e.g., for one of the two banks, three-quarters of the liabilities are toward banks outside this cycle? It may seem like this implies one of these values should be scaled down to $\frac{1}{4}$. However, for each bank in the cycle, only $\frac{1}{4}$ of the payments returns after the cycle has been traversed.

Consider the matrix $A_{SC} = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{bmatrix}$ for the circular component $SC = \{1, 2\}$ in the network represented by Table 7. The corresponding characteristic polynomial is $\det(\lambda I - A_{SC}) = \lambda^2 - \frac{1}{4}$, which has the roots $\lambda = \frac{1}{2}, -\frac{1}{2}$ as indicated within the complex unit circle in red in Table 7.

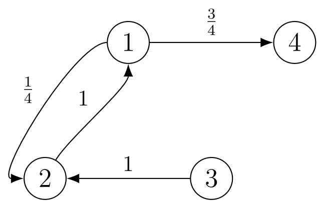
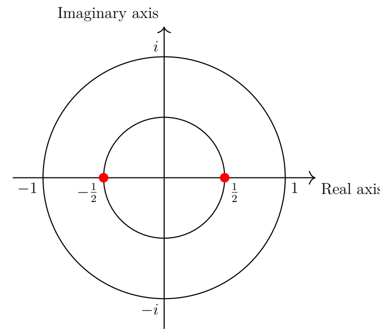
Thus, this uniformity in the fraction of their payment that returns to these banks, is reflected in the size of both eigenvalues which are $\frac{1}{2}$ times the eigenvalues of the circular sink component in the running example.

The following proposition formally establishes the eigenvalues of a general circular component.

Proposition 2 (Eigenvalues of a Circular Component)
Assume SC is a circular strong component with period $d = |SC|$. Without loss of generality, assume that $SC = \{1, ..., d\}$, and each bank $j \in SC$ contributes $x_j = \frac{L_{jj+1}}{l_j}$ of proportional liabilities to

[12] This concept is referred to as leakage for Markov chain applications and extensive discussions are available (Huisinga and Schmidt, 2005).

Table 7: Circular Component of Length 2 in a Network with 4 Banks

A	G[A]	Complex Unit Circle
$\begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$		

the cycle. Let $\pi_x = \left(\prod_{j=1}^d x_j\right)^{\frac{1}{d}}$. Then,

(i) The eigenvalues of A_{SC} are $\forall j \in \{1, ..., d\} : \lambda_j = \left(\prod_{j=1}^d x_j\right)^{\frac{1}{d}} \exp\left(\frac{2j\pi}{d} i\right)$.

(ii) If $x_1 \in (0, 1]$, and $\forall j > 1 : x_j = 1$, then $\pi_x \underset{d\to\infty}{\to} 1$.

Proof. (i) The characteristic polynomial is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -x_1 & 0 & \cdots & 0 \\ 0 & \lambda & -x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -x_{d-1} \\ -x_d & 0 & \cdots & 0 & \lambda \end{vmatrix}$

$$= \begin{vmatrix} \lambda & -x_1 & 0 & \cdots & 0 \\ \frac{\lambda^2}{x_1} & 0 & -x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\lambda^{d-1}}{\prod_{j=1}^{d-2} x_j} & 0 & \cdots & 0 & -x_{d-1} \\ \frac{\lambda^d}{\prod_{j=1}^{d-1} x_j} - x_d & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{d+1}\left(\frac{\lambda^d}{\prod_{j=1}^{d-1} x_j} - x_d\right)\prod_{j=1}^{d-1} -x_j = \lambda^d - \prod_{j=1}^d x_j.$$

Take $\lambda_j = \pi_x \exp\left(\frac{2j\pi}{d} i\right)$.

(ii) $\pi_x = x_1^{\frac{1}{d}} \underset{d\to\infty}{\to} 1$.

□

(i) tells us that the amount of proportional liabilities that remain within the circular component are evenly distributed among the eigenvalues via the geometric mean, i.e., π_x. This is an application of the Perron-Frobenius Theorem because the circular component of length d has period d.

(ii) implies that the size of the eigenvalues is affected by the amount of proportional liabilities that leave the circular component only in relation to the length of the circular component. Thus, if the cycle is not very long and only one bank has a large proportion of liabilities leaving the cycle, then π_x will be significantly smaller in comparison to the case where there are twice as many banks. This motivates the consideration of the eigenvalues in relation to the liabilities that are exposed to the risk of this component to measure the contribution of a circular component to systemic risk. Otherwise, the average eigenvalues may be the same for a very large circular component and a very small circular component with the same average of liabilities that leave these components per bank.

This property generalizes to more complex configurations of strong components.

Another comment on the Perron-Frobenius Theorem is that, if cycles of different lengths are likely it is reasonable for some strong components to be a-periodic. Take, e.g., a cycle of length 2 connected to a cycle of length 3 by one node. The periodicity of 1 does not tell us anything about the sizes of almost all eigenvalues. However, the rotational pattern on the complex unit circle is still roughly present for more complex configurations of strong components.

Consider the following proposition to establish the difference between the circular component and a strong component of two cycles connected at one bank.

Proposition 3 (Two-Cycle Strong Sink Component)
Let $SC \in \mathcal{SC}$ be a strong sink component and let $|SC| = 2k + u - 1$, SC consists of two dependency cycles of lengths k and $k + u$. The following properties hold for the spectrum of A_{SC}.

(i) The spectrum of A_{SC} contains $k - 1$ eigenvalues 0.

(ii) There are as many complex eigenvalues as the larger cycle has in isolation.

(iii) -1 is an eigenvalue of A_{SC} if and only if k and u are both even.

The proof is provided in Appendix A.2. Consider the following implications that are represented in Table 8.

Properties $(i) - (iii)$ indicate the largest cycle dominates the pattern along the complex unit circle. There are as many non-zero eigenvalues as there are for the larger cycle in isolation. Furthermore, the non-zero eigenvalues are adjusted to move closer to the eigenvalues of the smaller cycle. This is clear in the examples in

Table 8: Eigenvalues of two-cycle strong components. Green dots indicate common eigenvalues among the cycles of different lengths in isolation.

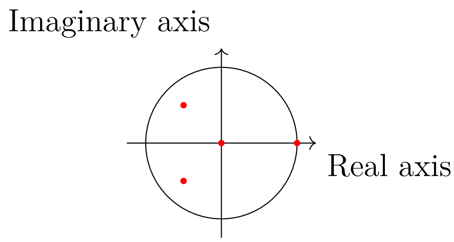
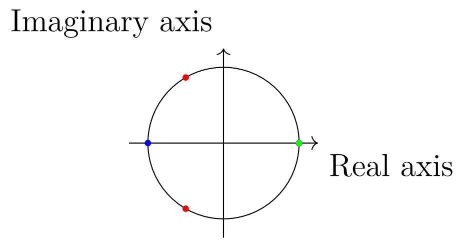
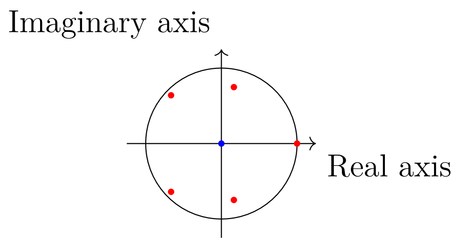
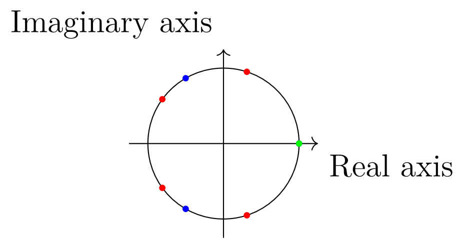
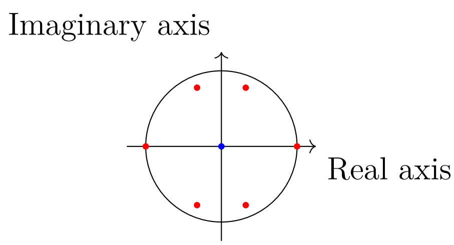
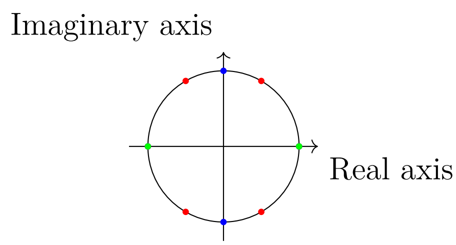
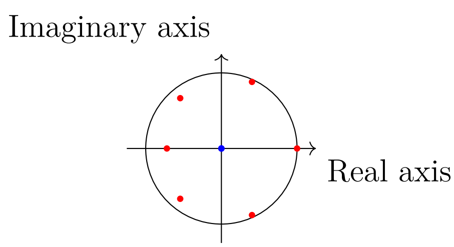
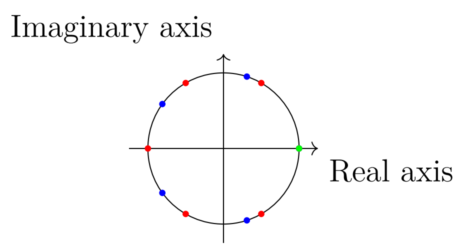
Cycle Sizes	Connected Eigenvalues	Separate Eigenvalues
2, 3		
3, 5		
4, 6		
5, 6		

Table 8.

This dominance of the larger cycle is reinforced by the fact that there is only a negative eigenvalue if the larger cycle has one in isolation, and it is only -1 if the smaller cycle has an eigenvalue -1 in isolation as well.[13]

[13]The latter property is a consequence of the Perron-Frobenius Theorem. If both k and u are even, then the period of the strong component must be even, and an even period implies $-\rho(A_{SC}) = -1$ is an eigenvalue. This follows from the bank that connects the nodes being able to reach itself in only even steps and all other banks in even and even plus even steps. In contrast, if both u and k are uneven, then $k + u$ is even and the bank that connects the

There is a clear distinction between the one-cycle strong component in Proposition 3. The sizes for the two-cycle strong component are not all on the complex unit circle even if the component is a sink. This highlights that the reduction in the size of the eigenvalues reflects liabilities from one dependency cycle toward another. Roughly speaking, this means that the more connected the banks are within the strong components, the smaller the eigenvalues are.

The interaction between the sizes of the individual cycles and the sizes of the eigenvalues is complex. Note, e.g., that the distribution of sizes can be uneven, as demonstrated in the 3 and 5, and 5 and 6 cycle size example in Table 8.[14] Still, the reduction in average size is a clear indication of a more complex strong component then the case of a circular strong component.

It should not be surprising that for even more complex configurations of strong components with more cycles intersecting at different banks, the sizes or typically much smaller. The 'most fragile' strong component is formalized in the following proposition. This is a slight deviation of a theorem by Mezić et al. (2019) to fit the context of financial networks.

Proposition 4 (Fully Connected Symmetric Strong Sink Component)
Consider $SC \in \mathcal{SC}$. Assume $SC = \{1, ..., h\}$ such that $\forall j \in SC$: $\frac{L_{ij}}{l_i} = \frac{1}{h-1}$. Then $\lambda_1 = 1$ and $\forall j \in SC \setminus \{1\} : \lambda_j = -\frac{1}{h-1}$ are the eigenvalues of A_{SC}.

Proof. A_{SC} is row stochastic and thus $\lambda_1 = 1$ is an eigenvalue. Furthermore, note that $\lambda I - A_{SC}$ has h linearly depended rows such that only 1 in isolation can be linearly independent if $\lambda = -\frac{1}{h-1}$. That is, the characteristic polynomial has $h-1$ roots $-\frac{1}{h-1}$. □

One immediate consequence of this observation is that $\frac{|\lambda_1|+...+|\lambda_h|}{h} = \frac{1+\frac{h-1}{h-1}}{h} = \frac{2}{h} \underset{h\to\infty}{\rightarrow} 0$. The average size of the eigenvalues of a fully connected symmetric strong sink component approaches 0 fast as the size of the strong component increases. This implies that even though all liabilities are within the strong component, the complexity of this component causes the eigenvalues to be on average very close to zero. As one can imagine, this effect is exacerbated

cycle can reach itself in odd steps and all other banks in both odd and even steps.

