Optimal Investment in an Oligopoly: Analyzing the effect of the Configuration of the Demand Function on investment.

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Abstract

In this thesis, we delve into extending the framework proposed by Huisman and Kort (2015) and Faninam et al. (2022). Interestingly, they found that in a monopoly, duopoly, and triopoly the investment quantity and investment size were equal, independent of how many potential entrants. Considering exogenous firm order entry and a linear inverse demand function, it is intriguing to determine whether this behavior persists when more firms can enter the market, an arbitrarily large number of firms n. Extending the number of firms to n, this is found to be the case, meaning that no matter how many firms can enter the market at a subsequent time, the i-th firm will always invest the same.

This is a remarkable outcome. Questioning whether this is due to the linear nature of the inverse demand function, delving into more complex formulas seems intriguing, trying to reveal some more insights into what leads to this outcome. We start by fixing the investment timing and investigating how different inverse demand functions influence the investment decisions of firms in both a monopoly and a duopoly. Here, every convex inverse demand function displays a similar behavior, whereas the concave inverse demand function exhibits the complete opposite behavior. Looking at these results, there seems to be a correlation between the convexity/concavity of an inverse demand function and the way the duopoly investment size and monopoly investment size differ. This is, however, for a fixed investment timing. Taking this investment timing as a decision variable, the result for our two selected inverse demand functions, one convex and one concave, persists. So, considering a duopoly we observe that when the inverse demand function is convex, the duopoly size of the first firm is larger than the monopoly output, and in the concave the opposite holds. Tweaking some parameters and observing their effect, some irregularities in the results are discovered. Our results, however, still seem promising when only taking some probably more sound values for these parameters.

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Contents

1 Introduction

Entering a market is a precarious endeavor. There are a lot of external factors that can, and will, change an investing decision of a potential entrant. These investments generally have four characteristics, as discussed by Faninam et al. (2022). These four characteristics are irreversibility, uncertainty, timing, and size.

Irreversibility refers to the fact that once an investment has been executed, the investor has limited or no opportunity to reverse the investment, which was already recognized by Dixit (1993). The uncertain nature of the investor's future income is another problem when dealing with these investment decisions. The investor deals at least with some uncertainty, but more likely with considerable uncertainty. The timing is another issue in these settings. The investor has an opportunity to invest, but it can postpone its investment if that will benefit him more. Lastly, the quantity of its investment is also an element to consider. How much of the product or goods will it produce? These are all elements that need addressing when undertaking such an investment.

When considering these investment decisions, it is essential information how many firms can, and possibly will, enter the market. If there is only one entrant, so there aren't any competitors for this lone firm, we are dealing with a monopoly. In this setting, this firm does not need to take into account other decisions of other firms and can solely focus on itself. This makes the choices for this entrant relatively straightforward. If, however, there is another firm that is also interested in entering the market, the first firm should take into account the decisions that the second firm will take, when considering its options.

Huisman and Kort (2015) already showed what will happen in the monopoly and duopoly setting. Now, Faninam et al. (2022) looked into the effect of an extra potential entrant, hence dealing with a triopoly. Here, the results were similar to the results of the monopoly and duopoly case. In these three settings, it was found that the investment decisions are not influenced by the entrance of an extra firm. Now, the question that arises from these outcomes is whether or not this result will prevail if more and more firms can enter the market. This question will be addressed in the upcoming sections.

A drawback of the previously mentioned studies is that they mainly focus on a linear inverse demand function. This, however, might not be the most comprehensive one. Huisman and Kort (2015) briefly treated the iso-elastic inverse demand function. It seems that only in the linear case, the monopolist will invest in the same quantity and at the same time as the first entrant in the duopoly setting. There are two effects of an extra potential investor. Firstly, the price will go down, if more firms enter, since the total invested quantity will go up. Hence, if this was the only relevant consequence of an extra firm, the first entrant would invest less. Secondly, the first entrant might want to apply a deterrence strategy in order to make it less profitable for the new entrant to invest, and will therefore want to invest more. These two effects cancel out when dealing with a linear inverse demand function. Therefore, it seems interesting to delve into various other functions, both concave, and convex, and see what will happen to the investment decisions.

Interestingly, it seems to be the case that all convex functions behave similarly, and all concave functions do too. In a convex setting, we find that, for our chosen demand functions, the optimal monopoly investment size is lower than the optimal duopoly investment size for the first entrant. It is intriguing that in a concave setting, the opposite seems to be true. Since it was already found that in a linear setting, these two are equal, this appears to be the tipping point. Also, here various parameter values are tested and the optimal investment decisions for different values are observed.

In the upcoming sections only exogenous firm order entry is considered. Hence, the order of firms entering the market is predetermined. It might be interesting to extend this into endogenous order entry, where the order is unknown, but that is beyond the scope of this thesis. Another limitation is that in this thesis, only the deterrence strategy is investigated, not the accommodation strategy. It is assumed that at the start of the investment window, the price is sufficiently low that no firm will invest immediately. In further research, the accommodation strategy can be considered, too. Earlier research by Huisman and Kort (2015) and Faninam et al. (2022) gives promising results, but this will not be researched here.

There is quite some history in the investigation of real options models. Dixit and Pindyck (1994) and Trigeorgis (1996) both delved into this. They, however, investigated the optimal investment timing but they did not consider the optimal investment quantity. The quantity was assumed to be given, and the optimal investment timing was determined based on this quantity.

Dixit (1993) did, however, investigate the opportunity to determine both the optimal investment quantity and timing. This builds on Pindyck (1990) where irreversibility, capacity choice, and the value of the firm were introduced, and on McDonald and Siegel (1986) where the value of waiting was introduced. Dixit (1993) restricted to a monopoly case exclusively. Bar-Ilan and Strange (1999) considered both timing and intensity separately, but also together, however, they also did not examine competitors. The same is the case for Dangl (1999) and Décamps et al. (2006). While all the previously mentioned paper have their own interesting insights into this problem, none of them cover the setting with a potential new entrant.

There has also been done research on frameworks where there are competitors. For example, Huisman et al. (2003) conducted research on this, but they only considered the investment timing for a fixed quantity.

There is also literature where competition is taken into account, here, however, the investment size is taken to be given, and the timing is investigated. Grenadier (2000), Chevalier-Roignant and Trigeorgis (2011), and, Azevedo and Paxson (2014) have done similar research in the fact that they all investigated the investment timing with competition, but with the investment size given.

The first literature revolving around entry deterrence and entry accommodation was conducted by Spence (1977), Dixit (1979), and Dixit (1980), where, e.g., Dixit (1979) did not prioritize preventing new entries from the outset, and it lets current companies determine their optimal strategy, considering the potential responses of future entrants. These papers have been clearly summarized by Tirole (1988).

Yang and Zhou (2007) combine both deterrence and stochastic pricing, taking both entry and quantity-setting into account. They showed that the expected profit of a new entrant is affected by the incumbent's capacity. A higher capacity for the incumbent leads to lower profit for the new entrant. Yang and Zhou (2007) extends the model of Dangl (1999), where it was concluded that the optimal installed capacity increases with uncertainty.

The sections below are arranged in the following manner. We start by delving into solely the linear inverse demand framework, mathematically investigating a monopoly, a duopoly, a triopoly, and finally an oligopoly. Afterwards, a numerical analysis will be done, looking at the effects of different standard deviations and visualizing some of the results.

Next, we will diverge away from the linear inverse demand function, looking into a more general case. First, some mathematical results are computed, after which several inverse demand functions, as proposed by Balter et al. (2022), are explored. In this section, we will not consider an oligopoly and only will look into the monopoly and duopoly, hoping to find some regularities between the functions.

In the last section, we study two inverse demand functions, for which an optimal investment strategy can be found. The quadratic and the squared-root inverse demand functions are examined. These two have been chosen since the quadratic function is concave and the square-root function is convex. After the mathematical computations have been done, a numerical analysis is executed in order to study these results.

2 Linear inverse demand function

We consider a framework where firms can decide to undertake an investment to enter the market. At the start, there are no firms present in the market. We will define the inverse demand function as

$$P(t) = X(t)(\alpha - Q(t))$$

where $Q(t)$ is the total market output at time t, α is the market size and X follows a geometric Brownian motion which is defined by

$$dX(t) = \mu X(t)dt + \sigma X(t)dz(t)$$

where $z(t) \sim \mathcal{N}(0,1)$. μ is the drift rate, $dz(t)$ is the increment of a Wiener process, and $\sigma > 0$ is a constant. Investors face an investment cost which will be defined as δK_i, where K_i is the capacity that the firm decides to invest and δ is a fixed cost per unit of capacity. The inverse demand function is an explicit function which is referred to by Dixit and Pindyck (1994) as $P = XD(Q)$. In this case, we have specified $D(Q)$ as $\alpha - Q(t)$. Comparing it against the inverse demand function adopted by Huisman and Kort (2015), which they defined as $P(t) = X(t)(1 - \eta Q(t))$, it can be observed that the two functions are fairly similar. Both are linear in $Q(t)$, however, in our case, we have a fixed maximum capacity α, instead of using their η.

We will assume that each firm will produce up to capacity, hence, $Q_i = K_i$. From now on, the invested capacity will be referred to by Q_i, because of this assumption. As shown by Dangl (1999), it is possible to make calculations relaxing this, however, this makes the computations considerably more difficult. It can be argued that an investor will most likely always produce up to capacity, as done by Goyal and Netessine (2007), however, there is also claiming the opposite, e.g., Chicu (2013). Here, we will assume the two are equivalent.

Note that if there are m active firms in the market we have the total output of the market is equal to $Q = Q_1 + \cdots + Q_m$. We will also assume that $r > \mu$, otherwise, the value of the function will tend to infinity, in which case the problem is not coherent. r refers to the discount rate. Another assumption we make is that the firms are risk-neutral, hence, they will always try to optimize their expected value.

We will only consider the deterrence strategy. Hence, the value of $X(0)$ is sufficiently low that the next firm will not invest immediately. In further research, investigating what will happen in the accommodation case is interesting too, but will not be dealt with here.

Each firm will possess a value function denoted as $V_i(X, Q)$, representing their unique value based on the investment quantity Q_i and investment timing X_i. The subscript i corresponds to the position of the firm as an entrant in the market. Thus, the value function for the first entrant is expressed as $V_1(X,Q)$, the second entrant's value function is denoted as $V_2(X,Q)$, and so on, reflecting the sequential order of entry into the market. Assumed are homogeneous products, hence, investors are unable to differentiate themselves based on product quality.

2.1 Monopoly

First, we will consider a monopoly. It is a monopoly in the sense that there is only one potential investor, who does not have to consider investment decisions of new entrants, since there aren't any new entrants. We are interested in two investment choices, namely the investment timing and investment quantity. Timing refers to a level of the stochastic process X. When it has reached a particular level, the investor will start its investment.

In a monopoly setting, we can use a similar computation as done by Huisman and Kort (2015). The expected value of the entrant is defined as follows

$$V\left(X\right) = E\left[\int_{t=0}^{\infty} QX\left(t\right)\left(\alpha - Q\right)\exp\left(-rt\right)dt - \delta Q \middle| X\left(0\right) = X\right]$$

The calculations have been conducted in subsection 6.1.1, resulting in the following decisions:

$$\begin{aligned} X^* &= \frac{\beta+1}{\beta-1}\frac{\left(r-\mu\right)\delta}{\alpha} \\ Q^* &= \frac{\alpha}{\beta+1}. \end{aligned} \tag{1}$$

where β is the positive root of the quadratic polynomial $\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0$, as shown by Dixit and Pindyck (1994). There are two separate cases. One is that the standard deviation of the geometric Brownian motion is zero, so there is no uncertainty in the market, and one case is where $\sigma > 0$.

We find the following solution to β.

$$\beta = \begin{cases} \frac{r}{\mu}, & \sigma = 0 \\ \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, & \sigma > 0 \end{cases} \tag{2}$$

If we compare (1) to the result from Huisman and Kort (2015), it can be noticed that these are reasonably similar. This is to be expected since they are also applying a linear inverse demand function in a similar setting.

2.2 Duopoly

In the duopoly case, two firms can decide on entering the market. We will compute this backwards, starting with the value for the second entrant, followed by computing the value for the first entrant.

The value function for the second potential entrant is defined by

$$V_2^*\left(X_2, Q_1, Q_2\right) = \frac{X_2 Q_2 \left(\alpha - \left(Q_1 + Q_2\right)\right)}{r - \mu} - \delta Q_2$$

Examining this function, there are 3 non-constants here. A crucial observation is that Q_1 is a given value for the second investor since the second entrant can only invest once the first entrant already has invested. This is due to the assumed exogenous firm order.

Now, taking the derivative with respect to Q_2, and applying value matching and smooth pasting, which has been computed in subsection 6.1.2, results in these investment decisions, dependent on Q_1,

$$\begin{aligned} X_2^*\left(Q_1\right) &= \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha-Q_1} \\ Q_2^*\left(Q_1\right) &= \frac{\alpha-Q_1}{\beta+1} \end{aligned} \tag{3}$$

The value function of the first investor in this duopoly setting is given by

$$V_1^{\text{det}}\left(X, Q_1\right) = \frac{Q_1\left(\alpha - Q_1\right)X}{r-\mu} - \delta Q_1 - \left(\frac{X}{X_2^*}\right)^{\beta_1}\frac{Q_1 Q_2^* X_2^*}{r-\mu}$$

Inserting (3) into this value function gives us a value function for the first investor only depending on Q_1 and X_1. Therefore, again, the derivative with respect to Q_1 is computed, and value matching and smooth pasting are applied. Resulting in the following optimal investment quantity and timing for the first investor,

$$\begin{aligned} X_1^* &= \frac{\delta}{\alpha}\left(r-\mu\right)\frac{\beta+1}{\beta-1} \\ Q_1^* &= \frac{\alpha}{\beta+1} \end{aligned} \tag{4}$$

where the value β is the same as in the monopoly case (2). The performed computations have been reported in subsection 6.1.2.

Notably, both the investment timing and size for the first entrant in a duopoly correspond to the investment decisions in a monopoly. This is similar to the result of Huisman and Kort (2015).

2.3 Triopoly

Earlier, we assumed only two firms. Now, we will add another one. Similar to the previous section, we will start off with the last entrant, working our way backwards.

Faninam et al. (2022) already analyzed this, however, for a slightly different inverse demand function. The function that they considered was $P(t) = X(t)(1 - \eta Q(t))$, which is identical to the one analyzed by Huisman and Kort (2015).

