Optimal Consumption, Investment, and Insurance Strategy Over the Life-Cycle for a Regret-Averse Investor

by

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Abstract

This thesis examines the optimal consumption-investment-insurance strategy for a risk- and regret-averse investor who is subject to biometric risks affecting her income and life expectancy. Various behavioral models have been implemented into an optimal asset allocation problem. However, there has been done relatively little research on optimal asset allocation in combination with regret theory. This thesis aims to analytically assess the optimal consumption-investment-insurance problem for a regret-averse investor with Epstein-Zin preferences. This thesis contributes to the literature by combining regret theory and Epstein-Zin preferences. To model regret-averse Epstein-Zin preferences, an alternative multiplicative regret utility function and a regret-averse normalized aggregator function are proposed. The agent can invest in a risk-free asset and in a risky asset. To hedge herself against biometric risks, she can buy continuously-adjustable life-insurance contracts. Closed-form solutions are derived for an arbitrary value of elasticity of intertemporal substitution (EIS) utilizing a dynamic optimization approach solving the Hamilton-Jacobi-Bellman (HJB) equation. The optimal consumption-investment-insurance strategy is determined by first solving an auxiliary model for an infinitely regret-averse investor and then substituting the foregone control processes and wealth into the HJB of the regret-averse investor. The special cases of constant relative risk aversion (CRRA) and unit EIS are highlighted. It will be shown that the optimal consumption-investment-insurance strategy for a regret-averse investor does not directly depend on foregone wealth, but only on realized wealth. Additionally, this thesis will proof that it is optimal for a both risk- and regret-averse agent to invest more into the risky asset than for a purely risk-averse agent under a weak condition. This observation could be taken into consideration when assessing fund managers' risk-taking behavior.

Keywords: Optimal life-cycle investment, regret theory, stochastic differential utility, health shock risk, health insurance, Hamilton-Jacobi-Bellman equation

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Contents

1 Introduction

The main challenge for any institutional as well as private investor is to allocate her wealth optimally. Optimal portfolio choice has been concerned with determining the optimal investment strategy to maximize expected utility from consumption and terminal wealth. In most economical and financial theories, it is assumed that the agents (investors) are rational. However, in practice, agents do not always behave in a fully rational manner, i.e. they are bounded by rationality. In other words, investors are affected by their emotions and as a result, they might invest differently as rationally expected. To model the bounded rationality of the investors, this thesis incorporates a behavioral model in the optimal portfolio choice problem. There are various behavioral models, but this thesis focuses on regret theory as independently introduced by Bell (1982), and Loomes and Sugden (1982). Regret theory is based on investors feeling regret over missing out on a more profitable investment or not investing at all. Studies conducted by Zeelenberg et al. (1998) and Zeelenberg (1999) showed that regret is inherently different from other emotions, such as disappointment, since regret is very persistent and widely experienced by investors. Furthermore, as shown by Goossens (2022), regret theory explains many stylized facts of the financial literature and more than most other behavioral theories. Among others, regret theory is able to explain the risk-free rate puzzle and long-term reversal. This might indicate that regret theory is a promising behavioral theory to incorporate in the optimal investment problem. To model the feeling of regret, a multiplicative regret-based utility function (Quiggin, 1994) which consists of a standard CRRA power-utility term and a multiplicative term that reflects the disutility from feeling regret (Goossens, 2022) is proposed. This regret-based utility function can then be utilized in the existing asset allocation theories. The model is further extended by including Epstein-Zin preferences (Epstein & Zin, 1989). Therefore, a regret-adjusted stochastic differential utility specification (Duffie & Epstein, 1992) is proposed in this thesis. Stochastic differential utility is a continuous-time version of recursive utility (Epstein & Zin, 1989) and it allows to differentiate between risk aversion and elasticity of intertemporal substitution. Moreover, the regret-averse investor is subject to biometric risks affecting her income and life expectancy (Hambel et al., 2022). Closed-form solutions will be derived for an arbitrary value of the EIS parameter. The special cases of CRRA-regret-utility and unit-EIS-regret-utility preferences will be highlighted. This thesis is concerned with analytically investigating the optimal consumption-investment-insurance strategy for an investor with regret-averse Epstein-Zin preferences who is subject to biometric risks.

This thesis contributes to the literature by being one of the very few papers investigating the optimal consumption-investment-insurance strategy in combination with regret theory and Epstein-Zin preferences. Moreover, this thesis adds to the literature by proposing an alternative multiplicative regret-utility function and a regret-adjusted normalized aggregator function for the Epstein-Zin preferences. The main findings of this thesis can be summarized as follows. It is found that closed-form solutions can be derived for an arbitrary EIS parameter value. It will be shown that the optimal investment and insurance strategies only directly depend on the risk and regret aversion, but not on the EIS parameter. However, these optimal control processes do depend on the EIS parameter indirectly via an age- and state-dependent function. The optimal consumption strategy depends on the risk and regret aversion parameter, and on the EIS parameter. Furthermore, an important finding is that a regret-averse agent invests more into the risky asset than a purely risk-averse agent under the weak condition that the well-known Merton investment fraction satisfies $\pi = \frac{\lambda}{\sigma\gamma} < 1$. Hence, the aversion to regret drives the agent to invest more into the risky asset.

The results of the model can be of particular interest for e.g. households, pension funds and fund managers. Barber and Odean (2008) observed that retail investors, i.e. households, tend to display attention-driven buying behavior. As a result, if many attention-driven investors buy stocks, then this could temporarily inflate the stock price, leading to lower than expected subsequent returns. The retail investors might then feel regret about those subsequent returns. This can indicate the importance of including regret theory into an optimal investment problem. In Section 5, the economic relevance of the model for the pension funds and fund managers is discussed.

In summary, Section 2 reviews the existing literature in the field of dynamic asset allocation and behavioral finance. Thereafter, Section 3 will describe the mathematical model and derive theoretical results for the dynamic asset allocation problem for a regret-averse investor living in a Black-Scholes world with Epstein-Zin preferences subject to biometric risks. The optimal consumption-investment-insurance strategies are determined by a dynamic programming approach. Given the preferences of the investor and the (foregone) wealth dynamics, the so-called Hamilton-Jacobi-Bellman (HJB) equation corresponding to the dynamic optimization problem can be constructed. To determine the optimal control processes of a regret-averse investor, first the optimal foregone consumption and insurance strategy of an auxiliary investor are derived. The determined foregone control strategies and the foregone wealth dynamics are then substituted into the HJB equation of the regret-averse investor. Based on this two-step approach, closed-form solutions will be derived. Section 4 will show numerical results obtained from a Monte-Carlo simulation for a benchmark regret-averse investor subject to mortality risk, i.e. the survival model of Richard (1975). Moreover, a sensitivity analysis for the benchmark survival model will be performed. Numerical results will be shown for various EIS parameter values ψ, regret aversion parameter values κ, risk aversion parameter values γ, and time preference rate δ. For each parameter value considered will the results be discussed. Subsequently, Section 5 summarizes and highlights the most profound results of the research. At last, modeling choices, limitations of the model, and possible recommendations for further research are discussed in Section 6.

2 Literature

One of the leading questions in the financial literature is how to optimally invest over an agent's life cycle. For decades, this question has been widely studied both in discrete- and in continuous-time. *Dynamic asset allocation problem*, *optimal portfolio selection problem*, *optimal portfolio choice problem*, and *optimal life-cycle investment strategy problem* all refer to the same type of problem concerning this question. The discrete-time models have been studied by Phelps (1962), Samuelson (1969), Hakansson (1970), and Fama (1970), among others. The first to write an article on optimal portfolio choice in continuous-time was Merton (1969). In his seminal article, Merton considered a risk-averse agent who maximizes expected utility from intermediate consumption and terminal wealth. The agent's utility preferences were given by the power-utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Merton showed for this problem (*Merton's portfolio problem*) that the optimal investment fraction is the well-known Merton investment fraction $\pi = \frac{\lambda}{\sigma\gamma}$ (Merton, 1969). Since Merton (1969), there have been various extensions to the original consumption-investment problem.

As Merton assumed constant investment opportunities, a logical extension is to assume instead a stochastic investment opportunity set. Assuming time-varying investment opportunities improves the realism of the model. The optimal investment fraction and its implications for time-varying investment opportunities has been documented by, e.g. Ocone and Karatzas (1991), Detemple et al. (2003), Nielsen and Vassalou (2006), and Liu (2007). Liu (2007) derived the optimal consumption-investment choice for general affine or quadratic market structures.

Instead of assuming all investment opportunities to be stochastic (depending on some underlying dynamics), one could also assume only the interest rate to be stochastic. Closed-form expressions can be found in case the interest rate dynamics are of an affine form. To the class of affine interest rate dynamics belong the Vasicek (1977) model, Hull and White (1990) model, and Cox-Ingersoll-Ross (1985) model. Sørensen (1999) examined the optimal investment choice for an expected utility maximizing investor with utility from terminal wealth only. Sørensen (1999) showed that changes in the opportunity set can be hedged by the zero-coupon bond with maturity at investment horizon. A downside of using affine interest rate models is their lack of realism. They are insufficient to model the whole Term Structure of Interest Rates (TSIR). Hence, the Heath-Jarrow-Morton (HJM) (1992) framework has been proposed. This framework does specify the whole TSIR. Munk and Sørensen (2004) investigate the optimal portfolio choice problem when the term structure of interest rates evolves according to the HJM framework.

Predictable asset returns have been one of the leading topics of empirical finance for decades. Based on the portfolio diversification theorem of Markowitz (1952), independently introduced Treynor (1961), Sharpe (1964), Lintner (1965), and Mossin (1966) the Capital Asset Pricing Model (CAPM). This particular model intends to describe the returns of a portfolio or stock by using the returns of the market as a whole. In 1992, Fama and French built upon the CAPM by including market capitalization and book-to-market ratio as additional risk factors. Fama and French (2015) refined their Famam-French 3-factor model further by additionally including profitability and investment as risk factor parameters. The idea of predictable asset returns has also been applied in the dynamic asset allocation problem. Both Kim and Omberg (1996) and Wachter (2002), among others, study the optimal investment strategy by applying mean-reverting stock returns.

Instead of stochastic interest rates, one could also investigate the affect of stochastic volatility. Most commonly used, is the tractable Heston (1993) model. In this model, the instantaneous volatility follows a Cox-Ingersoll-Ross (1985) process and the stock price depends on the square root of the instantaneous volatility. Liu and Pan (2003), Kraft (2005), and Liu (2007) determine closed-form solutions for various specifications of the stochastic volatility model. Liu and Pan (2003) allow for jumps of predetermined size in the stock price based on empirically observed stock market crashes. Liu et al. (2003) and Branger et al. (2008) extended upon Liu and Pan (2003) and considered models where both the stock price as well as the volatility may include jumps.

Up to now, most models that have been discussed were defined in real terms. However, inflation itself is uncertain and therefore, this uncertainty can be incorporated into the asset allocation problem. Optimal asset allocation with inflation risk has been studied by e.g. Campbell and Viceira (2001), Brennan and Xia (2002), Munk et al. (2004), and Munk and Sørensen (2007).

Most portfolio choice problems abstract from labor income. Naturally, it is more realistic if the investor earns labor income. The labor income process can be assumed to be a spanned or unspanned exogenous labor income, or an endogenous labor income. Closed-form solutions can be found for all cases, but only under strict assumptions. Cocco et al. (2005), Munk and Sørensen (2010), and Munk (2017) discuss the results when the investor earns spanned exogenous labor income. Munk and Sørensen (2010) further discuss the relationship between stochastic labor income and interest rates. Svensson and Werner (1993), Henderson (2005), and Christensen et al. (2012) derive closed-form solutions under strict assumptions in case of unspanned exogenous labor income. Cocco et al. (2005), Koijen et al. (2010), and Munk and Sørensen (2010) were able to derive numerical results for a more general model with unspanned income process. Bodie et al. (1992) show results in a setting with endogenous labor income.

An important extension to the previous models is to incorporate biometric risk into the model. Previously, it was assumed that the investor lives with certainty for a predetermined time. However, mortality risk is an important factor for retail investors, insurance companies, and pension funds. Richard (1975) was the first to introduce mortality risk to the original Merton problem (Merton, 1969). Steffensen and Kraft (2008) extended the idea of

Richard (1975) by representing the biometric sates of the investor by a general finite-state Markov chain. Cocco et al. (2005), Cocco and Gomes (2012), Hambel et al. (2017), Hambel (2020), Hambel et al. (2022), Hambel et al. (2023), and Steffensen and Søe (2023) further discuss several life-cycle investment problems including biometric risk. A recent survey on biometric risk in life-cycle investment problems has been written by Gomes (2020).

It has been the standard in optimal portfolio choice problems to consider an agent that maximizes expected utility with a Von Neumann-Morgenstern (1947) utility function. However, expected utility theorem conflicts with social experiments such as the Allais paradox (Allais, 1953). As a result, adaptations of expected utility theory have been proposed. One of the most well-known alternatives to expected utility theory is prospect theory by Kahneman and Tversky (1979) and the improved cumulative prospect theory (Tversky & Kahneman, 1992). Prospect theory and cumulative prospect theory aim to explain an agent's behavior by introducing the notion of prospects. According to Kahneman and Tversky (1979), all decisions under uncertainty can be modeled as prospects with different values and probabilities. Prospect theory has been applied in the framework of an optimal consumption and investment choice problem by e.g. Van Bilsen and Laeven (2020).

Alternatively, Abel (1990) argued that agents form a habit regarding their behavior. Abel (1990) postulated the habit formation theory which states that an agent always consumes at least her habit, e.g. a weighted average of past consumption levels. Munk (2008) and Kraft et al. (2017), among others, discuss the optimal consumption and investment choice for an investor with habit formation in preferences.

In contrast to defining prospects or a habit, Bell (1985), Loomes and Sugden (1986), and Gul (1991) challenged expected utility theory by arguing that an agent feels disappointment over outcomes that are considered worse than the expected outcome. Disappointment theory is developed around the emotion of disappointment of an agent. The theory states that the disappointment over the outcome of uncertain events influences the behavior of the agent. A disappointment-averse agent will behave differently as she experiences disutility from disappointments. Disappointment theory has been incorporated into dynamic asset allocation problems by e.g. Saltari and Travaglini (2009) and Kontosakos et al. (2018). A result of optimal portfolio choice with disappointment aversion is that under disappointment aversion, it might be better for an agent not to invest in the risky asset, even when the expected return is positive as the disutility from the expected disappointment might overshadow the utility from the expected return (Saltari & Travaglini, 2009).

A more commonly used behavioral model in optimal portfolio choice problems is the so-called Epstein-Zin preferences model (Epstein & Zin, 1989). In case of the classical power-utility function, both the risk aversion and the elasticity for intertemporal substitution are given by the same parameter. Epstein and Zin disentangled the agent's risk aversion and elasticity for intertemporal substitution. As a result, expected utility theory with a power-utility function is a special case of the recursive utility specification (Epstein & Zin, 1989) (discrete-time) and stochastic differential utility specification (Duffie & Epstein, 1992) (continuous-time). The application of Epstein-Zin preferences in life-cycle investment problems has been done by e.g. Schroder and Skiadas (1999), Cocco et al. (2005), and Bhamra and Uppal (2006).

At last, regret theory has been proposed independently by Bell (1982), and Loomes and Sugden (1982) to explain an agent's behavior under uncertainty. An agent might feel regret over her decision when the realized outcome does not exceed the anticipated foregone outcome. An investor experiences more regret if the realized outcome is much lower than the anticipated foregone outcome. A similarity between disappointment theory and regret theory is that they are both based on a negative emotion. Regret can either be included as an additive term to the utility function (see e.g. Bell (1982), Loomes and Sugden (1982), and Quiggin (1994)) or as a multiplicative term (see e.g. Quiggin (1994), Goossens (2021), and Goossens (2022)). Goossens (2022) showed that regret theory is able to explain many of the stylized facts known in asset pricing literature. Therefore, regret theory is shown to be a particularly promising behavioral model. The impact of regret aversion on the asset allocation decisions of an investor has been investigated by e.g. Muermann et al. (2006) and Blanchett (2023). Braun and Muermann (2004) examined the impact of regret on the demand for insurance.

For an extensive review on dynamic asset allocation and the models previously discussed is the reader referred to Munk (2017). This literature review took great inspiration of the work done by Munk (2017).

3 Model

In this thesis, an investor living in a world with constant investment opportunities who is not only risk-averse, but also regret-averse, is considered. The investor is subject to biometric risks affecting her labor income. The regret-averse investor's preferences are specified by a stochastic differential utility specification (Duffie & Epstein, 1992) adapted to include regret aversion. Closed-form solutions are derived for an arbitrary EIS parameter value ψ. The special cases of CRRA and unit EIS are highlighted.

First, the regret-averse utility specification will be discussed in Section 3.1. Second, the dynamics of the model will be specified in Section 3.2. Third, Section 3.3 depicts the results for an arbitrary EIS parameter value. In Section 3.4, the results for the special case of CRRA-regret-utility preferences will be highlighted and in Section 3.5, the results for the special case of unit-EIS-regret-utility preferences will be discussed.

3.1 Regret-averse utility specification

Merton (1969) showed in his seminal paper the optimal investment and consumption strategy for a risk-averse investor living in a Black-Scholes world. Merton considered a power-utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with γ being the risk aversion parameter. It is well known that the power-utility function is part of the class of CRRA utility specifications. Since Merton (1969), the power-utility function has been the leading utility specification for dynamic asset allocation problems as it allows for tractable models. However, this specification only models the investor's risk aversion and it neglects other behavioral aspects of the investor's choice set such as the marginal rate of time preference, i.e. Epstein-Zin preferences (Epstein & Zin, 1989), or habit formation in the consumption choice (Abel, 1990). An alternative behavioral aspect, which was found to be relevant to model an investor's preferences, is regret aversion.

The feeling of regret is based on a comparison between "what is" and "what might have been" (Lin et al., 2006). An investor experiences regret when her investments turn out to be less profitable than the foregone investments. Bell (1982), and independently Loomes and Sugden (1982) developed regret theory as a behavioral utility specification to model an agent's utility while taking regret into account. They originally formalized regret theory as an additive utility specification of the form

$$u(x, y) = v(x) + g(v(x) - v(y)) \tag{1}$$

with x being chosen by the investor and y being the foregone alternative. $v(x)$ is a Bernoulli utility function over monetary positions with $v' > 0$ and $v'' < 0$, and $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is a regret function which represents the regret or rejoicing (the positive feeling of having chosen a better option than the foregone alternative, i.e $x > y$) an agent experiences. Regret is modeled over the ex-post realized alternatives (Lin et al., 2006). Braun and Muermann (2004) modified the original specification by proposing a two-attribute utility function of the form

$$u(w) = v(w) - k \cdot g(v(w^{max}) - v(w)) \tag{2}$$

where $v(\cdot)$ and $g(\cdot)$ are defined as previously with $g' > 0$, $g'' > 0$, and $g(0) = 0$. In this specification, the investor gets utility over realized wealth w and experiences regret when $w^{max} > w$ with w^{max} being the wealth that the individual could have received by having made the optimal choice knowing the ex-post realized state of the world.

Quiggin (1994) proposed a different type of regret theory, namely multiplicative regret theory. The utility function in multiplicative regret theory has the following general functional form

$$u(x, y) = V(x) \cdot v(r) \tag{3}$$

where again x is the chosen outcome and r is the ex-post best foregone outcome. Goossens (2021) and Goossens (2022) proposed, based on properties as defined by Gollier (2018), Gabillon (2020), and Goossens (2021), the following multiplicative regret-utility function

$$u(x, r) = \frac{x^{1-\gamma}}{1-\gamma} \cdot r^{\kappa}, \quad \gamma - 1 \geq \kappa \geq 1,\ x > 0,\ \text{and}\ y > 0 \tag{4}$$

where γ is the risk aversion parameter and κ the regret aversion parameter. This utility function $u(x, r)$ can be decomposed into the CRRA power-utility function $V(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and regret function $v(r) = r^{\kappa}$. It should be noted that this utility specification does not allow for rejoicing as $r \geq x \geq 0$ by definition.

Multiplicative regret-utility functions are more tractable than additive regret-utility functions. This is similar to habit formation. A multiplicative habit function is in general more tractable than an additive habit function. However, the multiplicative utility specification (4) as proposed by Goossens (2022) does not allow for rejoicing. Rejoicing would be a desired property in a dynamic asset allocation problem as it is particularly difficult to define the best foregone alternative. Using the idea of Braun and Muermann (2004), the best foregone alternative will be defined by the consumption/wealth an investor could have had by investing all her available wealth in the stock market.[1] This is under the assumption that for typical values of μ and r ($\mu > r$) the stock market outperforms a composite of the stock market and the money market in the long run. However, the composite might outperform the maximum investment in the short term and hence, the model should allow for rejoicing. It will be shown that an investor who invests all her available wealth into the stock market is infinitely regret-averse given her fixed risk aversion level γ. This can be explained by the fact that an infinitely regret-averse investor fears to feel regret over missing out on a higher consumption/wealth and therefore invests all her available wealth into the stock market. The investor's regret aversion overshadows her risk aversion. Given this definition

[1] The investor will invest all her financial wealth in absence of any labor income. If the investor earns labor income, then she will invest her entire total wealth corrected by an amount corresponding to the future value of her labor income into the risky asset.

of foregone consumption/wealth, the following CRRA multiplicative regret-utility function for a regret-averse investor is assumed

$$u(x, y) = \frac{x^{1-\gamma}}{1-\gamma}\left(\frac{y}{x}\right)^{\kappa}, \quad \gamma - 1 \geq \kappa \geq 1,\ x > 0,\ \text{and}\ y > 0 \tag{5}$$

with γ being the time-independent risk aversion parameter and κ the time-independent regret aversion parameter. In this setting, x is either the realized consumption or wealth and y is the foregone consumption or wealth. Gollier (2018) argued that it is essential for modeling regret that the following conditions are satisfied (Goossens, 2022):

(i) marginal utility of consumption increases as foregone consumption increases

(ii) aversion to the foregone alternative

Goossens (2022) induced these conditions by assuming $\gamma - 1 \geq \kappa \geq 1$ for the multiplicative regret-utility function (4). Following Goossens (2022), it is therefore assumed that the risk and regret parameters should satisfy $\gamma - 1 \geq \kappa \geq 1$. Diecidue and Somasundaram (2017) showed in their article that regret theory is in line with their behavioral foundation if the inequalities are strict, i.e. $\gamma - 1 > \kappa > 1$. These inequalities can easily be imposed for the parameter values of the proposed regret-averse utility function (5). The proposed multiplicative regret-function (5) is in line with the usual multiplicative regret function as described by Quiggin (1994) and Goossens (2022), but differs as it models regret over the relative fraction between consumption/wealth and foregone consumption/wealth instead directly over foregone consumption/wealth. This way, the function allows foregone consumption/wealth to be less or equal than realized consumption/wealth and therefore, it allows for rejoicing in contrast to ordinary multiplicative regret-utility functions. Moreover, Lin et al. (2006) found that regret is mainly driven by a loss or gain relative to the reference point (foregone outcome), rather than by the size of that loss or gain. Hence, modeling regret over the relative difference between realized and foregone outcome seems to be empirically supported. Note, for $\kappa = 0$ the model reduces to the regular power-utility function $V(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Loomes and Sugden (1982) and Bell (1982) denoted this function as the choiceless utility function. It is the utility one would get independently of any choice-related feeling (Gabillon, 2020). Comparing the choiceless utility function and the proposed regret-utility function (5), the following cases can be distinguished:

1. $y > x \Rightarrow u(x, y) < V(x)$ as $\left(\frac{y}{x}\right)^{\kappa} > 1$ for $\kappa \geq 1$
   The investor experiences regret (disutility) over missing out on the foregone choice and hence the regret-utility is lower than the choiceless utility.

2. $y = x \Rightarrow u(x, y) = V(x)$ as $\left(\frac{y}{x}\right)^{\kappa} = 1$ for $\kappa \geq 1$
   The investor does not experience any regret or rejoicing as the realized choice is equal to the foregone choice. The regret-utility equals the choiceless utility.

3. $y < x \Rightarrow u(x, y) > V(x)$ as $\left(\frac{y}{x}\right)^{\kappa} < 1$ for $\kappa \geq 1$
   The investor experiences rejoicing over choosing the better option compared to the foregone choice and hence the regret-utility is higher than the choiceless utility.

The analysis of the proposed regret-utility function is concluded by stating that the regret-utility function satisfies several desired properties as defined by Goossens (2021). Table 1 shows the desired properties a multiplicative regret-utility function should satisfy according to Goossens (2021). The multiplicative regret-utility function (5) proposed in this thesis satisfies all properties, except property *P2c*. Property *P2c* states that a regret-utility function should be globally increasing. It can be verified that this property is only satisfied by the proposed regret-utility function (5) if $\frac{y}{x} \geq \frac{\kappa}{\gamma+\kappa-1}$. The property is thus by definition satisfied in case of regret, i.e. $x \leq y$, but it is only satisfied if the relative rejoicing, i.e. $x > y$, is bounded from below. As a result, the proposed regret-utility function would satisfy all desired properties if one would exclude rejoicing. Rejoicing would be excluded if one would be able to define the stochastic dynamics of maximum wealth in all ex-post realized states of the world, i.e. a stochastic representation of w^{max} as specified in equation 2. However, this is mathematically challenging, if even achievable. Therefore, the assumption about fully investing into the risky asset to determine foregone wealth dynamics is made.

Furthermore, it should be noted that the derived lower bound approaches 0 for $\gamma \rightarrow \infty$. Thus, the proposed regret-utility function satisfies all properties for an infinitely risk-averse investor, but in that case the investment problems becomes redundant as the agent would only invest into the risk-free asset.

The verification of the desired properties is given in the Appendix 7.2.

At last, by choosing to model regret over the relative effect between the foregone option and chosen option, the model becomes insensitive to scaling for the regret part $\nu(x, y) = \left(\frac{y}{x}\right)^{\kappa}$ of the regret-utility function. However, the choiceless utility part $V(x) = \frac{x^{1-\gamma}}{1-\gamma}$ is typically still sensitive to scaling.

Table 1: This table shows the desired properties of a multiplicative regret-utility function $u(x, y)$ as described by Goossens (2021). The partial derivative with respect of x and y are respectively denoted by $\frac{\partial u(x,y)}{\partial x} = u_1(x, y)$ and $\frac{\partial u(x,y)}{\partial y} = u_2(x, y)$. The second order derivatives are given by $\frac{\partial^2 u(x,y)}{\partial x^2} = u_{11}(x, y)$, $\frac{\partial^2 u(x,y)}{\partial y^2} = u_{22}(x, y)$, $\frac{\partial^2 u(x,y)}{\partial xy} = u_{12}(x, y)$, and $\frac{\partial^2 u(x,y)}{\partial yx} = u_{21}(x, y)$.

Property	
P1a: The choiceless utility is increasing	$\frac{\partial u(x,x)}{\partial x} = u_1(x, x) + u_2(x, x) \geq 0$
P1b: The choiceless utility is concave	$\frac{\partial^2 u(x,x)}{\partial x^2} = u_{11}(x, x) + u_{12}(x, x) + u_{21}(x, x) + u_{22}(x, x) \leq 0$
P2a: The regret-utility is increasing in x	$u_1(x, y) \geq 0$
P2b: The regret-utility is decreasing in y	$u_2(x, y) \leq 0$
P2c: The regret-utility is globally increasing	$u_1(x, y) + u_2(x, y) \geq 0$
P3: The regret-utility is supermodular	$u_{12}(x, y) = u_{21}(x, y) \geq 0$
P4a: The regret-utility exhibits payoff risk aversion	$u_{11}(x, y) \leq 0$
P4b: The regret-utility exhibits regret aversion	$u_{22}(x, y) \leq 0$

The proposed regret-utility function (5) is the utility specification in case of CRRA. To extend the model further, a regret-averse stochastic differential utility specification will be considered. Stochastic differential utility (Duffie & Epstein, 1992) is the continuous-time version of recursive utility as proposed by Epstein and Zin (1989). The utility index $V_t^{c,\theta}$ at time t for consumption process c and portfolio process θ over the remaining lifetime $[t, T]$ is given by (Munk, 2017)

$$V_t^{c,\theta} = \mathbb{E}_t \left[\int_t^T f(c_s, V_s^{c,\theta}) \, \mathrm{d}s + \bar{V}_T^{c,\theta} \right] \tag{6}$$

The investor maximizes $V_t^{c,\theta}$ for any $t < T$ over all admissible control processes given the state variables at time t. Hence, the indirect utility is given by

$$J_t = \sup_{(c,\theta)\in\mathcal{A}_t} V_t^{c,\theta} \tag{7}$$

with $\mathcal{A}_t$ being the set of all admissible control processes.

For a risk-averse investor is the so-called normalized aggregator f given by

$$f(c, V) = \begin{cases} \frac{\delta}{1-\frac{1}{\psi}} c^{1-\frac{1}{\psi}} ([1-\gamma]V)^{1-\frac{1}{\varphi}} - \delta\varphi V & \text{for } \psi \neq 1 \\ \delta(1-\gamma)V \ln(c) - \delta V \ln([1-\gamma]V) & \text{for } \psi = 1 \end{cases} \tag{8}$$

with $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$ (Munk, 2017). The time preference of the investor is given by δ, the degree of relative risk aversion by $\gamma > 1$ and the elasticity of intertemporal substitution is characterized by $\psi > 0$. The term $\bar{V}_T^{c,\theta}$ is given by $\bar{V}_T^{c,\theta} = \varepsilon \frac{W_T^{1-\gamma}}{1-\gamma}$ with $\varepsilon \geq 0$ and this term represents the utility from terminal wealth.

This utility specification is the continuous-time version of the Kreps-Porteus-Epstein-Zin recursive utility specification (Kreps and Porteus, 1978, and Epstein and Zin, 1989). For the special case that $\psi = \frac{1}{\gamma}$, then equation (6) collapses to standard CRRA utility

$$V_t^{c,\theta} = \delta \left(\mathbb{E}_t \left[\int_t^T \mathrm{e}^{-\delta(s-t)} \frac{1}{1-\gamma} c_s^{1-\gamma} \,\mathrm{d}s + \frac{1}{\delta} \mathrm{e}^{-\delta(T-t)} \frac{\varepsilon}{1-\gamma} W_T^{1-\gamma} \right] \right)$$

which is a positive multiple of the time-additive power-utility specification as originally considered by Merton (1969) (Munk, 2017).

To model both regret aversion and Epstein-Zin preferences, the stochastic differential utility specification is modified to incorporate regret aversion. First of all, the terminal utility term $\bar{V}_T^{c,\theta}$ is now given by $\bar{V}_T^{c,\theta} = \varepsilon \frac{W_T^{1-\gamma}}{1-\gamma} \left(\frac{\hat{W}_T}{W_T} \right)^{\kappa}$ with $\varepsilon \geq 0$. Second, an alternative normalized aggregator function $\mathcal{F}$ is considered. The regret-averse normalized aggregator function $\mathcal{F}$ is defined as

$$\mathcal{F}(c, \hat{c}, V) = \begin{cases} \frac{\delta}{1-\frac{1}{\psi}} c^{1-\frac{1}{\psi}} \left(\frac{\hat{c}}{c} \right)^{\frac{\kappa}{\varphi}} \left([1-\gamma] V \right)^{1-\frac{1}{\varphi}} - \delta \varphi V & \text{for } \psi \neq 1 \\ \delta V \left[(1-\gamma-\kappa) \ln(c) + \kappa \ln(\hat{c}) - \ln([1-\gamma] V) \right] & \text{for } \psi = 1 \end{cases} \tag{9}$$

where again $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$. In line with Epstein and Zin (1989) and Duffie and Epstein (1992) denotes δ the time preference parameter, γ the relative risk aversion parameter, $\psi > 0$ the EIS parameter, and κ the regret aversion parameter. The risk and regret aversion parameters should satisfy following condition: $\gamma - 1 \geq \kappa \geq 1$. This condition is the same as introduced in equation (5). The limiting case $\psi = 1$ is derived using the rule of l'Hopitâl. The proof can be found in the Appendix 7.1.