[14] The symmetry in the real axis follows from the property that for real matrices eigenvalues come in conjugate pairs.

if this component is not a sink but there is a rather large fraction of total liabilities per bank that leave the component.

6.3 Remarks on the Sizes of Eigenvalues

There is some nuance to the notion that smaller sizes indicate a larger fragility of circular components. This nuance mainly concerns special cases where the structure of the financial network mainly has circular components in $\mathcal{SC}$.

Consider again a version of the running example to illustrate this fact. If there are two banks in a dependency cycle represented by the following submatrix $\begin{bmatrix} 0 & \frac{1}{100} \\ \frac{1}{100} & 0 \end{bmatrix}$. Assume that the operational cash flow is zero for both banks. Assume furthermore that both banks hold the same total liabilities and both banks receive twice as much in liability payments as they owe to other banks if all banks are solvent. This implies that $\eta_i = 2l_i$, for $i = 1, 2$. Finally, assume for the sake of simplicity that default costs are total upon default, i.e., $\beta = \iota$. Then it is very well possible that the first bank defaults and only $\frac{1}{200}$ of the liabilities paid to the second bank are lost. This in turn implies that the second will not default and the default of the first bank behaves as if there is no dependency cycle. It is thus unlikely that the size in the eigenvalues, which must be $\lambda = \frac{1}{100}, -\frac{1}{100}$ by Theorem 3, are indicative of the self-fulfilling defaults inherent to the model. This is unlikely because only a small amount of proportional liabilities is lost for one of the two banks that causes the default. In that sense, it is unlikely to have defaults propagate through the system from these two banks because of this dependency cycle.

Luckily, unless a large proportion of non-trivial strong components are circular components, this is unlikely to be a problem. Still, suppose this scenario holds a significant likelihood. In that case, one possibility is to ensure these occurrences are ignored by discriminating between circular and non-circular strong components and determining some threshold on the liabilities.[15]

7 Spectral Fragility

By identifying the non-trivial strong components and separating them from the individual nodes, the rotational aspect is always

[15] For more complex configurations the eigenvalues will be generally much smaller and such a threshold may be much harder to establish.

present in the eigenvalues of the corresponding principle submatrices and is less relevant to quantifying the fragility of a component, as the previous section indicates. Instead, the size tells us a lot more about how many connections are within the strong component and the proportion of liabilities that leave the strong component.

Thus, the average size of the eigenvalues tells us how fragile a strong component and its neighbors are to large systematic shocks in terms of proportional liabilities. For this reason, one minus the average size of the eigenvalues corresponding to $SC \in \mathcal{SC}$, is referred to as the *fragility* of SC.

In addition, the contribution of a strong component to the fragility of the entire network depends both on the liabilities within the strong component and on paths pointing toward other banks in this network. If the size of the liabilities is not considered, two strong components that are identical in proportional liabilities, but one component holds 10^6 as much liabilities as the other would be considered equal in contribution to the fragility of the network.

Furthermore, it is possible to represent the network as hierarchical in case all non-trivial strong components are replaced by individual nodes. This hierarchy tells us how much liabilities are exposed to the fragility of the individual strong components.

These arguments motivate a measure that weighs the fragility of individual strong components by the proportion of total liabilities in the network exposed to the fragility of the component.

This measure can be calculated based on the liability matrix alone and allows for adjustments depending on how important the presence of strong components in the network is compared to the number of connections within the strong components themselves.

Because there exist efficient algorithms to find all strong components and all nodes reachable by each strong component, and more complex calculations are only performed on the strong components themselves, this measure is computationally efficient and a viable indicator of systemic risk that incorporates both network topology and the size of the total liabilities per bank.

Due to its nature based on the established literature in financial network topology, this indicator should measure the potential for a large gap in best and worst clearing payments and can thus potentially indicate whether it is beneficial to organize a costly central clearing mechanism or whether decentralized clearing suffices via local bankruptcy proceedings.[16]

Even if such a central clearing mechanism is generally too costly as motivated in literature (Csóka and Herings, 2018), the contribution of this measure could be to price systemic risk in terms of fragility per strong component.[17]

Based on the arguments throughout this thesis, we propose the following measure of financial network fragility.

Definition 8 (Spectral Fragility of a Financial Network)
The *spectral fragility* of a financial network (N, L, c) is a function $SF : \mathbb{R}_+^{n \times n} \mapsto [0, 1]$, defined by,

$$SF(L) = \frac{1}{|\mathcal{SC}|} \sum_{SC \in \mathcal{SC}} \frac{\sum_{j \in R(SC)} w_j(SC) l_j}{\iota^T l} \left(1 - r(SC) + r(SC)\left(1 - \frac{||\lambda\left(A_{SC}\right)||_{l_1}}{|SC|}\right)\right), \tag{27}$$

where $\mathcal{SC}$ is the set of non-trivial strong components of the financial network (N, L, c), $w_j(SC) \in [0, 1]$ is a discount factor, $r(SC) \in [0, 1]$ weighs the presence of strong components versus their fragility, $\lambda\left(A_{SC}\right)$ is the vector that contains the eigenvalues of A_{SC}, and $R(SC)$ contains all nodes reachable by the set SC.

The discount factor is chosen to be just a constant factor $w \in [0, 1]$ to the power of the shortest path of liabilities between the strong component and a bank. If e.g., the shortest path from SC to j is k, then $w_j = w^k$.[18] This allows us to control for the ideas in literature that centrality is far less important than the liabilities to and from neighboring nodes (Acemoglu et al., 2015).[19]

As discussed at the start of this section, once all non-trivial strong components are replaced by nodes, the resulting network must be hierarchical. This implies that there is a hierarchy of strong components in the following sense. Consider a financial network (N, L, c). Suppose there are 3 strong components and $\mathcal{SC} = \{SC_1, SC_2, SC_3\}$ is in order of the hierarchy. The hierarchy of strong components in the network can be represented by

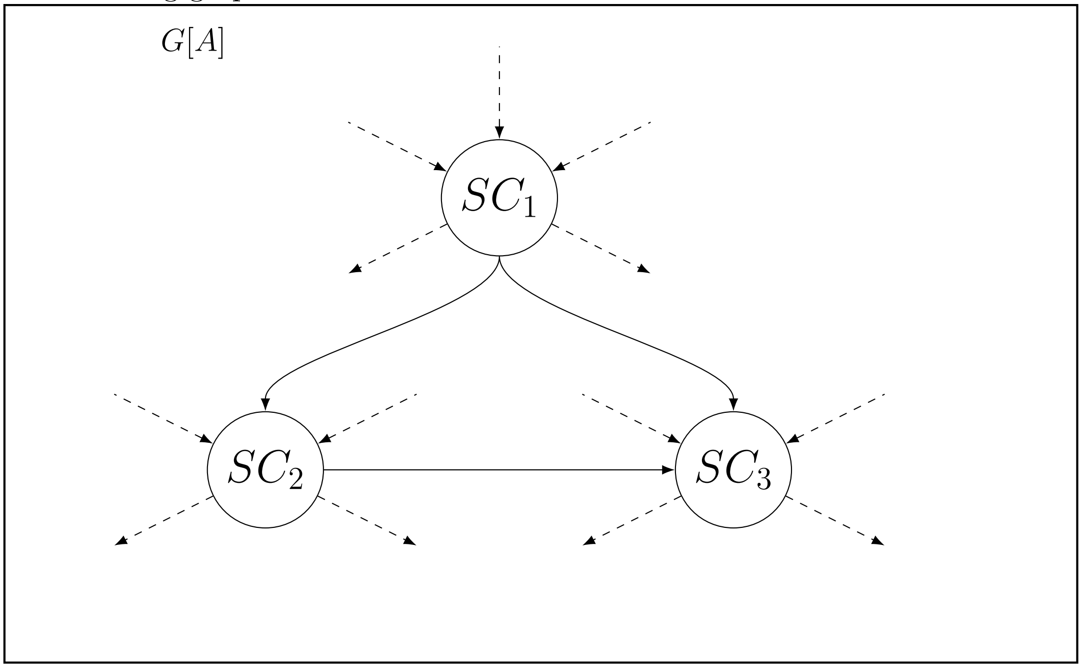
[16] The importance of the worst clearing payments is discussed in Section 4.1.

[17] There is already extensive literature available on the allocation of risk capital. See, e.g., Bauer and Zanjani (2013) and Baione et al. (2018). These tools are perhaps suitable to allocate risk capital based on the fragility of strong components.

[18] To make $w = 0$ meaningful, define $0^0 = 1$. Thus, if $w = 0$, the weight of the strong components is merely the fraction of total liabilities in the network owed by banks in the strong component.

[19] Network centrality refers to the role a node plays in a network. This concept plays a crucial role in network analysis that transcends scientific fields. Such readily available tools are not always appropriate, as argued by Acemoglu et al. (2015). To control for centrality concepts, an alternative measure based on established spectral centrality measures is proposed in Section 8.

the following graph.



Any path that is not in between elements of $\mathcal{SC}$ is either to sink nodes or from source nodes. The arrows from and to elements of $\mathcal{SC}$ reflect the hierarchy that must exist between the strong components. The paths between strong components may contain many nodes which contribute to the weight of the strong component the path leaves from, via the total liabilities of these nodes.

The expression (27) reflects this structure as follows. After the average size of the eigenvalues of each $SC \in \mathcal{SC}$ have been calculated, their assigned weight is the fraction of cumulative total liabilities of the entire network that are exposed to their default risk.

Intuitively, a less complex strong component can induce more systemic risk at the top of the hierarchy than a very complex strong component at the bottom of the hierarchy.

To understand this, consider the following example.

Example 6 Because the Spectral Fragility measure only depends on proportional and total liabilities, the example is stated only in terms of (N, A, l).

The first strong component is an example of a two-node circular component and the second strong component is a fully connected strong sink component of 3 nodes.

By Proposition 2, the eigenvalues of SC_1 are $\lambda_1 = \frac{3}{4}$ and $\lambda_2 = -\frac{3}{4}$. The fragility of SC_1 is thus $1 - \frac{2\times\frac{3}{4}}{2} = \frac{1}{4}$. All nodes

Table 9: Spectral Fragility Example

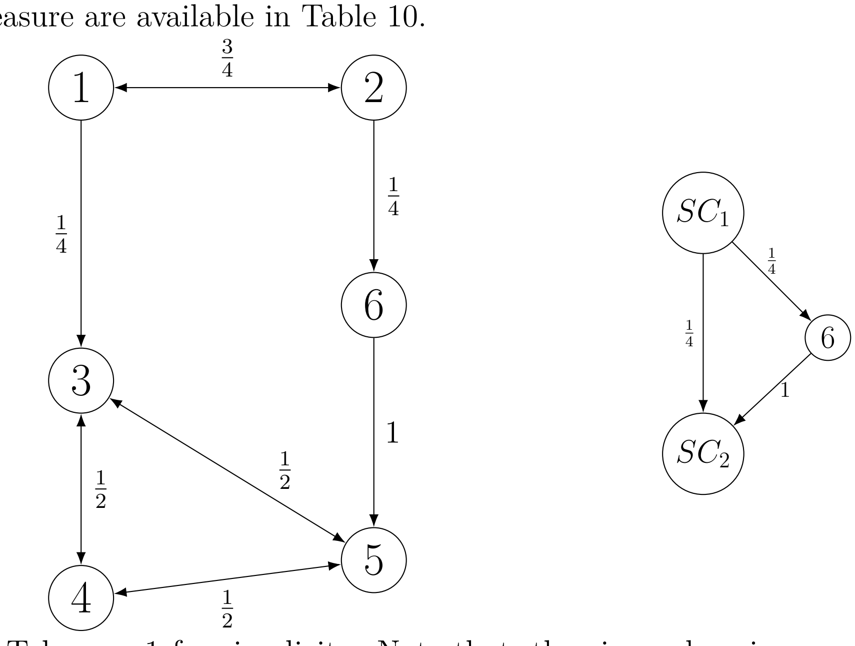
N	A	l
$\{1,2,3,4,5,6\}$	$\begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

are reachable from this component and thus, the weight of this strong component is $\frac{6}{6} = 1$.