The value function for the third entrant is defined by

$$V_3\left(X, Q_3\right) = \frac{Q_3 X \left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu} - \delta Q_3$$

The optimal investment decisions, as computed in subsection 6.1.3, for the third entrant are

$$\begin{aligned} X_3^* &= \frac{\beta + 1}{\beta - 1} \frac{\delta \left(r - \mu\right)}{\alpha - \left(Q_1 + Q_2\right)} \\ Q_3^* &= \frac{\alpha - \left(Q_1 + Q_2\right)}{\beta + 1} \end{aligned} \tag{5}$$

Next, the investment decisions for the second entrant will be computed. To this end, the value function of the second firm is defined by

$$V_2\left(X, Q_2\right) = \frac{Q_2 \left(\alpha - \left(Q_1 - Q_2\right)\right) X}{r - \mu} - \delta Q_2 - \left(\frac{X}{X_3^*}\right)^{\beta_1} \frac{Q_2 Q_3^* X_3^*}{r - \mu}$$

Plugging in the results (5) leads to the following optimal investment quantity and size

$$\begin{aligned} X_2^* &= \frac{\beta + 1}{\beta - 1} \frac{\delta \left(r - \mu\right)}{\alpha - Q_1} \\ Q_2^* &= \frac{\alpha - Q_1}{\beta + 1} \end{aligned} \tag{6}$$

Notably, we observe here that the optimal investment quantity and size of the second investor in a triopoly seems to correspond to the second investor in a duopoly. This is, however, not the case if the outcome for Q_1 is not the same in a triopoly as in a duopoly.

To evaluate the optimal investment in a triopoly, the value function for the first investor is necessary.

$$V_1\left(X, Q_1\right) = \frac{Q_1 \left(\alpha - Q_1\right) X}{r - \mu} - \delta Q_1 - \left(\frac{X}{X_2^*}\right)^{\beta_1} \frac{Q_1 Q_2^* X_2^*}{r - \mu} - \left(\frac{X}{X_3^*}\right)^{\beta_1} \frac{Q_1 Q_3^* X_3^*}{r - \mu}$$

Inserting our previously found (6) and (5), taking the derivative with respect to Q_1 and applying value matching and smooth pasting gives us the following

$$\begin{aligned} X_1^* &= \frac{\beta+1}{\beta-1}\frac{\delta\,(r-\mu)}{\alpha} \\ Q_1^* &= \frac{\alpha}{\beta+1} \end{aligned} \tag{7}$$

which corresponds to the decisions of the first investor in a monopoly and duopoly. Hence, similar to Faninam et al. (2022), we find that the decisions of the first two investors are not influenced by the introduction of a third potential entrant. Note that all of the above-mentioned computations are written down in subsection (6.1.3).

2.4 Oligopoly

The results in our previous sections exhibit an interesting pattern. It appears to be the case that no matter how many potential new entrants are included, the optimal investment timing and size remain the same. In this section, we will try to prove this.

Let us assume that in this oligopoly, there are n potential entrants. If we observe the results from the previous sections, an educated guess for the optimal investment decisions for the i-th entrant, where $i \in \{1, \ldots, n\}$, is

$$\begin{aligned} X_i^* &= \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha-\sum_{j=1}^{i-1} Q_j} \\ Q_i^* &= \frac{\alpha-\sum_{j=1}^{i-1} Q_j}{\beta+1} \end{aligned} \tag{8}$$

In order to prove this, strong induction is utilized. First, the base case $i = n$ will be proven. n refers to the last entrant of the market. Next, we will take into consideration the i-th entrant, where $1 \leq i < n$. Here, it is assumed (8) holds for all $i < j \leq n$, since strong induction is used. The proof for this is contained in subsection 6.1.4.

(8) can actually be rewritten into a non-recursive formula. The proof for this is in subsection 6.1.4 as well. The results are the following.

$$\begin{aligned} Q_i^* &= \left(\frac{\beta}{\beta+1}\right)^{i-1}\frac{\alpha}{\beta+1} \\ X_i^* &= \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha}\left(\frac{\beta+1}{\beta}\right)^{i-1} \end{aligned}$$

As expected, it can noticed that the investment decisions remain the same, independent of the number of potential entrants. This applies to all investors.

2.5 Analysis

With the computations in the previous subsections, we notice that no matter how many potential entrants will enter the market, the first entrant will always make the same decisions, i.e., it will always invest the same quantity at the same time. This is a very intriguing result. There are two effects that the first entrant should take into consideration when deviating from its monopoly strategy to deter potential entrants. The first effect is that it will be less profitable to invest, due to the entrance of new competitors. New competitors will increase the total output and, therefore, decrease the price. For this reason, the first entrant will want to invest less. The second consequence of deviating is that, if the first entrant invests more, the second investor will invest less, which will drive the price up. Apparently, these two effects cancel out, which is likely due to the linear nature of the chosen inverse demand function. This holds true for all entrants, not only for the first. To explore the effect of linearity of the inverse demand function on the investment decisions, different functions will be investigated to find regularities. This will be done in the upcoming sections.

In table (1), the outcomes for the different optimal investment quantities for firms $1, ..., 20$ are computed. Since we have proved that the optimal investment does not depend on the number of firms, this is valid for an arbitrary number of potential entrants.

As we can see, for the first two firms increasing uncertainty leads to higher investment. If we observe the values for firms that enter the market later than the first two, we can see that the value actually decreases in σ. There are two effects at play in this situation. On one hand, larger volatility leads to a larger investment, as observed by Dixit (1993). On the other hand, there is a reduction in the remaining market size due to the increase in earlier investments. If the uncertainty becomes larger, the preceding entrants will already have invested more, and therefore the new entrants will invest less.

Figure (2) is a graph representing the effects of σ. We can see that, for the first two firms, the optimal investment quantity increases, which was already observed by Dixit (1993), who found that when σ is larger, investors will take on larger projects. For all following entrants, the optimal quantity is actually less. If we look at the relative effect of the volatility by looking in table (1), we can see that the standard deviation has a larger effect on the investment quantity from firms that enter the market later, relatively speaking.

Now, we will take a look at the optimal investment timing for different entrants. These are denoted in table (2). Here we can see that X_i^* is increasing in σ, which is similar to the conclusions by Huisman and Kort (2015), and increases as more firms have already entered. Here, similar to our observation for the optimal investment quantity, it can be observed that the effect of the volatility will delay the investment of later entrants more significantly.

Figure (3) shows how the investment timing behaves over each firm for different values of σ. This shows the previously mentioned effect of the volatility very clearly. For $\sigma = 0.40$, the function seems to explode when looking at later entrants.

Another interesting thing to analyze is the evolution of the value function for each entrant and various σ's. This is denoted in table (3). Notably, the value appears to converge as the number of entrants approaches infinity. This observation is reasonable, considering that Q_i^* diminishes to a negligible value as i becomes larger.

Now, it seems interesting to see how the profit for the first entrant evolves over time. We will denote expected profit at time t as Π_t^n. Here, the n represents the number of potential entrants.

We can define the function for this profit function as follows:

$$\Pi_{\hat{t}}^n = \mathbb{E}\left[\int_{t=0}^{\hat{t}} Q_1^* X^*(t)(\alpha - Q_1^*)e^{-rt}dt - \delta Q_1^* - \sum_{k=i+1}^{n} \int_{t=T_k^*}^{\hat{t}} Q_1^* X^*(t) Q_k^* e^{-rt}dt \middle| X(0) = X^*\right]$$

At the point of investment, $t = 0$, we can see that the current profit is equal to $-\delta Q_1^*$. This is because the firm has not made any profit, however, has made the investment.

We need an assumption for the standard deviation, For now, we will set this to $\sigma = 0$, removing all uncertainty. Even though this simplifies the problem drastically, I can give us a general idea of how the value function behaves. In figure (1), the profit over time of the first entrant can be observed for various numbers of potential entrants.

Here, we can see that in the monopoly case, i.e., $n = 1$, we have that the profit follows a differentiable path. If we observe the function for $n = 2$, the duopoly, we see that there is a kink in the graph line. This is because, when the second firm has entered the market, the price will go down compared to before the investment of the second entrant, due to a larger total output. There, the value will suddenly increase less steeply. Afterwards, the line will follow a differentiable path, since there won't be any new entrants. The market is now only influenced by the stochastic process, which is not a stochastic process anymore due to our assumption that $\sigma = 0$. Since before the investment of the second entrant, the first firm will invest its monopoly optimal quantity, the function completely coincides with the monopoly function, which is the line for $n = 1$. If we look at $n = 3$, we see that there are two kinks, one when the second investor enters the market, and one when the third firm enters the market. This behavior resumes as more and more firms will enter the market. We can also see that the value converges as the number of firms approaches infinity. As the number of firms increases, the kinks are barely observable, due to the size of the investment of the new entrant. This minuscule investment, in comparison with the size of the first entrant, influences the price marginally. The investment size of new entrants approaches 0 as n goes to infinity. The lines in this figure converge to the values denoted in table (3).

	Q_1^*	Q_2^*	Q_3^*	Q_4^*	Q_5^*	Q_6^*	Q_7^*	Q_8^*	Q_9^*	Q_{10}^*
$\sigma = 0.00$	7.5000	4.6875	2.9297	1.8311	1.1444	0.7153	0.4470	0.2794	0.1746	0.1091
$\sigma = 0.05$	7.5626	4.7030	2.9246	1.8187	1.1310	0.7033	0.4374	0.2720	0.1691	0.1052
$\sigma = 0.10$	7.7258	4.7414	2.9098	1.7858	1.0960	0.6726	0.4128	0.2533	0.1555	0.0954
$\sigma = 0.15$	7.9400	4.7878	2.8871	1.7409	1.0498	0.6330	0.3817	0.2302	0.1388	0.0837
$\sigma = 0.20$	8.1650	4.8316	2.8591	1.6919	1.0012	0.5925	0.3506	0.2075	0.1228	0.0726
$\sigma = 0.25$	8.3785	4.8685	2.8290	1.6438	0.9552	0.5550	0.3225	0.1874	0.1089	0.0633
$\sigma = 0.30$	8.5714	4.8980	2.7988	1.5993	0.9139	0.5222	0.2984	0.1705	0.0974	0.0557
$\sigma = 0.35$	8.7413	4.9208	2.7701	1.5594	0.8778	0.4942	0.2782	0.1566	0.0882	0.0496
$\sigma = 0.40$	8.8889	4.9383	2.7435	1.5242	0.8468	0.4704	0.2613	0.1452	0.0807	0.0448

	Q_{11}^*	Q_{12}^*	Q_{13}^*	Q_{14}^*	Q_{15}^*	Q_{16}^*	Q_{17}^*	Q_{18}^*	Q_{19}^*	Q_{20}^*
$\sigma = 0.00$	0.0682	0.0426	0.0266	0.0167	0.0104	0.0065	0.0041	0.0025	0.0016	0.0010
$\sigma = 0.05$	0.0654	0.0407	0.0253	0.0157	0.0098	0.0061	0.0038	0.0024	0.0015	0.0009
$\sigma = 0.10$	0.0586	0.0359	0.0221	0.0135	0.0083	0.0051	0.0031	0.0019	0.0012	0.0007
$\sigma = 0.15$	0.0505	0.0304	0.0183	0.0111	0.0067	0.0040	0.0024	0.0015	0.0009	0.0005
$\sigma = 0.20$	0.0430	0.0254	0.0151	0.0089	0.0053	0.0031	0.0018	0.0011	0.0006	0.0004
$\sigma = 0.25$	0.0368	0.0214	0.0124	0.0072	0.0042	0.0024	0.0014	0.0008	0.0005	0.0003
$\sigma = 0.30$	0.0318	0.0182	0.0104	0.0059	0.0034	0.0019	0.0011	0.0006	0.0004	0.0002
$\sigma = 0.35$	0.0279	0.0157	0.0089	0.0050	0.0028	0.0016	0.0009	0.0005	0.0003	0.0002
$\sigma = 0.40$	0.0249	0.0138	0.0077	0.0043	0.0024	0.0013	0.0007	0.0004	0.0002	0.0001

Table 1: Optimal investment quantity Q_i^*
$r = 0.1, \mu = 0.6, \alpha = 20, \delta = 0.1$

	X_1^*	X_2^*	X_3^*	X_4^*	X_5^*	X_6^*	X_7^*	X_8^*	X_9^*	X_{10}^*
$\sigma = 0.00$	0.0008	0.0013	0.0020	0.0033	0.0052	0.0084	0.0134	0.0215	0.0344	0.0550
$\sigma = 0.05$	0.0008	0.0013	0.0021	0.0034	0.0055	0.0088	0.0142	0.0228	0.0367	0.0590
$\sigma = 0.10$	0.0009	0.0014	0.0023	0.0038	0.0062	0.0101	0.0165	0.0268	0.0437	0.0712
$\sigma = 0.15$	0.0010	0.0016	0.0027	0.0044	0.0073	0.0122	0.0202	0.0335	0.0555	0.0921
$\sigma = 0.20$	0.0011	0.0018	0.0031	0.0053	0.0089	0.0150	0.0254	0.0429	0.0725	0.1225
$\sigma = 0.25$	0.0012	0.0021	0.0037	0.0063	0.0108	0.0186	0.0320	0.0551	0.0949	0.1633
$\sigma = 0.30$	0.0014	0.0024	0.0043	0.0075	0.0131	0.0230	0.0402	0.0704	0.1231	0.2155
$\sigma = 0.35$	0.0016	0.0028	0.0050	0.0089	0.0158	0.0281	0.0499	0.0887	0.1576	0.2799
$\sigma = 0.40$	0.0018	0.0032	0.0058	0.0105	0.0189	0.0340	0.0612	0.1102	0.1984	0.3570

	X_{11}^*	X_{12}^*	X_{13}^*	X_{14}^*	X_{15}^*	X_{16}^*	X_{17}^*	X_{18}^*	X_{19}^*	X_{20}^*
$\sigma = 0.00$	0.0880	0.1407	0.2252	0.3603	0.5765	0.9223	1.4757	2.3612	3.7779	6.0446
$\sigma = 0.05$	0.0949	0.1526	0.2453	0.3945	0.6344	1.0201	1.6403	2.6378	4.2417	6.8208
$\sigma = 0.10$	0.1160	0.1891	0.3081	0.5020	0.8180	1.3328	2.1717	3.5387	5.7660	9.3954
$\sigma = 0.15$	0.1528	0.2533	0.4201	0.6967	1.1554	1.9160	3.1775	5.2695	8.7388	14.4923
$\sigma = 0.20$	0.2070	0.3498	0.5912	0.9990	1.6882	2.8529	4.8212	8.1473	13.7681	23.2667
$\sigma = 0.25$	0.2811	0.4837	0.8324	1.4326	2.4654	4.2429	7.3018	12.5661	21.6256	37.2166
$\sigma = 0.30$	0.3771	0.6600	1.1550	2.0213	3.5372	6.1901	10.8327	18.9572	33.1752	58.0565
$\sigma = 0.35$	0.4972	0.8832	1.5689	2.7869	4.9507	8.7943	15.6222	27.7513	49.2975	87.5719
$\sigma = 0.40$	0.6427	1.1568	2.0823	3.7481	6.7466	12.1440	21.8591	39.3464	70.8235	127.4824

Table 2: Optimal investment timing X_i^*
$r = 0.1, \mu = 0.6, \alpha = 20, \delta = 0.1$