The regret-averse normalized aggregator function $\mathcal{F}$ is constructed in such a way that it resembles the properties of the risk-averse normalized aggregator function f (8) and that it captures regret aversion. For the special case of CRRA, i.e. $\psi = \frac{1}{\gamma}$, it holds that $\mathcal{F}$ is given by

$$\mathcal{F}(c, \hat{c}, V) = \frac{\delta}{1-\gamma} c^{1-\gamma} \left(\frac{\hat{c}}{c} \right)^{\kappa} - \delta V$$

This special case yields that $V_t^{c,\theta}$ reduces to the time-additive regret-utility specification with regret-utility function (5).

Moreover, note that if $\kappa = 0$, then the regret-averse normalized aggregator function $\mathcal{F}$ reduces to the Epstein-Zin normalized aggregator function f.

This concludes the derivation of the regret-averse utility specification. In the upcoming sections, the biometric risk model is discussed and the optimal consumption-investment-insurance strategy for this model will be derived.

3.2 Investment opportunities and biometric risk model

In this section, the dynamics of the biometric risk model are explained. It is assumed that the investor lives in a so-called Black-Scholes world, i.e. the investment opportunities are constant over time. Thus the risk-free rate r, the expected stock return μ, and stock volatility σ are all constant over time. As a result, also the market price of risk λ is constant. The agent can invest into a stock (index)[2] S and a risk-free money market account M. The stock price evolves according to the following stochastic differential equation

$$\mathrm{d}S_t = S_t \left[\mu \, \mathrm{d}t + \sigma \, \mathrm{d}Z_t\right] \tag{10}$$

where $Z = (Z_t)_{t>0}$ is a standard Brownian motion under the physical probability measure $\mathbb{P}$. The stock price follows a geometric Brownian motion with the following closed-form expression

$$S_t = S_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t}$$

with S_0 being the stock price at time $t = 0$.

The money market account satisfies following stochastic differential equation

$$\mathrm{d}M_t = M_t r \, \mathrm{d}t \tag{11}$$

which has the following closed-form expression

$$M_t = M_0 \mathrm{e}^{rt}$$

with M_0 being the amount of money in the money market account at time $t = 0$. The agent receives continuously compounded interest with rate r over her money in the money market account.

Moreover, it is assumed that the investor can experience health shocks which influence her labor income. For example, the investor can become disabled or even die before terminal date T. As a result, the investor buys health insurance to hedge the possible loss of income due to a health shock. The extension of mortality risk to the Merton problem was firstly introduced by Richard (1975). As proposed by Richard (1975), and Steffensen and Kraft (2008), the investor can buy continuously adjustable short-term insurance contracts. This is a very strong and rather unrealistic assumption. However, this allows for a complete market and hence analytical results can be derived.

The biometric state of the investor is modeled by a finite-state Markov chain as was done by Steffensen and Kraft (2008), and Hambel et al. (2022). A probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is considered. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a discrete state variable $I = (I_t)_{t\geq 0}$ taking

[2] As only a single stock is considered, one could interpreted this stock as an index of for example the entire stock market.

values in a finite set $\mathcal{Q} = \{0, ..., Q\}$ of possible states and starting in state $I_0 = 0$ at time $t = 0$. The $(Q + 1)$-dimensional counting process $\mathbb{N}_t = (\mathbb{N}_t^0, ..., \mathbb{N}_t^Q)_{t\geq 0}$ is defined by

$$\mathbb{N}_t^q = \Big|\{s \in (0, t] | I_{s-} \neq q, I_s = q\}\Big|, \tag{12}$$

which counts the number of jumps into state q. Moreover, it is assumed there exist sufficiently smooth, age[3]-dependent, deterministic functions $h^{p,q} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, p, q \in \mathcal{Q}$, such that $\mathbb{N}^q$ admits the stochastic intensity process $(h^{I_{t-},q}(t))_{t\geq 0}$ for $q \in \mathcal{Q}$. The state $I_t = Q$ is the absorbing state and corresponds to the investor's death. The transition intensity from state p to state of death Q is given by $h^{p,Q}$, the so-called *hazard rate of death*.

As the time of death is uncertain, the time at which the model ends is uncertain as well. The time when the investor dies or the model ends, whichever occurs first, is given by

$$\tau = \inf_{t\geq 0}\{\mathbb{N}_t^Q > 0\} \wedge T \tag{13}$$

This framework is in line with the literature (see e.g. Steffensen and Kraft (2008), and Hambel et al. (2022)) and it allows for an optimal control problem for a risk- and regret-averse investor with unspanned biometric risk and a stochastic planning horizon with an uncertain time of death.

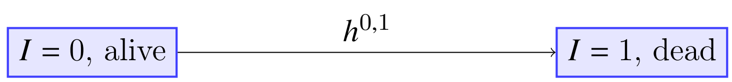


Figure 1: Illustration of the survival model adapted from Hambel et al. (2022).

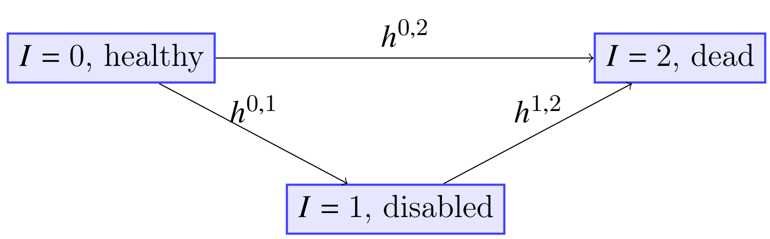


Figure 2: Illustration of the critical illness model adapted from Hambel et al. (2022).

The conventional model incorporating biometric risk into a consumption-investment-insurance problem is the so-called *survival model* as firstly introduced by Richard (1975). The survival model only includes two biometric states, *alive* and *dead*. The agent can transition given an age-dependent function $h_t^{0,1}$ from state 0, *alive*, to state 1, *dead*. Note that, as previously

[3] The words time and age are used interchangeably. However, formally age is given by the sum of time t and starting age at time $t = 0$, i.e. $age = t + age_{t=0}$.

explained, the state *dead* is an absorbing terminal state. The state diagram of the survival model is depicted in Figure 1.

A more involved biometric risk model as studied by Hambel (2020), Hambel et al. (2017) and Hambel et al. (2022) is the so-called *critical illness model*. In this elaborated model, the agent can additionally become critically ill. Mathematically speaking, this means that the Markov chain can transition from state 0, *healthy*, to state 1, *disabled*, with an age-dependent transition rate $h_t^{0,1}$. However, it is assumed that the agent cannot recover from her critical illness, i.e. the transition rate $h_t^{1,0} = 0$ for all t. Moreover, the agent can pass away both in the *healthy* state as well as in the *disabled* state. Thus, the Markov chain can transition to the state of *dead* with an age- and state-dependent transition rate $h_t^{p,2}$, $p \in \{0, 1\}$. Again, the state of *dead* denotes an absorbing terminal state. Figure 2 illustrates the state diagram of the critical illness model.

The investor receives an age- and state-dependent income stream $y = (y_t)_{t\geq 0}$ with $y_t \geq 0$ for all $t \in [0, T]$. The retirement age of the investor is predetermined to be T^r. Before retirement, the income process is defined by

$$\mathrm{d}y_t = y_{t-}[\alpha(t, I_{t-})\,\mathrm{d}t + \zeta(t, I_{t-})\,\mathrm{d}Z_t] + \sum_{q:q\neq I_{t-}} y_{t-}[P(t, I_{t-}, q) - 1]\,\mathrm{d}\mathbb{N}_t^q, \quad t < T^r \tag{14}$$

where $\alpha(t, I_{t-})$ and $\zeta(t, I_{t-})$ are age- and state-dependent deterministic functions, and $P(t, I_{t-}, q) \in (0, 1]$ is the fraction of income that remains after a transition from state p to state q for all $q \neq Q$. If the investor dies (changes into state Q), then the fraction of remaining income is $P(t, I_{t-}, Q) = 0$ for all states I_{t-} as the investor will not earn any money from labor after she passed away. The income process (14) closely resembles the income process as defined by Hambel et al. (2022). However, Hambel et al. (2022) assume that the labor income is not driven by a Brownian motion Z_t, i.e. $\zeta(t, I_{t-}) = 0$ for all t and $I_{t-} \in \mathcal{Q}$. This assumption is in line with the literature concerning optimal control processes with insurance contracts. Nevertheless, to show that the model allows the income process to be driven by the underlying Brownian motion of the stock market, this thesis assumes that the labor income stream is perfectly correlated with the stock market. This is a very strict assumption, but it allows for closed-form solutions as the market is complete and the income stream can be fully hedged. The income stream and stock market are positively (negatively) correlated if $\zeta > 0$ ($\zeta < 0$). Note that in case $\zeta = 0$, the income process reduces to the one considered by Hambel et al. (2022). The numerical results in section 4 are given for $\zeta = 0$.

In retirement, it is assumed that the investor earns a risk-free, but state-dependent fraction of her income shortly before retirement (Hambel et al., 2022). This fraction is called the replacement ratio $\Gamma(p) \in (0, 1]$. Hence, income after retirement is given by

$$y_t = \Gamma(I_{t-})y_{T^r}, \quad t \geq T^r \tag{15}$$

Given these dynamics of income, it can be concluded that the labor income stream can be valued as a financial asset. The income stream can be seen as a dividend stream from some trading strategy in the financial asset. The value of the present value of the future labor income stream at time t is referred to as the *human wealth* $H(y, t, p)$. Human wealth is known to be given by

$$H(y, t, p) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\tau} \mathrm{e}^{-r(s-t)} y_s \, \mathrm{d}s\right] = y_t F(t, p) \tag{16}$$

where $\mathbb{Q}$ denotes the risk-neutral measure. The risk-neutral measure is unique as the market is complete. The risk-neutral transition intensities are equal to the unit premiums (Hambel et al., 2022).

To hedge against the biometric risk, the investor is able to buy continuously-adjustable short-term insurance contracts. These insurance contracts give coverage for an infinitesimal time interval $[t, t + dt]$ at any point in time t. If at any point in time, say at time t^q, the investor suffers from a biometric shock of type q, i.e. $I_{t^q-} \neq q$ and $I_{t^q} = q$, then the contract pays out notional ι_t^q such that financial wealth is given by $W_{t^q} = W_{t^q-} + \iota_{t^q-}^q$. Following Hambel et al. (2022), the investor can buy these contracts at an insurance premium at a rate $\mathcal{P}_t^{p,q}$. This insurance premium is both age- and state-dependent. The insurance premium $\mathcal{P}_t^{p,q}$ is proportional to the notional ι_t^q. The *unit premium* for insurance against a transition into state q while being in state p at time t is given by $\hat{h}_t^{p,q} = \frac{\mathcal{P}_t^{p,q}}{\iota_t^q}$. In line with Hambel et al. (2017) and Hambel (2020), the unit premium is determined in such a way that the contract is actuarially fair and includes fees for the insurer, i.e. $\hat{h}_t^{p,q} = h_t^{p,q}(1+\phi_t^\iota)$ with $\phi_t^\iota \geq 0$. In case $\phi_t^\iota = 0$, then the unit premium is actuarially fair and otherwise the age-dependent function ϕ_t^ι incorporates all kind of additional fees. Given this specification, the unit premium $\hat{h}_t^{p,q}$ is a non-negative stochastic process satisfying suitable integrability conditions and

$$\hat{h}_t^{p,q} > 0 \text{ if and only if } h_t^{p,q} > 0.$$

It should be noted that in general, an admissible strategy does not guarantee positive financial wealth, i.e. $W_t > 0$, but it does guarantee the sum of positive wealth and life insurance to be positive, i.e. $W_t + \iota_t^Q > 0$ for all t (Hambel et al., 2022).

In conclusion, the agent invests θ_t amount of money into the stock market, she consumes c_t and buys health insurance with notional ι_t^q for all states $q \neq p$ at time t. The remaining wealth is invested into the money market account. Hence, given the investor is in biometric state $p \neq Q$, the investor's financial wealth evolves according to

$$\mathrm{d}W_t = \left[rW_t + \theta_t \lambda \sigma - c_t + y_t - \sum_{q \neq p} \iota_t^q \hat{h}_t^{I_t, q}\right] \mathrm{d}t + \theta_t \sigma \, \mathrm{d}Z_t \tag{17}$$

with $W_{t^q} = W_{t^q-} + \iota_{t^q-}^q$.

The regret-averse biometric risk model cannot be solved directly as foregone consumption and wealth are unknown to the investor. Hence, the following solution approach is utilized to determine the optimal control processes of the regret-averse investor.

① Solve the auxiliary model for an auxiliary investor who invests all her available wealth into the stock market with a choiceless utility function $V(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with the same risk aversion parameter γ as the regret-averse investor. As stated before, this auxiliary investor can be seen as an infinitely regret-averse investor. This provides a closed-form solution for the foregone consumption and notional strategy which only depends on foregone wealth and time.

② Solve the regret-averse model using the solution to the auxiliary model. The expressions for optimal foregone consumption, optimal foregone notional, foregone investment, and the known optimal dynamics of foregone wealth are utilized as the reference level for the regret-averse investor. This allows the Hamilton-Jacobi-Bellman (HJB) equation to be set up and solved for optimal investment, consumption and notional strategy.

This concludes the biometric risk model specification. In section 3.3, results for an arbitrary EIS parameter value ψ will be derived. Based on these results, the optimal consumption-investment-insurance strategy for the special case of CRRA-regret-utility preferences, $\psi = \frac{1}{\gamma}$, will be derived in Section 3.4. At last, in section 3.5, results for the limit case of unit EIS, $\psi = 1$, will be shown.

3.3 Arbitrary-EIS-regret-utility specification

This section discusses the model if the agent's preferences for elasticity of intertemporal substitution towards deterministic consumption plans are given by an arbitrary value of $\psi > 0$. Closed-form solutions for the optimal consumption-investment-insurance strategy will be derived. This general biometric risk model will be referred to as the arbitrary-EIS-regret model.

3.3.1 Auxiliary model

The auxiliary investor maximizes the utility over intermediate consumption and terminal wealth (bequest). The utility index $\tilde{J}(t, \tilde{W}, y, p)$ at time t for foregone consumption process $\tilde{c}$, portfolio strategy $\tilde{\theta}$, and notional choices $\tilde{\iota}^q$ over the remaining lifetime $[t, \tau]$ with τ as defined in equation (13) is given by

$$\tilde{J}(t, \tilde{W}, y, p) = \sup_{\left(\tilde{c}, (\tilde{\iota}^q)_{q=0}^{Q}\right) \in \tilde{\mathcal{A}}_t} \mathbb{E}_{t, \tilde{W}, y, p} \left[\int_t^\tau f(\tilde{c}_s, \tilde{J}_s) \, \mathrm{d}s + \tilde{\mathcal{J}}_\tau \right] \tag{18}$$

The investor maximizes $\tilde{J}(t, \tilde{W}, y, p)$ for any $t < \tau$ over all admissible control processes in set $\tilde{\mathcal{A}}_t$ given the state variables at time t.

The normalized aggregator function f for an arbitrary EIS parameter value $0 < \psi \neq 1$, as specified in equation (8), is given by

$$f(\tilde{c}, \tilde{J}) = \frac{\delta}{1 - \frac{1}{\psi}} \tilde{c}^{1-\frac{1}{\psi}} \left([1 - \gamma] \tilde{J} \right)^{1-\frac{1}{\varphi}} - \delta \varphi \tilde{J} \tag{19}$$

As explained previously, it holds that $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$. Moreover, δ denotes the time preference rate of the investor and $\gamma > 1$ the degree of relative risk aversion. The term $\tilde{\mathcal{J}}_\tau$ is given by $\tilde{\mathcal{J}}_\tau = \varepsilon \frac{\tilde{W}_\tau^{1-\gamma}}{1-\gamma}$ with $\varepsilon \geq 0$. This term represents the bequest motive with ε being the weight of the bequest motive.

To model foregone wealth, it is assumed that the investor invests all her available wealth into the financial asset. This is under the assumption that typically the stock market outperforms a composite of the money market and stock market in the long run (in case $\mu > r > 0$). Investors seek compensation for taking risks and hence the returns on risky assets should outperform the returns on riskless assets in the long run. The auxiliary investor has the same risk aversion level γ as the regret-averse investor, but she chooses to invest her total available wealth[4] into the stock market. From the point of view of the auxiliary investor, this can be seen as a suboptimal investment strategy, but it allows to model foregone consumption and wealth similar to the maximum ex post consumption and wealth the investor could have had.

The financial wealth dynamics of the auxiliary investor in biometric state $p \neq Q$ are given by

$$\mathrm{d}\tilde{W}_t = \left[r\tilde{W}_t + \tilde{\theta}_t \lambda \sigma - \tilde{c}_t + y_t - \sum_{q \neq p} \tilde{\iota}_t^q \hat{h}_t^{I_t,q} \right] \mathrm{d}t + \tilde{\theta}_t \sigma \, \mathrm{d}Z_t \tag{20}$$

with $\tilde{W}_{t^q} = \tilde{W}_{t^q-} + \tilde{\iota}^q_{t^q-}$ and $\tilde{c}$, $\tilde{\theta}$, $\tilde{\iota}^q$, and $\tilde{W}$ denoting the foregone consumption, investment amount, notional, and wealth, respectively.

Based on the indirect utility specification (18) and the foregone financial wealth dynamics (20) is the Hamilton-Jacobi-Bellman equation for an investor in state $p \neq Q$ given by

$$\begin{aligned} 0 =& \mathcal{L}^{\tilde{c}} + \mathcal{L}^{\tilde{\theta}} + \mathcal{L}^{\tilde{\iota}} + \tilde{J}_t \\ &+ \tilde{J}_{\tilde{W}}[(\tilde{W} + yF(t,p))r - yF(t,p)r + y] \\ &+ \tilde{J}_y y \alpha(t,p) + \frac{1}{2}\tilde{J}_{yy} y^2 \zeta(t,p)^2 - \sum_{q \neq p} h_t^{p,q} \tilde{J} \end{aligned} \tag{21}$$

with

$$\mathcal{L}^{\tilde{c}} = \sup_{\tilde{c} \geq 0} \left\{ \frac{\delta}{1-\frac{1}{\psi}} \tilde{c}^{1-\frac{1}{\psi}} ([1-\gamma]\tilde{J})^{1-\frac{1}{\varphi}} - \delta\varphi\tilde{J} - \tilde{c}\tilde{J}_{\tilde{W}} \right\}$$

$$\mathcal{L}^{\tilde{\theta}} = \tilde{J}_{\tilde{W}} \tilde{\theta}\sigma\lambda + \frac{1}{2}\tilde{J}_{\tilde{W}\tilde{W}} \tilde{\theta}^2 \sigma^2 + \tilde{J}_{\tilde{W}y} \tilde{\theta}\sigma y \zeta(t,p)$$

$$\mathcal{L}^{\tilde{\iota}} = \sup_{(\tilde{\iota}_t^q)_{q=0}^Q \in \mathbb{R}} \left\{ -\tilde{J}_{\tilde{W}} \sum_{q \neq p} \tilde{\iota}^q \hat{h}_t^{p,q} + \sum_{q \neq p,Q} h_t^{p,q} \tilde{J}(t, \tilde{W} + \tilde{\iota}^q, yP(t,p,q), q) + h_t^{p,q} \frac{\varepsilon}{1-\gamma} (\tilde{W} + \tilde{\iota}^Q)^{1-\gamma} \right\}$$

[4] Technically she will invest her total wealth minus a correction term for human wealth into the risky asset. Hence, it is referred to as *total available wealth*.

Subscripts of $\tilde{J}$ denote partial derivatives with respect to either the state variables or time t and the terminal condition $\tilde{J}(T, \tilde{W}, y, p) = \frac{\varepsilon}{1-\gamma}\tilde{W}^{1-\gamma}$.

As previously stated, the auxiliary investor does not maximize over all possible investment strategies, but chooses to invest all her available wealth into the stock market. Following Munk (2017), it can be conjectured that *total wealth* should evolve in the same way as *financial wealth* excluding labor income. Hence, it is conjectured that the dynamics for total wealth for the auxiliary investor are given by

$$\mathrm{d}(\tilde{W} + yF(t,p)) = \mu(t,y,p)\,\mathrm{d}t + \sum_{q \neq p} \nu(t,y,p,q)\,\mathrm{d}\mathbb{N}^q + \sigma\left(\tilde{W} + yF(t,p)\right)\,\mathrm{d}Z_t$$

with $\mu(t,y,p)$ and $\nu(t,y,p,q)$ being the drift rate and jump rate of total wealth, respectively, which can be determined using Itô's lemma for jump processes. These rates are unimportant for the derivations and hence they are omitted. Moreover, the investment fraction for the auxiliary investor without labor income is given by $\tilde{\theta}_t^{y=0} = \tilde{W}_t$ for all t (the superscript $y = 0$ denotes the model excluding labor income). Excluding any labor income, it is known that financial wealth equals total wealth ($yF(t,p) = 0$ for all t and p). Since it was conjectured that the auxiliary investor invests all her wealth into the stock market, it can be concluded that $\tilde{\theta}_t^{y=0} = \tilde{W}_t$ for all t. According to Itô's lemma, the dynamics of financial wealth should therefore be given by

$$\mathrm{d}\tilde{W} = \bar{\mu}(t,y,p)\,\mathrm{d}t + \sum_{q \neq p} \bar{\nu}(t,y,p,q)\,\mathrm{d}\mathbb{N}^q + \left[\sigma\left(\tilde{W} + yF(t,p)\right) - yF(t,p)\zeta(t,p)\right]\,\mathrm{d}Z_t$$

with $\bar{\mu}(t,y,p)$ and $\bar{\nu}(t,y,p,q)$ denoting the drift rate and jump rate of financial wealth. Again, these functions can be determined by Itô's lemma, but are omitted as they are irrelevant for the derivation.

Comparing these dynamics for foregone wealth with the one postulated in equation (20), it can be concluded that $\tilde{\theta}_t$ should satisfy

$$\tilde{\theta}_t = \left(\tilde{W}_t + yF(t,p)\right) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{22}$$

The optimal foregone consumption process can be derived by the first-order condition (FOC) of $\mathcal{L}^{\tilde{c}}$ with respect to $\tilde{c}$. The optimal foregone consumption process is given by

$$\tilde{c}^* = \delta^\psi \tilde{J}_{\tilde{W}}^{-\psi}([1-\gamma]\tilde{J})^{\psi(1-\frac{1}{\varphi})} \tag{23}$$

The FOC with respect to the foregone notional $\tilde{\iota}$ yields

$$\tilde{J}_{\tilde{W}}(t, \tilde{W} + \tilde{\iota}^q, yP(t,p,q), q) = \tilde{J}_{\tilde{W}}(t, \tilde{W}, y, p)\frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \tag{24}$$

To determine closed-form solutions, the conjecture is made that the indirect utility function $\tilde{J}(t, \tilde{W}, y, p)$ is of the form

$$\tilde{J}(t, \tilde{W}, y, p) = \frac{\tilde{G}(t, p)^{\gamma}}{1-\gamma}(\tilde{W} + yF(t, p))^{1-\gamma}$$

with the partial derivatives with respect to time t, foregone wealth $\tilde{W}$, and income y given in the Appendix 7.3.

Given the conjecture, the optimal foregone notional $(\tilde{\iota}^q)^*$ can be determined. The optimal foregone notional choice for state $q \neq p$ is given by

$$\begin{aligned}
\tilde{J}_{\tilde{W}}(t, \tilde{W} + \tilde{\iota}^q, yP(t, p, q), q) &= \tilde{J}_{\tilde{W}}(t, \tilde{W}, y, p) \frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \\
\iff \tilde{G}(t, p)^{\gamma} \left(\tilde{W} + \tilde{\iota}^q + yP(t, p, q)F(t, q)\right)^{-\gamma} &= \tilde{G}(t, p)^{\gamma} \left(\tilde{W} + yF(t, p)\right)^{-\gamma} \frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \\
\iff \tilde{W} + \tilde{\iota}^q + yP(t, p, q)F(t, q) &= \frac{\tilde{G}(t, q)}{\tilde{G}(t, p)} \left(\tilde{W} + yF(t, p)\right) \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\
(\tilde{\iota}^q)^* = \left(\frac{\tilde{G}(t, q)}{\tilde{G}(t, p)}\right) \left(\tilde{W} + yF(t, p)\right) &\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} - \left(\tilde{W} + yP(t, p, q)F(t, q)\right)
\end{aligned} \tag{25}$$

Similarly, given the conjecture, it holds that the optimal foregone consumption process is given by

$$\tilde{c}^*(t, \tilde{W}, y, p) = \delta^{\psi} \tilde{G}(t, p)^{-\psi \frac{\gamma}{\varphi}} (\tilde{W} + yF(t, p)) \tag{26}$$

Substituting the expressions for optimal foregone consumption $\tilde{c}^*$ and notional choice $(\tilde{\iota}^q)^*$, the foregone investment strategy $\tilde{\theta}$, and the conjecture into the HJB equation (21) yields a lengthy equation (see Appendix 7.4 equation (116)). Combining all terms that depend on $\tilde{G}(t, p)^{\gamma-1}(\tilde{W} + yF(t, p))^{1-\gamma}$ and rewriting, yields the following system of ordinary differential equations (ODE) for the time- and state-dependent function $\tilde{G}(t, p)$

$$\begin{aligned}
\frac{\partial \tilde{G}}{\partial t}(t, p) &= \left[\frac{1}{\gamma}\left(\varphi\delta + \sum_{q \neq p} h_t^{p,q}\right) + \frac{\gamma - 1}{\gamma}\left(r + \sum_{q \neq p} \hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2\right)\right] \tilde{G}(t, p) \\
&\quad + \frac{\delta^{\psi}(\gamma - 1)}{\gamma(\psi - 1)} \tilde{G}(t, p)^{\frac{\gamma\psi - 1}{\gamma - 1}} - \sum_{q \neq p} h_t^{p,q} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} \tilde{G}(t, q)
\end{aligned} \tag{27}$$

with boundary conditions $\tilde{G}(t, Q) = \tilde{G}(T, p) = \varepsilon^{\frac{1}{\gamma}}$. Note that all terms were divided by $\tilde{G}(t, p)^{\gamma}(\tilde{W} + yF(t, p))^{1-\gamma}$ which is by definition non-zero as $\tilde{G} > 0$ and $\left(\tilde{W} + yF(t, p)\right) > 0$.

In the same way, all remaining terms that depend on $\tilde{G}(t,p)^{\gamma}\left(\tilde{W}+yF(t,p)\right)^{-\gamma}$ can be combined into an ODE for $F(t,p)$. The purely time- and state-dependent function $F(t,p)$ is given by

$$\frac{\partial F}{\partial t}(t,p)=\left[r+\sum_{q\neq p}\hat{h}_t^{p,q}-\alpha(t,p)+\zeta(t,p)\lambda\right]F(t,p)-1-\sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F(t,q) \tag{28}$$

with boundary condition $F(t,Q)=F(T,p)=0$. Additionally, note that all terms were divided by $y>0$.

Moreover, as shown by Hambel et al. (2022), $F(t,p)$ can be separated based on whether the investor is active in the labor market or retired. This separation yields

$$F(t,p)=F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t)+F^r(t,p)\mathbb{1}_{\{t\geq T^r\}}(t)$$

with

$$\frac{\partial F^a}{\partial t}(t,p)=\left[r+\sum_{q\neq p}\hat{h}_t^{p,q}-\alpha(t,p)+\zeta(t,p)\lambda\right]F^a(t,p)-1-\sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^a(t,q)$$

and

$$\frac{\partial F^r}{\partial t}(t,p)=\left[r+\sum_{q\neq p}\hat{h}_t^{p,q}\right]F^r(t,p)-1-\sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^r(t,q)$$

with boundary conditions $F^r(t,Q)=F^r(T,p)=F^a(t,Q)=0$ and $F^a(T^r,p)=\Gamma(p)F^r(T^r,p)$.

In case the transition rates are directional, i.e. $h_t^{p,q}=0$ for $q\leq p$, then $F^a(t,p)$, and $F^r(t,p)$ have explicit expressions (Hambel et al., 2022) given by

$$\begin{aligned}
F^r(t,p)&=\int_{T^r}^{T}\mathrm{e}^{-\int_t^s(r+\sum_{q>p}\hat{h}^{p,q}(u))\,\mathrm{d}u}(1+\sum_{q>p}\hat{h}^{p,q}(s)F^r(s,q))\,\mathrm{d}s\\
F^a(t,p)&=\int_{t}^{T^r}\mathrm{e}^{-\int_t^s(r+\sum_{q>p}\hat{h}^{p,q}(u)-\alpha(u,p)+\zeta(u,p)\lambda)\,\mathrm{d}u}(1+\sum_{q>p}\hat{h}^{p,q}(s)p(s,p,q)F^a(s,q))\,\mathrm{d}s\\
&\quad+\Gamma(p)F^r(T^r,p)\mathrm{e}^{-\int_t^{T^r}(r+\sum_{q>p}\hat{h}^{p,q}(u)-\alpha(u,p)+\zeta(u,p)\lambda)\,\mathrm{d}u}.
\end{aligned}$$

It can be concluded that the HJB equation is solved by the derived expressions for the optimal foregone control processes, the conjectured foregone investment strategy, and the time- and state-dependent functions $\tilde{G}(t,p)$ and $F(t,p)$. Hence, the optimal foregone consumption process is denoted by

$$\tilde{c}^*(t,\tilde{W},y,p)=\delta^{\psi}\tilde{G}(t,p)^{-\psi\frac{\gamma}{\varphi}}\left(\tilde{W}+yF(t,p)\right) \tag{29}$$

Furthermore, the optimal foregone notional choice $(\tilde{\iota}^q)^*$ for biometric state q is given by

$$(\tilde{\iota}^q)^*(t, \tilde{W}, y, p) = \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)\left(\tilde{W} + yF(t,p)\right)\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} - \left(\tilde{W} + yP(t,p,q)F(t,q)\right) \tag{30}$$

and the foregone investment strategy is given by

$$\tilde{\theta}(t, \tilde{W}, y, p) = \left(\tilde{W} + yF(t,p)\right) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{31}$$

The optimal foregone consumption process, optimal notional choice, and foregone investment strategy will be used to determine the optimal investment strategy, consumption process and notional choice for the regret-averse investor. These results will be derived in the upcoming section.

3.3.2 Regret-averse model

The regret-averse investor maximizes utility over intermediate consumption and terminal wealth. The utility index $J(t, W, \hat{W}, y, p)$ at time t for consumption process c, investment strategy θ, and notional choices ι^q over the remaining lifetime $[t, \tau]$ with τ as defined by equation (13) is given by

$$J(t, W, \hat{W}, y, p) = \sup_{\left(c,\theta,(\iota^q)_{q=0}^{Q}\right)\in\mathcal{A}_t} \mathbb{E}_{t,W,\hat{W},y,p}\left[\int_t^\tau \mathcal{F}(c_s, \hat{c}_s, J_s)\,\mathrm{d}s + \mathcal{J}_\tau\right] \tag{32}$$

The investor maximizes $J(t, W, \hat{W}, y, p)$ for any $t < \tau$ over all admissible control processes in set $\mathcal{A}_t$ given the state variables and the (optimal) foregone state variables at time t. Note, the optimal foregone consumption process $\tilde{c}^*$ is denoted by $\hat{c}$, the optimal foregone notional choice $(\tilde{\iota}^q)^*$ by $\hat{\iota}^q$, and optimal foregone wealth $\tilde{W}^*$ by $\hat{W}$.