By Proposition 4, the eigenvalues of SC_2 are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -\frac{1}{3-1} = -\frac{1}{2}$. Thus, the fragility of SC_2 is $1 - \frac{1+2\times\frac{1}{2}}{3} = \frac{1}{3}$. The only liabilities exposed to the fragility of this component are the liabilities in this component. Thus, the weight of SC_2 is $\frac{3}{6} = \frac{1}{2}$.

Thus, the spectral fragility of this network is $SF(L) = \frac{1\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{3}}{2} = \frac{5}{24}$.

These components and their contribution to the spectral fragility measure are available in Table 10.



Take $r = 1$ for simplicity. Note that, the eigenvalue sizes on average indicate that the second component is more fragile then the first component. However, because there are twice as much liabilities that can be affected by defaults in the first component versus the amount of liabilities that can be affected by defaults in the second component, the first component contributes more to systemic risk than the second component. The first component contributes $\frac{3}{24}$ and the second component $\frac{2}{24}$ to the spectral fragility of the network. This score is not very high and this is mainly due to the small size of the fully connected sink component in combination with the position of this component at the bottom of the hierarchy. If the strong sink component would have contained, e.g., 10 banks, without altering the total liabilities in the component, the contribution of this component would have been $\frac{4}{5} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{5}$. This is a significant increase compared to $\frac{1}{12}$, but still not very large due to the position of the component.

Table 10: Spectral Fragility Components for Example 6

SC	$1 - \frac{\|\|\lambda(A_{SC})\|\|_{l_1}}{\|SC\|}$	$\frac{\sum_{j\in R(SC)} l_j}{\iota^T l}$	$\frac{1}{\|SC\|}\frac{\sum_{j\in R(SC)} l_j}{\iota^T l}\frac{\|\|\lambda(A_{SC})\|\|_{l_1}}{\|SC\|}$
$\{1,2\}$	$\frac{1}{4}$	1	$\frac{1}{8}$
$\{3,4,5\}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{12}$

In general, the larger the r value is, the smaller the spectral fragility measure is. If $r = 0$, the measure just counts the non-trivial strong components and assigns each component a weight depending on the fraction of total liabilities that are exposed to a default in the component.

The smaller the parameter $r \in [0, 1]$ is, the less the complexity differences between each strong component matters.

Furthermore, if it is suspected for some reason that the network consists mainly of a large number of circular strong components with few liabilities on outward pointing paths, their presence contributes barely anything to the overall fragility of the financial system if $r \approx 1$.

On the one hand, for a large network with many banks that are not directly lending to these components, their presence may not contribute much to systemic risk.

On the other hand, if the network consists mainly of these large circular structures then there is still a large fraction of total liabilities exposed to these fragile components. It may thus be useful to consider the spectral fragility for a wide range of values r. For the low values of r, the measure is indicative of the presence of strong components and the fraction of liabilities that are exposed to these strong components.

For larger values of r, the amount of liabilities from and to different cycles within the same strong component and banks outside the component plays a much larger role in the difference in values between two different networks. Luckily, all the complex operations to determine the relevant quantities are done before r is included in the calculations and thus, calculating the measure for a range of r values remains computationally efficient.

A natural question is whether it is possible to have a network with a spectral fragility of 1. This is indeed possible, at least in theory. If the network is just a fully connected symmetric strong component with a large number of banks, the weight of this component is 1, and by Proposition 4, the fragility is approximately 1. Thus, the spectral fragility of such a network is approximately 1. This provides an intuitive upper bound on the fragility of a financial network because such a network is the most fragile to large systematic shocks. This holds independent of the parameter $r \in [0, 1]$.

An important reason to include the parameter $r \in (0, 1)$ is to not have a zero spectral fragility for a network that is just a circular component. Even though such a network is unlikely to exist for any practical implementation if there are a lot of circular strong components, the network may be incredibly fragile to large systematic shocks. If one chooses $r = 1$, the spectral fragility may be almost 0.

The particular relevance of these parameters vary per network setup. In Section 8.2, a stylized example is provided to compare with some of the simulation results.

8 Random Networks and Regression

8.1 Random Networks

This section describes the simulation methods to arrive at random networks that will be used to generate relevant data for regression analysis.

That is, both the nominal liabilities and the operational cash flow become independent identically distributed random variables.

Much of the literature on network topology regarding contagion in financial networks assumes banks with identical roles in the network which allows for simple analysis. This assumption naturally extends to independent and identically distributed random liabilities. However, this assumption typically does not simulate a large variation of strong component configurations required to measure the effectiveness of the spectral fragility measure. The interaction between the hierarchy of strong components and the fragility of individual strong components is crucial in assessing the effectiveness of this spectral measure.

Regardless, the random networks described in the following pertain to independent and identically distributed liabilities.

The main reason is computational efficiency based on simple standard distributions for random variables.

Another important motivation for this approach is to highlight why this approach is ill-equipped to capture the potential predictive power of the measure and discuss directions for future research. The complex nature of network topology can obscure the true predictive efficacy for measures of systemic risk.

Another reason to keep the simulation approach simple is the exponential increase in configurations in the number of alternative parameter values. There is a strong interaction between the different parameters of the financial network and the effect of changing one parameter results in vastly different outcomes when controlling for the other parameters.

All random variables are assumed to be independent to allow for efficient simulations. The remainder of the exogenous quantities are identical to those in the pro-rata cost network.

Definition 9 (Random Financial Network)
A *random financial network* is a tuple $(N, B, \mathbf{L}, \hat{c}, \tilde{c}, \alpha, \beta)$, where $N = \{1, ..., n\}$ is the *set of banks*, $B \in \{0,1\}^{n\times n}$ is a *Bernoulli matrix*, $\mathbf{L} \in \{1, ..., k\}^{n\times n}$ is a *uniform matrix*, $\hat{c} \in \{0, z\}$ is the *systematic operational cash flow*, $\tilde{c} \in \{0, ..., z\}^n$ is the *idiosyncratic cash vector*, and $\alpha, \beta \in \mathbb{R}^n_+$ are *default parameters.*

The Bernoulli matrix contains random variables $\forall, i, j \in N, i \neq j : B_{ij} \overset{i.i.d}{\sim} \mathcal{B}(1, b)$, where $\mathcal{B}$ is the binomial distribution and $b \in (0, 1)$ is the interbank liability probability.

The uniform matrix has a zero diagonal and contains elements $\forall i \in N \forall j \in \setminus\{i\} : \mathbf{L}_{ij} \overset{i.i.d}{\sim} \mathcal{UD}\{1, k\}$, where $\mathcal{UD}$ is the discrete uniform distribution. The liability matrix is constructed by the element-wise product $L = B \otimes \mathbf{L}$.

Furthermore, the operational cash flows $\forall i \in N : c_i = \hat{c} + \hat{c}_i$ consists of a systematic part $\hat{c} \sim \mathcal{UD}\{0, z\}$ and an idiosyncratic part $\hat{c}_i \overset{i.i.d}{\sim} \mathcal{UD}\{0, z\}$. The distribution of c_i then satisfies $\forall x \in \{1, ..., z\} : \mathbb{P}(c_i = x) = \frac{x+1}{(z+1)^2}$ and $\forall x \in \{z+1, ..., 2z\} : \mathbb{P}(c_i = x) = \frac{2z+1-x}{(z+1)^2}$. Intuitively, this implies a distribution of c_i where $\mathbb{P}(c_i = z) = \frac{z+1}{(z+1)^2}$ and in both directions, the probability decreases linearly with slope $\frac{1}{(z+1)^2}$ until $\mathbb{P}(c_i = 0) = \mathbb{P}(c_i = 2z) =$

$\frac{1}{(z+1)^2}$. That is, c_i has a symmetric distribution around $c_i = z$ of which half is determined by a systematic component, i.e., $\hat{c}$, and half is determined by an idiosyncratic component, i.e., $\hat{c}_i$. Thus, the average operational cash flow equals $\mathbb{E}(c_i) = \mathbb{E}(\hat{c} + \hat{c}_i) = 2\frac{z}{2} = z$.

Because all idiosyncratic parts and the systematic part are independent, it holds that $\forall i, j \in N : cov(c_i, c_j) = \hat{c}$. This implies that a systematic shock that results in a low value $\hat{c}$ affects all banks in the random network.

Thus, the average operational cash flow equals $\mathbb{E}(c_i) = \mathbb{E}(\hat{c} + \hat{c}_i) = 2\frac{z}{2} = z$.

8.2 Regression Analysis

Throughout this section, there is much discussion on the two parameters of the spectral fragility measure regarding the effects in different network setups. These values are discussed mainly in terms of the eigenvalues corresponding to the strong components. Of course, the eigenvalues are only relevant concerning the total liabilities that are exposed to the fragility of a strong component. For the sake of brevity, the liabilities are not always mentioned unless there is a reason to highlight this. Consider, e.g., the core-periphery network where presumably a large fraction of total liabilities in the financial network are exposed to the fragility of one large strong component. This special case warrants the mention of the liabilities that are exposed to the fragility of the core.

Thus, arguments that describe how the main explanatory power lies in the eigenvalues of a strong component assume that the reader understands that the eigenvalues are only relevant to the extent the liabilities are exposed to the fragility of the strong component.

The dependent variable in all regressions pertains to the difference in the system-level costs for the least and greatest clearing vectors divided by the aggregate total liabilities in the network.[20] This dependent is from hereon referred to as the delta system-level costs. The choice for the delta system-level costs as the dependent is based on the necessity of strong components for a multiplicity of strong components. The spectral fragility measure

[20] The reason to favor the difference in system-level costs as a fraction of the total liabilities in the system is that the interpretation of the variance depends only on the interaction of variables. No information is lost because the size of the values in isolation is irrelevant.

is a weighted average of the fragility of strong components. This logic is controlled for in a later section.

The predictor of interest in this section is the spectral fragility measure. In the next section, the predictive power is compared to another spectral measure based on the Katz Borrower Centrality for different weight configurations.

To allow for complex regression models, this section considers a polynomial regression up to the sixth-order.

All regressions are performed for $m = 10000$ simulations of random networks, each with $n = 100$ banks. To examine the potential for over- and underfitting, the data is split into a train set of 7000 simulations and a test set of 3000 simulations. The accuracy of the predictors is assessed using the R^2 of the test set. This approach is appropriate because the actual values are not of interest but rather the explanatory power.

Before the results are provided, some limitations of the simulation method are discussed. These limitations dictate to a large extent what we can and cannot conclude from the results. After the results are provided, these limitations are elaborated on.

The interpretation and effect of changing one parameter on the prediction power of the measure depend severely on the values of the other parameters.

In addition, although $n = 100$ banks may seem like a sufficiently large number of banks, the required sparsity for our analysis necessitates the probability of interbank liabilities to be sufficiently small. This requirement rests on two arguments.

First of all, real-world financial networks are typically sparse (Bardoscia et al., 2021).

Second, the spectral fragility measure decomposes the network into a hierarchical network of strong components.

On the one hand, if the probability of interbank liabilities is too large, the likelihood of just one strong component is approximately 1.

On the other hand, the strong components need to have a sufficiently large amount of liabilities both within and outside the component to ensure sufficient variation in the spectral fragility measure. Finally, the number of simulations required to accurately assess the effect of the different parameters of the spectral fragility measure is large and results in a time-expensive computation.

The regression analysis starts with a base case. The parameters for the base case are available in Table 11.