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$\sigma=0.00$	1.1250	0.6110	0.3762	0.2689	0.2199	0.1975	0.1873	0.1826	0.1804	0.1795
$\sigma=0.05$	1.1733	0.6361	0.3901	0.2775	0.2260	0.2024	0.1916	0.1866	0.1844	0.1833
$\sigma=0.10$	1.3123	0.7081	0.4300	0.3019	0.2429	0.2158	0.2033	0.1975	0.1949	0.1937
$\sigma=0.15$	1.5302	0.8205	0.4913	0.3387	0.2679	0.2350	0.2198	0.2128	0.2095	0.2080
$\sigma=0.20$	1.8165	0.9674	0.5705	0.3850	0.2983	0.2577	0.2388	0.2299	0.2258	0.2239
$\sigma=0.25$	2.1647	1.1452	0.6651	0.4390	0.3325	0.2824	0.2587	0.2476	0.2424	0.2399
$\sigma=0.30$	2.5714	1.3521	0.7739	0.4997	0.3697	0.3081	0.2788	0.2650	0.2584	0.2553
$\sigma=0.35$	3.0352	1.5872	0.8963	0.5667	0.4095	0.3345	0.2987	0.2816	0.2735	0.2696
$\sigma=0.40$	3.5556	1.8502	1.0322	0.6399	0.4518	0.3615	0.3182	0.2975	0.2875	0.2827

	$n=11$	$n=12$	$n=13$	$n=14$	$n=15$	$n=16$	$n=17$	$n=18$	$n=19$	$n=20$
$\sigma=0.00$	0.1790	0.1788	0.1787	0.1787	0.1787	0.1787	0.1786	0.1786	0.1786	0.1786
$\sigma=0.05$	0.1828	0.1826	0.1825	0.1825	0.1825	0.1825	0.1824	0.1824	0.1824	0.1824
$\sigma=0.10$	0.1931	0.1929	0.1927	0.1927	0.1927	0.1926	0.1926	0.1926	0.1926	0.1926
$\sigma=0.15$	0.2072	0.2069	0.2068	0.2067	0.2067	0.2067	0.2066	0.2066	0.2066	0.2066
$\sigma=0.20$	0.2230	0.2225	0.2223	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222
$\sigma=0.25$	0.2388	0.2382	0.2379	0.2378	0.2378	0.2377	0.2377	0.2377	0.2377	0.2377
$\sigma=0.30$	0.2538	0.2531	0.2528	0.2526	0.2525	0.2525	0.2525	0.2525	0.2525	0.2525
$\sigma=0.35$	0.2677	0.2669	0.2664	0.2662	0.2661	0.2661	0.2661	0.2661	0.2661	0.2661
$\sigma=0.40$	0.2804	0.2793	0.2788	0.2786	0.2784	0.2784	0.2783	0.2783	0.2783	0.2783

Table 3: Value of the first firm for n potential entrants
$r = 0.1, \mu = 0.6, \alpha = 20, \delta = 0.1$

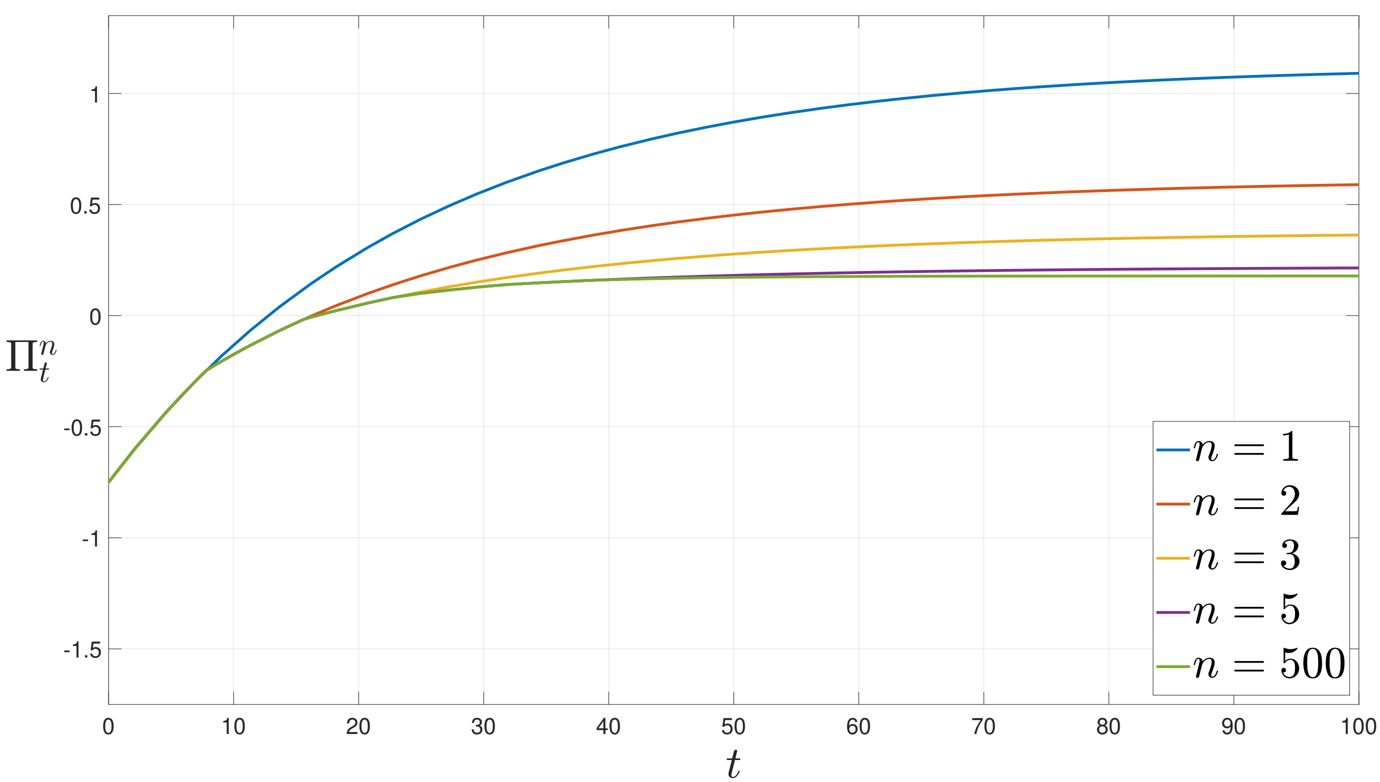


Figure 1: Profit $\Pi_{\hat{t}}^n$ over time t
$\sigma = 0, r = 0.1, \mu = 0.06, \alpha = 20, \delta = 0.1$

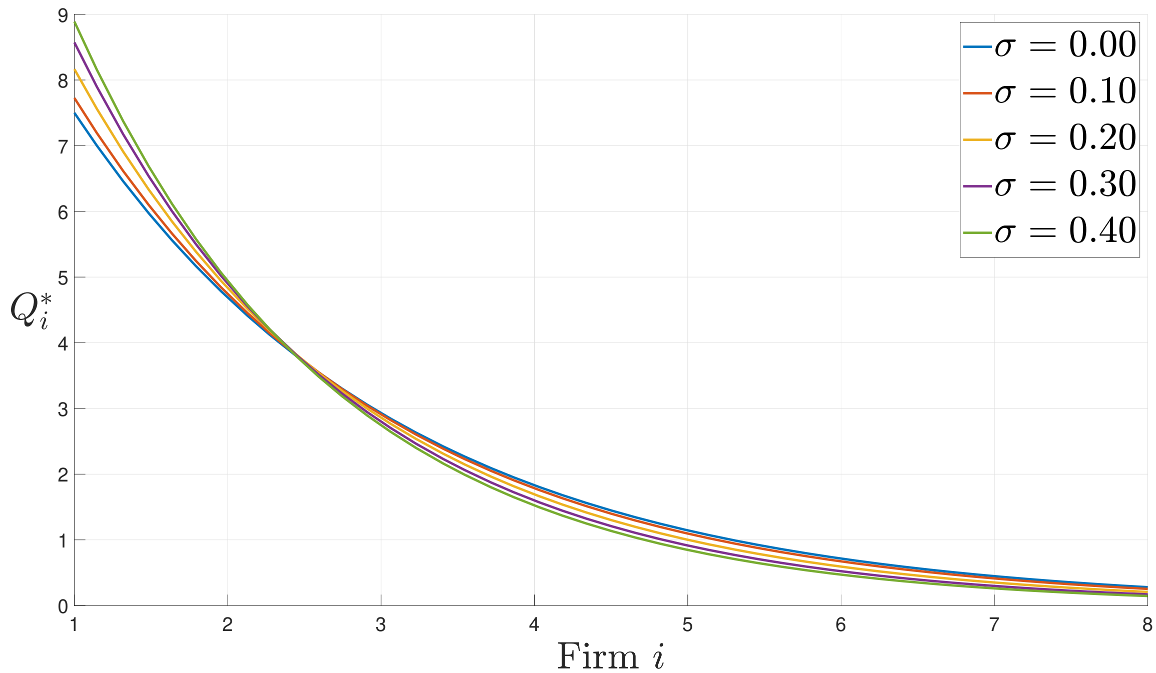


Figure 2: Optimal investment quantity Q_i^*
$r = 0.1, \mu = 0.06, \alpha = 20, \delta = 0.1$

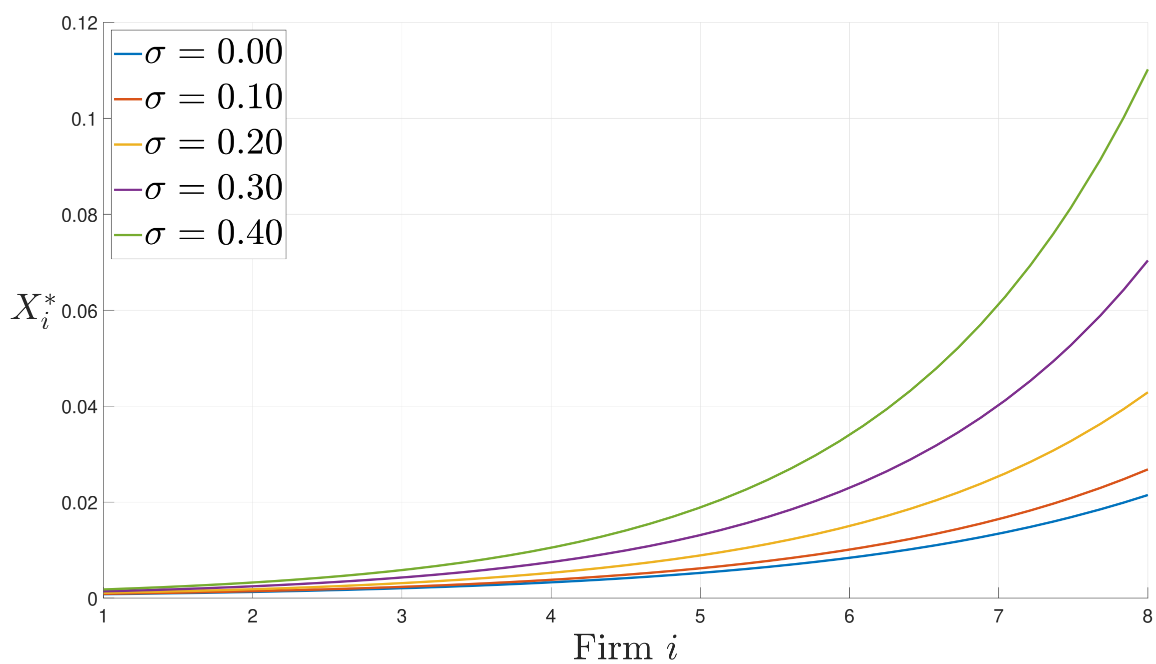


Figure 3: Optimal investment timing X_i^*
$r = 0.1, \mu = 0.06, \alpha = 20, \delta = 0.1$

3 General inverse demand function

So far, we have only considered the case where the price is a function of a geometric Brownian motion times a function that is decreasing and linear in Q, the linear inverse demand function. We will now evaluate a more general case, where the inverse demand function will be defined as follows

$$P = Xh(Q)$$

Note that we still assume $h(Q)$ to be decreasing, however, now the function is not necessarily linear. The rest of our assumptions will be as they were in the framework in section 2.

3.1 Monopoly

If we now apply the same calculations as we have done before, we obtain the following for the value function in the monopoly setting

$$V(X,Q) = \frac{Qh(Q)X}{r-\mu} - \delta Q$$

Taking the derivative with respect to Q, we obtain

$$\frac{\partial V(X,Q)}{\partial Q} = \frac{h'(Q)XQ}{r-\mu} + \frac{h(Q)X}{r-\mu} - \delta = 0$$

We, again, apply value matching and smooth pasting, which gives us the following

$$0 = -\frac{X}{\beta}\frac{Qh(Q)}{r-\mu} + \frac{Qh(Q)X}{r-\mu} - \delta Q$$

$$\left(\frac{\beta-1}{\beta}\right)\frac{h(Q)X}{r-\mu} = \delta$$

$$X = \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q)} \tag{9}$$

It is possible to simplify the above-mentioned equations by inserting X. This results in the following equation

$$\beta Q h'(Q) + h(Q) = 0 \tag{10}$$

Computing the formula for Q from this equation, we can easily find X, meaning we have our optimal investment quantity and investment timing for the monopoly setting.

3.2 Duopoly

Now, we will derive the optimal investment size and timing in the duopoly case. The value function for the second firm is defined by

$$V_2(X,Q) = \frac{XQ_2h(Q_1+Q_2)}{r-\mu} - \delta Q_2$$

taking the derivative with respect to Q_2 we obtain

$$\begin{aligned}\frac{\partial V_2(X,Q)}{\partial Q_2} &= \frac{X_2h(Q_2+Q_2)}{r-\mu} + \frac{X_2Q_2h'(Q_1+Q_2)}{r-\mu} - \delta \\ &= X\frac{h(Q_1+Q_2)+Q_2h'(Q_1+Q_2)}{r-\mu} - \delta \\ &= 0\end{aligned}$$

Applying value matching and smooth pasting

$$\begin{aligned}0 &= \frac{XQ_2h(Q_1+Q_2)}{r-\mu} - \frac{X}{\beta}\frac{Q_2h(Q_1+Q_2)}{r-\mu-\delta Q_2} \\ &= \frac{\beta-1}{\beta}\frac{Xh(Q_1+Q_2)}{r-\mu} - \delta\end{aligned}$$

$$X_2 = \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q_1+Q_2)} \tag{11}$$

We can simplify this further by plugging in X_2

$$\begin{aligned}\frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q_1+Q_2)}\frac{h(Q_1+Q_2)+Q_2h'(Q_1+Q_2)}{r-\mu} - \delta = 0 \\ \frac{\beta}{\beta-1}\frac{h(Q_1+Q_2)+Q_2h'(Q_1+Q_2)}{h(Q_1+Q_2)} - 1 = 0 \\ \beta(h(Q_1+Q_2)+Q_2h'(Q_1+Q_2)) = (\beta-1)h(Q_1+Q_2)\end{aligned}$$

$$\beta Q_2h'(Q_1+Q_2) + h(Q_1+Q_2) = 0 \tag{12}$$

With (11) and (12) we arrive at an obstacle. We are not always able to find an explicit solution for Q_2 for all $h(Q)$. An explicit formula for Q_2 is needed for value matching and smooth pasting.