The regret-adjusted aggregator function $\mathcal{F}$ for EIS parameter $0 < \psi$ as specified in equation (9) is given by

$$\mathcal{F}(c, \hat{c}, J) = \frac{\delta}{1-\frac{1}{\psi}} c^{1-\frac{1}{\psi}} \left(\frac{\hat{c}}{c}\right)^{\frac{\kappa}{\varphi}} \left([1-\gamma]J\right)^{1-\frac{1}{\varphi}} - \delta\varphi J \tag{33}$$

The bequest motive is given by $\mathcal{J}_\tau = \frac{\varepsilon}{1-\gamma} W_\tau^{1-\gamma}\left(\frac{\hat{W}_\tau}{W_\tau}\right)^\kappa$ with $\varepsilon \geq 0$. Again, ε denotes the weight for the bequest motive. As discussed in Section 3.1, $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$, $\delta > 0$ the time preference rate, γ the relative risk aversion, κ the regret aversion, and $0 < \psi \neq 1$ the EIS parameter. The risk and regret aversion parameter should satisfy the condition $\gamma - 1 \geq \kappa \geq 1$.

It should be noted that the labor income stream of the auxiliary investor and regret-averse investor are identical. This can be seen from the fact that the underlying investor is the same in both cases and labor income is an independent state variable. Neither the choices in the control processes of the auxiliary investor nor choices of the regret-averse investor affect the labor income stream. Hence, the dynamics of income and human wealth are identical in both cases. It will indeed be shown that for the regret-averse investor the time- and state-dependent function $F(t,p)$ is specified by the same ODE as for the auxiliary investor (see equation (28)).

Based on the indirect utility equation (32) and wealth dynamics (17), the following Hamilton-Jacobi-Bellman equation for an investor in biometric state $p \neq Q$ can be specified

$$\begin{aligned}
0 =& \mathcal{L}^c + \mathcal{L}^\theta + \mathcal{L}^\iota + J_t \\
&+ J_W\left[(W + yF(t,p))r - yF(t,p)r + y\right] \\
&+ J_{\hat{W}}\left[(\hat{W} + yF(t,p))r - yF(t,p)r + \tilde{\theta}\sigma\lambda + y - \hat{c} - \sum_{p\neq q} \hat{\iota}^q \hat{h}_t^{p,q}\right] \\
&+ \frac{1}{2}J_{\hat{W}\hat{W}}\tilde{\theta}^2\sigma^2 + J_y y\alpha(t,p) + \frac{1}{2}J_{yy}y^2\zeta(t,p)^2 \\
&+ J_{\hat{W}y}\tilde{\theta}\sigma y\zeta(t,p) - \sum_{q\neq p} h_t^{p,q} J
\end{aligned} \tag{34}$$

with

$$\begin{aligned}
\mathcal{L}^c &= \sup_{c\geq 0}\left\{\frac{\delta}{1-\frac{1}{\psi}} c^{1-\frac{1}{\psi}}\left(\frac{\hat{c}}{c}\right)^{\frac{\kappa}{\varphi}}\left([1-\gamma]J\right)^{1-\frac{1}{\varphi}} - \delta\varphi J - cJ_W\right\} \\
\mathcal{L}^\theta &= \sup_{\theta\in\mathbb{R}}\left\{J_W\theta\sigma\lambda + \frac{1}{2}J_{WW}\theta^2\sigma^2 + J_{Wy}\theta\sigma y\zeta(t,p) + J_{W\hat{W}}\theta\sigma^2\tilde{\theta}\right\} \\
\mathcal{L}^\iota &= \sup_{(\iota_t^q)_{q=0}^Q\in\mathbb{R}}\left\{-J_W\sum_{q\neq p}\iota^q\hat{h}_t^{p,q} + \sum_{q\neq p,Q} h_t^{p,q} J(t, W+\iota^q, \hat{W}+\hat{\iota}^q, yP(t,p,q), q)\right. \\
&\qquad\left. + h_t^{p,Q}\frac{\varepsilon}{1-\gamma}\left(W+\iota^Q\right)^{1-\gamma}\left(\frac{\hat{W}+\hat{\iota}^Q}{W+\iota^Q}\right)^\kappa\right\}
\end{aligned}$$

Like in the auxiliary model, subscripts denote the partial derivatives with respect to the state variables and time t, and the terminal condition is denoted by $J(T,W,\hat{W},y,p) = \frac{\varepsilon}{1-\gamma}W^{1-\gamma}\left(\frac{\hat{W}}{W}\right)^\kappa$.

The optimal control processes can be determined by the respective FOCs. The FOC of $\mathcal{L}^c$ with respect to c yields the following optimal consumption strategy

$$c^* = J_W^{-\frac{1}{\frac{1}{\psi}+\frac{\kappa}{\varphi}}}\left(\frac{1-\frac{1}{\psi}-\frac{\kappa}{\varphi}}{1-\frac{1}{\psi}}\right)^{\frac{1}{\frac{1}{\psi}+\frac{\kappa}{\varphi}}}\delta^{\frac{1}{\frac{1}{\psi}+\frac{\kappa}{\varphi}}}\hat{c}^{\frac{\frac{\kappa}{\varphi}}{\frac{1}{\psi}+\frac{\kappa}{\varphi}}}\left([1-\gamma]J\right)^{\frac{1-\frac{1}{\varphi}}{\frac{1}{\psi}+\frac{\kappa}{\varphi}}} \tag{35}$$

The FOC of $\mathcal{L}^\theta$ with respect to θ results in the following optimal investment strategy

$$\theta^* = -\frac{J_W\lambda}{J_{WW}\sigma} - \frac{J_{W\hat{W}}\tilde{\theta}}{J_{WW}} - \frac{J_{Wy}y\zeta(t,p)}{J_{WW}\sigma} \tag{36}$$

At last, the optimal notional choice for biometric state q, given that the investor is in biometric state p, can be derived by the FOC of $\mathcal{L}^\iota$ with respect to ι^q. The FOC yields

$$J_W(t, W+\iota^q, \hat{W}+\hat{\iota}^q, yP(t,p,q), q) = J_W(t,W,\hat{W},y,p)\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}$$

To determine closed-form solutions, it is conjectured that the indirect utility function $J(t,W,\hat{W},y,p)$ (32) has the following functional form

$$J\left(t,W,\hat{W},y,p\right) = \frac{G(t,p)^\gamma}{1-\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^\kappa$$

The partial derivatives with respect to time t, wealth W, foregone wealth $\hat{W}$, and income y can be found in the Appendix 7.3.

Given this conjecture and the optimal foregone consumption strategy (29), the optimal consumption strategy for a regret-averse investor is determined to be

$$c^*(t,W,y,p) = \delta^\psi \tilde{G}(t,p)^{\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)}} G(t,p)^{\frac{-\gamma\psi}{\varphi+\kappa\psi}}(W+yF(t,p)) \tag{37}$$

Furthermore, the optimal investment amount can be derived by substituting the conjecture together with the foregone investment strategy (31) into equation (36). The optimal investment strategy is given by

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{\sigma(\gamma+\kappa)}(W+yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{38}$$

Finally, using the conjecture, the optimal notional choice $(\iota^q)^*$ can be derived

$$\begin{aligned}
&\left(\frac{1-\gamma-\kappa}{1-\gamma}\right) G(t,q)^\gamma \left(W+\iota^q+yP(t,p,q)F(t,q)\right)^{-\gamma}\left(\frac{\hat{W}+\hat{\iota}^q+yP(t,p,q)F(t,q)}{W+\iota^q+yP(t,p,q)F(t,q)}\right)^\kappa = \\
&\left(\frac{1-\gamma-\kappa}{1-\gamma}\right) G(t,p)^\gamma \left(W+\iota^q+yF(t,p)\right)^{-\gamma}\left(\frac{\hat{W}+\hat{\iota}^q+yF(t,p)}{W+\iota^q+yF(t,p)}\right)^\kappa \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right) \\
\iff &(W+\iota^q+yP(t,p,q)F(t,q))^{-\gamma-\kappa} = \\
&\left(\frac{G(t,p)}{G(t,q)}\right)^\gamma (W+yF(t,p))^{-\gamma-\kappa}\left(\frac{\hat{W}+yF(t,p)}{\hat{W}+\hat{\iota}^q+yP(t,p,q)F(t,q)}\right)^\kappa\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right) \\
\iff (\iota^q)^* = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}}(W+yF(t,p))\left(\frac{\hat{W}+yF(t,p)}{\hat{W}+\hat{\iota}^q+yP(t,p,q)F(t,q)}\right)^{\frac{-\kappa}{\gamma+\kappa}}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma+\kappa}} \\
&-(W+yP(t,p,q)F(t,q))
\end{aligned}$$

This can be further reduced as the optimal foregone notional $\hat{\iota}^q$, as specified in section 3.3.1 in equation (30), yields that $\hat{W} + \hat{\iota}^q + yP(t,p,q)F(t,q) = \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)(\hat{W} + yF(t,q))\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}}$. Hence, the optimal notional for state q is given to be

$$\begin{aligned}(\iota^q)^* = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}} (W + yF(t,p)) \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t,p,q)F(t,q))\end{aligned} \tag{39}$$

Substituting the conjecture, the expressions for optimal consumption, notional, and investment strategy, and the expressions for (optimal) foregone consumption, notional, and investment strategy into the HJB equation (34) yields a very lengthy equation (see Appendix 7.4 equation (117)). After some tedious calculations, this equation can be decomposed into three separate ODEs.
First, all terms that consist of $G(t,p)^\gamma (W + yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^\kappa$ can be taken together. After dividing by $y \cdot G(t,p)^\gamma (W + yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^\kappa > 0$ and rewriting, the following system of ODEs for the purely time- and state-dependent function $F(t,p)$ is obtained

$$\frac{\partial F}{\partial t}(t,p) = \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} + \zeta(t,p)\lambda - \alpha(t,p)\right] F(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q)F(t,q) \tag{40}$$

with boundary conditions $F(t,Q) = F(T,p) = 0$. Note that this is indeed the same ODE as for the auxiliary investor as given in section 3.3.1 by equation (28).

Furthermore, all terms that depend on $G(t,p)^\gamma (W + yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^\kappa \frac{1}{\hat{W}+yF(t,p)}$ yield the following system of ODEs

$$\frac{\partial F}{\partial t}(t,p) = \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} + \zeta(t,p)\lambda - \alpha(t,p)\right] F(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q)F(t,q) \tag{41}$$

with again boundary conditions $F(t,Q) = F(T,p) = 0$. It should be noted that all terms are divided by $y \cdot G(t,p)^\gamma (W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^\kappa \frac{1}{\hat{W}+yF(t,p)} > 0$. This yields exactly the same equation as the previous equations (28) and (40). It can thus indeed be concluded that the ODE for $F(t,p)$ does not change for the regret-averse investor compared to the auxiliary investor. This is in line with expectation as both investors are the same person, but with different preferences. Hence, the underlying income process and human wealth should be the same for both investors.

As seen in section 3.3.1, following Hambel et al. (2022), $F(t,p)$ can be decomposed into

$$F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t \geq T^r\}}(t)$$

with

$$\frac{\partial F^a}{\partial t}(t,p) = \left[r + \sum_{q\neq p} \hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right] F^a(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^a(t,q)$$

and

$$\frac{\partial F^r}{\partial t}(t,p) = \left[r + \sum_{q\neq p} \hat{h}_t^{p,q}\right] F^r(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^r(t,q)$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ and $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

Finally, the system of ODEs for the time- and state-dependent function $G(t,p)$ for the regret-averse investor can be determined by combining all terms that depend on $G(t,p)^{\gamma-1}(W + yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}$. The function $G(t,p)$ should satisfy following non-linear ODE

$$\begin{aligned}
\frac{\partial G}{\partial t}(t,p) = &\left[\frac{1}{\gamma}\left(\delta\varphi + \sum_{q\neq p} h_t^{p,q}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \sum_{q\neq p}\hat{h}_t^{p,q}\right) - \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2}\right)\right. \\
&\left. - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 + \left(\frac{\kappa}{\gamma}\right)\left(\delta^{\psi}\tilde{G}(t,p)^{-\frac{\psi\gamma}{\varphi}} - \lambda\sigma + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right) h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\right)\right] G(t,p) \\
&- \left(\frac{\gamma+\kappa}{\gamma}\right)\sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}} G(t,p)^{\frac{\kappa}{\gamma+\kappa}} \\
&- \left(\frac{\delta^{\psi}}{\gamma}\right)(\varphi - 1 + \gamma + \kappa)\,\tilde{G}(t,p)^{\frac{-\psi^2\kappa\gamma}{\varphi(\varphi+\kappa\psi)}} G(t,p)^{\frac{\varphi+(\kappa-\gamma)\psi}{\varphi+\kappa\psi}}
\end{aligned} \tag{42}$$

with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

It can be verified that in case $\kappa = 0$ this ODE for $G(t,p)$ reduces to the ODE for $\tilde{G}(t,p)$ (27), but with the optimal investment strategy instead of the maximum investment strategy.

Purely time- and state-dependent functions for $F(t,p)$ and $G(t,p)$ have been found. From this it can be concluded that the conjecture was correct. In the remainder of this section, some remarks on the optimal control processes will be discussed.

First, it can be concluded that a regret-averse investor in biometric state p should at time t optimally invest θ_t^* amount of money into the stock market with θ_t^* being given by

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{43}$$

with $F(t,p)$ being specified by equation (40).

It is thus optimal for the regret-averse investor to invest a constant fraction of her total wealth into the stock market subtracted by a time- and state-dependent fraction of human wealth to correct for the dynamics of income. Interestingly, the optimal investment strategy does not depend on foregone wealth. Moreover, this investment fraction does not depend on the EIS parameter ψ. For every elasticity of intertemporal substitution value, invests the agent the same amount of money. However, the investment strategy does depend, as expected, on the risk and regret aversion parameters of the agent.

Comparing θ_t^* to the investment strategy $\tilde{\theta}_t$ of the auxiliary investor, it can be seen that the correction term is the same for both investors. Again, this is in line with expectation as the income dynamics are for both investors the same. The only difference is the constant investment fraction. For the auxiliary investor, it was imposed that she would invest all her total available wealth into the stock market, but the regret-averse investor invests optimally given her risk and regret aversion parameters. The classical optimal investment strategy for the Merton problem with spanned exogenous income and no biometric risks (see e.g. Merton (1969) and Munk (2017)) is given by

$$\theta^{Merton} = \frac{\lambda}{\sigma\gamma}(W + yF(t)) - yF(t)\frac{\zeta}{\sigma}$$

Based on this result and from the auxiliary model, it can easily be verified that the optimal investment strategy for the Merton problem with spanned exogenous income and biometric risks is given by

$$\theta^{Merton} = \frac{\lambda}{\sigma\gamma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma}$$

The regret-averse investor invests a similar, nevertheless, regret-adjusted fraction of total wealth corresponding to the stock dynamics $\frac{\lambda}{\sigma}$ and she invests additionally a fraction $\frac{\kappa\sigma}{(\gamma+\kappa)\sigma}$ of total wealth according to her risk aversion.

The regret-averse investment fraction of total wealth is typically larger than the classical Merton investment fraction. The following theorem states that a regret-averse investor invests a larger amount of money into the stock market than a purely risk-averse investor.

Theorem 3.1 (Regret-averse investment amount exceeds the Merton investment amount). *The difference between the optimal regret-averse investment fraction and Merton investment fraction is given by*

$$\begin{aligned}\theta^{Regret} - \theta^{Merton} =& \frac{\lambda + \kappa\sigma}{(\gamma+\kappa)\sigma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} - \left[\frac{\lambda}{\sigma\gamma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma}\right] \\ =& \left(1 - \frac{\lambda}{\sigma\gamma}\right)\frac{\kappa}{\gamma+\kappa}(W + yF(t,p)) > 0 \text{ if and only if } \gamma > \frac{\lambda}{\sigma}\end{aligned}$$

Theorem 3.1 shows that if and only if the Merton investment fraction $\pi^{Merton} = \frac{\lambda}{\sigma\gamma} < 1$, then the regret-averse investment amount exceeds the Merton investment amount. In Section 3.5.2, it will be shown that the optimal investment strategy for a regret-averse investor with unit EIS preferences invests according to the same strategy as a regret-averse investor with arbitrary EIS preferences. Hence, Theorem 3.1 holds for any general regret-averse investor with her preferences as given by (9). The proof for Theorem 3.1 can be found in the Appendix 7.5.

It is clear that for $\kappa = 0$, the regret-averse investment fraction reduces to the classical Merton investment fraction. Additionally for an infinitely regret-averse investor, i.e. $\kappa \to \infty$, with fixed risk aversion level γ[5], it holds that

$$\lim_{\kappa\to\infty} \frac{\lambda + \kappa\sigma}{(\gamma + \kappa)\sigma} \to 1$$

As a result, it also holds that

$$\lim_{\kappa\to\infty} \theta_t^* = \lim_{\kappa\to\infty} \frac{\lambda + \kappa\sigma}{(\gamma + \kappa)\sigma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \to W + yF(t,p) - yF(t,p)\frac{\zeta(t,p)}{\sigma} = \tilde{\theta}_t$$

This shows, as previously stated (see Section 3.1), that an infinitely regret-averse investor indeed invests all her total wealth into the stock market only corrected for her human wealth. It can thus be stated that the auxiliary investor corresponds to an infinitely regret-averse investor.

Furthermore, it should be noted that for an infinitely risk-averse investor with a fixed regret aversion level κ, it holds that the investment fraction corresponding to total wealth becomes zero as expected

$$\lim_{\gamma\to\infty} \frac{\lambda + \kappa\sigma}{(\gamma + \kappa)\sigma} \to 0.$$

This concludes the analysis of the optimal investment portfolio of a risk- and regret-averse investor.

Second, the investor should optimally consume c_t^* according to optimal consumption process

$$c^*(t, W, y, p) = \delta^{\psi} \tilde{G}(t,p)^{\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)}} G(t,p)^{\frac{-\gamma\psi}{\varphi+\kappa\psi}}(W + yF(t,p)) \tag{44}$$

with $F(t,p)$ being given by equation (40), $G(t,p)$ by equation (42), and $\tilde{G}(t,p)$ by equation (27).

The regret-averse investor should thus consume a purely time- and state-dependent fraction of total wealth. This fraction both depends on the time- and state-dependent function

[5]Note that this case violates the parameter value condition $\gamma - 1 \geq \kappa \geq 1$. Nevertheless, this condition can technically be violated if the weaker conditions $\gamma > 1$ and $\kappa \geq 1$ are satisfied.

$G(t,p)$ as well as on the function $\tilde{G}(t,p)$. Comparing c_t^* with $\tilde{c}_t^*$ from equation (26), it can be seen that the consumption process looks very similar. The regret-averse investor consumes based on a non-linear combination of $G(t,p)$ and the foregone function $\tilde{G}(t,p)$, whereas the auxiliary investor consumes purely depending on $\tilde{G}(t,p)$. Like for the optimal investment strategy, it should be noted that also the optimal consumption choice does not depend directly on foregone wealth. However, the $\tilde{G}(t,p)$ function is derived for the foregone wealth process $\tilde{W}$.

The investor consumes differently depending on her regret aversion parameter κ. If $\kappa = 0$, then the optimal consumption process for the regret-averse investor reduces to the optimal foregone consumption process of the auxiliary investor. This is in line with expectations. Furthermore, given γ and ψ, it holds that the larger κ, the more the agent consumes if she were the auxiliary investor. This can be seen from the following limits

$$\lim_{\kappa\to\infty} \frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)} \to \frac{-\psi\gamma}{\varphi} \qquad \text{and} \qquad \lim_{\kappa\to\infty} \frac{-\gamma\psi}{\varphi+\kappa\psi} \to 0$$

An economic interpretation of this result is that the regret aversion towards the foregone consumption drives her to consume as if she would have had the foregone total wealth. Similar to how an infinitely regret-averse agent invests all of her total available wealth into the risky asset.

The EIS parameter ψ affects the consumption process differently depending on whether $0 < \psi < 1$ or $\psi > 1$. For $0 < \psi < 1$, it can be shown that $\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)} < 0$ and also $\frac{-\gamma\psi}{\varphi+\kappa\psi} < 0$. Hence, the functions $\tilde{G}(t,p)$ and $G(t,p)$ are to the power of a negative value. In the auxiliary model, it also holds for $0 < \psi < 1$ that $\tilde{G}(t,p)$ is to the power of a negative value $\frac{-\gamma\psi}{\varphi} < 0$. Additionally, for $0 < \delta < 1$ (δ is typically between $(0, 0.15)$), it holds that $\delta^\psi > \delta$. However, for $\psi > 1$, the results are less unambiguous. It can be shown that $\varphi+\kappa\psi < 0$ if $1 < \psi < \frac{\gamma+\kappa-1}{\kappa}$. Under this condition it holds that $\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)} < 0$ and $\frac{-\gamma\psi}{\varphi+\kappa\psi} > 0$. Otherwise, the fractions satisfy $\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)} \leq 0$ and $\frac{-\gamma\psi}{\varphi+\kappa\psi} \leq 0$. The effect of ψ is partly mitigated by the effect of κ. This is in contrast to the results for the auxiliary model, where it was found that $\tilde{G}(t,p)$ is to the power of a positive value $\frac{-\gamma\psi}{\varphi} > 0$ for $\psi > 1$.

This concludes the analysis of the optimal consumption strategy for a regret-averse investor.

At last, the investor should buy life-insurance contracts against life shocks with optimal notional $(\iota^q)^*$ for all states $(q \neq p) \in \mathcal{Q}$ with $(\iota^q)^*$ being specified by

$$\begin{aligned}(\iota^q)^*(t,W,y,p) = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}} (W + yF(t,p)) \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t,p,q)F(t,q))\end{aligned} \tag{45}$$

again with $F(t,p)$ being given by equation (40), $G(t,p)$ by equation (42), and $\tilde{G}(t,p)$ by equation (27).

The regret-averse investor should buy life-insurance contracts with optimal notional $(\iota^q)^*$ for states $(q \neq p) \in \mathcal{Q}$ depending on her total wealth, the transition rates $\hat{h}^{p,q}$ and $h^{p,q}$, and the time- and state-dependent functions $\tilde{G}(t,p)$ and $G(t,p)$. The investor might short-sell her life-insurance contracts, i.e. $(\iota_t^q)^* < 0$, in case the total wealth in the new biometric state exceeds the time- and state-dependent fraction of current total wealth. Comparably to the previous optimal control processes, the optimal choice of the notional does not directly depend on foregone wealth $\tilde{W}$. The optimal notional $(\iota^q)^*$ is very similar to the optimal foregone notional $(\tilde{\iota}^q)^*$ of equation (30). The regret-averse investor decides on the notional, not only based on a time- and state-dependent fraction of $\frac{G(t,q)}{G(t,p)}$. but also based on the foregone fraction $\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}$. As expected, the regret-averse investor behaves like the auxiliary investor in case $\kappa \to \infty$. Furthermore, note that for $\kappa = 0$, the agent buys life-insurance contracts with a similar notional as in the auxiliary model, but the underlying function $G(t,p)$ is different compared to the auxiliary model. As explained previously, $G(t,p)$ depends on the optimal investment strategy, whereas $\tilde{G}(t,p)$ depends on a suboptimal investment strategy. Finally, the optimal insurance strategy for a regret-averse investor does not directly depend on the EIS parameter ψ. However, it should be noted that the underlying functions $\tilde{G}(t,p)$ and $G(t,p)$ depend on ψ.

This concludes the analysis of the optimal notional choice for biometric state q.

In conclusion, the results for this arbitrary-EIS-regret-utility specification with exogenous income subject to possible biometric shocks can be stated in the following theorem.

Theorem 3.2 (Biometric risk model for a regret-averse investor with arbitrary-EIS-regret-utility specifications). *For a regret-averse investor with arbitrary-EIS-regret-utility specifications living in a Black-Scholes world with exogenous labor income and who is subject to biometric shocks, it holds that the financial wealth dynamics of the investor in biometric state $p \neq \mathcal{Q}$ are given by*

$$\mathrm{d}W_t = \left[rW_t + \theta_t \lambda \sigma - c_t + y_t - \sum_{q \neq p} \iota_t^q \hat{h}_t^{I_t,q} \right] \mathrm{d}t + \theta_t \sigma \, \mathrm{d}Z_t$$

with $W_{t^q} = W_{t^q-} + \iota_{t^q-}^q$.

The value function is given by

$$J(t,W,\hat{W},y,p) = \frac{1}{1-\gamma} G(t,p)^\gamma (W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^\kappa$$

where both $F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t \geq T^r\}}(t)$ and $G(t,p)$ are satisfying a system of ordinary differential equations. The function $F(t,p)$ should satisfy the following system

of ordinary differential equations

$$\begin{aligned}\frac{\partial F^a}{\partial t}(t,p) &= \left[r + \sum_{q\neq p} \hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right] F^a(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^a(t,q) \\ \frac{\partial F^r}{\partial t}(t,p) &= \left[r + \sum_{q\neq p} \hat{h}_t^{p,q}\right] F^r(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^r(t,q)\end{aligned} \tag{46}$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ *and* $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

Moreover, the function $G(t,p)$ *should satisfy*

$$\begin{aligned}\frac{\partial G}{\partial t}(t,p) = &\left[\frac{1}{\gamma}\left(\delta\varphi + \sum_{q\neq p} h_t^{p,q}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \sum_{q\neq p}\hat{h}_t^{p,q}\right) - \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2}\right)\right. \\ &\left. - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 + \left(\frac{\kappa}{\gamma}\right)\left(\delta^\psi \tilde{G}(t,p)^{-\frac{\psi\gamma}{\varphi}} - \lambda\sigma + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right) h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\right)\right] G(t,p) \\ &- \left(\frac{\gamma+\kappa}{\gamma}\right)\sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}} G(t,p)^{\frac{\kappa}{\gamma+\kappa}} \\ &- \left(\frac{\delta^\psi}{\gamma}\right)(\varphi - 1 + \gamma + \kappa)\,\tilde{G}(t,p)^{\frac{-\psi^2\kappa\gamma}{\varphi(\varphi+\kappa\psi)}} G(t,p)^{\frac{\varphi+(\kappa-\gamma)\psi}{\varphi+\kappa\psi}}\end{aligned} \tag{47}$$

with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

The optimal consumption-investment-insurance strategy is given by the following expressions

$$c^*(t,W,y,p) = \delta^\psi \tilde{G}(t,p)^{\frac{-\psi^2\gamma\kappa}{\varphi(\varphi+\kappa\psi)}} G(t,p)^{\frac{-\gamma\psi}{\varphi+\kappa\psi}}(W + yF(t,p)) \tag{48}$$

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W_t + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{49}$$

$$\begin{aligned}(\iota^q)^*(t,W,y,p) = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}}(W + yF(t,p))\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t,p,q)F(t,q))\end{aligned} \tag{50}$$

where $\tilde{G}(t,p)$ *is determined by the auxiliary model and should satisfy the following system* *of ordinary differential equations*

$$\frac{\partial \tilde{G}}{\partial t}(t,p) = \left[\frac{1}{\gamma}\left(\delta\varphi + \sum_{q\neq p} h_t^{p,q}\right) + \frac{\gamma-1}{\gamma}\left(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2\right)\right]\tilde{G}(t,p) + \frac{\delta^{\psi}(\gamma-1)}{\gamma(\psi-1)}\tilde{G}(t,p)^{\frac{\gamma\psi-1}{\gamma-1}} - \sum_{q\neq p} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\tilde{G}(t,q) \tag{51}$$

with boundary conditions $\tilde{G}(t,Q) = \tilde{G}(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

This concludes the analysis of the optimal consumption-investment-insurance strategy for a regret-averse investor who is subject to biometric shocks with an arbitrary EIS parameter value. The sequential sections will first discuss the special case of $\psi = \frac{1}{\gamma}$, i.e. CRRA-regret-utility specification (Section 3.4), and thereafter the special case of $\psi = 1$, i.e. the unit-EIS-regret-utility specification (Section 3.5).

3.4 CRRA-regret-utility specification

This section discusses the special case of $\psi = \frac{1}{\gamma} < 1$[6]. It will be shown that for this specific value of the EIS parameter the model will reduce to a regret-averse time-additive power-utility model. This model is referred to as the CRRA-regret model.

3.4.1 Auxiliary model

As stated in Section 3.3.1, it is assumed that the auxiliary investor maximizes the utility over intermediate consumption and terminal wealth. Since $\psi = \frac{1}{\gamma} < 1$, equation (19) collapses to

$$f(\tilde{c},\tilde{J}) = \frac{\delta}{1-\gamma}\tilde{c}^{1-\gamma} - \delta\tilde{J} \tag{52}$$

as $\varphi = 1$. As a result, equation (18) reduces to

$$\tilde{J}(t,\tilde{W},y,p) = \sup_{\left(\tilde{c},(\tilde{\iota}^q)_{q=0}^{Q}\right)\in\tilde{\mathcal{A}}_t} \delta\left(\mathbb{E}_{t,\tilde{W},y,p}\left[\int_t^{\tau} \mathrm{e}^{-\delta(s-t)}\left(\frac{\tilde{c}_s^{1-\gamma}}{1-\gamma}\right)\mathrm{d}s + \frac{1}{\delta}\varepsilon\mathrm{e}^{-\delta(\tau-t)}\left(\frac{\tilde{W}_\tau^{1-\gamma}}{1-\gamma}\right)\right]\right) \tag{53}$$

This is a positive multiple of the classical time-additive power-utility specification (Munk, 2017). To show the most general results possible for the CRRA-regret model, this function is generalized to

$$\tilde{J}(t,\tilde{W},y,p) = \sup_{\left(\tilde{c},(\tilde{\iota}^q)_{q=0}^{Q}\right)\in\tilde{\mathcal{A}}_t} \mathbb{E}_{t,\tilde{W},y,p}\left[\int_t^{\tau} \mathrm{e}^{-\delta(s-t)}v(\tilde{c}_s)\,\mathrm{d}s + \mathrm{e}^{-\delta(\tau-t)}\bar{v}(\tilde{W}_\tau)\right] \tag{54}$$

[6]By definition, it holds that $\gamma > 1$ and as a result, $\psi = \frac{1}{\gamma} < 1$.

where the integral goes from t to τ with τ being specified by equation (13). The time preference of the investor is denoted by δ. The utility functions for intermediate consumption and terminal wealth are given by

$$v(\tilde{c}) = \varrho \frac{\tilde{c}^{1-\gamma}}{1-\gamma} \qquad\qquad \bar{v}(\tilde{W}_\tau) = \varepsilon \frac{\tilde{W}_\tau^{1-\gamma}}{1-\gamma}$$

The weight of intermediate consumption is given by ϱ and the weight of the bequest motive is given by ε. It should be noted that equation (53) is a special case of equation (54) with $\varrho = \delta$.

The HJB equation for the auxiliary investor in biometric state p is given by

$$\begin{aligned} 0 = &- \delta \tilde{J} + \mathcal{L}^{\tilde{c}} + \mathcal{L}^{\tilde{\theta}} + \mathcal{L}^{\tilde{\iota}} + \tilde{J}_t \\ &+ \tilde{J}_{\tilde{W}}[(\tilde{W} + yF(t,p))r - yF(t,p)r + y] \\ &+ \tilde{J}_y y\alpha(t,p) + \frac{1}{2}\tilde{J}_{yy}y^2\zeta(t,p)^2 - \sum_{q\neq p} h_t^{p,q}\tilde{J} \end{aligned} \tag{55}$$

with

$$\begin{aligned} \mathcal{L}^{\tilde{c}} &= \sup_{\tilde{c}\geq 0}\left\{\varrho\frac{\tilde{c}^{1-\gamma}}{1-\gamma} - \tilde{c}\tilde{J}_{\tilde{W}}\right\} \\ \mathcal{L}^{\tilde{\theta}} &= \tilde{J}_{\tilde{W}}\tilde{\theta}\sigma\lambda + \frac{1}{2}\tilde{J}_{\tilde{W}\tilde{W}}\tilde{\theta}^2\sigma^2 + \tilde{J}_{\tilde{W}y}\tilde{\theta}\sigma y\zeta(t,p) \\ \mathcal{L}^{\tilde{\iota}} &= \sup_{(\tilde{\iota}_t^q)_{q=0}^{Q}\in\mathbb{R}}\left\{-\tilde{J}_{\tilde{W}}\sum_{q\neq p}\tilde{\iota}^q\hat{h}_t^{p,q} + \sum_{q\neq p,Q} h_t^{p,q}\tilde{J}(t,\tilde{W}+\tilde{\iota}^q, yP(t,p,q),q) + h_t^{p,q}\frac{\varepsilon}{1-\gamma}(\tilde{W}+\tilde{\iota}^Q)^{1-\gamma}\right\} \end{aligned}$$

Subscripts of $\tilde{J}$ denote partial derivatives with respect to either the state variables or time t and the terminal condition is given by $\tilde{J}(T,\tilde{W},y,p) = \frac{\varepsilon}{1-\gamma}\tilde{W}^{1-\gamma}$.