The choice of the interbank liability probability $b = 0.062$

Table 11: Base Case

RNG	m	n	b	z	k	α	β
69	10000	100	0.062	10	10	0.5	0.75

is based on ad hoc analysis to balance the presence of multiple strong components versus sufficient variation in the fragility of the strong components.

The test R^2 values are plotted for the first to the sixth-order polynomial regression on the spectral fragility measure for weight values $w \in \{0, 0.5, 1\}$.

Consider Figure 1. The base case simulations are performed for two alternative RNG seed values to establish what remains consistent and what does not in Figures 2 and 3.

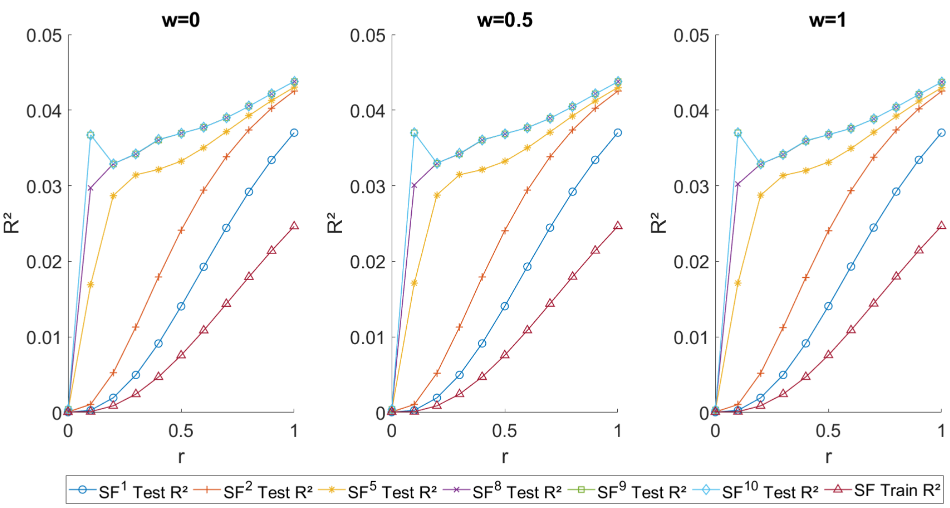


Figure 1: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.062$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. SF^k refers to the k-th order polynomial regression of the delta system-level costs on SF.

Consider the following implications of the results in figure 1.

First of all, the effect of different choices of the weight parameter is negligible. This remarkable consistency in predictive power is robust against different order polynomials.

Two potential explanations come to mind.

One reason could be that different choices of the weight parameter capture different equally relevant variations in the model parameters. This seems the most likely explanation.

However, another explanation could be that the simulated random networks that contribute to the predictive power of the spectral measure have a small probability of any semblance of a hierarchy of strong components.

In that case, the choice of weight parameter is irrelevant. However, as later results indicate, this is unlikely because later results show the relevance of this parameter.

Second, the performance of the polynomial increases with its order. After the ninth order, an increase in order does not alter the results. Importantly, the near identical test R^2 values for the polynomials of at least the ninth-order, clearly indicate a cut-off point in the complexity.

There is no reason to assume that the real-world predictive power does not further increase for more complex models due to the limited variation in the random network configurations. In all likelihood, the optimal amount of complexity depends to a significant degree on the variation of the configurations of strong components and the hierarchy of these components.

The effectiveness of higher-order polynomial regressions relative to a linear regression generalizes well to other parameter configurations. This is a strong indicator of a complex correlation between the spectral fragility measure and the delta system-level costs.

This complex correlation need not surprise us. After all, if the predictor and the dependent are both complex, their correlation is potentially complex as well. Intuitively, complex models should be better able to capture the complex interactions of strong components. This interaction is precisely what is captured by the hierarchy of strong components.

The choice of high-order polynomials as complex regression models need not be appropriate. Alternative, sophisticated methods may be more appropriate.

Note that the predictive power in terms of test R^2 is zero if $r = 0$.[21] This implies the presence of strong components cannot predict the delta system-level costs. After all, if $r = 0$ the eigenvalues that describe the fragility of the strong components do not affect the spectral fragility measure. In particular, $r = 0$ should only be relevant if the eigenvalues capture mainly noise, which is unlikely to hold for models that reflect real-world financial systems.

Consider the results of the base case in Figures 2 and 3, where

[21] An important nuance of the test R^2 is that it can be negative. This implies that the predictive power is worse than the average of the dependent. Thus, the test R^2 essentially compares the model's predictive power to a constant prediction, and an overfit can result in negative values.

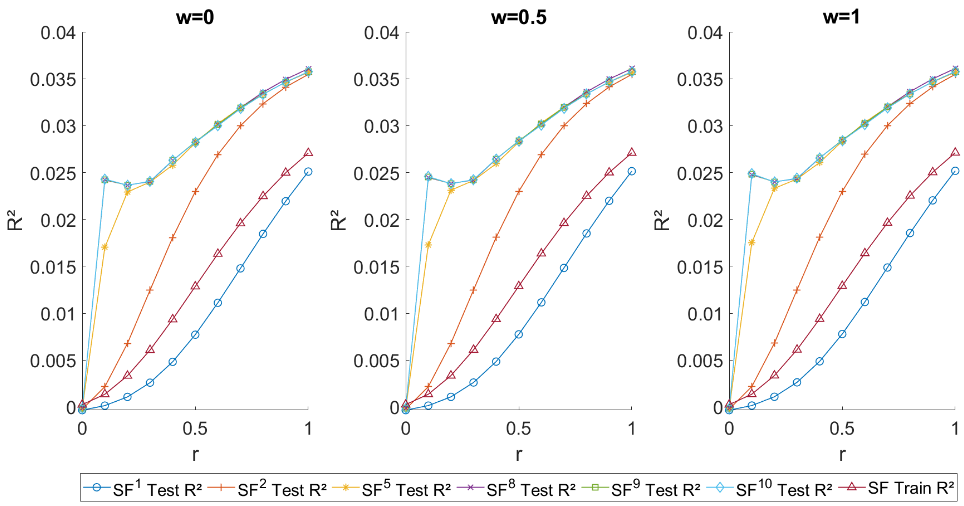


Figure 2: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.062$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 71. SF^k refers to the k-th order polynomial regression of the delta system-level costs on SF.

only the RNG seeds are altered.

Both these RNG seeds reduce the test R^2 in level, although the shape of the test R^2 values remains stable as a function of r.[22] Note that the dominance in the predictive power of the higher-order polynomials remains stable. The predictive power remains unaffected by the choice of the weight parameter $w \in \{0, 0.5, 1\}$.

Note that the local maximum for the lower range of r values for the fifth-order polynomial appears to move toward the left relative to the base case. This is likely due to the particular variance in strong components and the interaction with the hierarchy of strong components generated by these RNG seeds. If perchance these seeds result in financial networks where the noise is more prevalent for a slightly higher range of r values, the local maximum moves toward the left compared to the base case.

In all these examples, the predictive power vastly improves with the complexity of the model for low values of $r \in [0, 1]$. This implies that the complex correlation of the measure with the dependent is not just a consequence of the eigenvalues corresponding to a strong component in the measure. are there should be some compounding effect in the fraction of liabilities exposed to the credit risk via strong components. As the spectral fragility

[22]The choice of taking the RNG seed that maximizes the predictive power for the base case follows the assumption that these simulations perchance better capture the desired variation in the configurations of the strong components. Note that this is highly speculative.

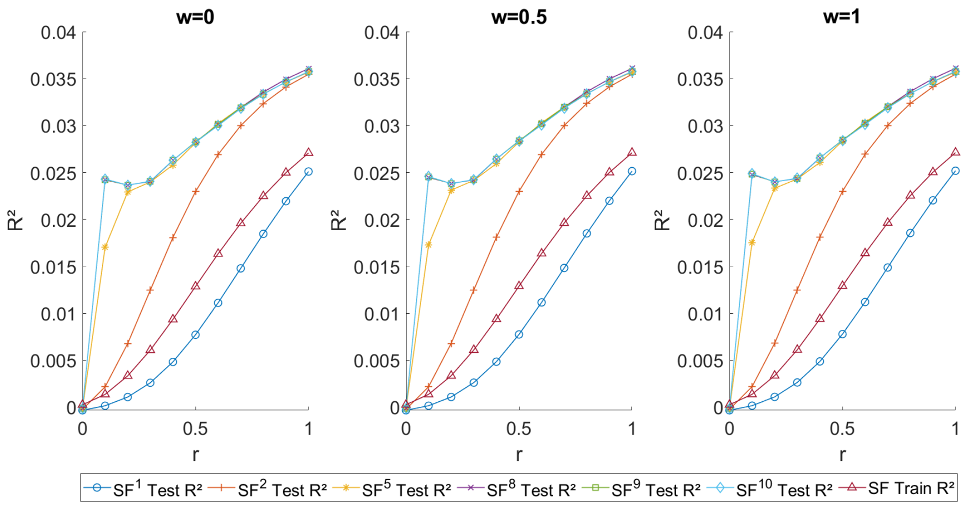


Figure 3: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.062$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 71. SF^k refers to the k-th order polynomial regression of the delta system-level costs on SF.

is an index in $[0, 1]$, the higher powers capture more nuanced variation in the spectral fragility which perhaps captures the effect of the

It is unlikely that low values of r, e.g., $r < 0.05$, are particularly relevant for realistic financial systems. To understand this argument, consider the following important example that considers both parameters of the spectral fragility measure.

Consider, one large strong component in a core-periphery structure of which the eigenvalues should tell a lot about the fragility of the entire network. A large fraction of total liabilities are affected by the interconnectivity of the core and its neighbors. Thus, ideally, r should be large to capture the connectivity of the strong component that reflects the fragility of the core to large shocks.

In such a framework the choice of the weight parameter should be particularly relevant.

On the one hand, the weight parameter should explain to what extent periphery banks are sensitive to credit risk originating in, or, before the core in the hierarchy of strong components.

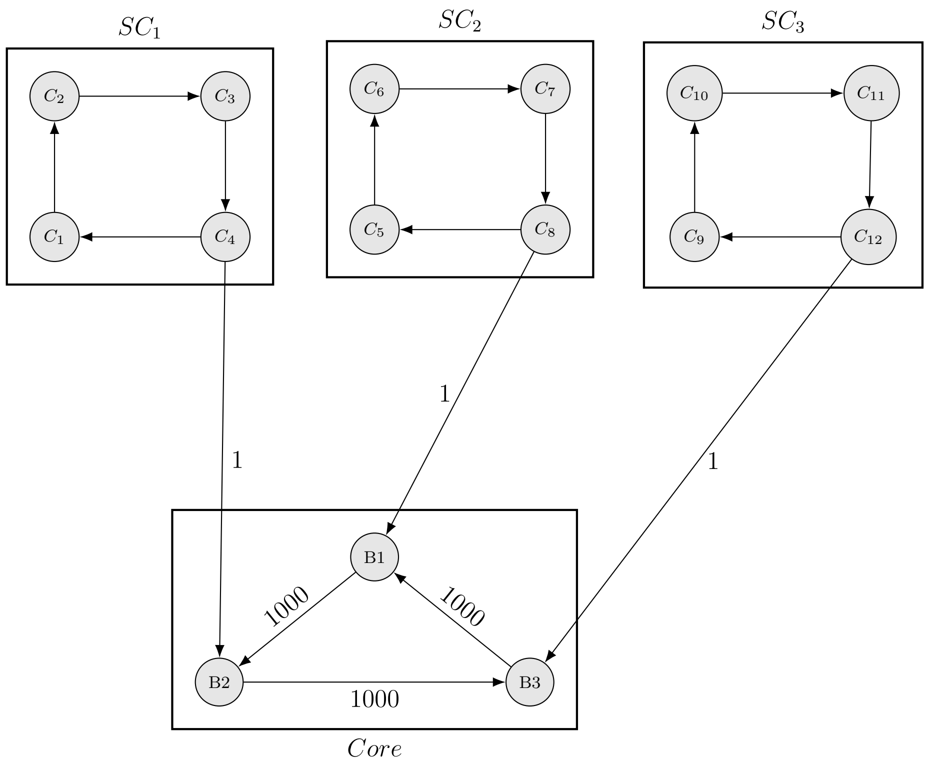
On the other hand, suppose that there are a lot of disjoint dependency cycles that hold direct liabilities toward the core of the financial system. If all these liabilities are particularly small, then all these dependency cycles could default without significantly affecting the core and thus the remainder of the network to a large extent. This implies that the optimal w should be small. If not, the spectral fragility measure is incredibly large due to the frequent occurrence of these large liabilities with high weights corresponding to these strong components.