Therefore, we will examine if the optimal investment timing and size of the first entrant will deviate from the monopoly case by fixing the investment timing, and then calculating the optimal investment quantity.

The value function for the first investor is defined as follows

$$
\begin{aligned}
V_1(X,Q) =&\mathbb{E}\left[\int_{t=0}^{T_2^*} Q_1 X(t) h(Q_1) e^{-rt} dt - \delta Q_1 + \int_{T_2^*}^{\infty} Q_1 X(t) h(Q_1+Q_2) e^{-rt} dt \middle| X(0)=X\right] \\
=&\mathbb{E}\Bigg[\int_{t=0}^{\infty} Q_1 X(t) h(Q_1) e^{-rt} dt - \int_{t=T_2^*}^{\infty} Q_1 X(t) h(Q_1) e^{-rt} dt - \delta Q_1 \\
&+ \int_{t=T_2^*}^{\infty} Q_1 X(t) h(Q_1+Q_2) e^{-rt} dt \Big| X(0)=X\Bigg] \\
=&\frac{Q_1 h(Q_1) X}{r-\mu} - \left(\frac{X}{X_2^*}\right)^{\beta} \frac{Q_1 h(Q_1) X_2^*}{r-\mu} + \left(\frac{X}{X_2^*}\right)^{\beta} \frac{Q_1 h(Q_1+Q_2) X_2^*}{r-\mu} - \delta Q_1 \\
=&\frac{Q_1 h(Q_1) X}{r-\mu} - \left(\frac{X}{X_2^*}\right)^{\beta} \frac{Q_1 X_2^*}{r-\mu}(h(Q_1) - h(Q_1+Q_2)) - \delta Q_1
\end{aligned}
$$

The optimal investment quantity of the first investor will be determined as follows. As mentioned before, we will fix the optimal investment timing to the monopoly investment timing, and we will deviate the investment quantity for the first entrant. From there, we can determine the optimal investment size and timing of the second entrant. The two equations that will be used for this are (11) and (12).

With the investment choices of the second entrant, we can determine the result for the value function of the first investor. We then try to find the optimum, and the Q_1 at the optimum is the optimal investment size given X_1.

This is, however, the case assuming that it is not possible to find an explicit solution for Q_2. If we assume that it is possible to find an explicit solution for Q_2, we can resume our calculations. What we assume is that there is a feasible, explicit solution to (12) in terms of Q_2. Since the only other decision variable present in this equation is Q_1, we know that Q_2 is only dependent on Q_1. If we were to plug Q_2 into (11), we can see that the only decision variable in this solution for X_2 is Q_1, hence, X_2 only depends on Q_1 as well.

The value function for the first investor is defined as

$$
V_1(Q,X) = \frac{Q_1 h(Q_1) X_1}{r-\mu} - \delta Q_1 - \left(\frac{X_1}{X_2}\right)^{\beta} X_2 \frac{Q_1\left(h(Q_1+Q_2) - h(Q_1)\right)}{r-\mu}
$$

which can be rewritten into

$$
V_1(Q,X) = \frac{Q_1 h(Q_1) X_1}{r-\mu} - \delta Q_1 - X_1^{\beta} g(Q_1)
$$

where

$$g\left(Q_1\right) = X_2^{1-\beta} \frac{Q_1\left(h(Q_1+Q_2)-h(Q_1)\right)}{r-\mu}$$

From this, we can take the derivative of Q_1, and apply value matching and smooth pasting, for which the calculations have been noted in subsection 6.2.2. Taking the results for the second investor, the following equations are left

$$\beta Q_2 h'(Q_1+Q_2) + h(Q_1+Q_2) = 0 \tag{13}$$

$$X_2 = \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q_1+Q_2)} \tag{14}$$

$$0 = \frac{\left(h\left(Q_1\right)+h'\left(Q_1\right)Q_1\right)}{r-\mu}X_1 - \delta + X_1^{\beta}\frac{\partial}{\partial Q_1}g\left(Q_1\right) \tag{15}$$

$$g\left(Q_1\right) = X_2^{1-\beta} \frac{Q_1\left(h(Q_1+Q_2)-h(Q_1)\right)}{r-\mu} \tag{16}$$

$$X_1 = \frac{\beta\delta(r-\mu)}{(\beta-1)h(Q_1)} \tag{17}$$

Here, there are four decision variables with four equations, hence solvable. The MathWorks, Inc. (2023) is used to compute the optimal investment decisions for both firms. Again, this procedure only works when there is an explicit solution for Q_2, which is generally not the case.

3.3 Analysis

As a start, the investment timing for the first investor will be fixed. In this way, it is possible to compute the optimal investment quantity for most inverse demand functions. The functions that will be considered are identical to the ones studied in Balter et al. (2022), except for the iso-elastic demand, which will not be treated here. The reason for this is that Huisman and Kort (2015) have proven that, using investment costs as δQ_i, the first firm will always invest at the start of the investment window. This gives some atypical results. They solved this by choosing an alternative investment cost, namely $\delta_0 + \delta_1 Q_i$. We have opted to exclude this because it has already been analyzed and three other convex demand functions will be discussed. If the iso-elastic demand function were to be included, the original framework should be altered without giving us much more insight. Therefore, it will be disregarded. In table (5), the inverse demand functions, the graph with the outcomes of the value function for different Q_1's, and how $h(Q)$ behaves are displayed. Here, σ has been set to 0.

As we can see, in the first row, we have the original $h(Q)$ which is linear and decreasing.

Here, as concluded before, we have that the investment quantity for the first entrant in a duopoly coincides with the investment quantity in a monopoly. What is interesting to observe is that, if $h(Q)$ is convex, the investment quantity for the first entrant is higher in a duopoly than in a monopoly. On the other hand, if $h(Q)$ is concave, the quantity is lower. It cannot be concluded that this result holds in general, however, this seems promising.

To interpret this, we should look at the two effects at play. As mentioned before, the investment size is decreased by the loss in value due to a potential new entrant and, on the other hand, it is increased because the firm wants to increase the size of the investment in order to delay the investment of the second investor.

If the function $h(Q)$ is convex, investment by the second investor influences the price less significantly. The first investor wants to be alone in the market as long as possible and will, therefore, invest more than it would in a monopoly. The effect of the loss of value has a less significant impact than the effect of the advantage of delaying the second firm's investment.

In the setting of a concave $h(Q)$, this works the opposite way. A larger investment reduces the price more significantly, hence the effect of the incentive to invest less due to the price reduction is more prevalent than the the incentive to delay the second investment.

As we have stated before, these effects completely balance each other out in the linear setting.

If we observe table (4), we see that this behavior persists over different values of σ. It can be observed that the difference between the monopoly and duopoly output actually amplifies, as σ increases.

Note that the duopoly Q_1's are not necessarily the optimal investment quantity. The investment timing is fixed to the monopoly timing, however, this might not be true in the optimum. What we can conclude is that in the specified functions, and with our assumed parameter values only in the linear demand we have that the investment quantity and timing are equal in the duopoly and monopoly setting. In all other functions that have been discussed, the monopoly strategy is not an optimal strategy for the first entrant in a duopoly.

We will investigate this further by looking into two inverse demand functions for which the optimal investment quantity and timing can be computed. Of the functions that will be considered, one is convex, and one is concave. In this manner, the effect of the convexity and concavity of the inverse demand function can be observed.

The functions that will be considered are the quadratic and square-root inverse demand functions, in sections 4.1 and 4.2, respectively. For both of these, we can compute an explicit solution for Q_2, meaning that we can actually find the optimal investment quantity and timing for both investors. This will be executed in the following section.

	Linear			Exponential		
	Q_1^{mon}	Q_1^{duo}	$Q_1^{\text{duo}} - Q_1^{\text{mon}}$	Q_1^{mon}	Q_1^{duo}	$Q_1^{\text{duo}} - Q_1^{\text{mon}}$
$\sigma = 0.00$	7.5000	7.5000	0.0000	1.3956	1.4524	0.0567
$\sigma = 0.05$	7.5626	7.5626	0.0000	1.4268	1.4870	0.0603
$\sigma = 0.10$	7.7258	7.7258	0.0000	1.5087	1.5789	0.0701
$\sigma = 0.15$	7.9400	7.9400	0.0000	1.6179	1.7027	0.0848
$\sigma = 0.20$	8.1650	8.1650	0.0000	1.7341	1.8366	0.1025
$\sigma = 0.25$	8.3785	8.3785	0.0000	1.8459	1.9676	0.1217
$\sigma = 0.30$	8.5714	8.5714	0.0000	1.9477	2.0892	0.1414
$\sigma = 0.35$	8.7413	8.7413	0.0000	2.0381	2.1989	0.1608
$\sigma = 0.40$	8.8889	8.8889	0.0000	2.1170	2.2965	0.1795

	Algebraic			Logarithmic			Logit		
	Q_1^{mon}	Q_1^{duo}	$Q_1^{\text{duo}} - Q_1^{\text{mon}}$	Q_1^{mon}	Q_1^{duo}	$Q_1^{\text{duo}} - Q_1^{\text{mon}}$	Q_1^{mon}	Q_1^{duo}	$Q_1^{\text{duo}} - Q_1^{\text{mon}}$
$\sigma = 0.00$	0.1317	0.1372	0.0056	0.5518	0.5440	-0.0078	0.0715	0.0745	0.0031
$\sigma = 0.05$	0.1361	0.1422	0.0061	0.5552	0.5472	-0.0080	0.0729	0.0761	0.0032
$\sigma = 0.10$	0.1478	0.1555	0.0077	0.5641	0.5555	-0.0086	0.0766	0.0802	0.0036
$\sigma = 0.15$	0.1636	0.1737	0.0101	0.5757	0.5662	-0.0094	0.0815	0.0857	0.0042
$\sigma = 0.20$	0.1806	0.1937	0.0131	0.5877	0.5774	-0.0104	0.0866	0.0915	0.0049
$\sigma = 0.25$	0.1970	0.2135	0.0165	0.5991	0.5878	-0.0114	0.0915	0.0971	0.0057
$\sigma = 0.30$	0.2121	0.2322	0.0201	0.6093	0.5970	-0.0123	0.0959	0.1023	0.0064
$\sigma = 0.35$	0.2255	0.2493	0.0238	0.6182	0.6050	-0.0132	0.0998	0.1068	0.0071
$\sigma = 0.40$	0.2373	0.2646	0.0273	0.6260	0.6120	-0.0140	0.1031	0.1109	0.0077

Table 4: Q_1^*'s for different σ's, $h(Q)$'s, and potential entrants.
The constants are the same as in (5)

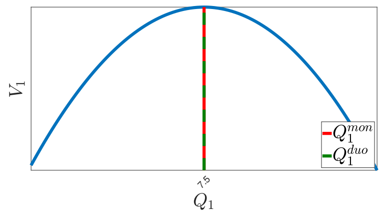
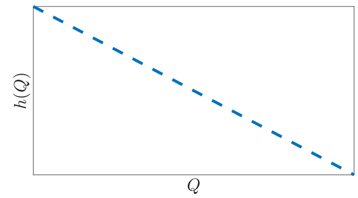
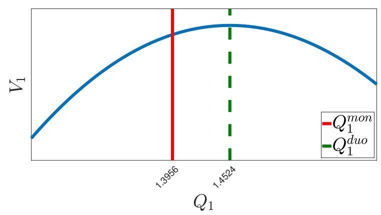
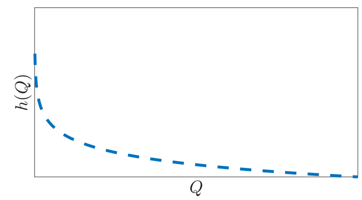
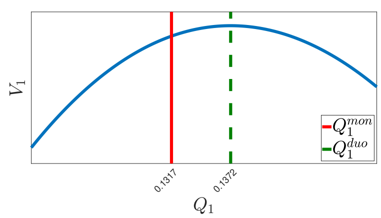
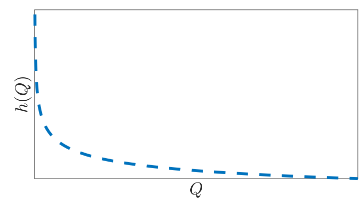
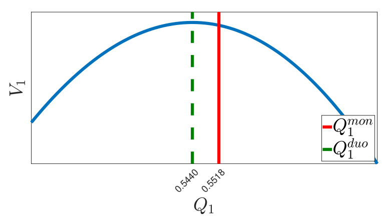
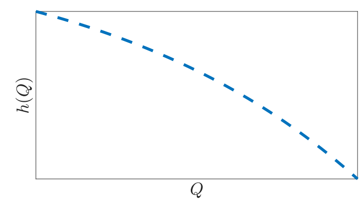
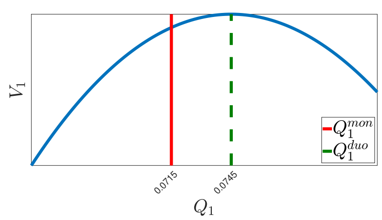
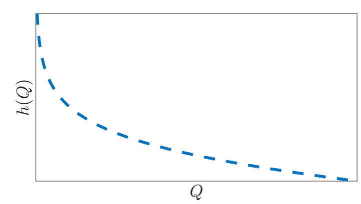
Linear demand $h(Q) = \alpha - Q$ $\alpha = 20$		
Exponential demand $h(Q) = 1 - \phi \ln Q$ $\phi = 0.5$		
Algebraic demand $h(Q) = \zeta Q^{-\theta} - 1$ $\zeta = 1, \theta = 0.2$		
Logarithmic demand $h(Q) = 1 - \xi e^{\kappa Q}$ $\xi = 0.3, \kappa = 1$		
Logit demand $h(Q) = \psi \ln\left(\frac{1}{Q} - 1\right) - 1$ $\psi = 1.3$		

Table 5: Optimal investment quantities for various price functions with $\sigma = 0$

4 Convex vs. Concave

Opposed to what we assumed in the previous section, i.e., the investment timing for the duopoly was fixed to the monopoly timing, in this section we will consider the case where the investment timing is a decision variable. We are hoping that the results from the previous section still hold. Namely, the duopoly output is higher than the monopoly output, while for the concave setting, the opposite is true.

4.1 Quadratic inverse demand function

In this section, the quadratic inverse demand function will be considered. We will define this inverse demand function as follows

$$P = X(\alpha - Q^2)$$

The reason for this is that we are looking for an explicit solution to Q_2, which is possible for this inverse demand function. The other assumptions from the previous sections will still be in place.

4.1.1 Monopoly

First, we will consider only one potential entrant, the monopoly setting. Using (9) and (10), we can easily derive the optimal investment decisions of the first, and only, investor. The computations are noted in subsection 6.3.1.1. The investment decisions are

$$Q = \sqrt{\frac{\alpha}{1+2\beta}}$$
$$X = \frac{1+2\beta}{\beta-1}\frac{\delta(r-\mu)}{2\alpha}$$

4.1.2 Duopoly

Next, the duopoly setting will be considered. We will again do this starting by optimizing the investment decision, assuming Q_1 and X_1 to be known. Our calculations in the previous section are again convenient.