In Section 3.3.1, it was conjectured that the indirect utility function $\tilde{J}(t,\tilde{W},y,p)$ has a solution of the form

$$\tilde{J}(t,\tilde{W},y,p) = \frac{\tilde{G}(t,p)^\gamma}{1-\gamma}(\tilde{W} + yF(t,p))^{1-\gamma}$$

This conjecture does not depend on ψ and hence remains the same. The partial derivatives can be found in the Appendix 7.3.

Based on the results of Section 3.3.1, the foregone investment strategy and the optimal foregone consumption process and notional choice can be determined. From equation (31), it is known that the foregone investment strategy is given by

$$\tilde{\theta}(t,\tilde{W},y,p) = \left(\tilde{W}_t + yF(t,p)\right) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{56}$$

Furthermore, from equation (30), it is known that the optimal foregone notional choice is given by

$$(\tilde{\iota}^q)^*(t,\tilde{W},y,p) = \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)\left(\tilde{W}+yF(t,p)\right)\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} - \left(\tilde{W}+yP(t,p,q)F(t,q)\right) \tag{57}$$

The optimal consumption process can be determined based on the FOC of $\mathcal{L}^{\tilde{c}}$ with respect to $\tilde{c}$. At the optimum, marginal utility from consumption should be equal to marginal utility from wealth. The envelope condition yields that optimal foregone consumption is given by

$$\tilde{c}^* = \varrho^{\frac{1}{\gamma}}\tilde{J}_{\tilde{W}}^{-\frac{1}{\gamma}} \tag{58}$$

Substituting in the conjecture yields the optimal foregone consumption strategy

$$\tilde{c}^*(t,\tilde{W},y,p) = \varrho^{\frac{1}{\gamma}}\frac{\tilde{W}+yF(t,p)}{\tilde{G}(t,p)} \tag{59}$$

As previously stated, the optimal foregone consumption process is identical to equation (29) for $\psi = \frac{1}{\gamma}$ and $\varrho = \delta$.

Substituting the foregone investment strategy, the optimal foregone consumption-insurance strategy, and the conjecture in the HJB equation (55) yields an ODE from which again the functions $\tilde{G}(t,p)$ and $F(t,p)$ can be determined. For completeness, the equation is given in the Appendix 7.4 by equation (116).

The time- and state-dependent function $\tilde{G}(t,p)$ should satisfy the following system of ODEs

$$\frac{\partial\tilde{G}}{\partial t}(t,p) = \left[\frac{1}{\gamma}\left(\delta+\sum_{q\neq p}h_t^{p,q}\right)+\frac{\gamma-1}{\gamma}\left(r+\sum_{q\neq p}\hat{h}_t^{p,q}+\sigma\lambda-\frac{1}{2}\gamma\sigma^2\right)\right]\tilde{G}(t,p) - \varrho^{\frac{1}{\gamma}} - \sum_{q\neq p}h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\tilde{G}(t,q) \tag{60}$$

with boundary condition $\tilde{G}(t,Q) = \tilde{G}(T,p) = \varepsilon^{\frac{1}{\gamma}}$. As expected, ODE (27) reduces for $\varrho = \delta$ and $\psi = \frac{1}{\gamma}$ to this ODE.

The purely time- and state-dependent function $F(t,p)$ is given by

$$\frac{\partial F}{\partial t}(t,p) = \left[r+\sum_{q\neq p}\hat{h}_t^{p,q}-\alpha(t,p)+\zeta(t,p)\lambda\right]F(t,p)-1-\sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F(t,q) \tag{61}$$

with boundary condition $F(t,Q) = F(T,p) = 0$.

Moreover, $F(t,p)$ can be separated based on whether the investor is active in the labor market or retired (Hambel et al., 2022). This separation yields

$$F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t\geq T^r\}}(t)$$

with

$$\frac{\partial F^a}{\partial t}(t,p) = \left[r + \sum_{q\neq p}\hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right]F^a(t,p) - 1 - \sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^a(t,q)$$

and

$$\frac{\partial F^r}{\partial t}(t,p) = \left[r + \sum_{q\neq p}\hat{h}_t^{p,q}\right]F^r(t,p) - 1 - \sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^r(t,q)$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ and $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

In case the transition rates are directional, i.e. $h_t^{p,q} = 0$ for $q \leq p$, then $\tilde{G}(t,p)$, $F^a(t,p)$, and $F^r(t,p)$ have explicit expressions given by (Hambel et al., 2022)

$$\tilde{G}(t,p) = \int_t^T \mathrm{e}^{-\int_t^s \Delta(u,p)\,\mathrm{d}u}(\varrho^{\frac{1}{\gamma}} + \sum_{q>p} h^{p,q}(s)\left(\frac{\hat{h}^{p,q}(s)}{h^{p,q}(s)}\right)^{1-\frac{1}{\gamma}}\tilde{G}(s,q))\,\mathrm{d}s + \varepsilon^{\frac{1}{\gamma}}\mathrm{e}^{-\int_t^T \Delta(u,p)\,\mathrm{d}u}$$

with $\Delta(t,p) = \frac{1}{\gamma}(\delta + \sum_{q\neq p} h_t^{p,q}) + \frac{\gamma-1}{\gamma}(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2)$ and

$$F^r(t,p) = \int_{T^r}^T \mathrm{e}^{-\int_t^s (r+\sum_{q>p}\hat{h}^{p,q}(u))\,\mathrm{d}u}(1 + \sum_{q>p}\hat{h}^{p,q}(s)F^r(s,q))\,\mathrm{d}s$$

$$F^a(t,p) = \int_t^{T^r} \mathrm{e}^{-\int_t^s (r+\sum_{q>p}\hat{h}^{p,q}(u)-\alpha(u,p)+\zeta(u,p)\lambda)\,\mathrm{d}u}(1 + \sum_{q>p}\hat{h}^{p,q}(s)p(s,p,q)F^a(s,q))\,\mathrm{d}s$$
$$+ \Gamma(p)F^r(T^r,p)\mathrm{e}^{-\int_t^{T^r}(r+\sum_{q>p}\hat{h}^{p,q}(u)-\alpha(u,p)+\zeta(u,p)\lambda)\,\mathrm{d}u}.$$

This concludes the results of the auxiliary investor for the general time-additive power-utility specification. These results will again be utilized in determining the optimal consumption-investment-insurance strategy for the CRRA-regret-utility specification.

3.4.2 Regret-averse model

The regret-averse investor maximizes her utility from intermediate consumption and terminal wealth. As described in section 3.1, for the special case of $\psi = \frac{1}{\gamma}$ reduces equation (33) to

$$\mathcal{F}(c,\hat{c},J) = \frac{\delta}{1-\gamma}c^{1-\gamma}\left(\frac{\hat{c}}{c}\right)^{\kappa} - \delta J \tag{62}$$

As a result, the indirect utility function (32) of the regret-averse investor collapses to the following expression

$$J(t,W,\hat{W},y,p)=\sup_{\left(c,\theta,(\iota^q)_{q=0}^{Q}\right)\in\mathcal{A}_t}\delta\left(\mathbb{E}_{t,W,\hat{W},y,p}\left[\int_t^\tau \mathrm{e}^{-\delta(s-t)}\frac{\delta}{1-\gamma}c_s^{1-\gamma}\left(\frac{\hat{c}_s}{c_s}\right)^\kappa \mathrm{d}s+\frac{1}{\delta}\varepsilon\mathrm{e}^{-\delta(\tau-t)}\frac{W_\tau^{1-\gamma}}{1-\gamma}\left(\frac{\hat{W}_\tau}{W_\tau}\right)^\kappa\right]\right)$$

Similar as in the auxiliary model, this can be seen as a positive multiple of a regret-averse time-additive power-utility specification. To generalize the results, it is assumed that the preferences of the investor are captured by the following time-additive expected utility function

$$J(t,W,\hat{W},y,p)=\sup_{\left(c,\theta,(\iota^q)_{q=0}^{Q}\right)\in\mathcal{A}_t}\mathbb{E}_{t,W,\hat{W},y,p}\left[\int_t^\tau \mathrm{e}^{-\delta(s-t)}u(c_s,\hat{c}_s)\,\mathrm{d}s+\mathrm{e}^{-\delta(\tau-t)}\bar{u}(W_\tau,\hat{W}_\tau)\right] \quad (63)$$

where again τ is given by equation (13). The optimal foregone consumption process, notional choice, and wealth are denoted by $\hat{c}$, $\hat{\iota}$, and $\hat{W}$, respectively. The utility functions for intermediate consumption and terminal wealth are given by

$$u(c,\hat{c})=\varrho\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\hat{c}}{c}\right)^\kappa \qquad \bar{u}(W_\tau,\hat{W}_\tau)=\varepsilon\frac{W_\tau^{1-\gamma}}{1-\gamma}\left(\frac{\hat{W}_\tau}{W_\tau}\right)^\kappa$$

with $\varrho\geq 0$ and $\varepsilon\geq 0$ being the relative weight of intermediate consumption and the bequest motive, respectively. Note that these are the regret-averse power-utility functions as specified in section 3.1 equation (5) with regret measured over the relative difference between foregone consumption/wealth and realized consumption/wealth.

Based on the indirect utility function (63) and wealth dynamics (17), the following Hamilton-Jacobi-Bellman equation for an investor in biometric state p can be specified

$$\begin{aligned}0=&-\delta J+\mathcal{L}^c+\mathcal{L}^\theta+\mathcal{L}^\iota+J_t\\&+J_W[(W+yF(t,p))r-yF(t,p)r+y]\\&+J_{\hat{W}}\left[(\hat{W}+yF(t,p))r-yF(t,p)r+\tilde{\theta}\sigma\lambda+y-\hat{c}-\sum_{p\neq q}\hat{\iota}^q\hat{h}_t^{p,q}\right]\\&+\frac{1}{2}J_{\hat{W}\hat{W}}\tilde{\theta}^2\sigma^2+J_y y\alpha(t,p)+\frac{1}{2}J_{yy}y^2\zeta(t,p)^2\\&+J_{\hat{W}y}\tilde{\theta}\sigma y\zeta(t,p)-\sum_{q\neq p}h_t^{p,q}J\end{aligned} \quad (64)$$

with

$$\mathcal{L}^c = \sup_{c\geq 0}\left\{\varrho\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\hat{c}}{c}\right)^{\kappa} - cJ_W\right\}$$

$$\mathcal{L}^\theta = \sup_{\theta\in\mathbb{R}}\left\{J_W\theta\sigma\lambda + \frac{1}{2}J_{WW}\theta^2\sigma^2 + J_{Wy}\theta\sigma y\zeta(t,p) + J_{W\hat{W}}\theta\sigma^2\tilde{\theta}\right\}$$

$$\mathcal{L}^\iota = \sup_{(\iota_t^q)_{q=0}^Q\in\mathbb{R}}\left\{-J_W\sum_{q\neq p}\iota^q\hat{h}_t^{p,q} + \sum_{q\neq p,Q}h_t^{p,q}J(t, W+\iota^q, \hat{W}+\hat{\iota}^q, yP(t,p,q), q)\right.$$
$$\left.+h_t^{p,q}\frac{\varepsilon}{1-\gamma}(W+\iota^Q)^{1-\gamma}\left(\frac{\hat{W}+\hat{\iota}^Q}{W+\iota^Q}\right)^{\kappa}\right\}$$

Like in the auxiliary model, subscripts denote the partial derivatives with respect to the state variables and time t and the terminal condition is specified to be $J(T,W,\hat{W},y,p) = \frac{\varepsilon}{1-\gamma}W^{1-\gamma}\left(\frac{\hat{W}}{W}\right)^{\kappa}$.

In Section 3.3.2, it was conjectured that the indirect utility function is given by the following functional form

$$J(t,W,\hat{W},y,p) = \frac{G(t,p)^\gamma}{1-\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\tilde{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}$$

Like it was the case for the auxiliary model, this functional form does not depend on ψ. For completeness, the partial derivatives can be found in the Appendix 7.3.

As explained in Section 3.3.1, the optimal investment strategy and the optimal notional choices do not directly depend on the EIS parameter value ψ. Hence, the optimal investment strategy for an agent with CRRA-regret-utility preferences is identical to the investment strategy determined by equation (43). Hence, the optimal investment amount for a regret-averse investor in biometric state p at time t is given by

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W+yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{65}$$

Furthermore, the optimal insurance strategy for biometric state $q \neq p$ for a regret-averse investor in biometric state p at time t was determined in equation (45). The optimal investment strategy was given to be

$$(\iota^q)^*(t,W,y,p) = \left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}}(W+yF(t,p))\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}}$$
$$-(W+yP(t,p,q)F(t,q)) \tag{66}$$

The optimal consumption strategy can be determined by the FOC of $\mathcal{L}^c$ with respect to c. The optimal consumption strategy for a regret-averse investor with CRRA-regret-utility preferences at time t is given by

$$c^* = \left(\frac{1-\gamma-\kappa}{1-\gamma}\right)^{\frac{1}{\gamma+\kappa}} \varrho^{\frac{1}{\gamma+\kappa}} J_W^{\frac{-1}{\gamma+\kappa}} \hat{c}^{\frac{\kappa}{\gamma+\kappa}} \tag{67}$$

Substituting the conjecture and the function for optimal foregone consumption as determined in equation (59) into equation (67) yields

$$c^*(t, W, y, p) = \varrho^{\frac{1}{\gamma}} \frac{W + yF(t,p)}{G(t,p)^{\frac{\gamma}{\gamma+\kappa}} \tilde{G}(t,p)^{\frac{\kappa}{\gamma+\kappa}}} \tag{68}$$

It should be noted that, similar to the auxiliary model with $\varrho = \delta$ and $\psi = \frac{1}{\gamma}$, equation (68) reduces to equation (44).

The expressions for the optimal consumption, investment and insurance strategies can be substituted into the HJB equation (64) together with the foregone investment strategy, optimal foregone consumption and investment strategies and the conjecture to yield a lengthy equation. This equation is given for completeness in the Appendix 7.4 equation (117). From this equation, the time- and state-dependent functions for $G(t,p)$ and $F(t,p)$ can be determined.

In Section 3.3, it was concluded that the function $F(t,p)$ is the same for the auxiliary investor as well as for the regret-averse investor. The underlying income process is for both investors the same, hence the human wealth should be the same as well. Therefore, it is known that the age- and state-dependent function $F(t,p)$ is given by equation (61).

The age- and state-dependent function $G(t,p)$ should satisfy the following non-linear system of ODEs

$$\begin{aligned}\frac{\partial G}{\partial t}(t,p) = &\left[\frac{1}{\gamma}\left(\delta + \sum_{q\neq p} h_t^{p,q}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \sum_{q\neq p} \hat{h}_t^{p,q}\right) - \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2}\right)\right.\\ &\left. - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 + \left(\frac{\kappa}{\gamma}\right)\left(\varrho^{\frac{1}{\gamma}}\tilde{G}(t,p)^{-1} - \lambda\sigma + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right) h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\right)\right] G(t,p)\\ &- \left[\left(\frac{\gamma+\kappa}{\gamma}\right)\left(\varrho^{\frac{1}{\gamma}}\tilde{G}(t,p)^{-\frac{\kappa}{\gamma+\kappa}} + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}}\right)\right] G(t,p)^{\frac{\kappa}{\gamma+\kappa}}\end{aligned} \tag{69}$$

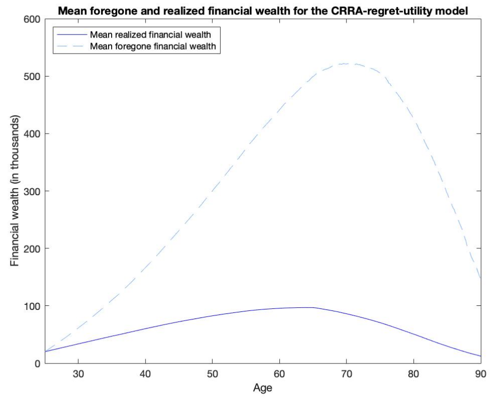
with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

Hence, purely time- and state-dependent functions $F(t,p)$ and $G(t,p)$ have been found. It can be concluded that an investor with CRRA-regret-utility preferences invests θ_t^* amount of money into the risky asset according to equation (65), buys life-insurance contracts for biometric state $q \neq p$ with notional $(\iota_t^q)^*$ as given by equation (66), and consumes c_t^* as specified by equation (68) at time t. It can easily be seen that for all these optimal control processes the functions reduce to the optimal control processes of the auxiliary model for $\kappa = 0$. In absence of regret aversion, all results reduce to purely risk-averse results. This is in line with the expectation.

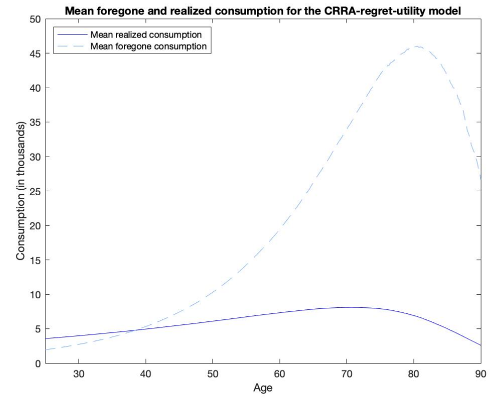
To further illustrate the retrieved results for the special case of CRRA-regret-utility preferences, some graphical representations are shown. The survival model of Richard (1975) (see Figure 1) has been considered. Hence, the investor is only subject to mortality risk. The results for the graphs have been retrieved by utilizing a Monte-Carlo simulation for parameter values as described in Section 4 Table 2 with $\psi = \frac{1}{\gamma}$ and the weight for intermediate consumption is given by $\varrho = \delta$. The reader is referred to Section 4.1 for a more in-depth explanation of the Monte-Carlo simulation.

Figure 3 illustrates the optimal foregone and realized wealth and control process for a regret-averse investor with CRRA-regret-utility preferences in the survival model. From Figure 3 (a), it can be seen that on average foregone wealth exceeds realized wealth, as conjectured. Thus, the investor does experience regret over her investment, as expected. Figure 3 (b) shows that the foregone consumption process of the regret-averse investor exceeds the realized consumption process on average at an age of approximately 40 years. The realized consumption is slightly higher at the beginning of the life cycle than the foregone consumption on average. This is due to the difference in underlying age- and state-dependent functions $G(t, p)$ and $\tilde{G}(t, p)$. The realized (foregone) investment fraction π is given by the investment amount θ^* ($\tilde{\theta}$) divided by financial wealth W ($\hat{W}$). As the optimal investment fraction is very sensitive to financial wealth values close to zero, Figure 3 (c) shows the trimmed mean of the investment fraction where the 5% highest and lowest observed π's have been trimmed. This gives a smoother function of the investment fraction. It can be seen that the optimal regret-averse investment fraction exceeds one in the younger ages and approaches $\frac{\lambda+\kappa\sigma}{\sigma(\gamma+\kappa)}$ as the human wealth decreases to zero. At younger ages it is advised to invest more wealth into the stock market. Figure 3 (d) shows the optimal foregone and realized notional choice for the biometric state *dead*. The average foregone notional choice is very negative at older ages as financial wealth greatly exceeds the time- and state-dependent fraction of total wealth. Realized optimal notional choice is less negative, but is on average still negative over the life-cycle showing that agents short-sell their life-insurance contracts to generate more wealth.

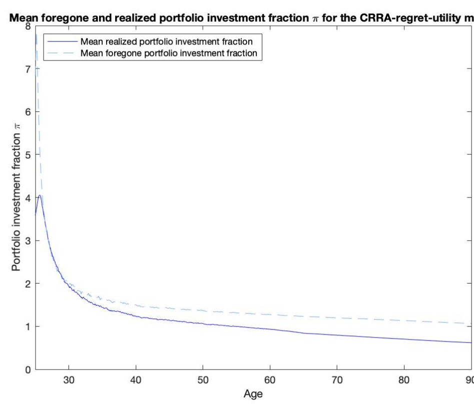
As previously described in section 3.1, the CRRA-regret-utility specification satisfies all desired properties of a multiplicative regret-utility function as described by Goossens (2021) if the ratio between foregone and realized wealth is at least $\frac{\kappa}{\gamma+\kappa-1}$. Figure 4 shows the 1%-, median and 99%-quantiles of this ratio retrieved from the Monte-Carlo simulation for the survival model. It can be seen that the 1%-quantile is above the lower bound $\frac{\kappa}{\gamma+\kappa-1} = 0.375$ for all ages. Hence, the proposed regret-utility function satisfies property $P2c$ in almost all sample paths. However, it should be noted that in some paths the ratio is smaller than the lower bound. Thus this property is not guaranteed to be satisfied for all states of the world.



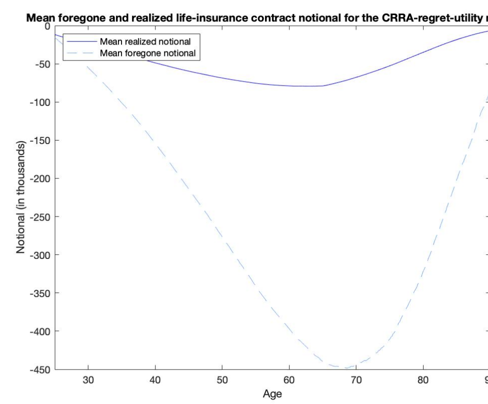
(a) Average foregone and realized financial wealth for an investor with CRRA-regret-utility preferences in the survival model.



(b) Average foregone and realized consumption for an investor with CRRA-regret-utility preferences in the survival model.



(c) Average foregone and realized investment fraction π for an investor with CRRA-regret-utility preferences in the survival model.



(d) Average foregone and realized life-insurance contract notional for an investor with CRRA-regret-utility preferences in the survival model.

Figure 3: The figure shows the by Monte-Carlo simulation obtained average optimal financial wealth and control processes for an investor with CRRA-regret-utility preferences who is only subject to mortality risk and has access to perfect life-insurance contracts in line with Richard (1975).

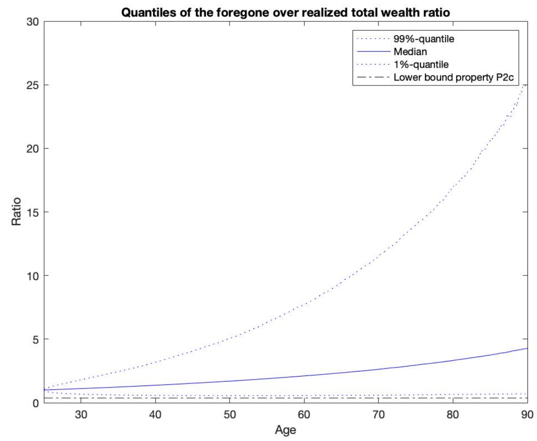


Figure 4: This figure shows the by Monte-Carlo simulation obtained median, the 1%- and 99%-quantile of the ratio between foregone and realized total wealth in the survival model. The 1%-quantile exceeds the lower bound of property $P2c$ as determined in section 3.1.

In conclusion, the results for this special case CRRA-regret-utility specification for an agent with exogenous income who is subject to biometric risk can be stated in the following theorem.

Theorem 3.3 (Biometric risk model for a regret-averse investor with CRRA-regret-utility specifications). *For a regret-averse investor with CRRA-regret-utility specifications living in a Black-Scholes world with exogenous labor income and who is subject to biometric risks, it holds that the financial wealth dynamics of the investor in biometric state $p \neq Q$ are given by*

$$\mathrm{d}W_t = \left[rW_t + \theta_t \lambda \sigma - c_t + y_t - \sum_{q \neq p} \iota_t^q \hat{h}_t^{I_t,q} \right] \mathrm{d}t + \theta_t \sigma \, \mathrm{d}Z_t$$

with $W_{t^q} = W_{t^q-} + \iota^q_{t^q-}$.

The value function is given by

$$J(t, W, \hat{W}, y, p) = \frac{1}{1-\gamma} G(t,p)^\gamma (W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^\kappa$$

where both $F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t \geq T^r\}}(t)$ and $G(t,p)$ are satisfying a system of ordinary differential equations. The function $F(t,p)$ should satisfy the following system of ordinary differential equations

$$\begin{aligned}
\frac{\partial F^a}{\partial t}(t,p) &= \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda \right] F^a(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q) F^a(t,q) \\
\frac{\partial F^r}{\partial t}(t,p) &= \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} \right] F^r(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q) F^r(t,q)
\end{aligned} \tag{70}$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ *and* $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

Moreover, the function $G(t,p)$ *should satisfy*

$$\begin{aligned}
\frac{\partial G}{\partial t}(t,p) = \Bigg[& \frac{1}{\gamma}\left(\delta + \sum_{q\neq p} h_t^{p,q}\right) + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \sum_{q\neq p}\hat{h}_t^{p,q}\right) - \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2}\right) \\
& - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 + \left(\frac{\kappa}{\gamma}\right)\left(\varrho^{\frac{1}{\gamma}}\tilde{G}(t,p)^{-1} - \lambda\sigma + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right) h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\right)\Bigg] G(t,p) \\
& - \left[\left(\frac{\gamma+\kappa}{\gamma}\right)\left(\varrho^{\frac{1}{\gamma}}\tilde{G}(t,p)^{-\frac{\kappa}{\gamma+\kappa}} + \sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}}\right)\right] G(t,p)^{\frac{\kappa}{\gamma+\kappa}}
\end{aligned} \tag{71}$$

with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

The optimal consumption-investment-insurance strategy is given by the following expressions

$$c^*(t,W,y,p) = \varrho^{\frac{1}{\gamma}}\frac{W + yF(t,p)}{G(t,p)^{\frac{\gamma}{\gamma+\kappa}}\tilde{G}(t,p)^{\frac{\kappa}{\gamma+\kappa}}} \tag{72}$$

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W_t + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{73}$$

$$\begin{aligned}
(\iota^q)^*(t,W,y,p) = & \left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}}(W + yF(t,p))\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\
& - (W + yP(t,p,q)F(t,q))
\end{aligned} \tag{74}$$

where $\tilde{G}(t,p)$ *is determined by the auxiliary model and should satisfy the following system of ordinary differential equations*

$$\begin{aligned}
\frac{\partial \tilde{G}}{\partial t}(t,p) = & \left[\frac{1}{\gamma}(\delta + \sum_{q\neq p} h_t^{p,q}) + \frac{\gamma-1}{\gamma}(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2)\right]\tilde{G}(t,p) \\
& - \varrho^{\frac{1}{\gamma}} - \sum_{q\neq p} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\tilde{G}(t,q)
\end{aligned} \tag{75}$$

with boundary conditions $\tilde{G}(t,Q) = \tilde{G}(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

This concludes the analysis of the optimal portfolio, consumption, and notional choices of a regret-averse investor with CRRA-regret-utility preferences who experiences possible health shocks affecting her future income process. The following corollary provides the results for a restricted model where the regret-averse investor earns spanned exogenous income without being subject to biometric risks. The utility preferences of the regret-averse investor are assumed to be given by the CRRA-regret-utility function (5). The model is solved using the same method of first deriving a closed-form solution to the auxiliary model and then substituting this solution into the regret-averse model. Results for the classical Merton portfolio problem with spanned exogenous income (Merton, 1969, and Munk, 2017) for a regret-averse investor will be shown. The corresponding proof is in the Appendix 7.6.

Corollary 3.1 (Merton portfolio problem with spanned exogenous labor income for a CRRA regret-averse investor). *The regret-averse investor earns exogenous labor income without biometric risk. The labor income dynamics are specified by the following stochastic differential equation*

$$\mathrm{d}y_t = y_t \left[\alpha \, \mathrm{d}t + \zeta \, \mathrm{d}Z_t\right]$$

The labor income process follows a geometric Brownian motion (GBM) with the same underlying Brownian motion as the stock market. Hence, the market is complete.

The financial wealth dynamics of the investor are given by

$$\mathrm{d}W_t = \left[rW_t + \theta_t \lambda \sigma - c_t + y_t\right] \, \mathrm{d}t + \theta_t \sigma \, \mathrm{d}Z_t$$

The value function of the Merton problem with exogenous labor income for a CRRA regret-averse investor is given by

$$J(t, W, \hat{W}, y) = \frac{1}{1-\gamma} g(t)^{\gamma} \left(W + H(t, y)\right)^{1-\gamma} \left(\frac{\hat{W} + H(t, y)}{W + H(t, y)}\right)^{\kappa}$$

where $g(t)$ is a purely time-dependent function satisfying the following ODE

$$g'(t) = \left[A + B(t)\right] g(t) - C(t) g(t)^{\frac{\kappa}{\gamma+\kappa}} \tag{76}$$

with terminal condition $g(T) = \varepsilon^{\frac{1}{\gamma}}$ and A, $B(t)$, and $C(t)$ as given by the following expressions

$$\begin{aligned} A = \frac{1}{\gamma(\gamma+\kappa)} \Bigg\{ &\delta(\gamma+\kappa) + r(\gamma-1)(\gamma+\kappa) - \kappa(\gamma+\kappa)\sigma\lambda - \frac{1}{2}\kappa(\kappa-1)(\gamma+\kappa)\sigma^2 \\ &- (1-\gamma-\kappa)\left(\frac{1}{2}\lambda^2 + \frac{1}{2}\kappa^2\sigma^2 + \kappa\sigma\lambda\right)\Bigg\} \end{aligned}$$

$$B(t) = \left(\frac{\kappa}{\gamma}\right) \varrho^{\frac{1}{\gamma}} \tilde{g}(t)^{-1}$$

$$C(t) = \left(\frac{\gamma+\kappa}{\gamma}\right) \varrho^{\frac{1}{\gamma}} \tilde{g}(t)^{\frac{-\kappa}{\gamma+\kappa}}$$

with $\tilde{g}(t)$, determined from the auxiliary model, being given by

$$\tilde{g}(t) = \varrho^{\frac{1}{\gamma}} \frac{1 - \mathrm{e}^{-\tilde{A}(T-t)}}{\tilde{A}} + \varepsilon^{\frac{1}{\gamma}} \mathrm{e}^{-\tilde{A}(T-t)} \tag{77}$$

with $\tilde{A} = \frac{\delta + r(\gamma-1) + \sigma\lambda(\gamma-1) + \frac{1}{2}\sigma^2\gamma(1-\gamma)}{\gamma}$.

Equation (76) is solved by the following time-dependent function $g(t)$

$$g(t) = \left[\frac{\gamma}{\gamma+\kappa} \mathrm{e}^{\frac{\gamma}{\gamma+\kappa}\int_0^t (A+B(s))\,\mathrm{d}s} \int_t^T \mathrm{e}^{-\frac{\gamma}{\gamma+\kappa}\int_0^s (A+B(u))\,\mathrm{d}u} C(s)\,\mathrm{d}s + \mathrm{e}^{-\frac{\gamma}{\gamma+\kappa}\int_t^T (A+B(s))\,\mathrm{d}s} \varepsilon_2^{\frac{1}{\gamma+\kappa}}\right]^{\frac{\gamma+\kappa}{\gamma}} \tag{78}$$

The human wealth function $H(t,y)$ *satisfies following PDE*

$$\frac{\partial H}{\partial t}(t,y) + (\alpha - \zeta\lambda) y H_y(t,y) + \frac{1}{2}\zeta^2 y^2 H_{yy}(t,y) - rH(t,y) + y = 0 \tag{79}$$

The optimal consumption and investment strategy for the CRRA regret-averse investor are given by

$$c^*(t,W,y) = \frac{\varrho^{\frac{1}{\gamma}}\,(W + H(t,y))}{g(t)^{\frac{\gamma}{\gamma+\kappa}}\,\tilde{g}(t)^{\frac{\kappa}{\gamma+\kappa}}} \tag{80}$$

$$\theta^*(t,W,y) = \frac{\lambda + \kappa\sigma}{(\gamma+\kappa)\sigma}\,(W + H(t,y)) \tag{81}$$

The proof is shown in the Appendix 7.6.