Consider an example of such a core-periphery network configuration with 3 dependency cycles, each of 4 banks. Take, e.g., total liabilities of 1 per each small strong component above the core in the hierarchy of strong components. Assume the core itself is a cycle of 3 banks that each hold 1000 in nominal liabilities.

Each liability from the small cycles to the core cycle is towards a different bank. Take a weight of, e.g., $w = 0.9$. It is unlikely that the core of 3000 in liabilities is severely affected if 3 in total liabilities are not paid. For the sake of simplicity, first assume $r = 0$. Then, the 3 irrelevant strong components jointly contribute at least $\frac{3\times(0.9+0.9^2+0.9^3)\times 1000}{\iota^T l} = \frac{\times(0.9+0.9^2+0.9^3)\times 3000}{\iota^T l}$ to the spectral fragility measure. Suppose no arcs leave the core, then the core itself contributes only $\frac{3000}{\iota^T l}$ to the spectral fragility measure. This is problematic because this would indicate that this network is much more fragile compared to the same network without the 3 cycles. This is highly misleading. Preferably, the network with the dependency cycles should have only a slightly larger spectral fragility measure.

If $r \neq 0$ it is possible that the proportional liabilities that leave the cycles before the core are sufficiently small to vastly reduce the problematic interpretation of the high weight for the spectral fragility measure. The weights may be preferably high for the banks that receive liability payments from the core. A simple and perhaps naive approach could be to have small weights before and high weights after the core in the hierarchy of strong components.

This example is represented in the following digraph. The nodes in the cycles are labeled $C_1, .., C_{12}$. The banks in the core are labeled B_1, B_2, B_3.

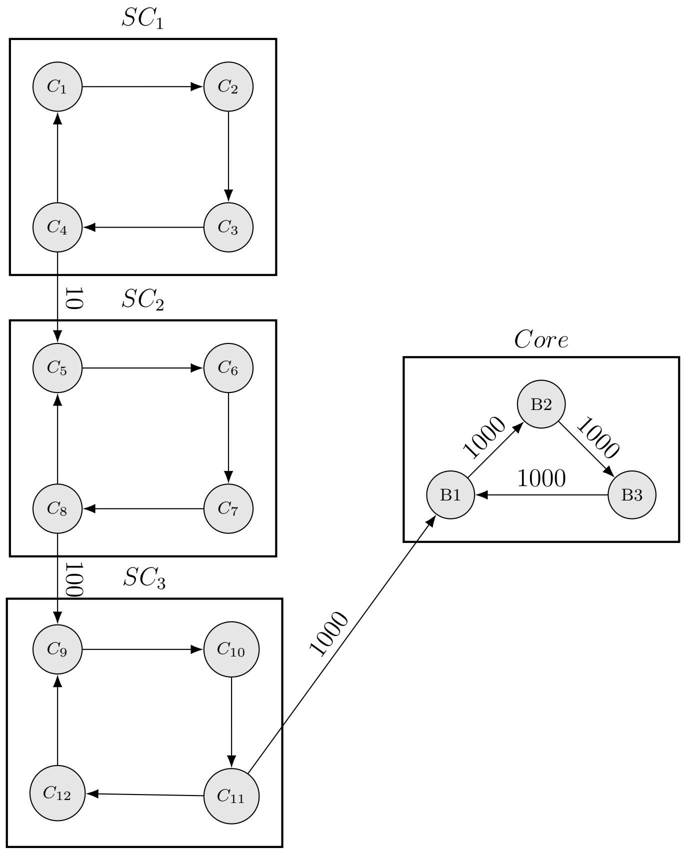


It is possible to come up with stylized examples that motivate that the r parameter need not be 1 in a core-periphery network to optimize the predictive power of the spectral fragility measure.

For instance, suppose there are a large number of smaller strong components that occur sequentially on a path of liabilities toward the core. Due to decentralized clearing, the self-fulfilling default cycles in these smaller strong components could result in a large level of system-level costs for the least clearing payments. These high system-level costs could follow the amplification of a small shock through the sequential strong components that, upon reaching the largest strong component, behave as a large systematic shock to the core of the network.

Suppose these strong components before the core in the hierarchy are, e.g., large circular components with only one large bank with large liabilities that leave the cycle. As a consequence, the contribution of the fragility of these strong components to the overall fragility of the network is not captured by the eigenvalues of these smaller cycles.

An example of such a core-periphery network configuration is the following. Consider 3 dependency cycles of 4 banks sequential in the hierarchy of strong components and suppose only the final component in this sequence reaches the core. The liabilities that leave the dependency cycles are in order 10, 100, and 1000, to allow for the amplification of a small shock in the first dependency cycle to a large systematic shock to the core. This is represented in the following graph.



Based on the arguments above, it is likely preferable to use existing knowledge on the shape of financial systems to assign distinctive parameters to individual strong components.

The lack of accurate data on financial institutions implies that it may not be possible or desirable to optimize these parameters via data science methods.

It is possible to assign personal parameters to different strong components by adding simple heuristics to the algorithm that obtains the spectral fragility measure. These heuristics should preferably be determined by those knowledgeable on the shape of financial systems and potentially improved by data science methods. This adjustment to the algorithm should not significantly affect the computational efficiency of obtaining the measure.[23]

[23]This computational efficiency can be decreased entirely by choosing infinitely complex heuristics. The question arises as to whether such complex heuristics are beneficial. If these heuristics are too complex, setting up regulation based on this measure may become infeasible.

8.3 Katz Bonacich Spectral Centrality Measure

Acemoglu et al. (2015) find that traditional eigenvector-based centrality measures such as the Fiedler vector (Fiedler, 1973; Montenegro and Tetali, 2006) and the Bonacich centrality measure (see, e.g., Glasserman and Young (2016)), are less able to capture non-symmetric relationships among banks present in financial systems. To fix this problem, Acemoglu et al. (2015) propose the harmonic distance to measure the position of banks to a distressed bank. The harmonic distance from bank $i \in N$ to $j \in N$ is the average number of arcs on a path from i to j. The total harmonic distance from any bank $i \in N$ to all other banks does not depend on the bank. It is rather a property of the entire network, just like spectral fragility.

In the following, established spectral measures are adjusted to fit the current model. This is done briefly such that in the next section linear regression can be performed to compare the explanatory power of the individual measures and the explanatory power for combinations of these measures.

The following explanation can be found in, e.g., Katz (1953) and Puhr et al. (2012).

The Katz-Bonacich centrality determines how central banks are to the network in terms of liabilities toward or from other banks. This is a variation of the Bonacich centrality where the more direct links are more important than the indirect links. This is established via a discount factor factor w. Consider the following motivation.

The geometric sum $I + A + A^2 + ...$ may converge, depending on the eigenvalues of A. In particular, $I + A + A^2 + ... = (I - A)^{-1}$ if $\rho(A) < 1$. This is not guaranteed for A, in particular, if A is row stochastic.

In addition, one may wish to discount the effect of A^k for higher powers k. That is, the direct liabilities between banks should weigh much heavier than the indirect path of 10 liabilities between two banks.

To ensure both convergence of the geometric sum and the reduced importance of indirect liabilities, Katz (1953) chooses a discount factor $w \in (0, 1)$ such that $w\rho(A) < 1$. This ensures that $I + wA + (wA)^2 + ... = (I - wA)^{-1}$ and the Katz vector $\varsigma = (I - wA)^{-1}\iota$ exists. Each element in this vector is indicative of how central a bank is in the network in terms of proportional liabilities.

The *Katz-Bonacich Borrower Centrality* is a function, $KBBC$: $\mathbb{R}_+^{n\times n} \mapsto \mathbb{R}$, defined by

$$KBBC(L) = \iota^T(I - wA)^{-1}l - \iota^T l. \tag{28}$$

The regression of the delta-system level costs on the Katz-Bonacich borrower centrality consists of a linear and fifth-order polynomial form.

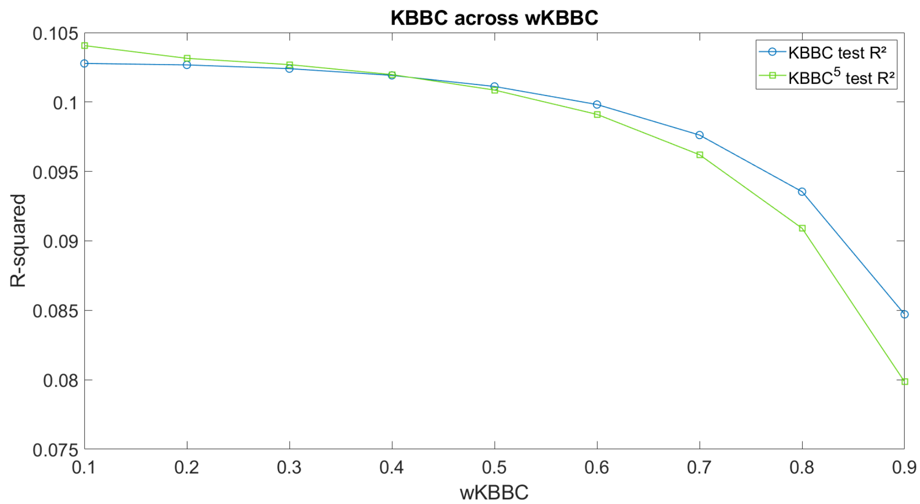


Figure 4: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.62$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. $KBBC^k$ refers to the k-*th* order polynomial regression of the delta system-level costs on $KBBC$.

One notable aspect of these simulation results is the high test R^2 values. This high level of predictive power generalizes well to different parameter configurations.

There can be several reasons for these high test R^2 values. On the one hand, this may be a natural consequence of oversimplified simulation methods. In light of poor empirical performance in literature and theoretical arguments at the individual bank level from literature, this explanation seems plausible.

One remarkable implication of these results is that the performance of this measure is maximized for an incredibly small weight parameter.[24] This supports earlier observations in the literature on financial networks (Acemoglu et al., 2015) that direct liability relations are far more important in explaining contagion. To understand why, consider the following.

[24] The measure is zero for $w = 0$ and after refinement of the interval several times it was not possible to determine for what w values the test R^2 jumps from 0 to over 0.1. This implies that the w that optimizes the test R^2 for this measure is exceptionally small.

Note, $\frac{w}{w^2} = \frac{1}{w} \underset{w \downarrow 0}{\to} \infty$. Thus, the predictive power of the measure is optimized in case the direct liabilities far outweigh the closest indirect liability relations. That is, the weights of liability paths of at least 2 steps in the network should optimally be infinitesimal relative to the direct liability weights.

This result motivates controlling for alternative *RNG* seeds. Consider the results in Figure 6 for the *RNG* seed set to 70.

Note that the fifth-order polynomial performs similarly to the linear regression. This is likely due to the linear nature of the Katz-Bonacich centrality measure. This could explain the poor empirical performance of these measures. If the predictive power of this measure is optimal for linear regression models, it is unlikely that this predictive power translates well to real-world financial systems.

Based on the arguments in this section, the likely reason for the superior predictive power of this measure is that its simple nature better suits the simple simulation setup. Although not included, parameter setups that allow for very little variation in financial networks typically improve the performance of this measure.