$h(Q)$ will be plugged into (13) and (14), which results in the subsequent investment quantity and timing for the second investor.

$$Q_2 = \frac{\sqrt{\beta^2 Q_1^2 + (2\beta+1)\alpha}}{2\beta+1} - \frac{\beta+1}{2\beta+1}Q_1$$

$$X_2 = \frac{\beta}{\beta - 1} \frac{\delta(r - \mu)}{\alpha - (Q_1 + Q_2)^2}$$

The derivation of these investment decisions is reported in subsection 6.3.1.2.

With these two equations, and the previously calculated equations (15) and (17), we end up with the following set of equations which can be solved to determine the optimal investment quantity and timing for both the first and second investor.

$$Q_2 = \frac{\sqrt{\beta^2 Q_1^2 + (2\beta + 1)\alpha}}{2\beta + 1} - \frac{\beta + 1}{2\beta + 1} Q_1$$

$$X_2 = \frac{\beta}{\beta - 1} \frac{\delta(r - \mu)}{\alpha - (Q_1 + Q_2)^2}$$

$$0 = \frac{\alpha - 3Q_1^2}{r - \mu} X_1 - \delta + X_1^\beta \frac{\partial}{\partial Q_1} g(Q_1)$$

$$g(Q_1) = X_2^{1-\beta} \frac{Q_1(Q_1^2 - (Q_1 + Q_2)^2)}{r - \mu}$$

$$X_1 = \frac{\beta}{\beta - 1} \frac{\delta(r - \mu)}{\alpha - Q_1^2}$$

We will not bother ourselves with calculating the mathematical equations. It is clear that it is possible to take the derivative of $g(Q_1)$ with respect to Q_1, by first plugging in our results for Q_2 and X_2. The MathWorks, Inc. (2023) will take care of the computations, the analysis is carried out in subsection 4.3.

4.2 Square-root inverse demand function

In this segment, we'll examine the square-root inverse demand function. This formulation facilitates an explicit derivation of

$$P = X(\alpha - \sqrt{Q})$$

making our task straightforward. Again, the assumptions from the previous sections are still in place.

4.2.1 Monopoly

Starting with a situation where there's only a single potential entrant, we delve into the monopoly framework. Building upon the analytical insights from section 3, the optimal investment choices for this lone investor can be determined. The calculations behind these investment decisions can be found in 6.3.2.1.

$$Q = \left(\frac{\alpha}{1+\frac{1}{2}\beta}\right)^2$$
$$X = \frac{2+\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha}$$

4.2.2 Duopoly

Transitioning to a scenario with two potential entrants, we'll decode the duopoly framework. We start our analysis by honing in on the optimal investment decisions, assuming Q_1 and X_1 to be known. Our previous calculations play a pivotal role here. By integrating $h(Q)$ into (13) and (14), we derive the following investment quantity and timing for our second entrant.

$$Q_2 = \frac{2\alpha^2 + 2\alpha\sqrt{\alpha^2 + (\beta^2+2\beta)Q_1} + (2\beta+4)Q_1}{(\beta+2)^2}$$
$$X_2 = \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha - \sqrt{Q_1+Q_2}}$$

Melding these derivations with the previously established equations (15) and (17), we are presented with a comprehensive equation system. This system paves the way for determining the optimal investment quantity and timing for both investors.

4.3 Analysis

Now, we try to analyze what this section is about, does a convex inverse demand function function behave differently than a convex alternative? As earlier discussed in subsection 3.3, when setting the duopoly timing of the first entrant to the monopoly timing, we observed that the duopoly output was higher when the inverse demand function was convex. The opposite was found to be true for a concave inverse demand function, namely, that the monopoly output was higher than the duopoly output for the first entrant. This seems like an exciting result. That is why now we will analyze the investment timing and size for the linear, quadratic, and square-root inverse demand functions, where the latter two are concave and convex, respectively. The linear setting is already extensively discussed in earlier research and earlier in this thesis.

As already concluded, the linear inverse demand function yields the same investment quantity and investment timing for the first entrant in both a monopoly and a duopoly. This result is, again, shown in table (6).

Besides these values that have already been reported, this table displays the results for the optimal investment quantity and size of a convex and a concave function. The investment quantity for the first entrant for a concave function is indeed larger in a monopoly than it is in a duopoly. Conversely, in a setting where the inverse demand function is a convex function, the investment quantity is smaller in the monopoly than it is in a duopoly. We have already seen this in subsection 3.3. Now, however, we also take the investment timing as a decision as opposed to setting it to the monopoly timing.

This is a compelling result, which would, together with the results in subsection 3.3, suggest that this behavior is consistent for all convex and concave functions. To show that this behavior is consistent for all inverse demand functions, there would be a mathematical proof needed, which is beyond the scope of this thesis. It is, however, promising, especially taking into consideration that for the linear inverse demand function, the monopoly and duopoly quantity are equal. This seems like the tipping point, where convex inverse demand functions fall into one cluster and concave ones into another.

If the function $h(Q)$ exhibits convexity, the impact of the second investor's investment on the inverse demand function is relatively minor. The first investor, aiming to maintain a monopoly in the market for as long as possible, will consequently increase their investment beyond the usual amount.

Conversely, in the context of a concave $h(Q)$, a larger investment leads to a more substantial reduction in price. Consequently, the incentive to invest less due to the price reduction outweighs the incentive to delay the second investment. In the linear scenario, these effects cancel each other out entirely.

Here, the investment timing is also taken as a decision variable, as opposed to our computations in subsection 3.3. As we had hoped, the results from this section persist when determining the investment timing.

It should be noted that the investment quantity and timing values cannot be compared directly, as α is held constant at a value of 20 across all three functions. While this may not reflect real-world scenarios, an examination of the values within each table reveals intriguing patterns in the reciprocal values.

If we look at the investment timing in the linear setting, we see that the monopoly and duopoly for the first entrant are equal, as we have seen before. This is different in the concave and convex setting.

It can be observed that when using a concave inverse demand function, the duopoly timing is earlier than the monopoly timing. This was to be expected, since looking at (17), it can be observed that X_1^{duo} is increasing in Q_1^{duo}.

What is important to note here, is that since we are considering exogenous firm entry, the second entrant will always wait with his investment until the first investor has entered. Hence, because the second firm cannot preempt, and the investment timing of the first entrant's timing does not influence the second firm, the leading decision variable here is the investment quantity of the first entrant. A higher investment quantity, results in a later investment timing.

Because a higher investment quantity automatically results in later investment, the duopoly investment timing is later than the monopoly timing, since the investment quantity is higher.

The amplification of the effect is observed to be greater with an increase in volatility, not only in absolute terms but also in relative terms. This phenomenon is observed in both the concave and convex settings. To visualize this, let us observe the relative effect of σ on the duopoly investment quantity. This is done by dividing Q_1^{duo} by Q_1^{mon} for the three inverse demand functions and varying values for σ. This is shown in figure (4).

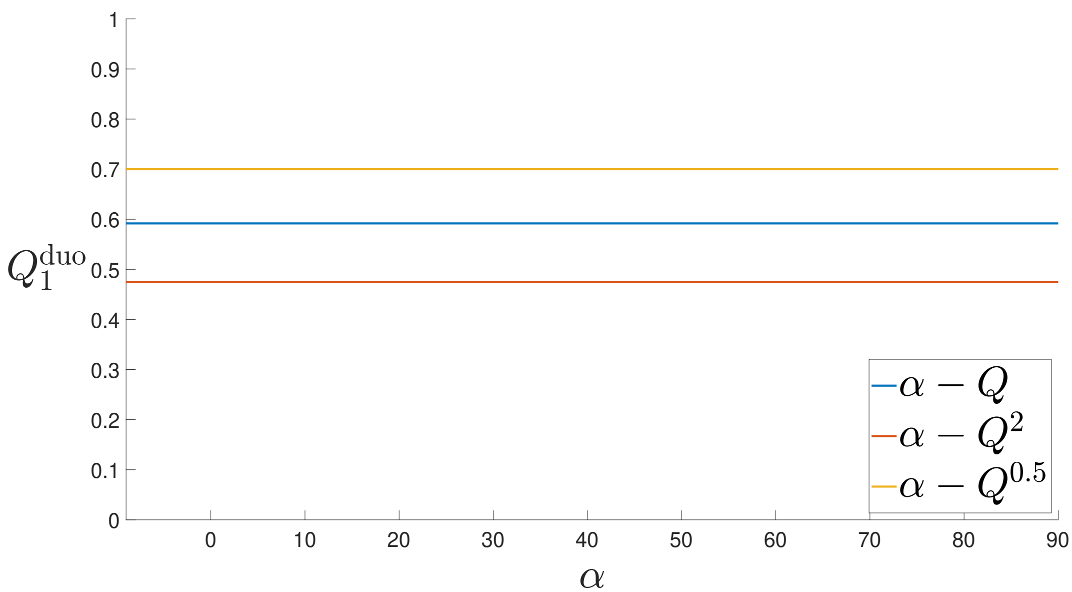


Figure 4: Relative effect of σ on $\frac{Q_1^{\text{duo}}}{Q_1^{\text{mon}}}$.
$\alpha = 20$

As we can see, there is an increase in the divergence of the duopoly quantity comparing it to the monopoly quantity in σ, however, it is very minor, nearly negligible. This applies to the inverse demand functions which are quadratic and square-root, concave and convex, respectively. As we have already seen in our results from 2.2, for the linear inverse demand function this remains 1 for various values of the standard deviation, since this means that the duopoly output for the first investor is equal to the investment size when dealing with a monopoly.

Next, we will analyze the effect of some parameters on the investment decisions. To check the effects of the parameters, we need some default values for the parameters. We use the same values as we have done so far, i.e., $\alpha = 20, r = 0.1, \mu = 0.06$, and $\delta = 0.1$. Lastly, in the previous sections, we observed the effect of σ. In order to evaluate the effect of the other parameters, we need to set the default value for σ, too. We will set σ to 0.2.

Starting with observing the effect of α on the investment decisions, the effect of α on the investment size is linear, as we have shown in subsection 2. This can be observed in figure (5), too. Of course, the monopoly quantity fully coincides with the duopoly output for the first entrant. In a duopoly, it appears that the ratio between the investment size of the first entrant and the investment size of the second entrant remains constant over α.

Now, considering the quadratic inverse demand function, figure (7), we see that, as concluded before, the monopoly investment quantity and the duopoly investment quantity of the first firm are not equal, and the difference slightly gets amplified by α. Also, it can be observed that the investment quantity follows an increasing and concave path. This can be explained by delving into the inverse demand function. If α is very large, we need less extra Q to accommodate this. Looking at the duopoly setting, it appears to be the case that the ratio between the investment size of the first and the second entrant actually decreases.

Lastly, looking at the square-root inverse demand function, figure (9), we see that, again as we have seen before, the monopoly output and the duopoly output for the first entrant are not equal. Similar to the quadratic setting, the difference between the two slightly increases as α increases. Also, where the quadratic inverse demand function follows a concave path, here, the optimal investment actually follows a convex path. The logic here is similar to the quadratic setting, however, reversed. Since the effect of investing more decreases the price less, the investor can increase its investment size more to accommodate this when α becomes larger.

It appears interesting to delve into the relative effect of α on the investment size. Therefore, in figure (11) this ratio between $Q_2^{\text{duo}}/Q_1^{\text{duo}}$ is depicted for the three functions.

Interestingly, the ratio between the investment size of the second and first entrant our completely constant over α, for all functions. This tells us that α does not impact the investment decisions of the investors relative to each other. The second investor will always invest the same fraction of the investment of the first entrant. This seems to be the case because of our choice for the inverse linear demand function. All three functions are linear in α. We would expect that this behavior would not persist if the relationship between the inverse linear demand function was not linear in α. This could be researched in later research but this is not within the scope of this thesis.

Next, we will delve into the effect of the market size on the investment timing. Observing figures (6), (8), and (10), we see that the investment timing behaves similarly over the various inverse demand functions. For smaller values of α, both firms will invest relatively late, while if α gets larger, the firms will invest earlier. This is due to our assumed inverse demand function, which is increasing in α. If α is large, the price will be higher earlier on, and it will be profitable to execute the investment earlier.

Subsequently, we will look at the relative effect of α on the investment timing. What happens to the ratio of the investment timing of the first and second entrant as α increases? This is plotted in figure (12). Similar to the results seen for the investment quantity, here we observe that the ratio is constant over all values of α, as well.

When looking at the effect of the interest rate on the optimal investment quantity and timing for the various inverse demand functions, we make some notable observations. Note that μ is fixed to 0.06, hence, when we increase the value of r, the difference between the interest rate and the drift rate increases. First, we will examine the linear inverse demand function by looking at figures (13) and (14). We observe that when the interest rate increases, the optimal investment quantity decreases for all investment quantities. This seems logical since it will be, relatively speaking, less profitable to make the investment, and the entrants will invest less. As r increases, the investors will invest later. This can be attributed to the following. If the interest rate is larger, the inverse demand function needs to be at a higher level to make the investment profitable. This behavior, for both the investment quantity and the investment timing, persists for the various inverse demand functions.

There is a troubling result when looking at the effect of the interest rate on the optimal investment when dealing with a square-root inverse demand function, visualized in figure (17). Here we see that the investment size of the first entrant actually becomes smaller than that of the second entrant when the interest rate becomes considerably larger in comparison to the drift term of the geometric Brownian motion.

From this figure, it is difficult to draw conclusions on what will happen to the optimal investment size for the first entrant in a monopoly and a duopoly. Therefore, in figure (19) the ratio between the two is plotted. Here we see that our result from the previous sections, namely that when the inverse demand function is convex the monopoly output is smaller than the duopoly output, does not persist over all values of r. Even though the values for the interest rate are probably not realistic, it does raise questions about the robustness of the results. Our predicted general result, that the duopoly output for the first entrant is larger than the monopoly output for a convex inverse demand function, does not seem to hold in general. In this case, it does not appear to be a significant concern, since this is only when the interest rate is unreasonably high compared to μ. The main threat is that it could become a concern when different concave functions behave in this manner for more realistic parameter values. This should be explored further, but this result is discouraging for a general conclusion for the behavior of convex and concave inverse demand functions.

Next, we will look a the effect of μ on the optimal investment decisions. First, we fix r to 0.1 and note that we, by assumption, have $\mu < r$. Figure (21), (23), and (25) display the effect of μ on the optimal investment quantity. We see that the quantity increases as μ increases. We see that, for the square-root inverse demand function, the optimal investment quantity is larger for the second entrant than for the first when μ is small. Also looking at the results from the analysis of r, we can conclude that, when we are dealing with the square-root inverse demand function and the difference between r and μ is large, the optimal investment quantity for the second entrant is larger than for the first. The question arises whether this holds for all convex functions or only for the square-root setting. This could be investigated further, but will not be done here.

Looking at figures (22), (24), and (26), we will now discuss the effect μ on the optimal investment timing. We observe that for very small values of μ the investment timing actually decreases, and as μ becomes increasing. This result holds for all three inverse demand functions.