The results for the Merton portfolio problem with exogenous income for a regret-averse investor are very similar to the results for the Merton problem with biometric risk (see Theorem 3.3). The optimal investment strategy and consumption choice are in both models given by almost identical formulas with the only difference being the underlying ODEs. In case of possible biometric shocks, these ODEs depend on both time and state, whereas for spanned exogenous income these only depend on time as the biometric state is the same for all t. It should be noted that, as expected, the biometric risk model reduces to the exogenous labor model if only a single biometric state is considered. Indeed, the ODE of the then purely time-dependent function $G(t)$ reduces to the same ODE as for the time-dependent function $g(t)$.

In case the investor does not earn any labor income (i.e. $y_t = 0$ for all t), then the model resembles the classical Merton problem for a regret-averse investor. The derivation and results of this model are omitted as these can be easily retrieved by fixing $y = 0$. The results for the classical Merton problem for a CRRA-regret-utility investor yield the same relationship between with and without exogenous income as in case of a classical power-utility investor (see e.g. Munk (2017)). However, it should be noted that in contrary to the model without labor income, the financial wealth W can turn negative, but total wealth $W + H(t,y)$ remains strictly positive. At younger ages, it is advised to invest more into the stock market than at older ages as the human wealth is larger at an earlier age. This can turn investments negative, i.e. it is advised to borrow money and invest into the stock market.

This concludes the analysis for the CRRA regret-averse utility specification. In the next section, results for the unit-EIS regret-averse utility specification will be shown.

3.5 Unit-EIS-regret-utility specification

In this section, it is assumed that the preferences for the elasticity of intertemporal substitution (EIS) of the investor are given by the unit value, i.e. $\psi = 1$. The regret-aversion-adjusted normalized aggregator function $\mathcal{F}$ (9) is given by the limiting case. This model specification is referred to as the unit-EIS-regret-utility specification. Note that the results shown for this unit-EIS-regret model could also have been derived by taking the limit of the results shown in Section 3.3. Hence, the unit-EIS-regret model is a special case of the arbitrary-EIS-regret model.

3.5.1 Auxiliary model

The auxiliary investor maximizes the utility over intermediate consumption and terminal wealth. The utility index $\tilde{J}(t, \tilde{W}, y, p)$ at time t for foregone consumption process $\tilde{c}$, investment strategy $\tilde{\theta}$, and notional choices $\tilde{\iota}^q$ over the remaining lifetime $[t, \tau]$ with τ as defined in equation (13) is given by

$$\tilde{J}(t, \tilde{W}, y, p) = \sup_{\left(\tilde{c}, (\tilde{\iota}^q)_{q=0}^{Q}\right) \in \tilde{\mathcal{A}}_t} \mathbb{E}_{t, \tilde{W}, y, p} \left[\int_t^\tau f(\tilde{c}_s, \tilde{J}_s) \, \mathrm{d}s + \tilde{\mathcal{J}}_\tau \right] \tag{82}$$

The investor maximizes $\tilde{J}$ for any $t < \tau$ over all admissible control processes in set $\tilde{\mathcal{A}}_t$ given the state variables at time t.

The normalized aggregator function f for unit-EIS as specified in equation (8) is given by

$$f(\tilde{c}, \tilde{J}) = \delta(1-\gamma)\tilde{J} \ln(\tilde{c}) - \delta \tilde{J} \ln([1-\gamma]\tilde{J}) \tag{83}$$

As explained previously, the time preference of the investor is denoted by δ, and the degree of relative risk aversion by $\gamma > 1$. The term $\tilde{\mathcal{J}}_\tau$ is given by $\tilde{\mathcal{J}}_\tau = \varepsilon \frac{\tilde{W}_\tau^{1-\gamma}}{1-\gamma}$ with $\varepsilon \geq 0$. This term represents the utility from terminal wealth.

Based on the indirect utility specification (82) and the foregone financial wealth dynamics (20) is the Hamilton-Jacobi-Bellman equation for an investor in state $p \neq Q$ given by

$$\begin{aligned} 0 = &\mathcal{L}^{\tilde{c}} + \mathcal{L}^{\tilde{\theta}} + \mathcal{L}^{\tilde{\iota}} + \tilde{J}_t \\ &+ \tilde{J}_{\tilde{W}} [(\tilde{W} + yF(t,p))r - yF(t,p)r + y] \\ &+ \tilde{J}_y y \alpha(t,p) + \frac{1}{2} \tilde{J}_{yy} y^2 \zeta(t,p)^2 - \sum_{q \neq p} h_t^{p,q} \tilde{J} \end{aligned} \tag{84}$$

with

$$\mathcal{L}^{\tilde{c}} = \sup_{\tilde{c}\geq 0}\left\{\delta(1-\gamma)\tilde{J}\ln(\tilde{c}) - \delta\tilde{J}\ln([1-\gamma]\tilde{J}) - \tilde{c}\tilde{J}_{\tilde{W}}\right\}$$

$$\mathcal{L}^{\tilde{\theta}} = \tilde{J}_{\tilde{W}}\tilde{\theta}\sigma\lambda + \frac{1}{2}\tilde{J}_{\tilde{W}\tilde{W}}\tilde{\theta}^2\sigma^2 + \tilde{J}_{\tilde{W}y}\tilde{\theta}\sigma y\zeta(t,p)$$

$$\mathcal{L}^{\tilde{\imath}} = \sup_{(\tilde{\imath}_t^q)_{q=0}^{Q}\in\mathbb{R}}\left\{-\tilde{J}_{\tilde{W}}\sum_{q\neq p}\tilde{\imath}^q\hat{h}_t^{p,q} + \sum_{q\neq p,Q} h_t^{p,q}\tilde{J}(t,\tilde{W}+\tilde{\imath}^q, yP(t,p,q),q) + h_t^{p,Q}\frac{\varepsilon}{1-\gamma}\left(\tilde{W}+\tilde{\imath}^Q\right)^{1-\gamma}\right\}$$

Subscripts of $\tilde{J}$ denote partial derivatives with respect to either the state variables or time t and the terminal condition $\tilde{J}(T,\tilde{W},y,p) = \frac{\varepsilon}{1-\gamma}\tilde{W}^{1-\gamma}$.

Comparing $\mathcal{L}^{\tilde{\theta}}$ and $\mathcal{L}^{\tilde{\imath}}$ with the expressions for $\mathcal{L}^{\tilde{\theta}}$ and $\mathcal{L}^{\tilde{\imath}}$ in Section 3.3.1, it is clear that the foregone investment strategy and optimal foregone notional choice do not change as the underlying optimization problems are the same. However, the foregone consumption process does change compared to the arbitrary-EIS-regret-utility specification.

Like in Section 3.3.1 and in Section 3.4.1, the conjecture is made that the indirect utility function $\tilde{J}(t,\tilde{W},y,p)$ has the following functional form

$$\tilde{J}(t,\tilde{W},y,p) = \frac{\tilde{G}(t,p)^\gamma}{1-\gamma}\left(\tilde{W}+yF(t,p)\right)^{1-\gamma}$$

The partial derivatives can again be seen in the Appendix 7.3.

The investor invests $\tilde{\theta}_t$ amount of wealth into the stock market at time t based on the same assumption as in Section 3.3.1. The foregone investment amount for the auxiliary investor is thus given by

$$\tilde{\theta}_t = \left(\tilde{W}+yF(t,p)\right) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{85}$$

Additionally, the optimal foregone notional choice remains the same for this model specification and was determined in equation (30) to be

$$(\tilde{\imath}^q)^* = \frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}(\tilde{W}+yF(t,p))\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} - (\tilde{W}+yP(t,p,q)F(t,q)) \tag{86}$$

The optimal foregone consumption choice for the auxiliary investor can be derived from the FOC of $\mathcal{L}^{\tilde{c}}$ with respect to $\tilde{c}$. The optimal foregone consumption choice is given by

$$\tilde{c}^* = \frac{\delta(1-\gamma)\tilde{J}}{\tilde{J}_{\tilde{W}}} \tag{87}$$

Substituting the expression of $\tilde{c}$ into $\mathcal{L}^{\tilde{c}}$ yields

$$\mathcal{L}^{\tilde{c}} = \delta(1-\gamma)\tilde{J}[\ln(\delta) + \ln([1-\gamma]\tilde{J}) - \ln(\tilde{J}_{\tilde{W}}) - 1] - \delta\tilde{J}\ln([1-\gamma]\tilde{J})$$

From the conjecture, it follows that $\mathcal{L}^{\tilde{c}}$ can be written as

$$\mathcal{L}^{\tilde{c}} = \delta[\ln(\delta) - 1]\tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{1-\gamma} - \delta\frac{\gamma}{1-\gamma}\ln(\tilde{G}(t,p))\tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{1-\gamma}$$

This expression is the only adaption for the unit-EIS-regret-utility biometric risk model compared to the arbitrary-EIS-regret-utility biometric risk model of Section 3.3.1. The expression $\mathcal{L}^{\tilde{c}}$ for optimal foregone consumption $\tilde{c}^*$ was given in Section 3.3.1 by

$$\mathcal{L}^{\tilde{c}} = \frac{\delta^{\psi}}{\psi - 1}\tilde{G}(t,p)^{\gamma-1}\tilde{G}(t,p)^{\frac{\gamma\psi-1}{\gamma-1}}(\tilde{W} + yF(t,p))^{1-\gamma} - \frac{\delta\varphi}{1-\gamma}G(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{1-\gamma}$$

The optimal foregone control processes and the conjectures for the foregone investment amount and the indirect utility function can be substituted into the HJB equation (84). For completeness, this lengthy equation is given in the Appendix 7.4 by equation (116).

Note that as $\mathcal{L}^{\tilde{c}}$ only depends on $\tilde{G}(t,p)^{\gamma-1}(\tilde{W} + yF(t,p))^{1-\gamma}$, it can be concluded that the function $F(t,p)$ does not change compared to the arbitrary-EIS-regret-utility specification. Hence, it is known that the function $F(t,p)$ should satisfy following ODE

$$\frac{\partial F}{\partial t}(t,p) = \left[r + \sum_{q\neq p}\hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right]F(t,p) - 1 - \sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F(t,q) \tag{88}$$

with boundary condition $F(t,Q) = F(T,p) = 0$.

Moreover, $F(t,p)$ can again be separated based on whether the investor is active in the labor market or retired (Hambel et al., 2022). This separation yields

$$F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t\geq T^r\}}(t)$$

with

$$\frac{\partial F^a}{\partial t}(t,p) = \left[r + \sum_{q\neq p}\hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right]F^a(t,p) - 1 - \sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^a(t,q)$$

$$\frac{\partial F^r}{\partial t}(t,p) = \left[r + \sum_{q\neq p}\hat{h}_t^{p,q}\right]F^r(t,p) - 1 - \sum_{q\neq p}\hat{h}_t^{p,q}P(t,p,q)F^r(t,q)$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ and $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

It is in line with expectation that the ODE for human wealth does not change compared to the arbitrary-EIS-regret-utility specification as the underlying income process remains the same for both specifications. The investor's preferences do not affect the income process and hence do not affect human wealth.

The time- and state-dependent function $\tilde{G}(t,p)$ should satisfy the following non-linear ODE

$$\begin{aligned}\frac{\partial G}{\partial t}(t,p) = \Bigg[& \frac{\delta(\gamma-1)}{\gamma}[\ln(\delta)-1] + \delta\ln(\tilde{G}(t,p)) + \frac{1}{\gamma}\sum_{q\neq p} h_t^{p,q} \\ & + \frac{\gamma-1}{\gamma}\left(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2\right)\Bigg]\tilde{G}(t,p) - \sum_{q\neq p} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\tilde{G}(t,q)\end{aligned} \tag{89}$$

with boundary conditions $\tilde{G}(t,Q) = \tilde{G}(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

In conclusion, the optimal foregone consumption process is determined to be given by

$$\tilde{c}^*(t,\tilde{W},y,p) = \delta\left(\tilde{W} + yF(t,p)\right) \tag{90}$$

The investor consumes a predetermined fraction of her total wealth as given by her time preference rate δ.

The optimal foregone notional choice is given by

$$(\tilde{\iota}^q)^*(t,\tilde{W},y,p) = \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)\left(\tilde{W} + yF(t,p)\right)\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} - \left(\tilde{W} + yP(t,p,q)F(t,q)\right) \tag{91}$$

This concludes the derivations of the foregone consumption, notional choice and wealth. In the next section, the optimal investment strategy, consumption process and notional choice for a regret-averse investor with unit-EIS preference who is subject to health risks will be determined.

3.5.2 Regret-averse model

The regret-averse investor maximizes utility over intermediate consumption and terminal wealth. The utility index $J(t,W,\hat{W},y,p)$ at time t for consumption process c, investment amount θ and notional choices ι^q over the remaining lifetime $[t,\tau]$ with τ as defined by equation (13) is given by

$$J(t,W,\hat{W},y,p) = \sup_{\left(c,\theta,(\iota^q)_{q=0}^{Q}\right)\in\mathcal{A}_t} \mathbb{E}_{t,W,\hat{W},y,p}\left[\int_t^\tau \mathcal{F}(c_s,\hat{c}_s,J_s)\,\mathrm{d}s + \mathcal{J}_\tau\right] \tag{92}$$

The investor maximizes $J(t,W,\hat{W},y,p)$ for any $t<\tau$ over all admissible control processes in the set $\mathcal{A}_t$ given the state variables and the optimal foregone state variables at time t.

The regret-aversion-adjusted aggregator function $\mathcal{F}$ for unit-EIS as specified in equation (9) is given by

$$\mathcal{F}(c,\tilde{c},J) = \delta J\left[(1-\gamma-\kappa)\ln(c) + \kappa\ln(\hat{c}) - \ln([1-\gamma]J)\right] \tag{93}$$

The term $\mathcal{J}_\tau$ is assumed to be given by $\mathcal{J}_\tau = \frac{\varepsilon}{1-\gamma} W_\tau^{1-\gamma} \left(\frac{\hat{W}_\tau}{W_\tau}\right)^\kappa$ with again $\varepsilon \geq 0$.

Given the indirect utility function (92) and the wealth dynamics (17), the following HJB equation for an investor in biometric state $p \neq Q$ can be constructed

$$
\begin{aligned}
0 =& \mathcal{L}^c + \mathcal{L}^\theta + \mathcal{L}^\iota + J_t \\
&+ J_W[(W + yF(t,p))r - yF(t,p)r + y] \\
&+ J_{\hat{W}} \left[(\hat{W} + yF(t,p))r - yF(t,p)r + \tilde{\theta}\sigma\lambda + y - \hat{c} - \sum_{p \neq q} \hat{\iota}^q \hat{h}_t^{p,q} \right] \\
&+ \frac{1}{2} J_{\hat{W}\hat{W}} \tilde{\theta}^2 \sigma^2 + J_y y \alpha(t,p) + \frac{1}{2} J_{yy} y^2 \zeta(t,p)^2 \\
&+ J_{\hat{W}y} \tilde{\theta}\sigma y \zeta(t,p) - \sum_{q \neq p} h_t^{p,q} J
\end{aligned}
\tag{94}
$$

with

$$
\begin{aligned}
\mathcal{L}^c &= \sup_{c \geq 0} \{\mathcal{F}(c, \hat{c}, J) - cJ_W\} \\
\mathcal{L}^\theta &= \sup_{\theta \in \mathbb{R}} \left\{ J_W \theta\sigma\lambda + \frac{1}{2} J_{WW} \theta^2 \sigma^2 + J_{Wy} \theta\sigma y \zeta(t,p) + J_{W\hat{W}} \theta\sigma^2 \tilde{\theta} \right\} \\
\mathcal{L}^\iota &= \sup_{(\iota_t^q)_{q=0}^Q \in \mathbb{R}} \left\{ -J_W \sum_{q \neq p} \iota^q \hat{h}_t^{p,q} + \sum_{q \neq p,Q} h_t^{p,q} J(t, W + \iota^q, \hat{W} + \hat{\iota}^q, yP(t,p,q), q) \right. \\
&\qquad \left. + h_t^{p,q} \frac{\varepsilon}{1-\gamma} (W + \iota^Q)^{1-\gamma} \left(\frac{\hat{W} + \hat{\iota}^Q}{W + \iota^Q} \right)^\kappa \right\}
\end{aligned}
$$

As it was assumed in this model that $\psi = 1$, the expression for $\mathcal{L}^c$ is, by equation (9), given by

$$
\mathcal{L}^c = \sup_{c \geq 0} \{\delta J[(1-\gamma-\kappa)\ln(c) + \kappa \ln(\hat{c}) - \ln([1-\gamma]J)] - cJ_W\}
$$

Like in the auxiliary model, subscripts denote the partial derivatives with respect to the state variables and time t and the terminal condition is denoted by $J(T, W, \hat{W}, y, p) = \frac{\varepsilon}{1-\gamma} W^{1-\gamma} \left(\frac{\hat{W}}{W}\right)^\kappa$.

Like in Section 3.5.1 for the auxiliary model, it can be seen that the optimization functions $\mathcal{L}^\theta$ and $\mathcal{L}^\iota$ do not change compared to the ones specified in Section 3.3.2. Hence, the optimal investment strategy and notional choice are identical to the ones derived in Section 3.3.2. Furthermore, in line with Section 3.3.2, it will be conjectured that the indirect utility function $J(t, W, \hat{W}, y, p)$ has the following functional form

$$
J(t, W, \hat{W}, y, p) = \frac{G(t,p)^\gamma}{1-\gamma} (W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^\kappa
$$

Again, the partial derivatives are shown in the Appendix 7.3. As the conjecture for the functional form of the indirect utility function is the same as in Section 3.3.2, it can thus be concluded that the optimal solutions for θ and ι^q should be the same given that the conjecture is correct.

The optimal investment amount for a regret-averse investor was determined to be

$$\theta^*(t, W, y, p) = \frac{\lambda + \kappa\sigma}{(\gamma + \kappa)\sigma}(W_t + yF(t, p)) - yF(t, p)\frac{\zeta(t, p)}{\sigma} \tag{95}$$

The investor should buy life-insurance contracts against life shocks with optimal notional $(\iota^q)^*$ for all states $(q \neq p) \in \mathcal{Q}$ with $(\iota^q)^*$ being specified by

$$\begin{aligned}(\iota^q)^*(t, W, y, p) = &\left(\frac{G(t, q)}{G(t, p)}\right)^{\frac{\gamma}{\gamma+\kappa}} (W + yF(t, p)) \left(\frac{\tilde{G}(t, q)}{\tilde{G}(t, p)}\right)^{\frac{\kappa}{\gamma+\kappa}} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t, p, q)F(t, q))\end{aligned} \tag{96}$$

The optimal consumption process can be derived using the FOC of $\mathcal{L}^c$ with respect to c. The FOC yields

$$c^*(t, W, y, p) = \frac{\delta(1 - \gamma - \kappa)J}{J_W} \tag{97}$$

Substituting the found optimal expression for c into $\mathcal{L}^c$ yields

$$\mathcal{L}^c = \delta(1 - \gamma - \kappa)J[\ln(\delta) + \ln([1 - \gamma - \kappa]J) - \ln(J_W) - 1] + \delta J\kappa \ln(\hat{c}) - \delta J \ln([1 - \gamma]J)$$

Compared to the HJB equation in Section 3.3.2, only the terms including the consumption process c and the foregone consumption process $\hat{c}$ change. All other terms remain the same as it was already stated that the optimal (foregone) notional choice and optimal (foregone) investment strategy are identical in both settings. Based on this result and the conjecture, the terms depending on c and $\hat{c}$ can be rewritten to

$$\mathcal{L}^c - \hat{c}J_{\hat{W}} = \left[\delta[\ln(\delta) - 1] - \delta\frac{\gamma}{1 - \gamma}\ln(G(t, p))\right] G(t, p)^{\gamma}(W + yF(t, p))^{1-\gamma}\left(\frac{\hat{W} + yF(t, p)}{W + yF(t, p)}\right)^{\kappa}$$

For comparison, this equation is given for the arbitrary-EIS-regret-utility specification of Section 3.3.2 by

$$\begin{aligned}\mathcal{L}^c - \hat{c}J_{\hat{W}} = &\left[\frac{\delta^{\psi}}{1 - \gamma}(\varphi - 1 + \gamma + \kappa)\tilde{G}(t, p)^{\frac{-\psi^2\kappa\gamma}{\varphi(\varphi+\kappa\psi)}}G(t, p)^{\frac{\varphi+(\kappa-\gamma)\psi}{\varphi+\kappa\psi}} - \frac{\delta\varphi}{1 - \gamma}G(t, p)\right. \\ &\left. - \frac{\kappa}{1 - \gamma}\delta^{\psi}\tilde{G}(t, p)^{\frac{-\psi\gamma}{\varphi}}G(t, p)\right] G(t, p)^{\gamma-1}(W + yF(t, p))^{1-\gamma}\left(\frac{\hat{W} + yF(t, p)}{W + yF(t, p)}\right)^{\kappa}\end{aligned}$$

The optimal control processes, (optimal) foregone control processes, and the conjecture can be substituted into the HJB equation (94). Again, for completeness, this lengthy equation is given in the Appendix 7.4 equation (117).

As the expression only depends on $G(t,p)^{\gamma-1}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}$, it can be concluded that the time- and state-dependent function $F(t,p)$ remains the same as in Section 3.3.2. Hence, the function $F(t,p)$ should satisfy the following system of ODEs

$$\frac{\partial F}{\partial t}(t,p) = \left[r + \sum_{q\neq p} \hat{h}_t^{p,q} + \zeta(t,p)\lambda - \alpha(t,p)\right] F(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F(t,q) \tag{98}$$

with again boundary conditions $F(t,Q) = F(T,p) = 0$. Moreover, it was shown that $F(t,p)$ can be decomposed into

$$F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t\geq T^r\}}(t)$$

with

$$\frac{\partial F^a}{\partial t}(t,p) = \left[r + \sum_{q\neq p} \hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda\right] F^a(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^a(t,q)$$

and

$$\frac{\partial F^r}{\partial t}(t,p) = \left[r + \sum_{q\neq p} \hat{h}_t^{p,q}\right] F^r(t,p) - 1 - \sum_{q\neq p} \hat{h}_t^{p,q} P(t,p,q) F^r(t,q)$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ and $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

The function $G(t,p)$ should satisfy following non-linear system of ODEs

$$\begin{aligned}\frac{\partial G}{\partial t}(t,p) = \Bigg[&\delta \ln(G(t,p)) + \frac{1}{\gamma}\sum_{q\neq p} h_t^{p,q} + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \delta[\ln(\delta)-1]\right) \\ &- \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2}\right) - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 \\ &+ \left(\frac{\kappa}{\gamma}\right)\left(\sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right) h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} - \lambda\sigma\right)\Bigg] G(t,p) \\ &- \left[\left(\frac{\gamma+\kappa}{\gamma}\right)\sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}}\right] G(t,p)^{\frac{\kappa}{\gamma+\kappa}}\end{aligned} \tag{99}$$

with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$ and $\tilde{G}(t,p)$ as specified by equation (89).

In conclusion, it has been shown that there exist time- and state-dependent functions $F(t,p)$ and $G(t,p)$. Based on this result, it can be concluded that the optimal consumption process for a regret-averse investor with unit elasticity of intertemporal substitution who is subject to biometric risk is given by

$$c^*(t,W,y,p) = \delta(W + yF(t,p)) \tag{100}$$

The investor should thus consume a predetermined fraction of her total wealth depending on her time preference rate δ. Thus the optimal consumption process for a risk- and regret-averse investor is the same as the optimal consumption process for a purely risk-averse investor with unit-EIS preferences (see for example Munk (2017)). It can thus be concluded that the regret aversion does not affect the consumption strategy of the agent if the agent's elasticity of intertemporal substitution towards deterministic consumption plans is given by the unit value.

The optimal notional choice $(\iota^q)^*$ for all states $(q \neq p) \in Q$ was derived to be

$$\begin{aligned}(\iota^q)^*(t,W,y,p) = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}} (W + yF(t,p)) \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t,p,q)F(t,q))\end{aligned} \tag{101}$$

The insurance strategy is the same as for the arbitrary-EIS preferences, but the underlying $\tilde{G}(t,p)$ and $G(t,p)$ functions are different. The investor should buy life-insurance contracts with notional ι^q for all $q \neq p$ based on a age- and state-dependent fraction of total wealth.

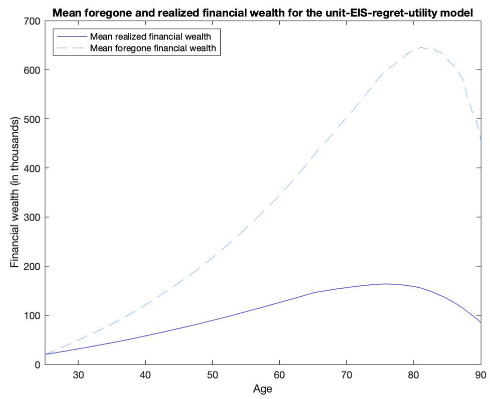
At last, the optimal investment strategy was found to be

$$\theta^*(t,W,y,p) = \frac{\lambda + \kappa\sigma}{(\gamma+\kappa)\sigma}(W + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{102}$$

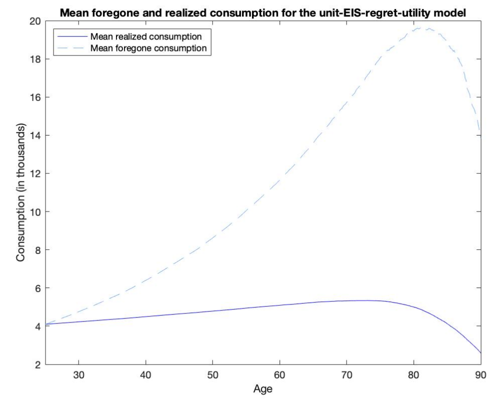
As a result, the investor should invest a constant fraction of her total wealth and a time- and state-dependent fraction of her human wealth into the stock market. This investment strategy is identical to the investment strategy derived in Section 3.3.2 as it is only dependent on the human wealth function $F(t,p)$ and not on $\tilde{G}(t,p)$ or $G(t,p)$. As it was stated that the human wealth function is not affected by the EIS parameter ψ, it can be concluded that the optimal investment strategies are indeed identical.

Like in Section 3.4.2, to further illustrate the retrieved results, some graphs are depicted. Again, the survival model of Richard (1975) (see Figure 1) is considered and the results are obtained from a Monte-Carlo simulation with the parameter values as stated in Table 2 with $\psi = 1$. Further details on the Monte-Carlo simulation are given in Section 4.1.

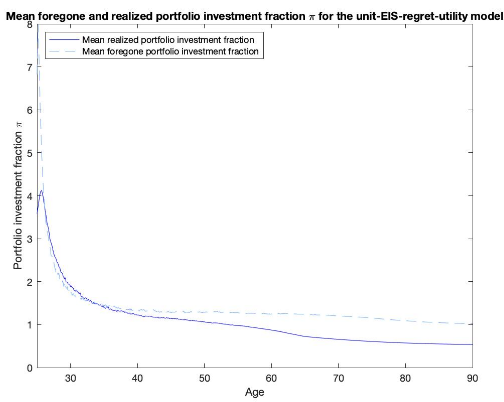
Figure 5 depicts the optimal financial wealth and control processes of a regret-averse investor with unit-EIS preferences who is subject to mortality risk. Similar to the case with CRRA-regret-utility specification, it holds true that on average the foregone financial wealth exceeds the realized financial wealth. Realized consumption very slightly increases until an age of approximately 75 years and then it starts to decrease. This can be explained from the fact that the regret-averse investor consumes a constant fraction of total wealth based on her time preference parameter δ and until an age of 75 years is the mortality risk rather small. The results for the investment fraction and notional choice are very similar as for the CRRA-regret-utility specification (see Figure 3). As expected, the optimal regret-averse investment fraction approaches the same value as for the CRRA-regret-utility specification and the foregone investment fraction approaches one, as expected. The notional choice is on average negative over the life-cycle. Thus, again investors choose to short-sell their life-insurance to generate additional wealth. The average foregone and realized notional choices are decreasing up to an age of approximately 75 years and 70 years, respectively.



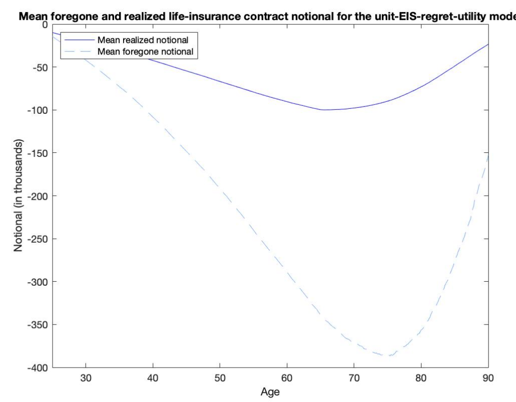
(a) Average foregone and realized financial wealth for an investor with unit-EIS-regret-utility preferences in the survival model.



(b) Average foregone and realized consumption for an investor with unit-EIS-regret-utility preferences in the survival model.



(c) Average foregone and realized investment fraction π for an investor with unit-EIS-regret-utility preferences in the survival model.



(d) Average foregone and realized life-insurance contract notional for an investor with unit-EIS-regret-utility preferences in the survival model.

Figure 5: The figure shows the by Monte-Carlo simulation obtained average optimal financial wealth and control processes for an investor with unit-EIS-regret-utility preferences who is only subject to mortality risk and has access to perfect life-insurance contracts in line with Richard (1975).

This concludes the derivation of the optimal consumption-investment-insurance strategy for a regret-averse investor with unit-EIS. The results for this specification can be concluded into the following theorem.

Theorem 3.4 (Biometric risk model for a regret-averse investor with unit-EIS-regret-utility specifications). *For a regret-averse investor with unit-EIS-regret-utility specifications living in a Black-Scholes world with exogenous labor income and who is subject to* *biometric risk, it holds that the financial wealth dynamics are given by*

$$\mathrm{d}W_t = \left[rW_t + \theta_t \lambda \sigma - c_t + y_t - \sum_{q \neq p} \iota_t^q \hat{h}_t^{I_t,q} \right] \mathrm{d}t + \theta_t \sigma \, \mathrm{d}Z_t$$

with $W_{t^q} = W_{t^q-} + \iota^q_{t^q-}$.

The value function is given by

$$J(t, W, \hat{W}, y, p) = \frac{1}{1-\gamma} G(t,p)^{\gamma} (W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^{\kappa}$$

where both $F(t,p) = F^a(t,p)\mathbb{1}_{\{t<T^r\}}(t) + F^r(t,p)\mathbb{1}_{\{t \geq T^r\}}(t)$ *and* $G(t,p)$ *are satisfying a system of ordinary differential equations. The function* $F(t,p)$ *should satisfy the following system of ordinary differential equations*

$$\begin{aligned} \frac{\partial F^a}{\partial t}(t,p) &= \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} - \alpha(t,p) + \zeta(t,p)\lambda \right] F^a(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q) F^a(t,q) \\ \frac{\partial F^r}{\partial t}(t,p) &= \left[r + \sum_{q \neq p} \hat{h}_t^{p,q} \right] F^r(t,p) - 1 - \sum_{q \neq p} \hat{h}_t^{p,q} P(t,p,q) F^r(t,q) \end{aligned} \tag{103}$$

with boundary conditions $F^r(t,Q) = F^r(T,p) = F^a(t,Q) = 0$ *and* $F^a(T^r,p) = \Gamma(p)F^r(T^r,p)$.