8.4 Controlling for the System-Level Costs as Dependent

The spectral measures can predict the system-level costs for the least clearing payments as well. In particular, the parameter choices determine to a large extent whether the largest and the smallest system-level costs are correlated. Furthermore, the size and sign of the correlation depend on the parameter choice as well.

Consider the configuration where the liabilities are on average large relative to the operational cash flow of banks and the probability of interbank liabilities is much smaller in Figure

.

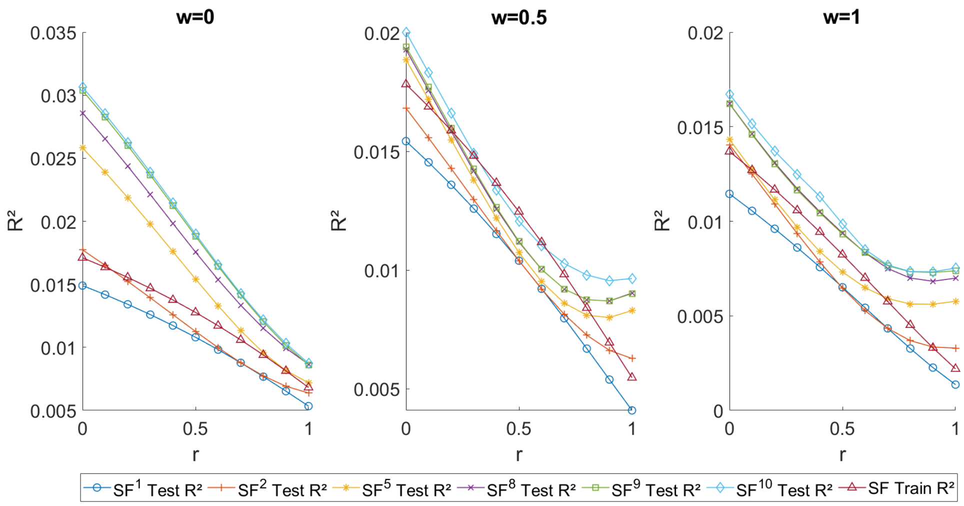


Figure 5: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.02$, $z = 1$, $k = 20$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. SF^k refers to the k-th order polynomial regression of the delta system-level costs on SF.

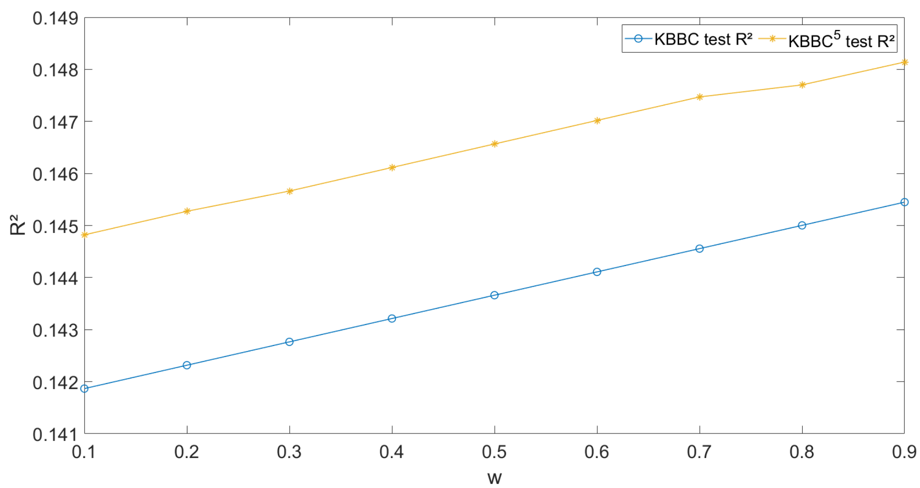


Figure 6: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.02$, $z = 1$, $k = 20$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. $KBBC^k$ refers to the k-th order polynomial regression of the delta system-level costs on $KBBC$.

The predictive power is maximized for $r = 0$. This result is somewhat misleading.

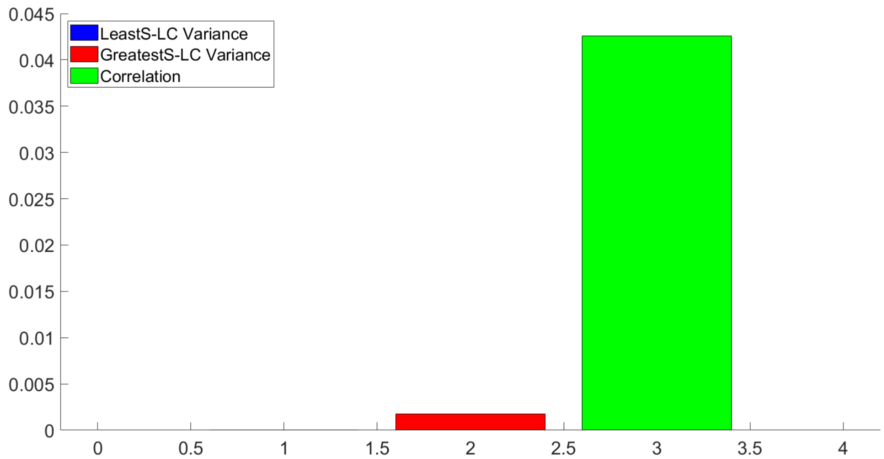


Figure 7: The variance of the system-level costs in case of the least and greatest clearing payments and the correlation of these costs for the simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.02$, $z = 1$, $k = 20$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69.

Consider the following. In this scenario, these results are determined entirely, within a precision of 5 decimal places, by the variation of the system-level costs in case of the greatest clearing payments. This is represented in Figure 7.

Intuitively, the Katz-Bonacich measure is better suited to predict the system-level costs because the non-linearities induced by default costs are less pronounced. In that sense, if there is a significant gap in the least and greatest clearing payments, the system-level costs behave more linearly than the system-level costs for the least clearing payments.

Thus, the larger the relative size of the variance of the system-level costs relative to the variance of the system-level costs for the least clearing payments, the larger the prediction power held by the Katz-Bonacich measure for the delta system-level costs.

This indicates that the arguments against linear centrality measures hold much more credence for the analysis of the least clearing payments.

The infinitesimal variance of the system-level costs for the least clearing payments is likely in part due to the frequency of interbank liabilities being so small that the fragility as captured by the eigenvalues generally captures a lot of noise.

If there even is a significant variation in the fragility of strong components, the rarity implies that the presence of these components can explain more than their configurations.

As a consequence, although it is unlikely for financial systems, if the r parameter is optimally set to 0, this is indicative of a relatively linear network configuration.

This is due to the sharp increase in the average liabilities and vastly reduced average operational cash flow. This results in the maximum system-level costs

In addition, the likelihood of small strong components implies that any liabilities that leave the strong component are relatively large compared to the total aggregate liabilities of the strong component.

In addition, the unlikely presence of banks that receive liabilities from the strong component may result these liabilities capturing noise.

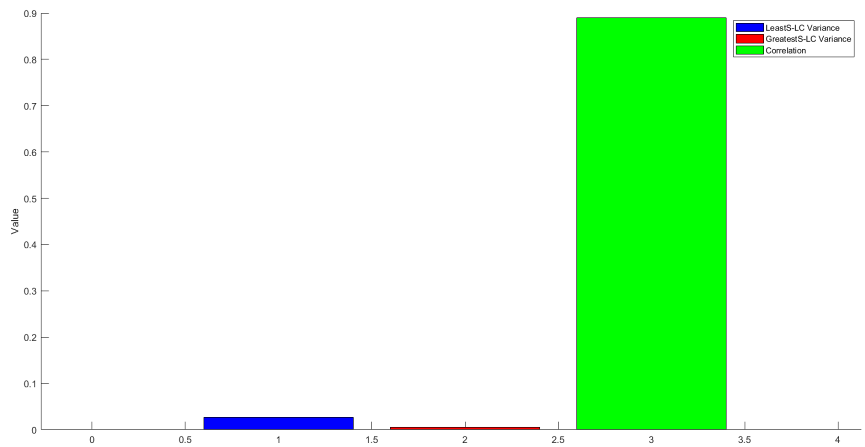


Figure 8: The variance of the system-level costs in case of the least and greatest clearing payments and the correlation of these costs for the simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.62$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69.

The complex models are again better able to predict for all cases. Thus, even if only the presence of strong components is particularly relevant relative to the fragility, the complex correlation with the dependent variable is present as well.

This is clear from the similar shapes in the test R^2 values as functions of r and the lack of variability in the w parameter.

The delta system-level costs instead of the system-level costs for the least clearing payments, the regressions are repeated for the base case with the latter as the dependent. Consider Figures 9 and 10 with the spectral fragility and Katz-Bonacich measures as the predictors, respectively.

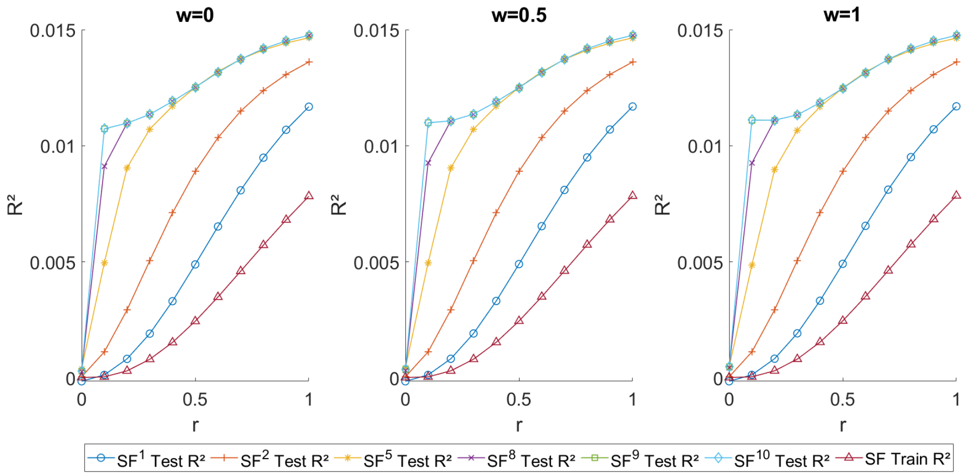


Figure 9: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.62$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. SF^k refers to the k-th order polynomial regression of the system-level costs for the least clearing payments on SF.

These results support the theoretical arguments that the spectral fragility measure primarily predicts the delta system-level costs.

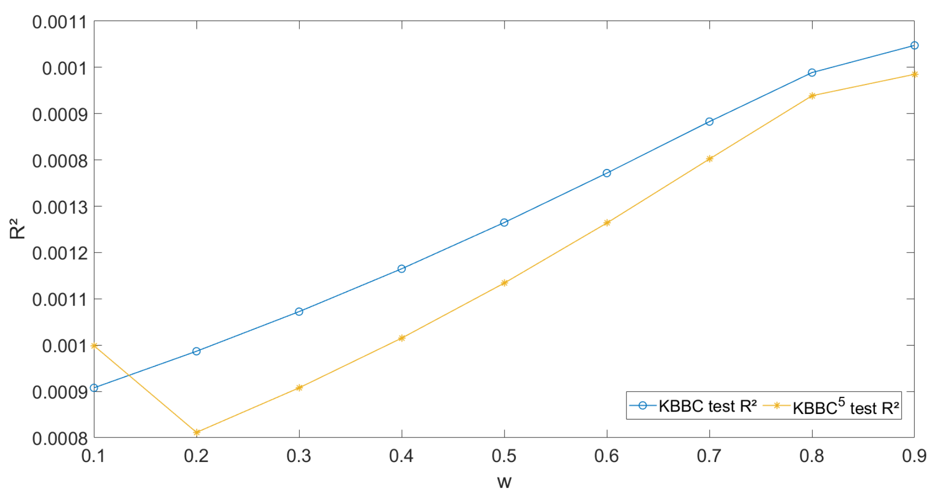


Figure 10: Simulation results with parameters: $m = 10000$, $n = 100$, $b = 0.62$, $z = 10$, $k = 10$, $\alpha = 0.5$, $\beta = 0.75$, RNG seed 69. $KBBC^k$ refers to the k-th order polynomial regression of the system-level costs for the least clearing payments on $KBBC$.