Looking into the effect of δ on the investment decisions, we have plotted the optimal investment quantity for the first entrant in a duopoly and a monopoly, and the investment quantity for the second entrant in a doupoly. For the three inverse demand function this is displayed in figure (27), (29), and (31) for the linear, quadratic, and square-root inverse demand function, respectively.

Interestingly, we see that the investment costs do not influence the optimal investment quantity at all. From this, we can conclude that in our assumptions on the investment costs, the profits outweigh the costs, and the cost of the investment is not relevant when determining the investment quantity. It seems interesting what happens when you would take an investment cost of, e.g., δQ^2 or even δQ^3. We would expect that when the costs grow faster in Q, δ would affect the optimal investment quantity. This, however, will not be discussed in this thesis but does seem interesting to delve into in future research. Making this adjustment will lead to more difficult mathematical computations, but can give more insight into the effect of the investment costs for the entrants.

This is only considering the optimal investment quantity. When looking at equation (17), we see that the cost per quantity is relevant when considering the optimal investment timing. This equation tells us that the optimal investment timing for the first entrant in a duopoly setting grows linear in δ. So when δ increases, the investor will delay its investment. The reason for this is when the investment costs become larger, the discounted cost for this investment becomes smaller as you delay your investment. This coincides with the figures (28), (30), and (32). Since there is no preemption, due to our exogenous firm order, the first investor can calmly wait to pick the optimal investment timing, without the threat of the second investor entering the market.

$$h(Q) = \alpha - Q$$

	Q_1^{mon}	Q_1^{duo}	Q_2^{duo}	X_1^{mon}	X_1^{duo}	X_2^{duo}
$\sigma = 0.00$	7.5000	7.5000	4.6875	8.0000e-04	8.0000e-04	1.2800e-03
$\sigma = 0.05$	7.5626	7.5626	4.7030	8.2056e-04	8.2056e-04	1.3195e-03
$\sigma = 0.10$	7.7258	7.7258	4.7414	8.7944e-04	8.7944e-04	1.4330e-03
$\sigma = 0.15$	7.9400	7.9400	4.7878	9.7088e-04	9.7088e-04	1.6101e-03
$\sigma = 0.20$	8.1650	8.1650	4.8316	1.0899e-03	1.0899e-03	1.8418e-03
$\sigma = 0.25$	8.3785	8.3785	4.8685	1.2335e-03	1.2335e-03	2.1227e-03
$\sigma = 0.30$	8.5714	8.5714	4.8980	1.4000e-03	1.4000e-03	2.4500e-03
$\sigma = 0.35$	8.7413	8.7413	4.9208	1.5889e-03	1.5889e-03	2.8225e-03
$\sigma = 0.40$	8.8889	8.8889	4.9383	1.8000e-03	1.8000e-03	3.2400e-03

$$h(Q) = \alpha - Q^2$$

	Q_1^{mon}	Q_1^{duo}	Q_2^{duo}	X_1^{mon}	X_1^{duo}	X_2^{duo}
$\sigma = 0.00$	2.1483	2.0749	1.0149	6.5000e-04	6.3715e-04	9.5666e-04
$\sigma = 0.05$	2.1594	2.0850	1.0171	6.6542e-04	6.5199e-04	9.8345e-04
$\sigma = 0.10$	2.1881	2.1111	1.0225	7.0958e-04	6.9447e-04	1.0603e-03
$\sigma = 0.15$	2.2256	2.1451	1.0292	7.7816e-04	7.6039e-04	1.1798e-03
$\sigma = 0.20$	2.2649	2.1806	1.0356	8.6742e-04	8.4611e-04	1.3359e-03
$\sigma = 0.25$	2.3020	2.2140	1.0413	9.7509e-04	9.4943e-04	1.5245e-03
$\sigma = 0.30$	2.3355	2.2440	1.0459	1.1000e-03	1.0692e-03	1.7437e-03
$\sigma = 0.35$	2.3649	2.2703	1.0497	1.2417e-03	1.2050e-03	1.9926e-03
$\sigma = 0.40$	2.3905	2.2930	1.0528	1.4000e-03	1.3567e-03	2.2712e-03

$$h(Q) = \alpha - Q^{0.5}$$

	Q_1^{mon}	Q_1^{duo}	Q_2^{duo}	X_1^{mon}	X_1^{duo}	X_2^{duo}
$\sigma = 0.00$	119.0083	123.0176	93.3756	1.1000e-03	1.1225e-03	1.8905e-03
$\sigma = 0.05$	120.4549	124.5623	93.8424	1.1308e-03	1.1546e-03	1.9545e-03
$\sigma = 0.10$	124.2339	128.6018	95.0002	1.2192e-03	1.2465e-03	2.1389e-03
$\sigma = 0.15$	129.2137	133.9342	96.3914	1.3563e-03	1.3894e-03	2.4274e-03
$\sigma = 0.20$	134.4653	139.5689	97.6946	1.5348e-03	1.5757e-03	2.8062e-03
$\sigma = 0.25$	139.4686	144.9473	98.7815	1.7502e-03	1.8007e-03	3.2666e-03
$\sigma = 0.30$	144.0000	149.8268	99.6370	2.0000e-03	2.0620e-03	3.8045e-03
$\sigma = 0.35$	147.9984	154.1386	100.2911	2.2833e-03	2.3585e-03	4.4180e-03
$\sigma = 0.40$	151.4793	157.8971	100.7842	2.6000e-03	2.6902e-03	5.1067e-03

Table 6: Optimal investment quantity and timing for the linear, quadratic and square-root inverse demand function. For all, $\alpha = 20$

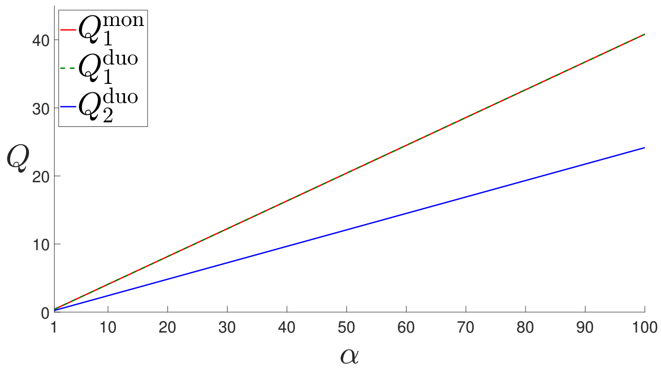


Figure 5: Optimal investment quantities
$h(Q) = \alpha - Q$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

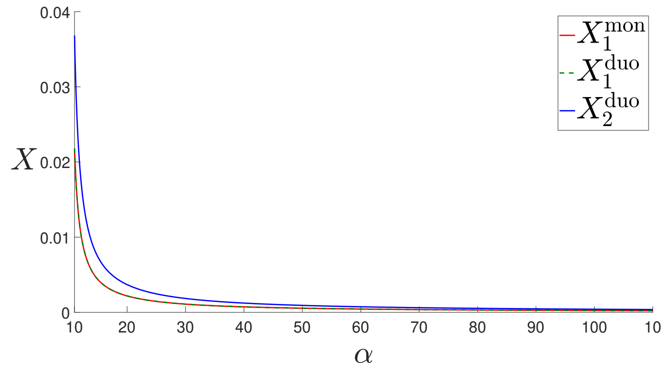


Figure 6: Optimal investment timing
$h(Q) = \alpha - Q$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

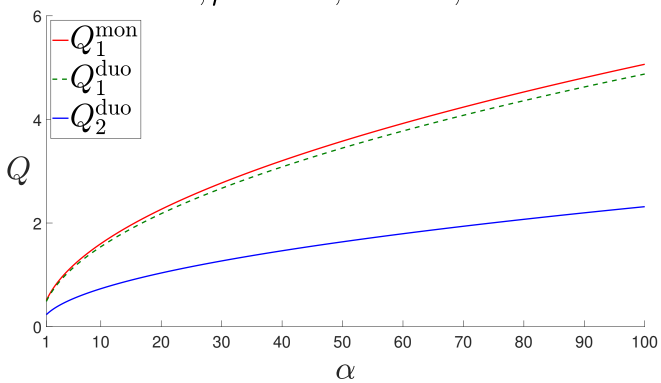


Figure 7: Optimal investment quantities
$h(Q) = \alpha - Q^2$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

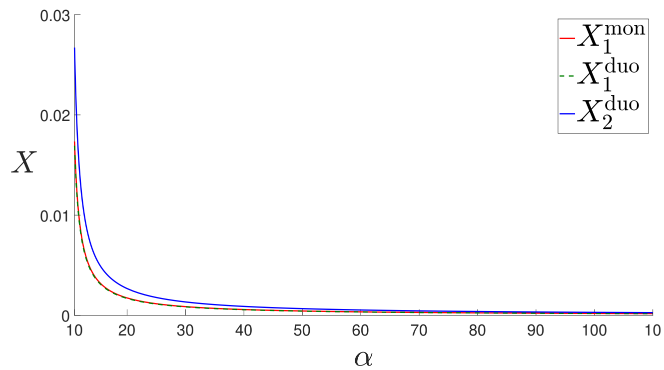


Figure 8: Optimal investment timing
$h(Q) = \alpha - Q^2$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

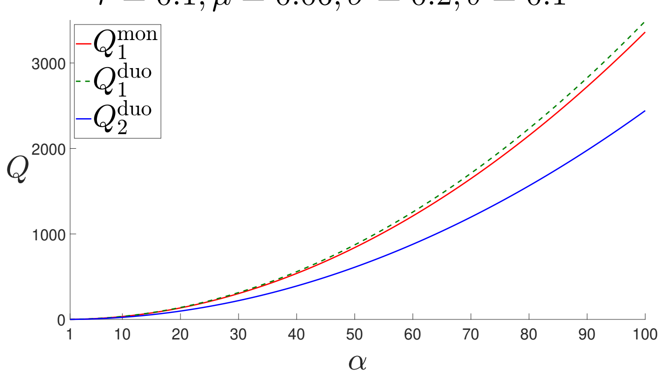


Figure 9: Optimal investment quantities
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

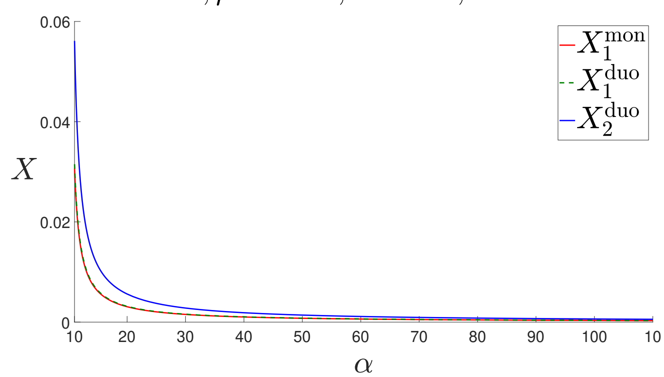


Figure 10: Optimal investment timing
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

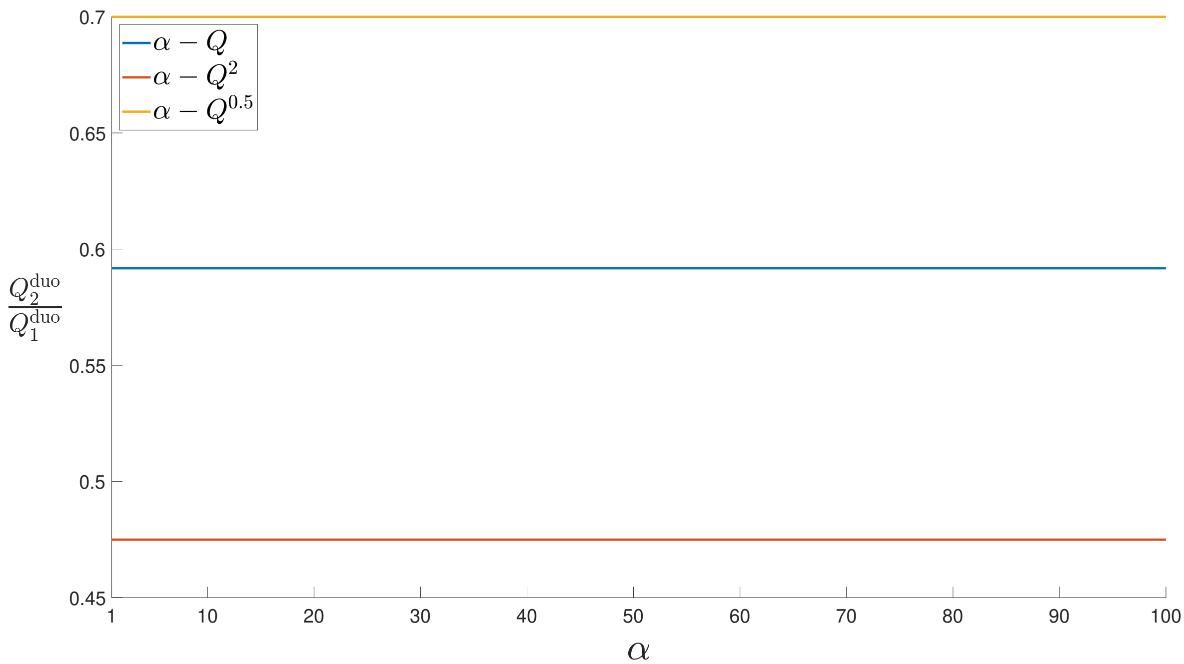


Figure 11: Investment quantity ratio
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

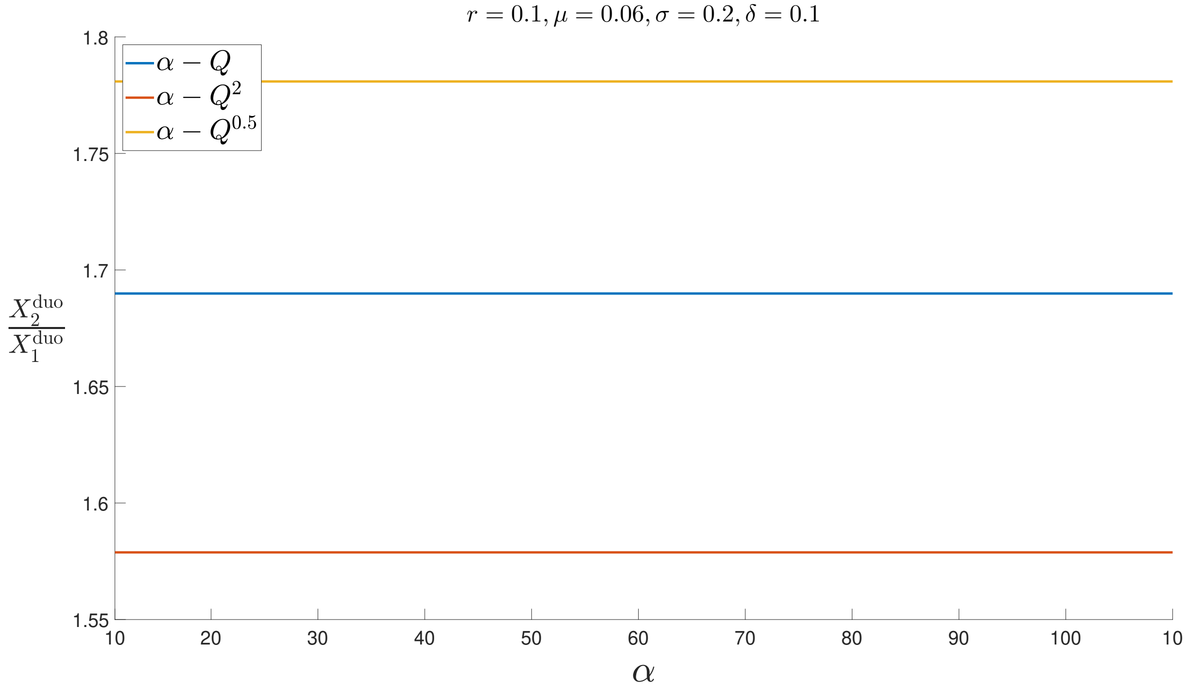


Figure 12: Investment quantity ratio
$r = 0.1, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