Moreover, the function $G(t,p)$ *should satisfy the following system of non-linear ordinary differential equations*

$$\begin{aligned} \frac{\partial G}{\partial t}(t,p) = \Bigg[& \delta \ln(G(t,p)) + \frac{1}{\gamma} \sum_{q \neq p} h_t^{p,q} + \left(\frac{\gamma - 1}{\gamma} \right) \left(r + \sum_{q \neq p} \hat{h}_t^{p,q} + \delta[\ln(\delta) - 1] \right) \\ & - \left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)} \right) \left(\frac{\lambda^2}{2} + \lambda\sigma\kappa + \frac{\sigma^2\kappa^2}{2} \right) - \frac{1}{2} \left(\frac{\kappa(\kappa-1)}{\gamma} \right) \sigma^2 \\ & + \left(\frac{\kappa}{\gamma} \right) \left(\sum_{q \neq p} \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)} \right) h_t^{p,q} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \right)^{1-\frac{1}{\gamma}} - \lambda\sigma \right) \Bigg] G(t,p) \\ & - \left[\left(\frac{\gamma+\kappa}{\gamma} \right) \sum_{q \neq p} \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)} \right)^{\frac{\kappa}{\gamma+\kappa}} h_t^{p,q} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \right)^{1-\frac{1}{\gamma}} G(t,q)^{\frac{\gamma}{\gamma+\kappa}} \right] G(t,p)^{\frac{\kappa}{\gamma+\kappa}} \end{aligned} \tag{104}$$

with boundary conditions $G(t,Q) = G(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

The optimal consumption-investment-insurance strategy is given by the following expressions

$$c^*(t,W,y,p) = \delta(W + yF(t,p)) \tag{105}$$

$$\theta^*(t,W,y,p) = \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W_t + yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} \tag{106}$$

$$\begin{aligned}(\iota^q)^*(t,W,y,p) = &\left(\frac{G(t,q)}{G(t,p)}\right)^{\frac{\gamma}{\gamma+\kappa}} (W + yF(t,p)) \left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{-\frac{1}{\gamma}} \\ &- (W + yP(t,p,q)F(t,q))\end{aligned} \tag{107}$$

where $\tilde{G}(t,p)$ is determined by the auxiliary model and should satisfy the following system of ordinary differential equations

$$\begin{aligned}\frac{\partial\tilde{G}}{\partial t}(t,p) = &\Bigg[\frac{\delta(\gamma-1)}{\gamma}[\ln(\delta)-1] + \delta\ln(\tilde{G}(t,p)) + \frac{1}{\gamma}\sum_{q\neq p} h_t^{p,q} \\ &+ \frac{\gamma-1}{\gamma}(r + \sum_{q\neq p}\hat{h}_t^{p,q} + \sigma\lambda - \frac{1}{2}\gamma\sigma^2)\Bigg]\tilde{G}(t,p) - \sum_{q\neq p} h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\tilde{G}(t,q)\end{aligned} \tag{108}$$

with boundary conditions $\tilde{G}(t,Q) = \tilde{G}(T,p) = \varepsilon^{\frac{1}{\gamma}}$.

This concludes the analysis of the optimal consumption, portfolio and notional choice of a regret-averse investor with unit-EIS-regret-utility preferences who is subject to biometric risk. The following corollary shows the results for the simplified model where the investor earns spanned exogenous labor income excluding biometric risk (Corollary 3.2). The proof for Corollary 3.2 is given in the Appendix 7.7.

Corollary 3.2 (Merton portfolio problem with spanned exogenous labor income for a unit EIS regret-averse investor). *The regret-averse investor earns exogenous labor income without biometric risk. The labor income dynamics are specified by the following stochastic differential equation*

$$\mathrm{d}y_t = y_t\,[\alpha\,\mathrm{d}t + \zeta\,\mathrm{d}Z_t] \tag{109}$$

The labor income process follows a geometric Brownian motion (GBM) with the same underlying Brownian motion as the stock market. Therefore, the market is complete.

As a result, the financial wealth dynamics of the investor are given by

$$\mathrm{d}W_t = [rW_t + \theta_t\lambda\sigma - c_t + y_t]\,\mathrm{d}t + \theta_t\sigma\,\mathrm{d}Z_t$$

The value function of the Merton problem with exogenous labor income for a unit EIS regret-averse investor is given by

$$J(t,W,\hat{W},y) = \frac{1}{1-\gamma}g(t)^{\gamma}(W + H(t,y))^{1-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^{\kappa}$$

where $g(t)$ is a purely time-dependent function satisfying the following ODE

$$g'(t) = [A + \delta \ln(g(t))]g(t) \tag{110}$$

with $g(T) = \varepsilon^{\frac{1}{\gamma}}$ and A as given by the following expression

$$\begin{aligned} A = \Bigg[&\frac{\delta(\gamma-1)}{\gamma}\ln(\delta) + \frac{\delta(1-\gamma)}{\gamma} - \frac{1}{2}\left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\lambda^2 - \frac{1}{2}\left(\frac{(1-\gamma-\kappa)\kappa^2}{\gamma(\gamma+\kappa)}\right)\sigma^2 \\ &- \left(\frac{(1-\gamma-\kappa)\kappa}{\gamma(\gamma+\kappa)}\right)\sigma\lambda + \frac{r(\gamma-1)}{\gamma} - \frac{\kappa}{\gamma}\sigma\lambda - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2 \Bigg] \end{aligned}$$

Equation (110) is solved by the following time-dependent function $g(t)$

$$g(t) = \mathrm{e}^{-\frac{A}{\delta}+\frac{1}{\delta}\mathrm{e}^{-\delta(T-t)}[\ln(\varepsilon)+A]^{\frac{\delta}{\gamma}}} \tag{111}$$

The human wealth function $H(t, y)$ satisfies following PDE

$$\frac{\partial H}{\partial t}(t, y) + (\alpha - \zeta\lambda)yH_y(t, y) + \frac{1}{2}\zeta^2 y^2 H_{yy}(t, y) - rH(t, y) + y = 0 \tag{112}$$

The optimal consumption and investment strategy for the unit-EIS regret-averse investor are given by

$$c^*(t, W, y) = \delta(W + H(t, y)) \tag{113}$$

$$\theta^*(t, W, y) = \frac{\lambda + \kappa\sigma}{(\gamma+\kappa)\sigma}(W + H(t, y)) - H(t, y)\frac{\zeta}{\sigma} \tag{114}$$

The proof for Corollary 3.2 is given in the Appendix 7.7.

A regret-averse investor with unit-EIS preferences and spanned exogenous labor income consumes a predetermined fraction of her total wealth defined by her time preference rate δ. As expected, the agent invests the same amount of total wealth into the stock market as the regret-averse CRRA investor. Again, this investment fraction consists of a predetermined fraction of total wealth given the regret and risk aversion parameters and a correction term for human wealth.

The results for the Merton problem for a regret-averse investor without labor income are again omitted, but these results can easily be derived. It can be verified that the results are identical to the results of Corollary 3.2 with $y = 0$ for all time periods t. Note that for $y = 0$, it holds that $H(t, y) = 0$ for all t and hence the correction term for human wealth drops out of the expression of the investment strategy. Thus, an agent without any labor income invests only a predetermined, time-independent fraction of financial wealth into the stock market. The consumption choice remains a given time-independent fraction of financial wealth. At last, as expected, the results of Theorem 3.4 reduce to the results of Corollary 3.2 if one assumes only one biometric state.

This concludes the analytical derivations of the unit-EIS regret-averse biometric risk model.

Closed-form solutions have been retrieved for an arbitrary EIS parameter value ψ. The sequential section will discuss the Monte-Carlo simulation utilized to illustrate the figures of the previous sections. Furthermore, additional results are shown and a sensitivity analysis for the parameters of a benchmark survival model will be performed.

4 Numerical results

In this section, numerical results for a benchmark survival model are shown. First, the Monte-Carlo simulation and chosen benchmark parameter values will be discussed. Second, results for the benchmark survival model will be shown. At last, a sensitivity analysis for the most prominent parameters is performed.

4.1 Numerical method

The numerical results are based on a Monte-Carlo simulation using $100,000$ sample paths and where the consumption-investment-insurance strategy is updated 12 times per year, i.e. once every month. The initial age of all investors is 25 years ($t = 0$) and the agents are known to retire at an age of 65 ($T^r = 40$). The biometric model considered is the survival model (see Figure 1) as introduced by Richard (1975). Hence, the agent is only subject to mortality risk. Stated differently, she might pass away before the terminal date T. The model ends either if the agent has passed away or when the agent is 150 years old ($T = 125$). The terminal date has been chosen in such a way that all agents have died with a exceptionally high probability before the terminal date. Hence, for every agent, the model will, very likely, end by an uncertain time of death. Following Hambel et al. (2022), it is assumed that the hazard rate of death $h^{0,1}(t)$ follows a Gompertz mortality law of the form

$$h^{0,1}(t) = \frac{1}{b} e^{\frac{a+t-m}{b}} \tag{115}$$

where a denotes the age of the agent at time $t = 0$, m the x-axis displacement parameter, and b the steepness parameter. The parameter values are taken from Hambel et al. (2022). Hambel et al. (2022) calibrated the model to life tables of Germany as of 2010 and they retrieved $a = 25$, $m = 84.56$, and $b = 8.8$. The agent can insure herself against the loss of income due to an early death. It is assumed that the unit premium satisfies $\hat{h}_t^{0,1} = (1 + \phi_t^\iota) h_t^{0,1}$. For simplicity, it is assumed that ϕ_t^ι is constant over time and the fees for the insurer are 20%, i.e. $\phi_t^\iota = 0.2$.

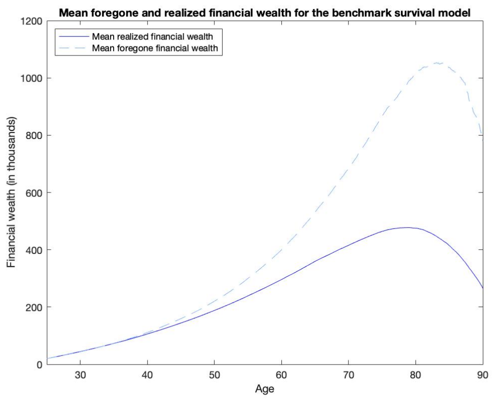
Following Hambel et al. (2022), the risk-free rate is assumed to be $r = 0.01$. The expected stock return is $\mu = 0.06$ and the stock volatility is given by $\sigma = 0.2$. The investor earns labor income with a constant expected income growth of $\alpha = 0.01$ and an income volatility of $\zeta = 0$. The replacement ratio at retirement is determined to be $\Gamma = 0.6$. The agent has a risk aversion parameter of $\gamma = 4$ and a regret aversion parameter of $\kappa = 1.8$. The regret aversion parameter is based on the empirical results of Bleichrodt et al. (2010). The elasticity of intertemporal substitution is given by $\psi = 1.5$ following the seminal paper of Bansal and Yaron (2004). The time preference rate of the agent is assumed to be $\delta = 0.03$ and the weight of the bequest motive of the agent is assumed to be $\varepsilon = 1$.
Table 2 summarizes all parameter values for the benchmark model.

General parameters		
γ	Risk aversion parameter	4
κ	Regret aversion parameter	1.8
ψ	Elasticity of intertemporal substitution	1.5
δ	Time preference rate	0.03
ε	Weight of the bequest motive	1
r	Risk-free rate	0.01
μ	Expected stock return	0.06
σ	Stock volatility	0.2
W_0	Initial financial wealth (in thousands)	20
α	Expected income growth	0.01
ζ	Income volatility	0
Γ	Replacement ratio of the income	0.6
T^r	Retirement date	40
T	Terminal date	125
y_0	Initial income (in thousands)	2.5
a	Age at $t = 0$	25
m	x-axis displacement	84.56
b	Steepness parameter	8.8
ϕ^ι	Additional fees of the insurer	0.2

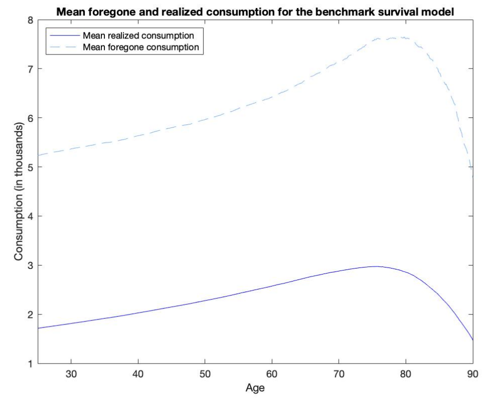
Table 2: Benchmark parameter values.

4.2 Results

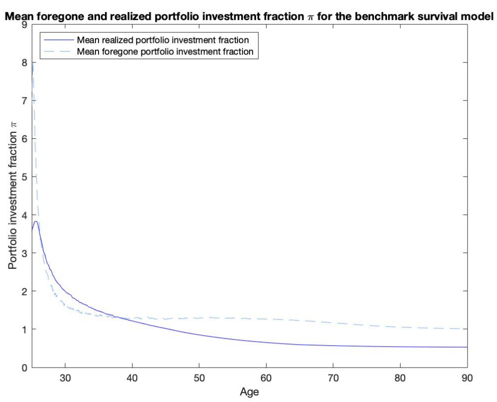
Given the model parameters of Table 2, the financial wealth and control processes of the benchmark model can be simulated. Figure 6 (a) shows the average foregone and realized financial wealth for the benchmark survival model. Figure 6 (b) depicts the average optimal foregone and realized consumption process for the benchmark survival model. The (optimal) foregone and realized investment fraction π is depicted in Figure 6 (c). As previously stated, the investment fraction π denotes the relative fraction of amount invested into the risky asset by financial wealth, i.e. $\pi = \frac{\theta}{W}$. Note, as it is assumed that $\zeta(t,p) = 0$, it holds that the correction term for human wealth $(-yF(t,p)\frac{\zeta(t,p)}{\sigma})$ equals zero for all time periods t and biometric states p. The average investment fractions are determined based on a trimmed mean where the highest and lowest 5% of the π's are trimmed. This reduces the noise in the average of π as π is very sensitive to financial wealth close to zero. At last, figure 6 (d) denotes the optimal foregone and realized notional choice for the life-insurance contracts. It should be noted that although the model ends ultimately at terminal date T, the graphs are shown up to an age of 90 years. At an age of 90 years, approximately 15.7% of the agents are still alive.



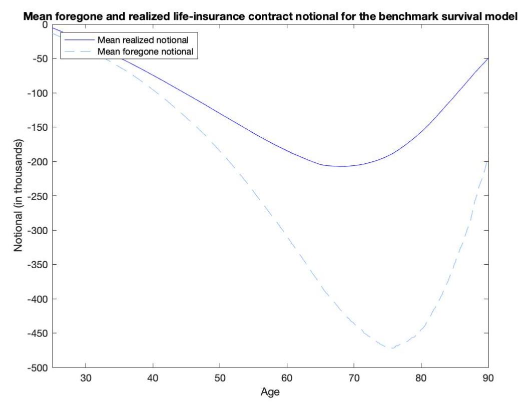
(a) Average foregone and realized financial wealth for an investor in the benchmark model.



(b) Average foregone and realized consumption for an investor in the benchmark model.



(c) Average foregone and realized investment fraction π for an investor in the benchmark model.



(d) Average foregone and realized life-insurance contract notional for an investor in the benchmark model.

Figure 6: The figure shows the by Monte-Carlo simulation obtained average optimal financial wealth and control processes for an investor in the benchmark model. The investor has access to actuarially fair priced life-insurance contracts in line with Richard (1975).

As expected, foregone financial wealth exceeds realized financial wealth on average. Both foregone and realized financial wealth is increasing up to an age of approximately 85 and 80 years, respectively. Thereafter, financial wealth decreases, but it remains on average positive at an age of 90 years. The foregone financial wealth exceeds realized financial wealth vastly at an age of 90 years. The dynamics of the average realized consumption over the life-cycle are very similar to the dynamics of average financial wealth. It increases up to an age of approximately 78 years and then decreases to below the consumption level at an age of 25 years ($t = 0$). Foregone consumption shows the same pattern, but is overall much higher. As the benchmark EIS parameter $\psi = 1.5$, it holds that ψ satisfies the condition of $\psi < \frac{\gamma+\kappa-1}{\kappa}$

(see the analysis of the optimal consumption process in Section 3.3.2). Hence, the optimal consumption process is differently affected by the age- and state-dependent function $\widetilde{G}(t,p)$ as by $G(t,p)$. Therefore, the foregone and realized consumption processes differ even when foregone total wealth equals realized total wealth, which is obvious the case at an age of 25 years. The average investment fractions are very similar to the average investment fractions as showed in Section 3.4.2 Figure 3 and Section 3.5.2 Figure 5. This is as expected, as the investment fractions only depend on total wealth and the functional forms were determined to be independent of ψ. The optimal notional choice is on average negative indicating that the investors choose to short-sell their life-insurance contracts to generate more wealth. The optimal realized notional is smaller than for the CRRA- and unit-EIS-regret-utility specifications. The optimal foregone notional choice is more similar compared to the previous specifications. The optimal realized notional choice decreases on average up to an age of 70 years and the optimal foregone notional choice up to an age of 75 years. The agents opt to short-sell their life-insurance contracts as their financial wealth greatly exceeds on average the time- and state-dependent fraction of total wealth resulting in negative notional choices.

This concludes the numerical results for the benchmark model. In the subsequent section, a sensitivity analysis for the most prominent parameters is carried out.

4.3 Sensitivity analysis

In this section a sensitivity analysis for various parameters is performed. The optimal realized financial wealth and consumption are compared for different values of a specific parameter, whereas all other parameter values remain given by Table 2. It is chosen to only show the results for optimal consumption and financial wealth as the impact of the various parameters is typically less obvious for consumption and financial wealth. Hence, a visual representation might give some insights in the effect of the various parameters.

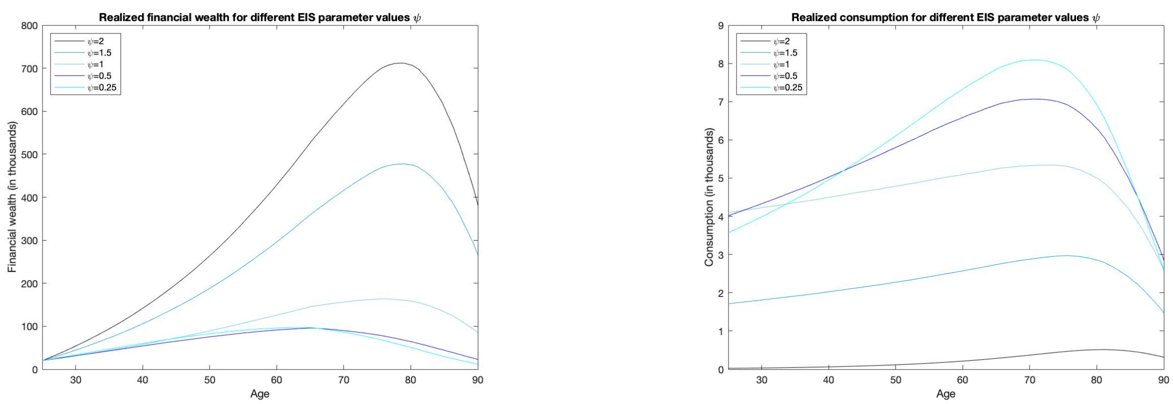
Figure 7 (a) and (b) show the average financial wealth and consumption for a regret-averse investor with different EIS parameter values. The different values considered are: $\psi = 0.25$ (CRRA), $\psi = 0.5$, $\psi = 1$ (unit EIS), $\psi = 1.5$ (benchmark model), and $\psi = 2$. Note that changing ψ while keeping γ fixed also changes $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$. From Figure 7 it can be seen that the CRRA-model $\psi = 0.25$ and $\psi = 0.5$ result in the lowest financial wealth on average. The results for these two models are very close on average. The highest financial wealth is achieved with an EIS parameter value of $\psi = 2$. For $\psi > 1$, it seems to hold true that having a higher EIS parameter value increases financial wealth on average. The optimal consumption process is on average the highest for a $\psi = 0.25$. The consumption process is clearly decreasing in ψ for an age older than 40 years. Up to an age of 40 years, the average consumption for $\psi = 0.5$ exceeds the average consumption for $\psi = 0.25$. For $\psi = 2$, the consumption process is almost zero for the earlier ages and slightly increases at older ages. At an age of 90 years, the consumption amount for $\psi \leq 1$ is approximately the same, but lower for $\psi = 1.5$ and $\psi = 2$.

The average financial wealth and consumption over the life-cycle for different values of the regret aversion parameter κ are depicted in Figure 8. The results are shown for $\kappa = 1.1$, $\kappa = 1.5$, $\kappa = 1.8$ (benchmark model), $\kappa = 2.5$, and $\kappa = 2.9$. These parameter values satisfy the necessary condition of $1 < \kappa < \gamma - 1 = 3$ (see section 3.1). Figure 8 (a) shows that increasing κ increases average financial wealth. This is especially prominent at an age of approximately 80 years. The difference in financial wealth is smaller at very young and very old ages. At an age of 90 years, a higher κ results in a higher financial wealth on average. The increase in financial wealth at the middle of the life-cycle could be explained by the fact that a regret-averse investor is inclined to invest a higher fraction of (total) wealth into the stock market. Therefore, this could increase the financial wealth on average. Figure 8 (b) illustrates the corresponding consumption amounts over the life-cycle. Up to an age of 55/60 years results a higher regret aversion parameter in lower consumption. After the age of 60 results a higher regret aversion parameter on average in more consumption. For $\kappa = 1.1$ the consumption peaks at an age of 78 years whereas for $\kappa = 2.9$ the consumption peaks at an age of 82 years.

Figure 9 illustrates the effect of the risk aversion parameter γ on the average financial wealth and consumption over the life-cycle. Similar as for the analysis of ψ is φ adjusted accordingly for the various risk aversion parameters γ. The risk aversion values considered are $\gamma = 3$, $\gamma = 4$ (benchmark model), $\gamma = 6$, $\gamma = 8$, and $\gamma = 10$. In line with the sensitivity analysis of the regret aversion parameter κ, satisfy these risk aversion values the necessary condition of $1 < \kappa = 1.8 < \gamma - 1$. From Figure 9 (a) it can be seen that a higher risk aversion parameter decreases financial wealth on average. A higher risk aversion level decreases the amount invested into the stock market. Hence, financial wealth is on average lower for a higher risk aversion. The amount consumed over the life-cycle varies with the risk aversion parameter γ as can be seen in Figure 9 (b). For a higher risk aversion parameter ($\gamma = 8$ and $\gamma = 10$) shows the average consumption process a decreasing pattern over the life-cycle, whereas for a lower risk aversion parameter shows the average consumption process an increasing pattern up to an age of approximately 78 years and decreasing afterwards. However, at a very early age (up to an age of 35 years) yields a higher risk aversion level in a higher consumption compared to a lower risk aversion level. A higher risk aversion parameter not only decreases investment into the risky asset and average financial wealth, but also decreases consumption over the life-cycle except for at a very early age.

The final parameter of interest for this sensitivity analysis is the time preference rate parameter δ. The results are shown in Figure 10. This sensitivity analysis considers a very small value of $\delta = 0.001$, $\delta = 0.01$, the benchmark value of $\delta = 0.03$, a moderate value of $\delta = 0.05$ and a high value of $\delta = 0.1$. From Figure 10 (a), it becomes clear that having a lower time preference rate yields on average a higher financial wealth. The average financial wealth for $\delta = 0.001$ and $\delta = 0.01$ are almost identical up to an age of 60 years. Thereafter, the average financial wealth for $\delta = 0.01$ exceeds the financial wealth for $\delta = 0.001$. It should also be noted that for $\delta = 0.1$, the average financial wealth turns negative for most of the life-cycle. From Figure 10 (b) this can be explained by the fact that the agent consumes a lot in the beginning of the life-cycle as the agent has a very high time preference rate. This results in negative financial wealth. Note that financial wealth is allowed to be negative in this model, but total wealth should remain positive. The consumption process for $\delta = 0.001$ is approximately zero over the entire life-cycle. The time preference rate of the investor is very low and therefore she chooses to consume only a very little. Moreover, a time preference rate of $\delta = 0.03$ yields a slightly higher average consumption at older ages than $\delta = 0.05$. For $\delta = 0.01$ and $\delta = 0.03$ is the consumption process increasing up to an age of 80 years, whereas for $\delta = 0.05$ and $\delta = 0.1$ the process is monotonically decreasing on average.

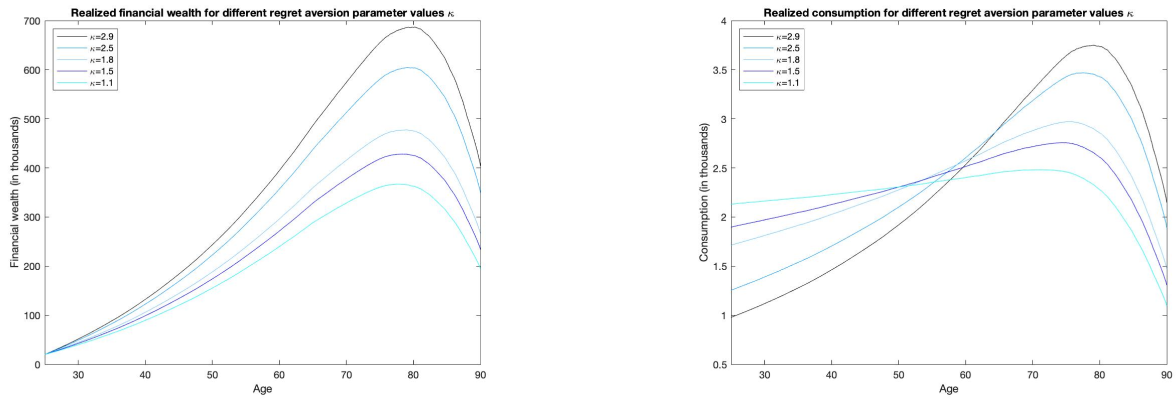
This concludes the sensitivity analysis of the numerical benchmark model. It is worth noting that the other parameter values such as e.g. the risk-free rate r, expected stock return μ, and expected income growth α also play a crucial role in the optimal wealth and consumption-investment-insurance strategy. However, the effect of these parameter values are not studied in this thesis.



(a) Average financial wealth for various values of ψ.

(b) Average consumption for various values of ψ.

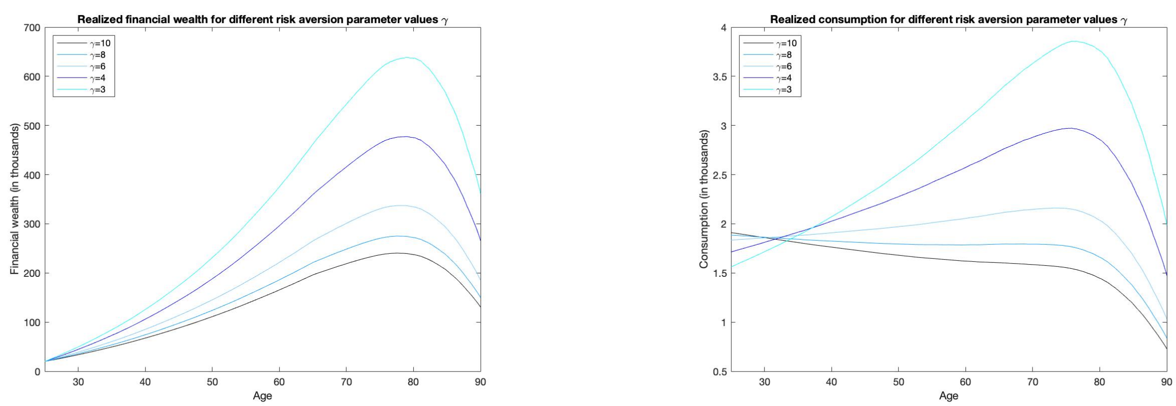
Figure 7: This figure shows the optimal financial wealth and consumption process for different values of the EIS parameter ψ. The values considered are $\psi = 0.25$ (CRRA), $\psi = 0.5$, $\psi = 1$ (unit EIS), $\psi = 1.5$ (benchmark model) and $\psi = 2$. Note that $\varphi = \frac{1-\gamma}{1-\frac{1}{\psi}}$ is adjusted accordingly.



(a) Average financial wealth for various values of κ.

(b) Average consumption for various values of κ.

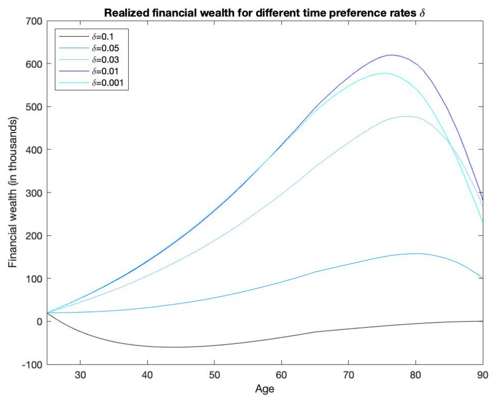
Figure 8: This figure shows the optimal financial wealth and consumption process for different values of the regret aversion parameter κ. The values considered are $\kappa = 1.1$, $\kappa = 1.5$, $\kappa = 1.8$ (benchmark model), $\kappa = 2.5$, and $\kappa = 2.9$.



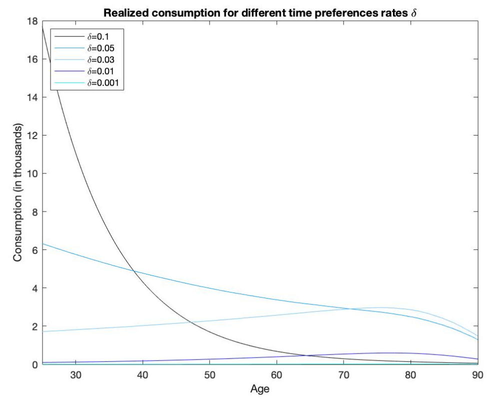
(a) Average financial wealth for various values of γ.

(b) Average consumption for various values of γ.

Figure 9: This figure shows the optimal financial wealth and consumption process for different values of the risk aversion parameter γ. The values considered are $\gamma = 3$, $\gamma = 4$ (benchmark model), $\gamma = 6$, $\gamma = 8$, and $\gamma = 10$. Note that $\varphi = \frac{1-\gamma}{1-\psi}$ is adjusted accordingly.



(a) Average financial wealth for various values of δ.



(b) Average consumption for various values of δ.

Figure 10: This figure shows the optimal financial wealth and consumption process for different values of the time preference rate δ. The values considered are $\delta = 0.001$, $\delta = 0.01$, $\delta = 0.03$ (benchmark model), $\delta = 0.05$, and $\delta = 0.1$.

5 Conclusion

This section summarizes and highlights the most profound findings of the previous sections.

This thesis incorporated a behavioral model into the optimal consumption-investment-insurance problem. The behavioral model considered in this thesis is a regret-adjusted Epstein-Zin preferences model. Goossens (2022) showed that regret theory can explain various stylized facts of the financial literature. Moreover, Zeelenberg (1999), and Zeelenberg et al. (1998) showed that regret is inherently different from other emotions as it is widely experienced among investors and very persistent. As a result, regret could affect investors decision making and could therefore be of importance in a dynamic asset allocation problem. Epstein-Zin preferences are utilized to distinguish between risk aversion and elasticity of intertemporal substitution.

In Section 3.1, an alternative multiplicative regret-utility function based on the regret-utility function of Goossens (2022) is proposed. This proposed regret-utility function allows for both regret as well as rejoicing, i.e. positive regret. The regret-utility function consists multiplicatively of a CRRA power-utility part and a regret part. The regret part models regret over the relative difference between the realized outcome and the foregone outcome. It is shown that the proposed regret-utility function satisfies all desired properties as stated by Goossens (2021) of a multiplicative regret-utility function if the ratio between foregone and realized wealth is bounded from below, i.e. $\frac{\hat{W}}{W} \geq \frac{\kappa}{\gamma+\kappa-1}$. It was numerically shown by a Monte-Carlo simulation that this condition is at least in 99% of the sample paths satisfied.

To further extend the model, a regret-averse differential utility specification was proposed. It was shown that the proposed regret-averse normalized aggregator function reduces to the normalized aggregator function as proposed by Duffie and Epstein (1992) for regret aversion parameter $\kappa = 0$. Two special cases can be distinguished based on the differential utility specification. Either the EIS parameter ψ equals $\psi = \frac{1}{\gamma}$ (CRRA) or $\psi = 1$ (unit EIS). For the first parameterization, the utility preferences are given by the regret-utility function proposed previously. The second parameterization is a limiting case of the regret-averse differential utility specification.