9 Conclusions

The spectral fragility measure is a new spectral measure based on theoretical results on the interconnectivity of strong components. This measure is based on the principle that highly interconnected financial networks are fragile to large shocks.

The theoretical building blocks of this spectral measure indicate its efficacy is best captured for a network with a large variance in strong component configurations and sizes.

As predicted, the simulation method that considers banks with independent and identically distributed random liabilities is likely unable to capture the predictive power of this measure. In particular, the sparsity of financial networks and the heterogeneous roles of different financial institutions in these networks require a large number of banks with a low average probability of interbank liabilities. As the number of banks increases, the required number of simulations to effectively predict the difference in the system-level costs for the least and the greatest clearing payments increases rapidly. Future research with a focus on simulation methods can greatly increase efficiency via the creation of a large number of random network distributions that each focus on different network shapes.

This approach allows for efficient simulation without fixing the outcome beforehand. Note that this is a complex task.

One important reason for this complexity is the exponential increase in scenarios to consider when controlling for different parameter values. That is, the consideration of one particular model with one alternative value for 5 parameters results in $2^5 = 32$ combinations.

This problem can be mitigated to some extent. Some of these combinations lead to similar results because the interaction of parameter values is far more important than the size of any particular coefficient if the focus is on theoretical modeling. For example, the effect of an increase in the average operational cash flow can have a similar effect as a decrease in the average nominal liability. Furthermore, the effect of an increase in the likelihood of liabilities between any two banks is likely mitigated if the number of banks increases as well. The importance of such a preliminary analysis should not be underestimated.

The literature on financial networks considers special cases of financial network structures to arrive at conclusions on systemic risk. These special cases enable us to understand different dimensions of systemic risk in isolation. In turn, these conclusions allow others to hypothesize and control for the interaction of these dimensions.

However, simple assumptions are unlikely able to capture the complexity of these financial networks.

One particular observation is that the Katz-Bonacich measure predicts a lot better in case of the variance of the system is incredibly small. In general, this measure is able to achieve more test R^2 because it is able to capture more variance in the system-level costs for the least clearing payments.

This is a natural consequence because the measure does not require the existence of non-trivial strong components to predict system-level costs.

To predict the delta system-level costs, this should be considered as overfitting the predictive power of the delta-system level costs. This is the main reason that even only a change in RNG can change the optimal weight parameter from one side of the interval toward the other side.

The incredible sensitivity of the optimal weight parameter for slight changes in the data under these incredibly simple assumptions reinforces the ineffectiveness of standard spectral measures that are linear.

Although the ninth-order polynomial performed the best, the effectiveness for the relevant parameter values in the base case indicates that a second-order polynomial is effective as well.

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A Details on Theoretical Results

A.1 Alternative Proof of Theorem 1

The following proof is faster for those familiar with the literature or particular knowledge of the underlying theory.

The following definitions are found in Gierz et al. (1980) and are simplified for this thesis.

A partially ordered set $(L, \leq)$ is called a lattice if any two elements in L have a supremum and infimum in L. That is, $\forall \{a, b\} \subseteq L : \{a \wedge b, a \vee b\} \subseteq L$. Note, $\vee$ is the maximum counterpart of $\wedge$. That is, this operator finds the maximum between the real numbers beside it, i.e., $a \vee b \equiv \max\{a, b\}$, for $a, b \in \mathbb{R}$. A complete lattice is a lattice in which any subset has a unique infimum and supremum inside the set. Importantly, 'any subset' implies any sequence contained in L must have the limit in L as well. The proof could be reduced to the following. Any line segment $[a, b] \subseteq \mathbb{R}^n$ is a complete lattice for the coordinate-wise $\leq$ operator because the set is a closed ball in $\mathbb{R}^n$. A set in $\mathbb{R}^n$ is a closed ball if and only if all sequences within the set have their limit within the set. This ensures that infima and suprema of any subset are contained within the set. In contrast, $(a, b) \subseteq \mathbb{R}^n$ can contain sequences with the limits being potentially a or b. Take, e.g., $a - \iota * \frac{1}{k} \xrightarrow[k \to \infty]{} a \notin (a,\ b)$. This implies a is the infimum and thus the greatest under bound is not contained within the set.

This does not mean $((a,\ b),\ \ \leq)$ is not a lattice. We can compare each v_i and u_i for any two vectors $u,\ v \in (a,\ b)$ in at most n steps. Each v_i and u_i are contained within (a_i, b_i). It must thus hold that the resulting infimum and supremum are contained within $(a,\ b)$. Tarski's fixed point implies that the set of fixed points of a monotone transformation from a complete lattice to the same complete lattice is itself a complete lattice, and thus there is a unique greatest and least fixed point (Zeidler, 1986) within the set of clearing vectors. That is, $\Phi([0, l]) \subseteq [0, l]$ and Φ is monotone by lemma 1.

Note, for any two elements $u,\ v \in [a, b] \subseteq \mathbb{R}^n$ it is not always required to satisfy $u \leq v$ or $u \geq v$. Thus, for the other complete lattice, the set of clearing vectors, it need not hold that one is necessarily weakly larger than the other. This possibility does require the set of clearing vectors to contain at least three elements because it is established that $p_* \leq p^*$.

A.2 Contraction Theorem

Rudin (1976) defines metric spaces and explains that the space of real vectors is an example. The distance of payment vectors is defined slightly differently than typical distances in $\mathbb{R}^n$. It is therefore useful to acknowledge what changes and what does not when switching norms in $\mathbb{R}^n$. Rudin (1976) states the contraction Theorem which provides a uniqueness condition for fixed points in some metric spaces. All the following definitions are provided by Rudin (1976). In particular, A set X, whose elements we shall call points, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, such that

(i) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

(ii) $d(p, q) = d(q, p)$;

(iii) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a distance function, or a metric. The most important examples of metric spaces, from our standpoint, are the Euclidean spaces $\mathbb{R}^k$, especially $\mathbb{R}^1$ (the real line) and $\mathbb{R}^2$ (the real plane); the distance in $\mathbb{R}^k$ is defined by

$$d(x, y) = ||x - y||, \;\; x, y \in \mathbb{R}^k. \tag{29}$$

Typically, the distance in R^n is chosen to be the L^2 norm. The Euclidean norm (or L^2 norm) of a vector $\mathbf{x}$ in $\mathbb{R}^n$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, is defined as:

$$||\mathbf{x}||_2 = \left(\sum_{u=1}^{n} x_i^2 \right)^{1/2}, \tag{30}$$

where x_i represents the i-th component of the vector $\mathbf{x}$. The Euclidean norm measures the "length" of the vector $\mathbf{x}$ in the Euclidean space $\mathbb{R}^n$. Given a vector $\mathbf{x} \in \mathbb{R}^n$, the L^1 norm of $\mathbf{x}$, denoted as $||\mathbf{x}||_1$, is defined as the sum of the absolute values of its components. Formally, this can be expressed as:

$$||\mathbf{x}||_1 = \sum_{u=1}^{n} |x_i|, \tag{31}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and x_i represents the i-th component of the vector $\mathbf{x}$ (Eisenberg and Noe, 2001). The L^1 norm is used when the vectors are payments because the cumulative absolute distance between payments is relevant instead of the geometric norm, i.e., the square root of the sum of squared payments. What does the length from in the origin in a two-dimensional graph to the point which is $p_1 > 0$ to the right and $p_2 > 0$ above the origin? The importance lies in the size of p_1 and p_2.

Definition: A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Definition: A metric space in which every Cauchy sequence converges is said to be complete.

It is also noteworthy that all compact metric spaces and all Euclidean spaces are complete.

Definition 10 (Contraction) Let (M, d) be a metric space. A function $T : M \to M$ is called a *contraction* if there exists a constant $0 < k < 1$ such that for all $x, y \in M$, the inequality

$$d(T(x),\ T(y)) \leq k \cdot d(x, y)$$

holds. The constant k is known as the *contraction constant*.

Theorem 4 (Contraction Mapping Theorem) Let (X, d) be a complete metric space. Suppose $T : X \to X$ is a contraction mapping on X, that is, there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(T(x),\ T(y)) \leq k \cdot d(x, y).$$

Then, the following assertions hold:

1. There exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.
2. For any $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^*.

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$, the triangle inequality for the L^1 norm is expressed as:

$$||\mathbf{a} + \mathbf{b}||_1 \leq ||\mathbf{a}||_1 + ||\mathbf{b}||_1. \tag{32}$$

This is thus property (iii) of the distance function.

From here on out it is assumed that $||x||$ denotes the L^1 norm of $x \in \mathbb{R}^n$. Let $A_{j\cdot}$ denote row j of matrix A. One important property of the norm for a row-stochastic matrix A is the following. $||A^T x|| = ||\sum_{u=1}^n x_i A_{i\cdot}|| \leq \sum_{u=1}^n ||x_i A_{i\cdot}|| = \sum_{u=1}^n ||x_i|| ||A_{i\cdot}|| = \sum_{u=1}^n ||x_i|| = ||x||$ because A is row-stochastic. This property need not hold for L^2 norms which is an important distinction.

This allows for the following prove of uniqueness as long as $\beta > 0$.

Let $\otimes$ denote the Hadamard, or piece-wise, vector product. Then, the solution in step 3 for the least algorithm can be written as $I_{U_\mu} x = I_{U_\mu}[(A^T I_{S_\mu} l + A^T I_{U_\mu} x) \otimes (\iota - b) - I_{U\mu} a] \vee 0$. Thus, for $\hat{x} \neq \tilde{x}$: $||I_{U_\mu}([(A^T I_{S_\mu} l + A^T \hat{x}) \otimes I_{U_\mu}(\iota - b) - I_{U\mu} a] \vee 0 - [(A^T I_{S_\mu} l + A^T I_{U_\mu} \tilde{x}) \otimes I_{U_\mu}(\iota - b) - I_{U\mu} a] \vee 0)|| \leq ||I_{U_\mu}[(A^T I_{S_\mu}(l - l) + A^T I_{U\mu}(\hat{x} - \tilde{x})) \otimes (\iota - b) - I_{U\mu}(a - a)]|| = ||I_{U_\mu} A^T (\hat{x} - \tilde{x}) \otimes (\iota - b)|| < ||I_{U_\mu} A^T I_{U_\mu}(\hat{x} - \tilde{x}) \otimes (\iota - 0)|| = ||I_{U_\mu} A^T I_{U_\mu}(\hat{x} - \tilde{x}) \otimes I_{U_\mu}(\iota - b)|| \leq ||I_{U_\mu} A^T I_{U_\mu}(\hat{x} - \tilde{x}) \otimes (\iota - 0)|| \leq ||I A^T I(\hat{x} - \tilde{x})|| \leq ||A^T(\hat{x} - \tilde{x})|| \leq ||\hat{x} - \tilde{x}||$. The right-hand side is thus a contraction of x. By the contraction Theorem, the solutions must be unique. The proof of uniqueness for the greatest algorithm if $\beta > 0$ is identical and thus omitted.

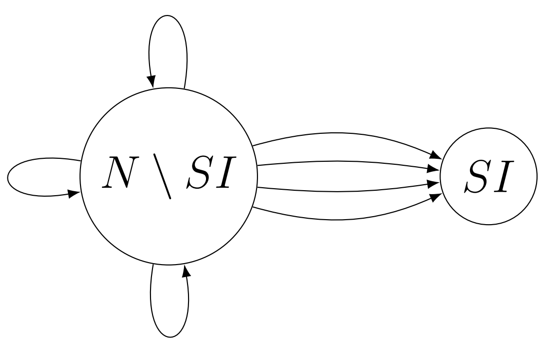
If $B \in \mathbb{R}^{n \times n}$ is column sub-stochastic and at least one column is strictly sub-stochastic and $x \in \mathbb{R}^n_+$, then $||Bx|| = \sum_{u=1}^n |(Bx)_i| = \sum_{u=1}^n \sum_{j=1}^n B_{ij} x_j = \sum_{j=1}^n x_j \sum_{u=1}^n B_{ij} < \sum_{j=1}^n x_j$ because at least one column of BV does not sum up to 1 (Karlin, 1959; Eisenberg and Noe, 2001).