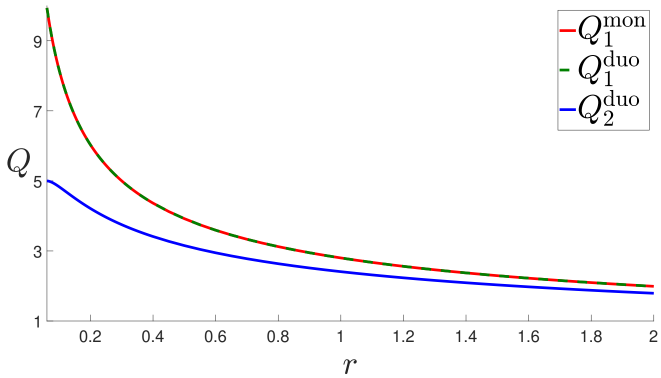


Figure 13: Optimal investment quantities
$h(Q) = \alpha - Q$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

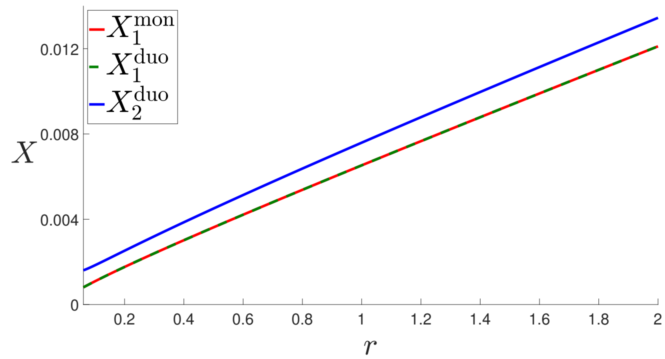


Figure 14: Optimal investment timing
$h(Q) = \alpha - Q$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

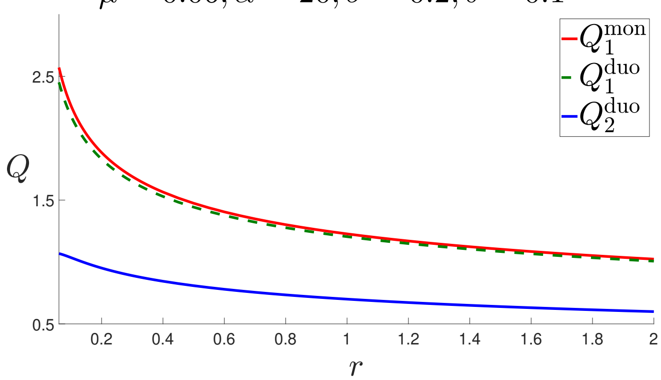


Figure 15: Optimal investment quantities
$h(Q) = \alpha - Q^2$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

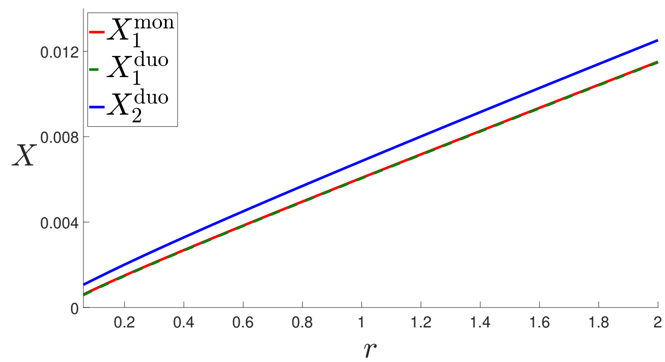


Figure 16: Optimal investment timing
$h(Q) = \alpha - Q^2$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

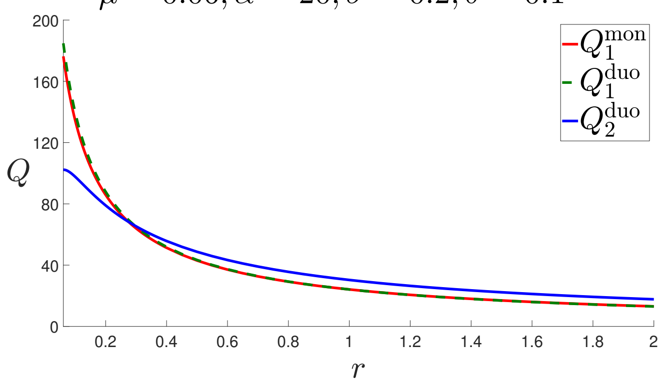


Figure 17: Optimal investment quantities
$h(Q) = \alpha - Q^{0.5}$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

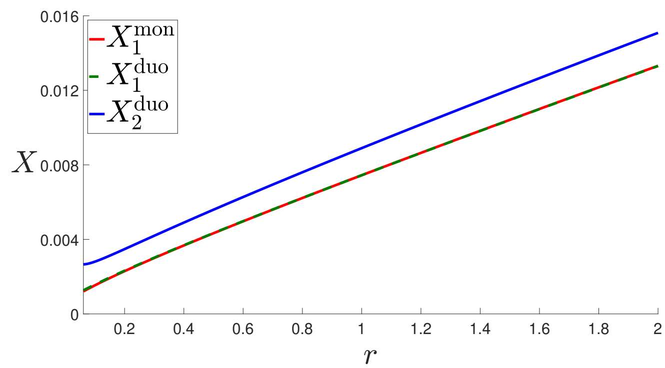


Figure 18: Optimal investment timing
$h(Q) = \alpha - Q^{0.5}$
$\mu = 0.06, \alpha = 20, \sigma = 0.2, \delta = 0.1$

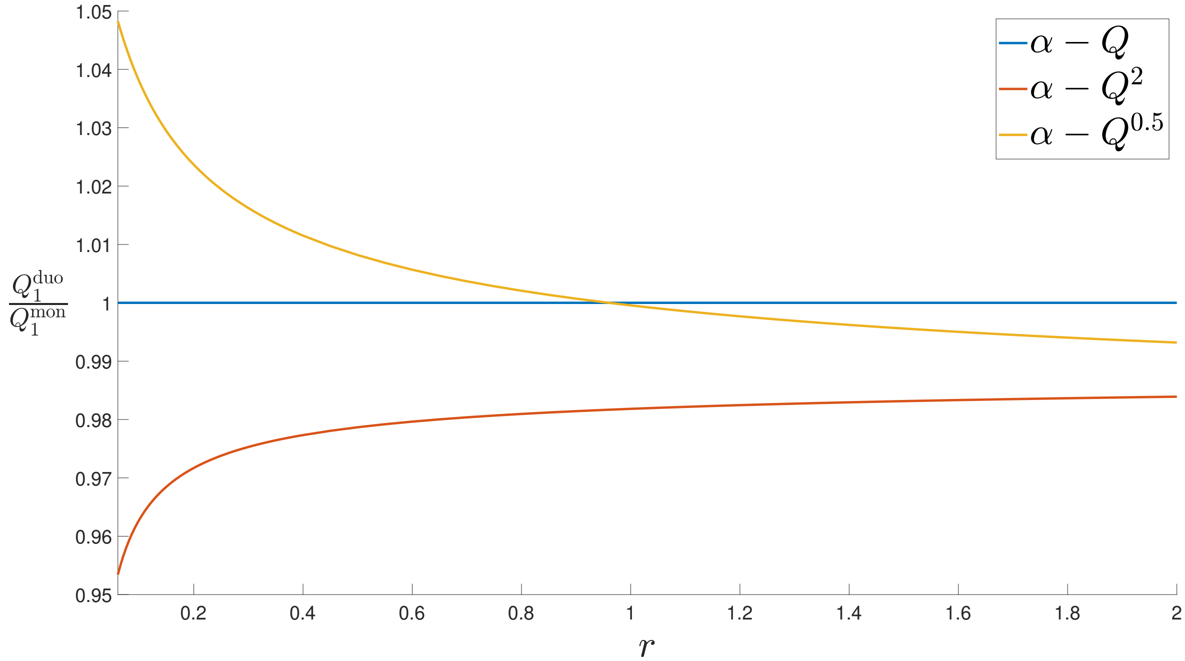


Figure 19: Investment quantity ratio
$\alpha = 20, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

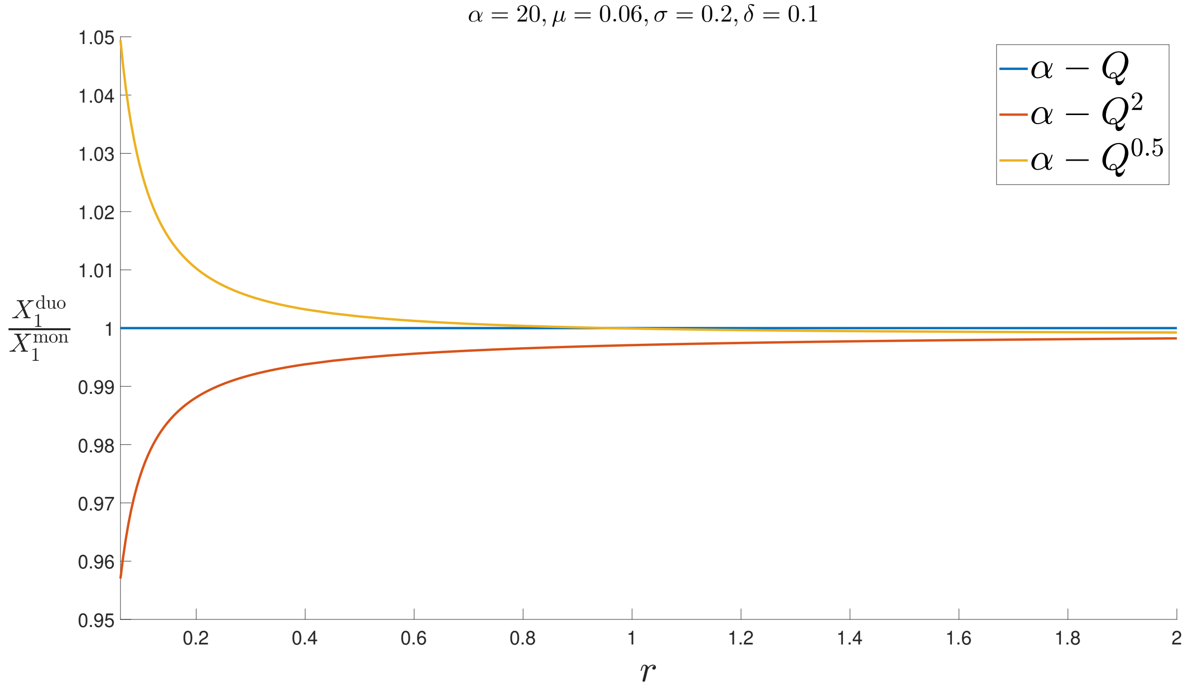


Figure 20: Investment quantity ratio
$\alpha = 20, \mu = 0.06, \sigma = 0.2, \delta = 0.1$

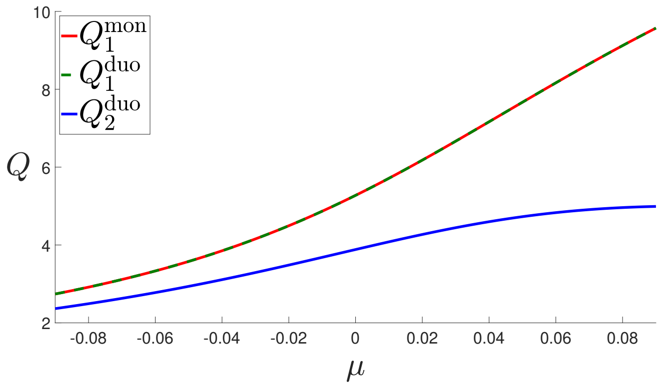


Figure 21: Optimal investment quantities
$h(Q) = \alpha - Q$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

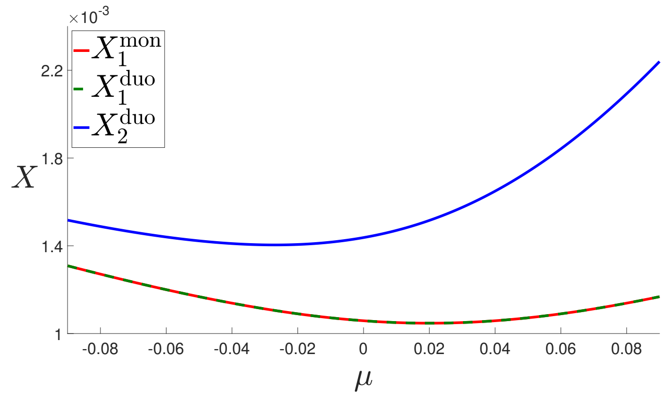


Figure 22: Optimal investment timing
$h(Q) = \alpha - Q$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

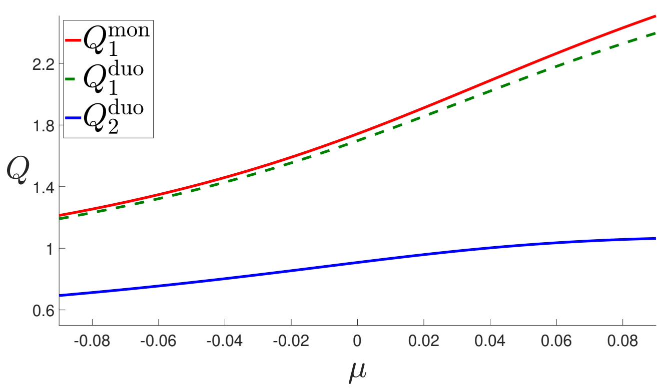


Figure 23: Optimal investment quantities
$h(Q) = \alpha - Q^2$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