This thesis thus considered a regret-averse investor with Epstein-Zin preferences. In line with Hambel et al. (2022), it is moreover assumed that the investor is subject to biometric risks. The investor has access to a perfect life-insurance market where she is able to buy continuously-adjustable life-insurance contracts to insure herself against a possible loss of income due to a biometric shock. This model builds upon the work of Hambel et al. (2022) as they assumed a purely risk-averse investor with a power-utility function. The investment opportunities and the biometric model specification were discussed in Section 3.2.

In Section 3.3, the optimal consumption-investment-insurance strategy for investor with an arbitrary-EIS-regret-utility specification is derived. First the optimal consumption-insurance strategy for an auxiliary investor who invests her total available wealth into the risky asset are derived. These optimal control strategies are then substituted into the regret-averse model to determine the optimal control strategies for the regret-averse investor. It was shown that the optimal control strategies do not depend on foregone wealth, only on realized wealth. The optimal consumption strategy depends non-linearly on the underlying ODE of the regret-averse model and the underlying ODE of the auxiliary model. The optimal investment amount was determined to be a constant fraction of total wealth and a correction term for human wealth. It was proven that the amount invested into the stock market by a regret-averse investor exceeds the amount invested by a purely risk-averse investor under the condition that the Merton investment fraction of the purely risk-averse investor does not exceed one. This condition is typically satisfied. Hence, it is optimal for a regret-averse investor to invest more into the risky asset to mitigate the possibility of missing out on returns in the long run. At last, the optimal insurance strategy also depends non-linearly on the underlying time- and state-dependent ODEs. Section 3.4 highlights the results in case of $\psi = \frac{1}{\gamma}$. For this special case, the model reduces to a regret-averse time-additive power-utility specification. Moreover, a closed-form solution for the Merton portfolio problem with spanned exogenous income and CRRA-regret-utility preferences has been shown. Section 3.5 described the optimal consumption-investment-insurance strategy for an investor with a unit-EIS-regret-utility specification. In line with the results of classical Epstein-Zin preferences are the optimal investment and insurance strategies unchanged compared to the optimal investment and insurance strategies for the arbitrary-EIS-regret-utility specification (Munk, 2017). It should be noted nonetheless that the underlying ODEs are different. The consumption strategy is in this case a constant fraction of total wealth depending on the investor's time preference rate. This consumption strategy is the same as for a purely risk-averse investor. Hence, the regret aversion parameter does not affect the consumption strategy in case of unit EIS. Similar as for the CRRA-regret-utility specification, a closed-form solution for the Merton portfolio problem with spanned exogenous income and unit-EIS-regret-utility preferences has been shown.

Section 4 illustrated numerical results for a benchmark model retrieved by utilizing a Monte-Carlo simulation. As expected, foregone financial wealth exceeded realized financial wealth on average. The average optimal consumption process was found to be increasing up to an age of approximately 78 years. The investment fractions decreased over the life-cycle. Thus, it is advised to invest more than a 100% of financial wealth into the stock market at a younger age and then invest less at an older age. The optimal notional choice was retrieved to be on average negative indicating that the agents short-sell their life-insurance contracts to generate more wealth. For completeness, a sensitivity analysis was performed. Results for various values of the EIS parameter ψ, the regret aversion parameter κ, the risk aversion parameter γ, and the time preference rate δ were discussed.

In conclusion, the aim of this thesis was to analytically assess the optimal consumption-investment-insurance strategy for a regret-averse investor with Epstein-Zin preferences. This thesis proposed a behavioral dynamic asset allocation model for a risk- and regret-averse investor with Epstein-Zin preferences who is subject to biometric risks. The model takes into account the feeling of regret an investor might have after missing out on higher ex-post foregone investment choice. Yet, the model is tractable and might therefore be a consideration opposed to the classical CRRA power-utility model. The fact that the model is tractable could have economic relevance for institutional investors such as pension funds. A pension fund assumes a general utility function for the underlying agents to optimally invest based on the preferences of the agents. For practical reasons, it is often desired that the utility function and model is tractable. The behavioral model proposed in this thesis would provide the pension funds with an alternative utility function which is still tractable, but also captures a relevant behavioral aspect of the underlying agents. This could improve the pension funds investment strategy and decision making. Moreover, an important finding was that the optimal investment amount into the risky asset is larger for a risk- and regret-averse investor than for a purely risk-averse investor. This finding could be taken into consideration for fund managers. Fund managers might personally be more affected by the feeling of regret towards their investments as their income (bonuses) is often performance-based. Benyelles and Arisoy (2019) find that fund managers who perform worse than their peers tend to take more risk (invest more into risky assets) in the subsequent period indicating that fund managers are indeed affected by regret. Even if they achieve large gains, missing out on even higher gains or being outperformed by peers worsens their performance status. For fund managers, it might be reasonable to think that relative gains are of much more importance than absolute gains. From this, it could be concluded that fund managers might have a higher than average regret aversion. As it was determined in this thesis that regret aversion increases the amount invested into the risky asset, this observation could explain the risk-taking behavior of fund managers. In the Introduction 1, it was already discussed that households might be affected by a feeling of regret. Hence, gaining insights from implementing regret theory into an optimal investment problem is for both institutional as well as retail investors of economic relevance.

6 Discussion and recommendations

To derive the closed-form solutions, some strong assumption were made. First of all, it was assumed, to determine the foregone consumption and wealth, that the auxiliary investor invests her total available wealth into the risky asset. This assumption allows for an explicit dynamic foregone wealth process and hence also for an optimal foregone consumption process and notional choice. However, this formulation does not resemble the ex-post highest amount of consumption or wealth an investor could have had at any given point in time. This issue was also addressed in Section 3.1. Therefore, one might criticize the optimal foregone processes and foregone wealth dynamics. If one would be able to define foregone consumption/wealth in such a way that it is larger or equal than realized consumption/wealth, almost surely, then the proposed regret-utility function would satisfy all desired properties of Table 1 as rejoicing would be excluded. Furthermore, it would also allow one to use a different regret-utility function such as the one proposed by Goossens (2022). Another possibility would be to implement an additive regret-utility function. In that case one could use for example the regret-utility function as proposed by Braun and Muermann (2004). For this, numerical solution methods are most likely required.

Additionally, the proposed regret-utility function is not guaranteed to satisfy all the desired properties of a multiplicative regret-utility function if the agent can experience rejoicing. It was shown that based on a Monte-Carlo simulation the lower bound is satisfied in at least 99% of the paths, but there were some sample paths where the lower bound exceeded the ratio between foregone and realized wealth. Nonetheless, as this was rarely the case, it is argued that the proposed regret-utility function should behave like a desired multiplicative regret-utility function. In line with previous critique, this issue could also be circumvented if foregone consumption/wealth is defined in such a way that it is larger or equal than realized consumption/wealth for all states of the world and all time periods.

Furthermore, it is assumed in this thesis that the investor is only able to invest into a risk-free asset and a risky asset with a constant investment opportunity set. One possibility of extending the model would be to include a bond market to the asset class. This would allow the investor to invest in the risky asset as well as in bonds to hedge stochastic interest rates. Instead of stochastic interest rates, the model could also be extended by stochastic volatility or stochastic market price of risk or a combination of the three. Different asset types might also be included in the asset selection of the investor. For example, similar to Hambel (2020), the asset class could be extended by including real estate or by including derivatives. For households, real estate is a crucial part of their equity and fluctuations might affect the consumption-investment-insurance strategy significantly. For these proposed model extensions, closed-form solutions might not be obtainable and hence, one should resort to numerical solutions.

A different extension to the model would be allowing for state-dependent risk and regret aversion parameters. Steffensen and Søe (2023) derived closed-form solutions for the optimal consumption-investment-insurance strategy for a state-dependent risk aversion model. Similar to their work, one could implement these techniques to allow for state dependence of the risk and regret aversion parameters in the model. Instead of state-dependent parameters, one could also assume time-dependent risk and regret aversion parameters. A state- and time-dependent regret aversion parameter could be of interest for the analysis of fund managers. Allowing the regret aversion parameter to increase (decrease) after a bad (good) state of the world could partly explain the risk-taking behavior of fund managers as Benyelles and Arisoy (2019) showed that fund managers tend to take on more risk in the subsequent period if they are outperformed by peers.

At last, the model could numerically be assessed with more realistic assumptions such as no access to continuously-adjustable life-insurance contracts, and short-selling and borrowing constraints. The assumption of having access to a perfect life-insurance market with continuously-adjustable life-insurance contracts is rather unrealistic. Hence, a numerical model relaxing such an assumption improves the real world credibility of the model. It can be shown that the indirect utility functions considered in this thesis satisfy the homogeneity property. As a result, one could numerically analyze the more realistic constrained model by implementing a two-dimensional grid-based finite difference scheme. In a follow up paper, such a finite difference scheme will be implemented. Inference based on such a model would be more creditable.

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7 Appendix

7.1 Proof limiting case $\psi = 1$ of the regret-averse normalized aggregator function

Proof limiting case $\psi = 1$ of the regret-averse normalized aggregator function. For $\psi = 1$, it holds that

$$\mathcal{F}(c, \hat{c}, V) = \lim_{\psi \to 1} \frac{\delta c^{1-\frac{1}{\psi}} \left(\frac{\hat{c}}{c}\right)^{\frac{\kappa}{\varphi}} ([1-\gamma]V)^{1-\frac{1}{\varphi}} - \delta(1-\gamma)V}{1 - \frac{1}{\psi}}$$

By rule of l'Hopitâl this equals

$$= \lim_{\psi \to 1} \frac{-\delta \hat{c}^{\frac{\kappa(1-\frac{1}{\psi})}{1-\gamma}} c^{1-\frac{1}{\psi}-\frac{\kappa(1-\frac{1}{\psi})}{1-\gamma}} V[\{\ln(c) - \ln(\hat{c})\}\kappa + \gamma \ln(c) - \ln(c) + \ln([1-\gamma]V)]}{(V[1-\gamma])^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \psi^2 \psi^{-2}}$$
$$= \delta V \left[(1-\gamma-\kappa)\ln(c) + \kappa \ln(\hat{c}) - \ln([1-\gamma]V)\right]$$

□

7.2 Verification of the desired properties of table 1

The multiplicative regret-utility function proposed in Section 3.1 is specified to be

$$u(x, y) = \frac{x^{1-\gamma}}{1-\gamma}\left(\frac{y}{x}\right)^{\kappa}, \quad \gamma - 1 \geq \kappa \geq 1, \; x > 0, \text{ and } y > 0$$

with γ being the time-independent risk aversion parameter and κ the time-independent regret aversion parameter. Note that both realized and foregone consumption/wealth are strictly positive.

It will be verified that this regret-utility specification satisfies all desired properties as defined by Goossens (2021) (Table 1) except property *P2c*. The derivations of the respective properties are given below:

- P1a: $\frac{\partial u(x,x)}{\partial x} = x^{-\gamma} > 0$
- P1b: $\frac{\partial^2 u(x,x)}{\partial x^2} = -\gamma x^{-\gamma} < 0$
- P2a: $u_1(x, y) = \frac{1-\gamma-\kappa}{1-\gamma} x^{-\gamma} \left(\frac{y}{x}\right)^{\kappa} > 0$
- P2b: $u_2(x, y) = \frac{\kappa}{1-\gamma} x^{1-\gamma} \left(\frac{y}{x}\right)^{\kappa} \frac{1}{y} < 0$

- P2c: $u_1(x,y) + u_2(x,y) = \frac{1-\gamma-\kappa}{1-\gamma} x^{-\gamma} \left(\frac{y}{x}\right)^{\kappa} + \frac{\kappa}{1-\gamma} x^{1-\gamma} \left(\frac{y}{x}\right)^{\kappa} \frac{1}{y}$
  $= \left(\frac{1}{x} + \frac{\kappa}{\gamma-1}\left(\frac{1}{x} - \frac{1}{y}\right)\right) x^{1-\gamma} \left(\frac{y}{x}\right)^{\kappa} = \begin{cases} \geq 0 & \text{if } x \leq y \\ < 0 & \text{if } \frac{y}{x} < \frac{\kappa}{\gamma+\kappa-1} \\ \geq 0 & \text{if } \frac{y}{x} \geq \frac{\kappa}{\gamma+\kappa-1} \end{cases}$
  as it holds that $\left(\frac{1}{x} + \frac{\kappa}{\gamma-1}\left(\frac{1}{x} - \frac{1}{y}\right)\right) \geq 0 \iff \frac{y}{x} \geq \frac{\kappa}{\gamma+\kappa-1}$. Note that by definition $x^{1-\gamma} \left(\frac{y}{x}\right)^{\kappa} > 0$.

- P3: $u_{12}(x,y) = \frac{(1-\gamma-\kappa)\kappa}{1-\gamma} x^{-\gamma} \left(\frac{y}{x}\right)^{\kappa} \frac{1}{y} = u_{21}(x,y) > 0$

- P4a: $u_{11}(x,y) = \frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma} x^{-\gamma-1} \left(\frac{y}{x}\right)^{\kappa} < 0$

- P4b: $u_{22}(x,y) = \frac{\kappa(\kappa-1)}{1-\gamma} x^{1-\gamma} \left(\frac{y}{x}\right)^{\kappa} \frac{1}{y^2} \leq 0$

This concludes the verification of the properties as defined by Goossens (2021) for the proposed multiplicative regret-utility function.

7.3 Conjecture and corresponding partial derivatives

Auxiliary model

It is conjectured that the indirect utility function of the auxiliary model has the following functional form

$$\tilde{J}(t,\tilde{W},y,p) = \frac{1}{1-\gamma}\tilde{G}(t,p)^{\gamma}\left(\tilde{W} + yF(t,p)\right)^{1-\gamma}$$

with the following corresponding derivatives with respect to $\tilde{W}$, y and t

$$\begin{aligned}
\tilde{J}_{\tilde{W}}(t,\tilde{W},y,p) &= \tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma} \\
\tilde{J}_{\tilde{W}\tilde{W}}(t,\tilde{W},y,p) &= -\gamma\tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma-1} \\
\tilde{J}_{y}(t,\tilde{W},y,p) &= \tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma}F(t,p) \\
\tilde{J}_{yy}(t,\tilde{W},y,p) &= -\gamma\tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma-1}F(t,p)^2 \\
\tilde{J}_{\tilde{W}y}(t,\tilde{W},y,p) &= -\gamma\tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma-1}F(t,p) \\
\frac{\partial\tilde{J}}{\partial t}(t,\tilde{W},y,p) &= \frac{\gamma}{1-\gamma}\frac{\partial\tilde{G}}{\partial t}(t,p)\tilde{G}(t)^{\gamma-1}(\tilde{W} + yF(t,p))^{1-\gamma} \\
&\quad + \tilde{G}(t,p)^{\gamma}(\tilde{W} + yF(t,p))^{-\gamma}y\frac{\partial F}{\partial t}(t,p)
\end{aligned}$$

where the subscripts denote the partial derivatives with respect to the state variables and time.

Regret-averse model
The conjecture is made that the indirect utility function of the regret-averse model has the following functional form

$$J(t,W,\hat{W},y,p)=\frac{1}{1-\gamma}G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}$$

with the following corresponding derivatives with respect to W, $\hat{W}$, y and t

$$\begin{aligned}
J_W(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
J_{WW}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
J_{\hat{W}}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{\kappa}{1-\gamma}\right)(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}\\
J_{\hat{W}\hat{W}}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}\\
J_{W\hat{W}}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}\\
J_y(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}F(t,p)\\
&\quad+G(t,p)^{\gamma}\left(\frac{\kappa}{1-\gamma}\right)(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}F(t,p)\\
J_{yy}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}F(t,p)^2\\
&\quad+2\cdot G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}F(t,p)^2\\
&\quad+G(t,p)^{\gamma}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}F(t,p)^2\\
J_{Wy}(t,W,\hat{W},y,p)&=G(t)^{\gamma}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}F(t,p)\\
&\quad+G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}F(t,p)\\
J_{\hat{W}y}(t,W,\hat{W},y,p)&=G(t,p)^{\gamma}\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}F(t,p)\\
&\quad+G(t,p)^{\gamma}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}F(t,p)
\end{aligned}$$

$$\frac{\partial J}{\partial t}(t, W, \hat{W}, y, p) = \frac{\gamma}{1-\gamma} G(t,p)^{\gamma-1} \frac{\partial G}{\partial t}(t,p)(W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^{\kappa}$$
$$+ G(t,p)^{\gamma} \left(\frac{1-\gamma-\kappa}{1-\gamma} \right) (W + yF(t,p))^{-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^{\kappa} y \frac{\partial F}{\partial t}(t,p)$$
$$+ G(t,p)^{\gamma} \left(\frac{\kappa}{1-\gamma} \right) (W + yF(t,p))^{1-\gamma} \left(\frac{\hat{W} + yF(t,p)}{W + yF(t,p)} \right)^{\kappa} \frac{1}{\hat{W} + yF(t,p)} y \frac{\partial F}{\partial t}(t,p)$$

where the subscripts denote the partial derivatives with respect to the state variables and time.

7.4 ODEs for the regret-utility specifications

This section shows the lengthy ODEs for the various regret-utility specifications considered in this paper. The ODEs are constructed by substituting the optimal (foregone) control processes and the conjecture into the respective HJB equations. For the auxiliary model, the equations only differ in $f(c, \tilde{J})$ given by equation (8). For the regret-averse models, the equations only differ in $\mathcal{F}(c, \hat{c}, J) - \hat{c} J_{\hat{W}}$ with $\mathcal{F}(c, \hat{c}, J)$ given by equation (9).

Auxiliary model

$$0 = \begin{cases} \left(\frac{\delta^{\psi}}{\psi-1} \tilde{G}(t,p)^{\frac{\gamma\psi-1}{\gamma-1}} - \frac{\delta\varphi}{1-\gamma} \tilde{G}(t,p) \right) \tilde{G}(t,p)^{\gamma-1} (\tilde{W} + yF(t,p))^{1-\gamma} & \text{for } \psi \neq 1 \\ \left(\varrho^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} - \frac{\delta}{1-\gamma} \tilde{G}(t,p) \right) \tilde{G}(t,p)^{\gamma-1} (\tilde{W} + yF(t,p))^{1-\gamma} & \text{for } \psi = \frac{1}{\gamma} \\ \left(\delta[\ln(\delta) - 1] - \delta \frac{\gamma}{1-\gamma} \ln(\tilde{G}(t,p)) \right) \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} & \text{for } \psi = 1 \end{cases}$$
$$+ \frac{\gamma}{1-\gamma} \frac{\partial G}{\partial t}(t,p) \tilde{G}(t,p)^{\gamma-1} (\tilde{W} + yF(t,p))^{1-\gamma} + \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} y \frac{\partial F}{\partial t}$$
$$+ \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} r - \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} r y F(t,p)$$
$$+ \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} \sigma\lambda - \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} \lambda \zeta(t,p) y F(t,p)$$
$$+ \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} y - \frac{1}{2} \gamma \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} \sigma^2 \tag{116}$$
$$+ \frac{\gamma}{1-\gamma} \sum_{q \neq p} h_t^{p,q} \left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}} \right)^{1-\frac{1}{\gamma}} \tilde{G}(t,q) \tilde{G}(t,p)^{\gamma-1} (\tilde{W} + yF(t,p))^{1-\gamma}$$
$$+ \sum_{q \neq p} \hat{h}_t^{p,q} \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} - \sum_{q \neq p} \hat{h}_t^{p,q} \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} y F(t,p)$$
$$+ \sum_{q \neq p} \hat{h}_t^{p,q} y P(t,p,q) F(t,q) \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma}$$
$$- \frac{1}{1-\gamma} \sum_{q \neq p} h_t^{p,q} \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{1-\gamma} + \tilde{G}(t,p)^{\gamma} (\tilde{W} + yF(t,p))^{-\gamma} y F(t,p) \alpha(t,p)$$

Regret-averse model

$$
0=\begin{cases}
\left(\frac{\delta^{\psi}}{1-\gamma}(\varphi-1+\gamma+\kappa)\tilde{G}(t,p)^{\frac{-\psi^2\kappa\gamma}{\varphi(\varphi+\kappa\psi)}}G(t,p)^{\frac{\varphi+(\kappa-\gamma)\psi}{\varphi+\kappa\psi}}-\frac{\delta\varphi}{1-\gamma}G(t,p)\right. & \text{for } \psi\neq 1\\
\quad\left.-\frac{\kappa}{1-\gamma}\delta^{\psi}\tilde{G}(t,p)^{\frac{-\psi\gamma}{\varphi}}G(t,p)\right)G(t,p)^{\gamma-1}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa} & \\
\left(\frac{\varrho^{\frac{1}{\gamma}}(\gamma+\kappa)}{1-\gamma}\tilde{G}(t,p)^{\frac{-\kappa}{\gamma+\kappa}}G(t,p)^{\frac{\kappa}{\gamma+\kappa}}-\frac{\delta}{1-\gamma}G(t,p)\right. & \text{for } \psi=\frac{1}{\gamma}\\
\quad\left.-\frac{\kappa}{1-\gamma}\varrho^{\frac{1}{\gamma}}\tilde{G}(t,p)^{-1}G(t,p)\right)G(t,p)^{\gamma-1}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa} & \\
\left(\delta[\ln(\delta)-1]-\delta\frac{\gamma}{1-\gamma}\ln(G(t,p))\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa} & \text{for } \psi=1
\end{cases}
$$

$$
\begin{aligned}
&+\left(\frac{\gamma+\kappa}{1-\gamma}\right)\sum_{q\neq p}h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)^{\frac{\kappa}{\gamma+\kappa}}G(t,p)^{\frac{\kappa}{\gamma+\kappa}}G(t,q)^{\frac{\gamma}{\gamma+\kappa}}\\
&\qquad\cdot G(t,p)^{\gamma-1}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)\sum_{q\neq p}\hat{h}_t^{p,q}G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
&-\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)\sum_{q\neq p}\hat{h}_t^{p,q}yF(t,p)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\qquad(117)\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)\sum_{q\neq p}\hat{h}_t^{p,q}yP(t,p,q)F(t,q)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)\left(\frac{\lambda^2+\lambda\sigma\kappa}{\gamma+\kappa}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
&-\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)\zeta(t,p)\lambda\\
&+\frac{1}{2}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)\left(\frac{\lambda+\sigma\kappa}{\gamma+\kappa}\right)^2G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\\
&+\frac{1}{2}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}y^2F(t,p)^2\zeta(t,p)^2\\
&-\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)\left(\frac{(\lambda+\sigma\kappa)\zeta(t,p)}{\gamma+\kappa}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)\\
&+\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)\left(\frac{\lambda\sigma+\sigma^2\kappa}{\gamma+\kappa}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)\zeta(t,p)\sigma\\
&-\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,P)}\left(\frac{\lambda+\sigma\kappa}{\gamma+\kappa}\right)yF(t,p)\zeta(t,p)\\
&+\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,P)}y^{2}F(t,p)^{2}\zeta(t,p)^{2}\\
&+\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)\left(\frac{(\lambda+\sigma\kappa)\zeta(t,p)}{\gamma+\kappa}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,P)\\
&-\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}y^{2}F(t,p)^{2}\zeta(t,p)^{2}\\
&+\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,P)}\left(\frac{\lambda+\sigma\kappa}{\gamma+\kappa}\right)yF(t,p)\zeta(t,p)\\
&-\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}y^{2}F(t,p)^{2}\zeta(t,p)^{2}\\
&+\left(\frac{\gamma}{1-\gamma}\right)G(t,p)^{\gamma-1}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{\partial G}{\partial t}(t,p)\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}y\frac{\partial F}{\partial t}(t,p)\\
&+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}y\frac{\partial F}{\partial t}(t,p)\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}r\\
&-\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)r\\
&+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}y\\
&+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}r\\
&-\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}yF(t,p)r\\
&+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\sigma\lambda\\
&-\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}yF(t,p)\zeta(t,p)\lambda
\end{aligned}
$$

$$+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}y$$

$$-\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\sum_{q\neq p}\left(\frac{\tilde{G}(t,q)}{\tilde{G}(t,p)}\right)h_t^{p,q}\left(\frac{\hat{h}_t^{p,q}}{h_t^{p,q}}\right)^{1-\frac{1}{\gamma}}$$

$$+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\sum_{q\neq p}\hat{h}_t^{p,q}$$

$$-\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}\sum_{q\neq p}\hat{h}_t^{p,q}yF(t,p)$$

$$+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}\sum_{q\neq p}\hat{h}_t^{p,q}yP(t,p,q)F(t,q)$$

$$+\frac{1}{2}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\sigma^2$$

$$+\frac{1}{2}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}y^2F(t,p)^2\zeta(t,p)^2$$

$$-\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}yF(t,p)\zeta(t,p)\sigma$$

$$+\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)\alpha(t,p)$$

$$+\left(\frac{\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}yF(t,p)\alpha(t,p)$$

$$+\frac{1}{2}\left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma-1}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}y^2F(t,p)^2\zeta(t,p)^2$$

$$+\frac{1}{2}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}y^2F(t,p)^2\zeta(t,p)^2$$

$$+\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}y^2F(t,p)^2\zeta(t,p)^2$$

$$+\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}yF(t,p)\zeta(t,p)\sigma$$

$$-\left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}y^2F(t,p)^2\zeta(t,p)^2$$

$$+\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{\hat{W}+yF(t,p)}yF(t,p)\zeta(t,p)\sigma$$

$$- \left(\frac{\kappa(\kappa-1)}{1-\gamma}\right) G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}\frac{1}{(\hat{W}+yF(t,p))^2}y^2F(t,p)^2\zeta(t,p)^2$$
$$- \sum_{q\neq p} h_t^{p,q} G(t,p)^{\gamma}(W+yF(t,p))^{1-\gamma}\left(\frac{\hat{W}+yF(t,p)}{W+yF(t,p)}\right)^{\kappa}$$

7.5 Proof of Theorem 3.1

Proof of Theorem 3.1. Then the difference between the optimal regret-averse investment amount and the Merton investment amount is given by

$$\begin{aligned}
\theta^{Regret} - \theta^{Merton} =& \frac{\lambda+\kappa\sigma}{(\gamma+\kappa)\sigma}(W+yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma} - \left[\frac{\lambda}{\sigma\gamma}(W+yF(t,p)) - yF(t,p)\frac{\zeta(t,p)}{\sigma}\right]\\
=& \left(\frac{\lambda+\sigma\kappa}{(\gamma+\kappa)\sigma} - \frac{\lambda}{\sigma\gamma}\right)(W+yF(t,p))\\
=& \frac{(\lambda+\sigma\kappa)\sigma\gamma - \lambda\sigma(\gamma+\kappa)}{\sigma^2\gamma(\gamma+\kappa)}(W+yF(t,p))\\
=& \frac{\sigma^2\kappa\gamma - \lambda\sigma\kappa + \lambda\sigma\gamma - \lambda\sigma\gamma}{\sigma^2\gamma(\gamma+\kappa)}(W+yF(t,p))\\
=& \frac{\sigma^2\kappa\gamma - \lambda\sigma\kappa}{\sigma^2\gamma(\gamma+\kappa)}(W+yF(t,p))\\
=& \left[\frac{\kappa}{\gamma+\kappa} - \frac{\lambda}{\sigma\gamma}\frac{\kappa}{\gamma+\kappa}\right](W+yF(t,p))\\
=& \left(1-\frac{\lambda}{\sigma\gamma}\right)\frac{\kappa}{\gamma+\kappa}(W+yF(t,p)) > 0 \text{ if and only if } \gamma > \frac{\lambda}{\sigma}
\end{aligned}$$

as $\gamma - 1 \geq \kappa \geq 1$ and $W + yF(t,p) > 0$ by definition. □

7.6 Proof of Corollary 3.1

Proof of Corollary 3.1. In this model it is assumed that there is a constant investment opportunity set, i.e. the expected rate of return μ and volatility of the stock σ, interest rate r, and market price of risk λ are constant over time. The regret-averse investor earns spanned exogenous labor income without biometric risks. The labor income dynamics are specified as follows

$$\mathrm{d}y_t = y_t\left[\alpha\,\mathrm{d}t + \zeta\,\mathrm{d}Z_t\right]$$

The labor income process follows a geometric Brownian motion with the same underlying Brownian motion as the stock market. Hence, the market is complete. The labor income can therefore be valued as a financial asset and it can be seen as a dividend stream from some trading strategy in the financial asset. The value of the labor income stream at time

t, *human wealth*, should be (Munk, 2017)

$$\begin{aligned} H(t,y) &= \mathbb{E}^{\mathbb{Q}}_{t,y}\left[\int_t^T \mathrm{e}^{-r(s-t)} y_s \,\mathrm{d}s\right] \\ &= \mathbb{E}_{t,y}\left[\int_t^T \mathrm{e}^{-(r-\frac{1}{2}\lambda^2)(s-t)-\lambda(Z_s-Z_t)} y_s \,\mathrm{d}s\right] \\ &= y\begin{cases} \frac{1}{r-\alpha+\zeta\lambda}\left(1-\mathrm{e}^{-(r-\alpha+\zeta\lambda)(T-t)}\right) & \text{if } r-\alpha+\zeta\lambda \neq 0 \\ (T-t) & \text{if } r-\alpha+\zeta\lambda = 0 \end{cases} \\ &= yM(t) \end{aligned}$$

Human wealth is given by the product of current labor income and a time-dependent multiplier $M(t)$. The investor can "sell" her future labor income at the financial market to get $H(t,y)$ such that the investor has a total wealth of $W + H(t,y)$.

Moreover, the function $H(t,y)$ satisfies, according to the famous Fayman-Kac theorem, the following PDE

$$\frac{\partial H}{\partial t}(t,y) + (\alpha-\zeta\lambda)yH_y(t,y) + \frac{1}{2}\zeta^2 y^2 H_{yy}(t,y) - rH(t,y) + y = 0$$

where $H_y(t,y)$ denotes the first-order derivative with respect to y and $H_{yy}(t,y)$ denotes the second-order derivative with respect to y. From $H(t,y) = yM(t)$, it can be easily verified that $H_y(t,y) = \frac{H(t,y)}{y} = M(t)$ and $H_{yy}(t,y) = 0$.

The financial wealth dynamics of the investor are given by

$$\mathrm{d}W_t = [rW_t + \theta_t\lambda\sigma - c_t + y_t]\,\mathrm{d}t + \theta_t\sigma\,\mathrm{d}Z_t$$

Auxiliary model
The indirect utility function of the auxiliary investor is the following

$$\tilde{J}(t,\tilde{W},y) = \sup_{(\tilde{c})\in\tilde{\mathcal{A}}_t} \mathbb{E}_{t,\tilde{W},y}\left[\int_t^T \mathrm{e}^{-\delta(s-t)} v(\tilde{c}_s)\,\mathrm{d}s + \mathrm{e}^{-\delta(T-t)}\bar{v}(\tilde{W}_T)\right]$$

where the supremum runs over all admissible consumption strategies and $v(\tilde{c}) = \varrho\frac{\tilde{c}^{1-\gamma}}{1-\gamma}$ and $\bar{v}(\tilde{W}_T) = \varepsilon\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma}$ with $\varrho \geq 0$ and $\varepsilon \geq 0$ denoting the relative preference for intermediate consumption and the bequest motive, respectively.

Based on the indirect utility function and the wealth dynamics, the following HJB equation can be constructed

$$\begin{aligned} \delta\tilde{J} =& \mathcal{L}^{\tilde{c}} + \frac{\partial\tilde{J}}{\partial t} + \tilde{J}_{\tilde{W}} r\tilde{W} + \tilde{J}_{\tilde{W}}\sigma\lambda\tilde{\theta} + \tilde{J}_{\tilde{W}} y \\ &+ \frac{1}{2}\tilde{J}_{\tilde{W}\tilde{W}}\sigma^2\tilde{\theta}^2 + \tilde{J}_y y\alpha + \frac{1}{2}\tilde{J}_{yy} y^2\zeta^2 + \tilde{J}_{\tilde{W}y} y\tilde{\theta}\sigma\zeta \end{aligned}$$

with $\mathcal{L}^{\tilde{c}} = \sup_{\tilde{c}\geq 0}\left\{\varrho\frac{\tilde{c}^{1-\gamma}}{1-\gamma} - \tilde{c}\tilde{J}_{\tilde{W}}\right\}$.