Note, $\beta > 0$ is a mild regularity requirement that should realistically hold. However, the result holds as well for $b = 0$ if just one column of $I_{U_\mu} A^T$ sums up to less than 1. If $a = 0$ and $\forall i \in N \exists j \in o(i) : c_j > 0$, then (ii) and (xi), and thus (iii) in Proposition 1 hold. This implies both the uniqueness of the clearing vector and the values x_i in step 3. By (ii) in Proposition 1, there is a solvent bank reachable from any defaulting node in step 3 of the algorithm. This implies that at least one column in $I_{U_\mu} A^T$ sums up to less than 1 because it is not possible to have both $\exists j \in o(i) : c_j > 0$ and $\forall s \in o(i) : e_s = 0$ (Eisenberg and Noe, 2001). It could seem odd that $a = 0$ is required because the a values cancel as seen in the proof of uniqueness for $\beta > 0$. However, $e = \iota^T c$ must hold for all clearing vectors to establish uniqueness. Constant equity need not hold if $a \gneq 0$.

These fundamental results are often omitted in the present for these types of financial network models because it is assumed to be standard knowledge that is proven in earlier work. For this reason, this appendix acts as a shortcut to understanding these results.

A.3 No Dependency Cycle

The following result is used, though not motivated, in a relevant proof by Jackson and Pernoud (2020). If there are no dependency cycles, there must be a subset of nodes X_0 which hold no nominal liability assets. To prove this result, assume there is no dependency cycle, and suppose for all nodes in N there is at least one bank that owes them nominal liabilities. This implies we can subdivide the nodes into two sets, one, denoted SI, which contains sink nodes that do not hold nominal liabilities, and one, possibly empty set, $N \setminus SI$, which contains nodes that receive and pay nominal liabilities. This can be represented in the following sketch of a digraph.



The amount of arcs that point towards a node $i \in N$ is referred to as the in-degree of node i. Let the number of nodes in the complement of the sink component be denoted $t = |N \setminus SI|$. Then, because there are t receivers in the set, we have t nodes connected by at least t directed arcs of which each must be a recipient. That is, each node has the in-degree of at least 1. The following fundamental property from undirected graph theory is useful. This property states that a graph with i nodes can only be acyclic if it contains at most $u-1$ arcs. Thus, a graph that contains at least i arcs for i nodes must immediately result in at least one undirected cycle. This result itself only tells us that there is at least a cycle without mandating for the direction of the arcs to align to form a directed cycle. In the case of t arcs, the fact that each node has in-degree at least 1 in the same set ensures that an equivalent directed cycle must hold for the $N \setminus SI$ set. Note, it is always possible to obtain a digraph of t nodes with in-degree 1 from any digraph where there are more than t arcs for t nodes with in-degree at least one 1 by only removing arcs if the receiving node has an in-degree larger than 1.

A.4 Proof of Proposition 3

Proof.

$$A_{SC} = \left[\begin{array}{cccc|c|cccc}
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 & 0 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 1 & 0 & \ddots & 0 & 0 \\
\hline
x & 0 & \cdots & 0 & 0 & 1-x & 0 & \ddots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0
\end{array}\right].$$

$$\text{Then } \det(\lambda I - A_{SC}) = \left|\begin{array}{cccc|c|cccc}
\lambda & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \lambda & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & -1 & 0 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \lambda & -1 & 0 & \ddots & 0 & 0 \\
\hline
-x & 0 & \cdots & 0 & \lambda & -(1-x) & 0 & \ddots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & \lambda & -1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & \lambda & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & -1 \\
0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & \lambda
\end{array}\right| =$$

$$
\left|\begin{array}{cccc|c|cccc}
\lambda & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\lambda^2 & 0 & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & -1 & 0 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & -1 & 0 & \ddots & 0 & 0 \\
\hline
\lambda^k - x & 0 & \cdots & 0 & 0 & -(1-x) & 0 & \ddots & 0 \\
\hline
\lambda\frac{\lambda^k-x}{1-x} & 0 & \cdots & 0 & 0 & \lambda & -1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & \lambda & \ddots & \vdots \\
\lambda^{k+u-2}\frac{\lambda^k-x}{1-x} & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & -1 \\
\lambda^{k+u-1}\frac{\lambda^k-x}{1-x} - \lambda^{k-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right|
$$

$= (-1)^{2k+u}(-1)^{2k-3+u}(-(1-x))\lambda^{k-1}\left[\frac{(\lambda^k-x)}{1-x}\lambda^u - 1\right] = \lambda^{k-1}[(\lambda^k - x)\lambda^u - (1 - x)]$. Thus, there are $k-1$ eigenvalues 0 due to the term λ^{k-1}, there is an eigenvalue 1 because the matrix is stochastic, and all other eigenvalues are either complex pairs or complex pairs plus a negative value.

The first step is to prove that 1 is the only positive eigenvalue. By the Perron-Frobenius Theorem, there is only one root 1. Thus it is possible to write the characteristic polynomial as $\det(\lambda I - A) = (\lambda - 1)(\cdots)$ but not $\det(\lambda I - A) = (\lambda - 1)^m(\cdots)$ for $m \in \mathbb{N} \setminus \{1\}$. Thus, if there are more positive roots, this implies $\lambda \in (0, 1)$. This implies $(\lambda^k - x)\lambda^u - (1 - x) < (1 - x) \times 1 - (1 - x) = 0$. Thus, the only positive eigenvalue is 1. If there are other real eigenvalues, these must be negative or zero. This is unsurprising. There is no other positive root than the spectral radius for an isolated cycle as demonstrated in the proof of Theorem 3. Thus, there is no reason for the presence of either cycle to create a new positive eigenvalue or push it down from the spectral radius.

Specifically, if k and u are even, then $\exists v, y \in \mathbb{N} : (\lambda^k - x)\lambda^u - (1 - x) = ((\lambda^2)^v - x)\lambda^{2y} - (1 - x) = 0$ for both $\lambda = 1$ and $\lambda = -1$. Note, for a circular graph with period $d = 2v$ for $v \in \mathbb{N}$, $\lambda = -1$ is a root as well. If u and k are both odd then $u + k$ is still even. This results in a negative root as well, however, now the negative root is not on but within the unit circle. Note, $\exists v \in \mathbb{N} : (\lambda^2)^v - \lambda^u x - (1-x) = (\lambda^2)^v + |\lambda|^u x - (1-x) = |\lambda|^{k+u} + |\lambda|^u x - (1-x)$. Note, if $|\lambda| = 0$ then $|\lambda|^{k+u} + |\lambda|^u x - (1-x) = -(1-x)$ whereas $|\lambda| = 1$ implies $|\lambda|^{k+u} + |\lambda|^u x - (1-x) = 1 - 1 + 2x = 2x > 0$. Thus by the intermediate value theorem $\exists |\lambda| \in (0, 1)$ such that $|\lambda|^{k+u} + |\lambda|^u x - (1-x) = 0$. Thus $\exists \lambda \in (-1, 0)$ such that $\det(\lambda I - A) = 0$. This is the only negative root. Suppose not , note that $|\lambda|^{k+u} + |\lambda|^u x - (1 - x)$ is strictly monotone in $\lambda \in [-1, 0]$. This provides the necessary contradiction.

Let $k + u$ be odd. Apart from 1 and 0, there are only complex roots. Suppose not and consider first the case where k is odd such that u must be even. If $\lambda \in [-1, 0)$. Then $(\lambda^k - x)\lambda^u - (1 - x) = (\lambda(\lambda^2)^v - x)\lambda^{2y} - (1 - x) < 0$ for $v, y \in \mathbb{N}$. Consider the case where k is even and thus u must be odd. Then

$$(\lambda^k - x)\lambda^u - (1-x) = \left((\lambda^2)^v - x\right)(\lambda^2)^y \lambda - (1-x) < 0. \qquad \square$$

B Implementation of The Spectral Fragility Measure

This section outlines how the spectral fragility of a financial network can be calculated efficiently. This section first outlines an important algorithm to find strong components in directed graphs. Other algorithms to find general strong components are available by Csóka and Herings (2024) and for core-periphery networks by Jackson and Pernoud (2023)» CHECK.

B.1 Time Complexity

To understand the computational efficiency of the spectral fragility measure it is important to quantify this efficiency.

Let t(n) denote the number of steps in the algorithm, where n is the number of banks. Then $t(n) = \mathcal{O}(g(n))$ if $\exists c, n_0 > 0 : t(n) \leq cg(n)$ for all $n \geq n_0$. $\mathcal{O}(g(n))$ is the *time complexity* of the spectral fragility algorithm. Once we have such a function $g(n)$, the number of steps in the algorithm cannot increase faster than linear in the function $g(n)$.

The first step in the algorithm consists of summing each row of L. This has time complexity $\mathcal{O}(n)$. Calculating l then has the same time complexity.

The second step is to divide each non-zero row of L by the row sums to obtain A. This has time complexity $\mathcal{O}(n)$ as well.

The next step is to apply Tarjan's algorithm to find strong components, this has time complexity $\mathcal{O}(n + |E[A]|)$ which is not worse than $\mathcal{O}(n^2)$ (Tarjan, 1972).

B.2 Worst Case Time Complexity

The following concepts closely follow the work by Tarjan (1972).

A stack is an arbitrary datatype with two operations. The push operation adds an element and the pop operation removes the latest element.

The Tarjan algorithm performs a depth-first search approach that starts at an arbitrary node and follows a directed path to nodes not in the stack and saves the earliest reachable node in the stack at each step.[25]

Once all arcs to nodes not in the stack have been exhausted, nodes are moved from the stack to a strong component array until the earliest reachable node in the stack is the final node in the stack. Then, the strong component is complete.

[25]For simplicity, the algorithm starts at node 1.

For this thesis, the algorithm is adjusted to distinguish between sink and non-sink trivial strong components. This distinction is relevant for both spectral complexity and network analysis.

The matlab function that performs the Tarjan algorithm is included

The next step is to sum all values of l per strong component. This has time complexity $\mathcal{O}(n)$.

The following step consists of collapsing strong components to individual nodes. First each node is assigned the new index, intuitively this implies time complexity $\mathcal{O}(n)$. Next, for each off-diagonal value of L, it is checked whether it is positive and if so it is assigned to the correct node in the new matrix. This has time complexity $\mathcal{O}(n^2)$ because it loops over each off-diagonal element of the matrix L.

The next step consists of creating an adjacency matrix where each arc is indicated by a 1 and otherwise 0. This has time complexity $\mathcal{O}(n^2)$.

Because the spectral fragility measure requires for each strong component to know what nodes are reachable, a matrix is required where each 1 indicates whether there is a path of any length to the component corresponding to the column. This can be done by the Floyd-Warshall algorithm with time complexity $\mathcal{O}(n^3)$ (Floyd and Warshall, 1962). Intuitively, the algorithm considers from each node, $n \times n$ potential paths. In particular, indirect paths from each node to $n-1$ other nodes through at most $n-2$ other banks. The time complexity for the weight parameters depends on the choice of this parameter. This can be as complex as desired. For the exponential weight parameter applied in this thesis, the weights are calculated efficiently by taking the weight parameter to the element-wise power of a matrix that contains the number of arcs on the shortest path.

The calculation of the eigenvalues a time complexity dependent on the properties of the matrix and the strong components.

The final calculations are rather simple and thus the dominating time complexity is $\mathcal{O}(n^3)$