The auxiliary investor has a total wealth of $\tilde{W} + H(t, y)$ with financial wealth $\tilde{W}$ and human wealth $H(t, y)$. Following Munk (2017), the conjecture is made that the investor invests in such a way that the dynamics of total wealth are similar to the dynamics of (financial) wealth in case of $y = 0$. By definition of the auxiliary investor, it is known that in a setting excluding labor income the agent invests all her wealth into the stock market, i.e. $\tilde{\theta}_t = \tilde{W}_t$ for all t. This results in the following dynamics for total wealth

$$\mathrm{d}(\tilde{W}_t + H(t, y)) = \mu(t, y)\,\mathrm{d}t + \sigma(\tilde{W}_t + H(t, y))\,\mathrm{d}Z_t$$

with $\mu(t, y)$ being the drift rate of total wealth, but this drift rate is not of any importance for the derivations. Hence, the functional form is omitted.

Applying Itô's Lemma to $H(t, y)$ yields that the dynamics of human wealth follow a stochastic differential equation as given by

$$\mathrm{d}H(t, y) = \left[\frac{\partial H}{\partial t}(y, t) + H_y(t, y)y\alpha + \frac{1}{2}H_{yy}(t, y)y^2\zeta^2\right]\mathrm{d}t + H_y(t, y)y\zeta\,\mathrm{d}Z_t$$

Hence, the dynamics of financial wealth should be given by

$$\begin{aligned} &\mathrm{d}W = \left[rW + \tilde{\theta}_t\sigma\lambda - c_t + y_t\right]\mathrm{d}t + \tilde{\theta}_t\sigma\,\mathrm{d}Z_t \\ \Longleftrightarrow\quad &\mathrm{d}W = \bar{\mu}(t, y)\,\mathrm{d}t + \left[\sigma(W + H(t, y)) - H_y(t, y)y_t\zeta\right]\mathrm{d}Z_t \end{aligned}$$

with $\bar{\mu}(t, y)$ being the drift rate of financial wealth.

These dynamics of financial wealth are satisfied by an investment strategy $\tilde{\theta}_t$ that satisfies

$$\tilde{\theta}_t\sigma = \sigma(W_t + H(t, y)) - H_y(t, y)y_t\zeta$$

Thus, the amount invested into the risky asset should be

$$\tilde{\theta}_t = W_t + H(t, y) - H(t, y)\frac{\zeta}{\sigma}$$

as $H_y(t, y) = \frac{H(t,y)}{y}$.

Solving $\mathcal{L}^{\tilde{c}}$ for $\tilde{c}$ by its first-order condition yields

$$c^* = (v')^{-1}\left(\tilde{J}_{\tilde{W}}\right)$$

and hence,

$$\mathcal{L}^{\tilde{c}} = v(I_v(\tilde{J}_{\tilde{W}})) - I_v(\tilde{J}_{\tilde{W}})\tilde{J}_{\tilde{W}}$$

where the inverse of the marginal utility is denoted by $I_v = (v')^{-1}$.

It is conjectured that the indirect utility function is given by

$$\tilde{J}(t, \tilde{W}, y) = \frac{\tilde{g}(t)^\gamma}{1-\gamma} \left(\tilde{W} + H(t, y)\right)^{1-\gamma}$$

The partial derivatives are the same as for the biometric risk model and can be seen in Appendix 7.3.

Substituting the conjecture into $\mathcal{L}^{\tilde{c}}$ and thereafter in the HJB yields

$$\begin{aligned}
\frac{\delta}{1-\gamma}\tilde{g}(t)^\gamma(\tilde{W} + H(t, y))^{1-\gamma} =& \varrho^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma}\tilde{g}^{\gamma-1}(\tilde{W} + H(t, y))^{1-\gamma} \\
&+ \frac{\gamma}{1-\gamma}\tilde{g}'(t)\tilde{g}(t)^{\gamma-1}(\tilde{W} + H(t, y))^{1-\gamma} \\
&+ \tilde{g}^\gamma(\tilde{W} + H(t, y))^{1-\gamma} r \\
&+ \tilde{g}(t)^\gamma(\tilde{W} + H(t, y))^{1-\gamma}\sigma\lambda \\
&- \frac{1}{2}\gamma\tilde{g}(t)^\gamma(\tilde{W} + H(t, y))^{1-\gamma}\sigma^2
\end{aligned}$$

This equation should hold for all $\tilde{W} + H(t, y)$ and all $t \in [0, T)$ (Munk, 2017). Hence, the following first-order differential equation can be identified

$$\tilde{g}'(t) = \tilde{A}\tilde{g}(t) - \tilde{B}$$

with terminal condition $\tilde{g}(T) = \varepsilon^{\frac{1}{\gamma}}$, and $\tilde{A}$ and $\tilde{B}$ being constants given by

$$\begin{aligned}
\tilde{A} &= \frac{\delta + r(\gamma - 1) + \sigma\lambda(\gamma - 1) + \frac{1}{2}\sigma^2\gamma(1-\gamma)}{\gamma} \\
\tilde{B} &= \varrho^{\frac{1}{\gamma}}
\end{aligned}$$

Here $\tilde{A}$ is assumed to be positive. This assumption is in line with the assumption made by Munk (2017). For reasonable values of δ, r, σ, λ and $\gamma > 2$ it holds that $\frac{\delta+\mu(\gamma-1)}{\gamma} > \frac{1}{2}\sigma^2$ and in that case the assumption $\tilde{A} > 0$ is satisfied.

The solution to the first-order ODE is known to be

$$\tilde{g}(t) = \varrho^{\frac{1}{\gamma}} \frac{1 - \mathrm{e}^{-\tilde{A}(T-t)}}{\tilde{A}} + \varepsilon^{\frac{1}{\gamma}}\mathrm{e}^{-\tilde{A}(T-t)}$$

with $\tilde{A} = \frac{\delta+r(\gamma-1)+\sigma\lambda(\gamma-1)+\frac{1}{2}\sigma^2\gamma(1-\gamma)}{\gamma}$.

From this it can be concluded that the optimal foregone consumption choice is given by

$$\tilde{c}^*(t, \tilde{W}, y) = \varrho^{\frac{1}{\gamma}} \frac{\tilde{W} + H(t, y)}{\tilde{g}(t)}$$

Regret-averse model
The regret-averse investor's indirect utility is given by

$$J(t,W,\hat{W},y) = \sup_{(c,\theta)\in\mathcal{A}_t} \mathbb{E}_{t,W,\hat{W},y}\left[\int_t^T \mathrm{e}^{-\delta(s-t)} u(c_s,\hat{c}_s)\,\mathrm{d}s + \mathrm{e}^{-\delta(T-t)}\bar{u}(W_T,\hat{W}_T)\right]$$

where $u(c) = \varrho\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\hat{c}}{c}\right)^{\kappa}$ and $\bar{u}(W_T) = \varepsilon\frac{W_T^{1-\gamma}}{1-\gamma}\left(\frac{\hat{W}_T}{W_T}\right)^{\kappa}$ with $\hat{c} = \tilde{c}^*$ and $\hat{W} = \tilde{W}^*$ being the optimal foregone consumption and wealth as determined by the auxiliary model and $\varrho \geq 0$ and $\varepsilon \geq 0$ denote the relative preference for intermediate consumption and the bequest motive, respectively.

For the given indirect utility function of the investor and the underlying dynamics for wealth, labor income, and foregone wealth the associated HJB equation can be constructed

$$\begin{aligned}\delta J =& \mathcal{L}^c + \mathcal{L}^\theta + \frac{\partial J}{\partial t} + J_W(rW+y)\\ &+ J_{\hat{W}}(r\hat{W} + \tilde{\theta}\sigma\lambda + y - \tilde{c}) + \frac{1}{2}J_{\hat{W}\hat{W}}\tilde{\theta}^2\sigma^2\\ &+ J_y y\alpha + \frac{1}{2}J_{yy}y^2\zeta^2 + J_{\hat{W}y}\tilde{\theta}\sigma y\zeta\end{aligned}$$

with

$$\mathcal{L}^c = \sup_{c\geq 0}\left\{\varrho\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{\hat{c}}{c}\right)^{\kappa} - cJ_W\right\}$$

and

$$\mathcal{L}^\theta = \sup_{\theta\in\mathbb{R}}\{J_W\theta\sigma\lambda + \frac{1}{2}J_{WW}\theta^2\sigma^2 + J_{W\hat{W}}\theta\sigma^2\tilde{\theta} + J_{Wy}\theta\sigma y\zeta\}$$

Note that $\tilde{c}$ and $\tilde{\theta}$ are given by the solutions to the auxiliary model as determined previously.

Solving $\mathcal{L}^c$ and $\mathcal{L}^\theta$ by their first-order conditions gives the following optimal consumption choice and investment strategy

$$c^* = (u')^{-1}(J_W)$$

and

$$\theta^* = -\frac{J_W\lambda}{J_{WW}\sigma} - \frac{J_{W\hat{W}}\tilde{\theta}}{J_{WW}} - \frac{J_{Wy}y\zeta}{J_{WW}\sigma}$$

The following conjecture about the functional form of the indirect utility function is made

$$J(t,W,\hat{W},y) = \frac{g(t)^\gamma}{1-\gamma}(W+H(t,y))^{1-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^{\kappa}$$

with the corresponding derivatives given in Appendix 7.3 (ignoring biometric state p).

Using the conjecture for the indirect utility function, the optimal consumption process can be determined.

$$
\begin{aligned}
c^* = (u')^{-1}(J_W) = &\left(\frac{1-\gamma}{1-\gamma-\kappa}\right)^{\frac{1}{-\gamma-\kappa}} \frac{J_W^{\frac{1}{-\gamma-\kappa}}}{\hat{c}^{\frac{\kappa}{-\gamma-\kappa}}} \varrho^{\frac{1}{\gamma+\kappa}} \\
= &\frac{\varrho^{\frac{1}{\gamma}}(W + H(t,y))}{g(t)^{\frac{\gamma}{\gamma+\kappa}} \tilde{g}(t)^{\frac{\kappa}{\gamma+\kappa}}}
\end{aligned}
$$

The optimal investment fraction is given by

$$
\begin{aligned}
\theta^* = &- \frac{J_W\lambda}{J_{WW}\sigma} - \frac{J_{W\hat{W}}\tilde{\theta}}{J_{WW}} - \frac{J_{Wy}y\zeta}{J_{WW}\sigma} \\
= &- \frac{g(t)^\gamma \left(\frac{1-\gamma-\kappa}{1-\gamma}\right)(W + H(t,y))^{-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \lambda}{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W + H(t,y))^{-\gamma-1}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \sigma} \\
&- \frac{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W + H(t,y))^{-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \frac{1}{\hat{W}+H(t,y)}\tilde{\theta}}{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W + H(t,y))^{-\gamma-1}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa} \\
&- \frac{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W + H(t,y))^{-\gamma-1}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa H_y(t,y)y\zeta}{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W + H(t,y))^{-\gamma-1}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \sigma} \\
&- \frac{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)\kappa}{1-\gamma}\right)(W + H(t,y))^{-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \frac{1}{\hat{W}+H(t,y)}H_y(t,y)y\zeta}{g(t)^\gamma \left(\frac{(1-\gamma-\kappa)(-\gamma-\kappa)}{1-\gamma}\right)(W + H(t,y))^{-\gamma-1}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa \sigma}
\end{aligned}
$$

using that $\tilde{\theta} = (\hat{W} + H(t,y)) - H(t,y)\frac{\zeta}{\sigma}$ and that $H_y(t,y) = \frac{H(t,y)}{y}$ this equation can be simplified into the following

$$
\begin{aligned}
= &\frac{1}{\gamma+\kappa}\frac{\lambda}{\sigma}(W + H(t,y)) + \frac{\kappa}{\gamma+\kappa}(W + H(t,y))\frac{1}{\hat{W} + H(t,y)}(\hat{W} + H(t,y) - H(t,y)\frac{\zeta}{\sigma}) \\
&- \frac{H(t,y)}{y}y\frac{\zeta}{\sigma} + \frac{\kappa}{\gamma+\kappa}(W + H(t,y))\frac{1}{\hat{W} + H(t,y)}\frac{H(t,y)}{y}y\frac{\zeta}{\sigma} \\
= &\frac{\lambda + \sigma\kappa}{(\gamma+\kappa)\sigma}(W + H(t,y)) - H(t,y)\frac{\zeta}{\sigma}
\end{aligned}
$$

The HJB equation can be solved by filling in the conjecture, all the partial derivatives, the already known optimal solution for foregone consumption and investment strategy, and the optimal control processes c^* and θ^*. A lot of terms will cancel out and some terms will be equal to zero as $H(t,y)$ should satisfy the Fayman-Kac PDE as previously denoted.

The terms that remain yield the following nonlinear first-order differential equation for $g(t)$

$$g'(t) = [A + B(t)]\, g(t) - C(t)g(t)^{\frac{\kappa}{\gamma+\kappa}}$$

with terminal condition $g(T) = \varepsilon^{\frac{1}{\gamma}}$ and A, $B(t)$ and $C(t)$ given by the following expressions:

$$\begin{aligned}
A &= \frac{1}{\gamma(\gamma+\kappa)}\left\{\delta(\gamma+\kappa) + r(\gamma-1)(\gamma+\kappa) - \kappa(\gamma+\kappa)\sigma\lambda - \frac{1}{2}\kappa(\kappa-1)(\gamma+\kappa)\sigma^2\right.\\
&\qquad \left. -\,(1-\gamma-\kappa)\left(\frac{1}{2}\lambda^2 + \frac{1}{2}\kappa^2\sigma^2 + \kappa\sigma\lambda\right)\right\}\\
B(t) &= \left(\frac{\kappa}{\gamma}\right)\varrho^{\frac{1}{\gamma}}\tilde{g}(t)^{-1}\\
C(t) &= \left(\frac{\gamma+\kappa}{\gamma}\right)\varrho^{\frac{1}{\gamma}}\tilde{g}(t)^{\frac{-\kappa}{\gamma+\kappa}}
\end{aligned}$$

with $\tilde{g}(t)$ again as given by the auxiliary model.

The solution to this nonlinear differential equation is given by

$$g(t) = \left[\frac{\gamma}{\gamma+\kappa}\mathrm{e}^{\frac{\gamma}{\gamma+\kappa}\int_0^t (A+B(s))\,\mathrm{d}s}\int_t^T \mathrm{e}^{-\frac{\gamma}{\gamma+\kappa}\int_0^s (A+B(u))\,\mathrm{d}u} C(s)\,ds + \mathrm{e}^{-\frac{\gamma}{\gamma+\kappa}\int_t^T (A+B(s))\,\mathrm{d}s}\varepsilon^{\frac{1}{\gamma+\kappa}}\right]^{\frac{\gamma+\kappa}{\gamma}}$$

with

$$\begin{aligned}
A &= \frac{1}{\gamma(\gamma+\kappa)}\left\{\delta(\gamma+\kappa) + r(\gamma-1)(\gamma+\kappa) - \kappa(\gamma+\kappa)\sigma\lambda - \frac{1}{2}\kappa(\kappa-1)(\gamma+\kappa)\sigma^2\right.\\
&\qquad \left. -\,(1-\gamma-\kappa)\left(\frac{1}{2}\lambda^2 + \frac{1}{2}\kappa^2\sigma^2 + \kappa\sigma\lambda\right)\right\}\\
B(t) &= \left(\frac{\kappa}{\gamma}\right)\varrho^{\frac{1}{\gamma}}\tilde{g}(t)^{-1}\\
C(t) &= \left(\frac{\gamma+\kappa}{\gamma}\right)\varrho^{\frac{1}{\gamma}}\tilde{g}(t)^{\frac{-\kappa}{\gamma+\kappa}}
\end{aligned}$$

From the existence of the purely time-dependent function $g(t)$ it can be concluded that the conjecture was correct. Hence, the optimal consumption choice is given by

$$c^*(t, W, y) = \frac{\varrho^{\frac{1}{\gamma}}(W + H(t, y))}{g(t)^{\frac{\gamma}{\gamma+\kappa}}\tilde{g}(t)^{\frac{\kappa}{\gamma+\kappa}}}$$

and the optimal investment strategy is given by

$$\theta^*(t, W, y) = \frac{\lambda + \kappa\sigma}{(\gamma+\kappa)\sigma}\,(W + H(t, y)) - H(t, y)\frac{\zeta}{\sigma}$$

This concludes the proof of Corollary 3.1. □

7.7 Proof of Corollary 3.2

Proof of Corollary 3.2. The proof for Corollary 3.2 is very similar as the proof for Corollary 3.1. Therefore parts of the proof are omitted and the reader is referred to the proof of Corollary 3.1.

In this model it is again assumed that the investor lives in a Black-Scholes world, i.e. the expected rate of return μ and volatility of the stock σ, interest rate r, and market price of risk λ are predetermined and constant over time.

The regret-averse investor earns exogenous labor income specified by the following dynamics

$$dy_t = y_t \left[\alpha\, dt + \zeta\, dZ_t\right]$$

Human wealth should satisfy following PDE

$$\frac{\partial H}{\partial t}(t,y) + (\alpha - \zeta\lambda) y H_y(t,y) + \frac{1}{2}\zeta^2 y^2 H_{yy}(t,y) - rH(t,y) + y = 0$$

where $H_y(t,y)$ denotes the first-order derivative with respect to y and $H_{yy}(t,y)$ denotes the second-order derivative with respect to y. From $H(t,y) = yM(t)$, it can be easily verified that $H_y(t,y) = \frac{H(t,y)}{y} = M(t)$ and $H_{yy}(t,y) = 0$.

The financial wealth dynamics are given by

$$dW_t = \left[rW_t + \theta_t \lambda\sigma - c_t + y_t\right] dt + \theta_t \sigma\, dZ_t$$

Auxiliary model
The auxiliary investor maximizes the utility over intermediate consumption and terminal wealth. The utility index $\tilde{J}(t,\tilde{W},y)$ at time t for foregone consumption process $\tilde{c}$ over the remaining lifetime $[t,T]$ is given by

$$\tilde{J}(t,\tilde{W},y) = \sup_{(\tilde{c})\in\tilde{\mathcal{A}}_t} \mathbb{E}_{t,\tilde{W},y}\left[\int_t^T f(\tilde{c}_s,\tilde{J})\,\mathrm{d}s + \tilde{\mathcal{J}}_T\right]$$

The investor maximizes $\tilde{J}(t,\tilde{W},y)$ for any $t < T$ over all admissible control processes in set $\tilde{\mathcal{A}}_t$ given the state variables at time t.

The normalized aggregator function f for unit EIS as specified in equation (8) is given by

$$f(\tilde{c},\tilde{J}) = \delta(1-\gamma)\tilde{J}\ln(\tilde{c}) - \delta\tilde{J}\ln([1-\gamma]\tilde{J})$$

The time preference of the investor is denoted by δ, and the degree of relative risk aversion by $\gamma > 1$.
The term $\tilde{\mathcal{J}}_T$ is given by $\tilde{\mathcal{J}}_T = \varepsilon\frac{\tilde{W}_T^{1-\gamma}}{1-\gamma}$ with $\varepsilon \geq 0$. This term represents the utility from

terminal wealth with ε being the preference for the bequest motive.

With this specification and the given wealth dynamics, the HJB equation can be formulated as

$$\begin{aligned}0 =&\mathcal{L}^{\tilde{c}} - \delta\tilde{J}\ln([1-\gamma]\tilde{J}) + \frac{\partial \tilde{J}}{\partial t} + \tilde{J}_{\tilde{W}}\tilde{W}(r+\sigma\lambda\tilde{\theta}) + \frac{1}{2}\tilde{J}_{\tilde{W}\tilde{W}}\tilde{W}^2\sigma^2\tilde{\theta}^2 \\ &+\tilde{J}_y y\alpha + \frac{1}{2}\tilde{J}_{yy}y^2\zeta^2 + \tilde{J}_{\tilde{W}y}y\tilde{\theta}\sigma\zeta\end{aligned}$$

with

$$\mathcal{L}^{\tilde{c}} = \sup_{\tilde{c}\geq 0}\left\{\delta(1-\gamma)\tilde{J}\ln(\tilde{c}) - \tilde{c}\tilde{J}_{\tilde{W}}\right\}$$

Following the proof of Corollary 3.1, it is given that the auxiliary investor invests $\tilde{\theta}_t$ amount of money into the risky asset with $\tilde{\theta}_t$ given by

$$\tilde{\theta}_t = W_t + H(t,y) - H(t,y)\frac{\zeta}{\sigma}$$

The optimal foregone consumption is determined by the first-order condition of $\mathcal{L}^{\tilde{c}}$ with respect to $\tilde{c}$. The optimal foregone consumption process is found to be

$$\tilde{c}^* = \frac{\delta(1-\gamma)\tilde{J}}{\tilde{J}_{\tilde{W}}}$$

The conjecture is made the indirect utility function is given by

$$\tilde{J}(t,\tilde{W},y) = \frac{\tilde{g}(t)^\gamma}{1-\gamma}\left(\tilde{W} + H(t,y)\right)^{1-\gamma}$$

and the corresponding derivatives can again be found in Appendix 7.3.

Substituting the conjecture into the HJB equation and simplifying yields the following ODE for $g(t)$

$$\tilde{g}'(t) = \left[\frac{\delta(\gamma-1)}{\gamma}\{\ln(\delta)-1\} + \delta\ln(\tilde{g}(t)) + \frac{r(\gamma-1)}{\gamma} + \sigma\lambda\frac{\gamma-1}{\gamma} + \frac{1}{2}(1-\gamma)\sigma^2\right]\tilde{g}(t)$$

with $\tilde{A} = \frac{\delta(\gamma-1)}{\gamma}\{\ln(\delta)-1\} + \frac{r(\gamma-1)}{\gamma} + \sigma\lambda\frac{\gamma-1}{\gamma} + \frac{1}{2}(\gamma-1)\sigma^2$ and $\tilde{g}(T) = \varepsilon^{\frac{1}{\gamma}}$.

Note that all terms have been divided by $\frac{\gamma}{1-\gamma}\tilde{g}(t)^{\gamma-1}(\tilde{W}+H(t,y))^{1-\gamma} > 0$.

The non-linear ODE is solved by the following time-dependent function $\tilde{g}(t)$

$$\tilde{g}(t) = \mathrm{e}^{(-\tilde{A}+\mathrm{e}^{\delta(t+c_1)})/\delta}$$

Solving for $\tilde{g}(t) = \varepsilon^{\frac{1}{\gamma}}$ gives

$$
\begin{aligned}
&\mathrm{e}^{(-\tilde{A}+\mathrm{e}^{\delta(t+c_1)})/\delta} = \varepsilon^{\frac{1}{\gamma}} \\
\Longleftrightarrow\ &(-\tilde{A} + \mathrm{e}^{\delta(t+c_1)})/\delta = \ln(\varepsilon^{\frac{1}{\gamma}}) \\
\Longleftrightarrow\ &\mathrm{e}^{\delta(t+c_1)} = \frac{\delta}{\gamma}\ln(\varepsilon) + \tilde{A} \\
\Longleftrightarrow\ &\delta(T + c_1) = \ln(\frac{\delta}{\gamma}\ln(\varepsilon) + \tilde{A}) \\
\Longleftrightarrow\ &c_1 = \frac{1}{\delta}\ln(\frac{\delta}{\gamma}\ln(\varepsilon) + \tilde{A}) - T
\end{aligned}
$$

Filling the found expression for the constant c_1 into the function for $\tilde{g}(t)$ and simplifying, gives a closed-form expression for the time-dependent function $\tilde{g}(t)$

$$
\tilde{g}(t) = \mathrm{e}^{-\frac{\tilde{A}}{\delta}+\frac{1}{\delta}\mathrm{e}^{-\delta(T-t)}[\ln(\varepsilon)+\tilde{A}]\frac{\delta}{\gamma}}
$$

There exists thus an only time-dependent function $\tilde{g}(t)$ and hence the conjecture is correct. As a result, the optimal foregone consumption process is found to be

$$
\tilde{c}^*(t, \tilde{W}, y) = \frac{\delta(1-\gamma)\frac{1}{1-\gamma}\tilde{g}(t)^{\gamma}(\tilde{W} + H(t,y))^{1-\gamma}}{\tilde{g}(t)^{\gamma}(\tilde{W} + H(t,y))^{-\gamma}} = \delta(\tilde{W} + H(t,y))
$$

Regret-averse model

The regret-averse investor maximizes utility over intermediate consumption and terminal wealth. The utility index $J(t, W, \hat{W}, y)$ at time t for consumption process c and investment process θ over the remaining lifetime $[t, T]$

$$
J(t, W, \hat{W}, y) = \sup_{(c,\theta)\in\mathcal{A}_t} \mathbb{E}_{t,W,\hat{W},y}\left[\int_t^T \mathcal{F}(c_s, \hat{c}_s, J)\,\mathrm{d}s + \mathcal{J}_T\right]
$$

The investor maximizes $J(t, W, \hat{W}, y)$ for any $t < T$ over all admissible control processes in set $\mathcal{A}_t$ given the state variables and the optimal foregone state variables at time t.

The regret-aversion-adjusted aggregator function $\mathcal{F}$ for unit-EIS as specified in equation (9) is given by

$$
\mathcal{F}(c, \tilde{c}, J) = \delta J\left[(1-\gamma-\kappa)\ln(c) + \kappa\ln(\hat{c}) - \ln([1-\gamma]J)\right]
$$

The term $\mathcal{J}_T$ is assumed to be given by $\mathcal{J}_T = \varepsilon\frac{W_T^{1-\gamma}}{1-\gamma}\left(\frac{\hat{W}_T}{W_T}\right)^{\kappa}$ with $\varepsilon \geq 0$ being the preference for the bequest motive.

Given the wealth dynamics and the indirect utility specification, the HJB can be written as

$$\begin{aligned} 0 =& \mathcal{L}^c + \mathcal{L}^\theta + \frac{\partial J}{\partial t} - \delta J \ln([1-\gamma]J) + \delta\kappa J \ln(\hat{c}) + J_W(rW + y) \\ &+ J_{\hat{W}}(r\hat{W} + \tilde{\theta}\sigma\lambda + y - \tilde{c}) + \frac{1}{2}J_{\hat{W}\hat{W}}\tilde{\theta}^2\sigma^2 \\ &+ J_y y\alpha + \frac{1}{2}J_{yy}y^2\zeta^2 + J_{\hat{W}y}\tilde{\theta}\sigma y\zeta \end{aligned}$$

with

$$\mathcal{L}^c = \sup_{c\geq 0}\{\delta J(1-\gamma-\kappa)\ln(c) - cJ_W\}$$

and

$$\mathcal{L}^\theta = \sup_{\theta\in\mathbb{R}}\{J_W\theta\sigma\lambda + \frac{1}{2}J_{WW}\theta^2\sigma^2 + J_{W\hat{W}}\theta\sigma^2\tilde{\theta} + J_{Wy}\theta\sigma y\zeta\}.$$

Note that $\tilde{c}$ and $\tilde{\theta}$ are given by the solutions to the auxiliary model as determined previously.

As $\mathcal{L}^\theta$ is the same as in Corollary 3.1, it is known that the optimal investment amount is given by

$$\theta^*(t, W, y) = \frac{\lambda + \sigma\kappa}{(\gamma+\kappa)\sigma}(W + H(t, y)) - H(t, y)\frac{\zeta}{\sigma}$$

Solving $\mathcal{L}^c$ by the first-order conditions yields the following expression for optimal consumption choice

$$c^* = \frac{\delta(1-\gamma-\kappa)J}{J_W}$$

In line with the conjecture of the auxiliary model and the regret-averse utility specification, the following conjecture about the functional form of the indirect utility function is made

$$J(t, W, \hat{W}, y) = \frac{g(t)^\gamma}{1-\gamma}(W + H(t, y))^{1-\gamma}\left(\frac{\hat{W} + H(t, y)}{W + H(t, y)}\right)^\kappa.$$

The reader is referred to Appendix 7.3 for the corresponding partial derivatives. The biometric state p should then be ignored.

The optimal consumption process is found to be

$$\begin{aligned} c^* =& \frac{\delta(1-\gamma-\kappa)J}{J_W} \\ =& \frac{\delta\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)g(t)^\gamma(W + H(t, y))^{1-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa}{\left(\frac{1-\gamma-\kappa}{1-\gamma}\right)g(t)^\gamma(W + H(t, y))^{-\gamma}\left(\frac{\hat{W}+H(t,y)}{W+H(t,y)}\right)^\kappa} = \delta(W + H(t, y)) \end{aligned}$$

After substituting the conjecture into the HJB equation and simplifying, the following equation is derived

$$
\begin{aligned}
0 = \Bigg[&\delta \ln(\delta) - \delta\left(\frac{1-\gamma-\kappa}{1-\gamma}\right) - \delta\left(\frac{\gamma}{1-\gamma}\right)\ln(g(t)) \\
&+ \frac{1}{2}\left(\frac{1-\gamma-\kappa}{(1-\gamma)(\gamma+\kappa)}\right)\lambda^2 + \frac{1}{2}\left(\frac{(1-\gamma-\kappa)\kappa^2}{(1-\gamma)(\gamma+\kappa)}\right)\sigma^2 + \left(\frac{(1-\gamma-\kappa)\kappa}{(1-\gamma)(\gamma+\kappa)}\right)\sigma\lambda \\
&+ r - \left(\frac{\kappa}{1-\gamma}\right)\delta + \left(\frac{\kappa}{1-\gamma}\right)\sigma\lambda + \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{1-\gamma}\right)\sigma^2\Bigg] g(t) + \left(\frac{\gamma}{1-\gamma}\right) g'(t)
\end{aligned}
$$

This can be rewritten as a non-linear first-order differential equation of the same form as for the auxiliary investor.

$$g'(t) = [A + \delta \ln(g(t))]g(t)$$

with

$$
\begin{aligned}
A = \Bigg[&\frac{\delta(\gamma-1)}{\gamma}\ln(\delta) + \frac{\delta(1-\gamma)}{\gamma} - \frac{1}{2}\left(\frac{1-\gamma-\kappa}{\gamma(\gamma+\kappa)}\right)\lambda^2 - \frac{1}{2}\left(\frac{(1-\gamma-\kappa)\kappa^2}{\gamma(\gamma+\kappa)}\right)\sigma^2 \\
&- \left(\frac{(1-\gamma-\kappa)\kappa}{\gamma(\gamma+\kappa)}\right)\sigma\lambda + \frac{r(\gamma-1)}{\gamma} - \frac{\kappa}{\gamma}\sigma\lambda - \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\gamma}\right)\sigma^2\Bigg]
\end{aligned}
$$

From the auxiliary model, it is known that the solution to this nonlinear first-order differential equation is given by

$$g(t) = \mathrm{e}^{-\frac{A}{\delta} + \frac{1}{\delta}\mathrm{e}^{-\delta(T-t)}[\ln(\varepsilon)+A]^{\frac{\delta}{\gamma}}}$$

as $g(T) = \varepsilon^{\frac{1}{\gamma}}$. Hence, there exists a purely time-dependent function $g(t)$.

The optimal consumption choice is given by

$$c^*(t, W, y) = \delta(W + H(t, y))$$

which is a time-independent function only depending on the current total wealth $W+H(t,y)$. Thus a regret-averse investor with unit-EIS regret-averse utility specifications and exogenous labor income consumes a constant fraction of wealth depending on the investor's time preference δ.

As previously determined, the optimal investment strategy is given by

$$\theta^*(t, W, y) = \frac{\lambda + \sigma\kappa}{(\gamma+\kappa)\sigma}(W + H(t, y)) - H(t, y)\frac{\zeta}{\sigma}$$

This concludes the derivation of the optimal consumption-investment strategy of a regret-averse investor with unit-EIS preferences living a Black-Scholes model who earns spanned exogenous labor income. □