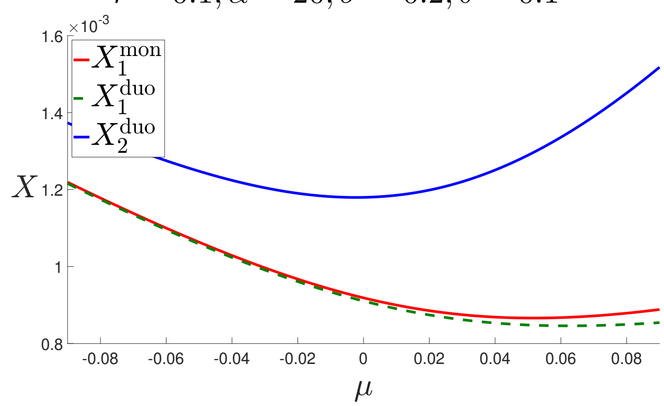


Figure 24: Optimal investment timing
$h(Q) = \alpha - Q^2$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

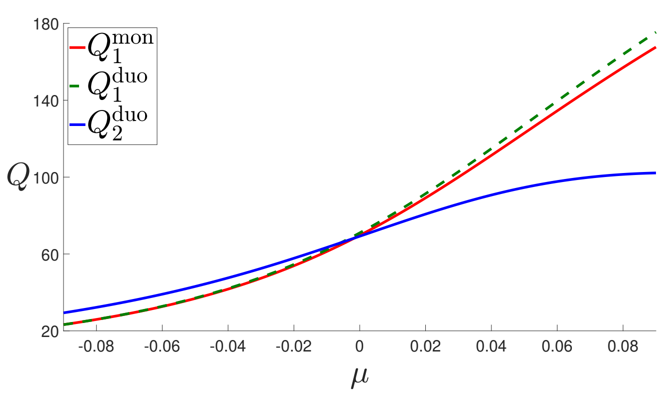


Figure 25: Optimal investment quantities
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

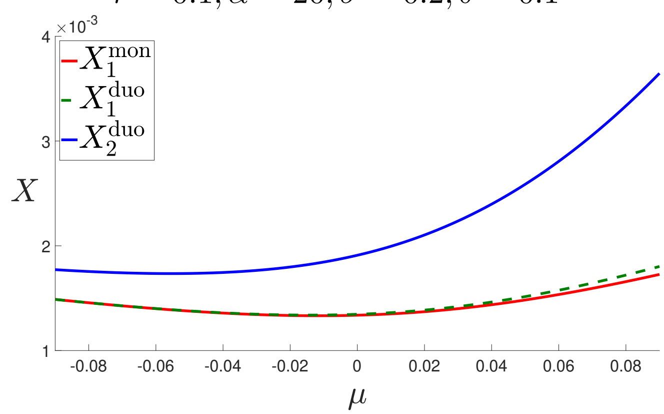


Figure 26: Optimal investment timing
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \alpha = 20, \sigma = 0.2, \delta = 0.1$

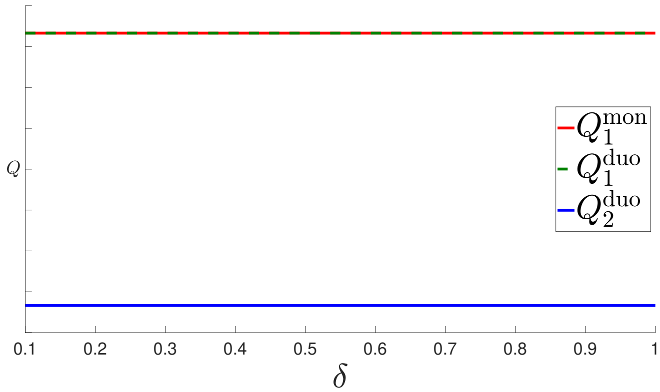


Figure 27: Optimal investment quantities
$h(Q) = \alpha - Q$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

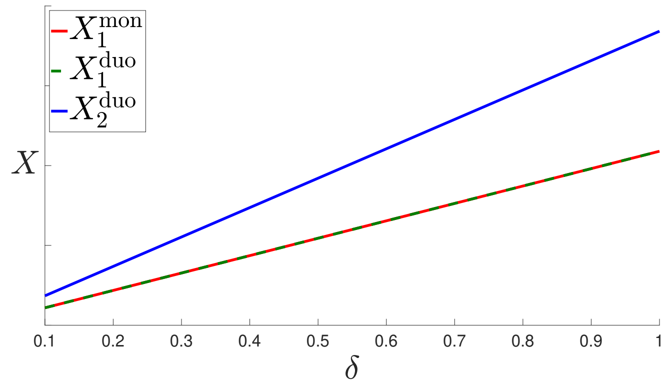


Figure 28: Optimal investment timing
$h(Q) = \alpha - Q$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

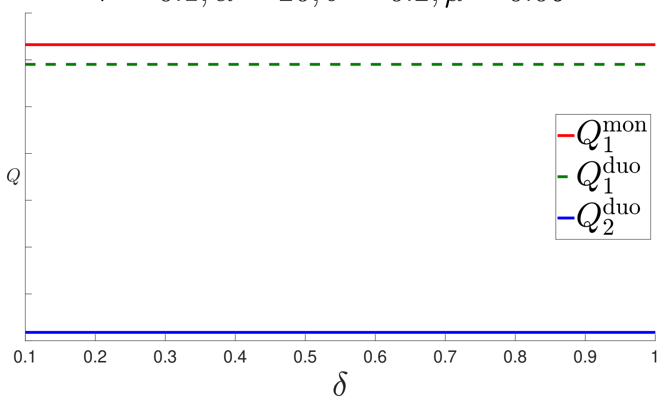


Figure 29: Optimal investment quantities
$h(Q) = \alpha - Q^2$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

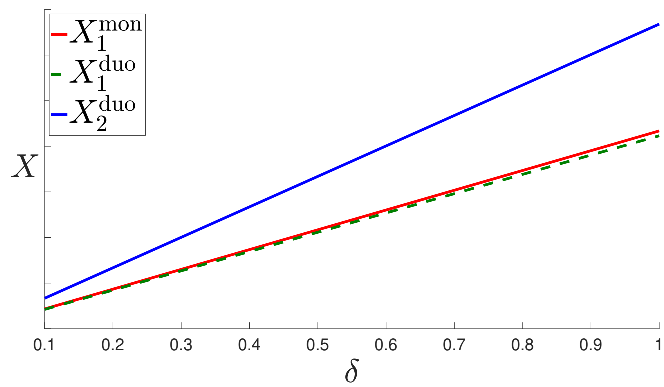


Figure 30: Optimal investment timing
$h(Q) = \alpha - Q^2$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

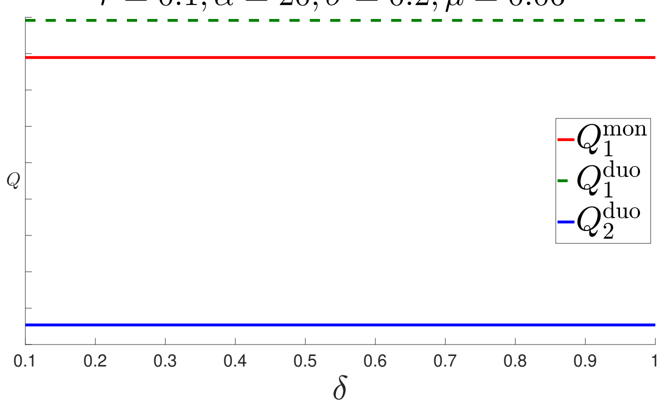


Figure 31: Optimal investment quantities
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

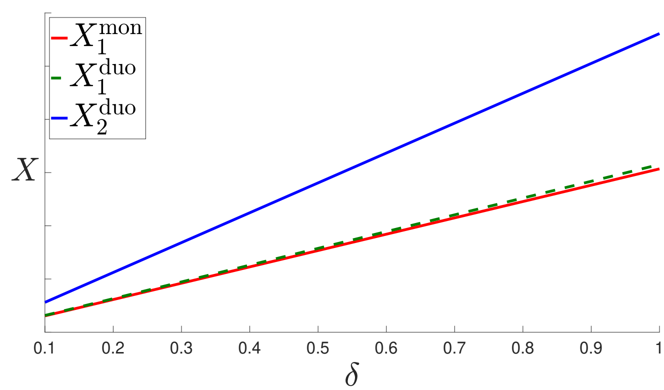


Figure 32: Optimal investment timing
$h(Q) = \alpha - Q^{0.5}$
$r = 0.1, \alpha = 20, \sigma = 0.2, \mu = 0.06$

5 Conclusion

In this thesis, the model proposed by Huisman and Kort (2015) and Faninam et al. (2022) was looked into and extended. Only considering exogenous firm order and assuming only the deterrence strategy, it was found that in a monopoly and duopoly case, similar results were found. The investment quantity and size were found to be exactly equal for the first entrant in both a monopoly and a duopoly, which was also found by Huisman and Kort (2015).

Next, a triopoly was considered, comparable to Faninam et al. (2022). Here, the same result was found, i.e., the first entrant will take the same investment decisions in a monopoly, a duopoly, and a triopoly. This did not come as a surprise, since this was already discovered by Huisman and Kort (2015) and Faninam et al. (2022).

Nonetheless, it appeared promising that such behavior extends to a greater number of potential entrants. We have proven that this is actually the case. An entrant will 'ignore' new entrants and will always invest the same amount at the same time. This is likely due to the linear nature of the inverse demand function. The reason for this is there are two effects present when there is a potential new entrant. Firstly, the firm that enters first in the market will make less since the price will go down due to the increased total output. On the other hand, if the first firm decides to invest more, it can decrease the quantity that the next firm will invest and it can delay its investment. In the linear setting, these two apparently cancel out. Therefore, it seemed interesting to delve into other inverse demand functions to observe this effect.

Subsequently, this was performed, looking into multiple inverse demand functions. Balter et al. (2022) already touched on this subject, however, they did not look into the optimal investment quantity. Here, the duopoly was reconsidered. A problem that arose was that it was not always possible to construct an explicit solution for the optimal investment quantity and investment timing. Therefore, in the first part, the optimal investment timing in a duopoly was fixed to the optimal investment timing in the monopoly setting. This resulted in some interesting insights. Namely, all convex and concave inverse demand functions behave similarly. In the convex setting, the optimal duopoly quantity for the first entrant was found to be higher than the monopoly investment quantity. When dealing with a concave inverse demand function, it was observed that the opposite was true. This was, however, for a fixed investment timing.

Looking at these results, deviating the investment timing for various inverse demand functions seems interesting, mainly focusing on the convexity or concavity of the functions. Therefore, two functions have been selected for which it is possible to determine both the optimal investment timing and size. The inverse demand functions chosen for analysis are those of the quadratic and square-root forms.

It was proven that indeed we were able to compute an optimal investment quantity for the entrants in this duopoly setting. It was also observed that the behavior observed earlier persists. It is too early to conclude that this prevails over all convex and concave functions, this should be mathematically proven. This thesis does not cover this particular aspect, however, our results seem promising.

If we, however, start to examine the effect of some of the parameters, we observe some irregularities. Analyzing the market size and investment costs, α and δ, respectively, we observe results as we would expect, there are no inconsistencies with our earlier results. If we alter the values for parameters μ and r, we do find some irregularities. Taking the difference between the drift term of the geometric Brownian motion and the interest rate too large, we see that, for our chosen convex inverse linear demand function, the duopoly output for the first firm is larger than the monopoly output. This contradicts our expectation that in a convex setting the optimal investment quantity for the first entrant in a duopoly will be lower than in a monopoly. Even though our results become slightly distorted, this does not seem like a major issue, since it is probably not realistic to assume these kinds of parameter values. It is, however, worrying for our general result. It does not completely reject our results, since for presumably more realistic values, we have not found irregularities, but this might be a problem for various other inverse demand functions.

There are other limitations to our research. Firstly, only a deterrence strategy has been examined. Considering the accommodation strategy, looking at the results from Huisman and Kort (2015) and Faninam et al. (2022), we would expect similar results as we have found for the deterrence strategy. This could be examined in future research. Besides this, another drawback of our research is that only exogenous firm order has been dealt with. It seems interesting what will happen to these results when more than three firms can enter the market and firms have the possibility to preempt.

For our more general inverse demand function, there is also some interesting future research. As mentioned before, it might be possible to mathematically prove that our result, that concave and convex functions behave differently, holds in general. For this, some parameter assumptions might have to be made, to avoid the difference between μ and r becoming too large. We have looked into some specific inverse demand functions, but we are not able to say that the result holds in general. It is promising and delving further into the calculations might give some more insights. Also, for the general inverse demand function, only the duopoly has been considered. Extending this model, taking into account more potential entrants, could give some compelling results.

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6 Appendix

6.1 Linear inverse demand function

6.1.1 Monopoly

We want to compute the following maximization problem.

$$V(X) = E\left[\int_{t=0}^{\infty} QX(t)(\alpha - Q)\exp(-rt)dt - \delta Q \middle| X(0) = X\right] \tag{18}$$

First, we maximize with respect to Q. We we get

$$\begin{aligned} V(X) &= E\left[\int_{t=0}^{\infty} QX(t)(\alpha - Q)\exp(-rt)dt - \delta Q \middle| X(0) = X\right] \\ &= \int_{t=0}^{\infty} QE[X(t)|X(0) = X](\alpha - Q)\exp(-rt)dt - \delta Q \end{aligned} \tag{19}$$

Since $X \sim GBM(\mu, \sigma)$, we have $E[X(t)|X(0) = X] = Xe^{\mu t}$. Therefore

$$\begin{aligned} V(X) &= \int_{t=0}^{\infty} QXe^{\mu t}e^{-rt}(\alpha - Q)dt - \delta Q \\ &= \int_{t=0}^{\infty} QXe^{(\mu - r)t}(\alpha - Q)\,dt - \delta Q \\ &= Q(\alpha - Q)X\int_{t=0}^{\infty} e^{(\mu - r)t}dt - \delta Q \\ &= Q(\alpha - Q)X\left[\frac{1}{\mu - r}e^{(\mu - r)t}\right]_{t=0}^{t=\infty} - \delta Q \\ &= Q(\alpha - Q)X\frac{1}{r - \mu} - \delta Q \end{aligned} \tag{20}$$

Now, taking the derivative with respect to Q

$$\begin{aligned} 0 &= (-Q^*X + (\alpha - Q^*)X)\frac{1}{r - \mu} - \delta \\ &= -\frac{2}{r - \mu}Q^*X + \alpha X\frac{1}{r - \mu} - \delta \end{aligned}$$

$$\begin{aligned} \Rightarrow Q^* &= \frac{1}{2}\alpha - \frac{1}{2X}\delta(r - \mu) \\ &= \frac{1}{2}\left(\alpha - \frac{1}{X}\delta(r - \mu)\right) \end{aligned} \tag{21}$$

$F(X)$ is the expected net present value of the cash flow. We have by Huisman and

Kort (2015) that $F(X) = AX^{\beta}$, where β is the positive root of the quadratic polynomial $\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0$.

We apply value matching and smooth pasting

$$F\left(X^*\right) = V\left(X^*, Q\right) \tag{22}$$

$$\left.\frac{\partial F\left(X\right)}{\partial X}\right|_{X=X^*} = \left.\frac{\partial V\left(X, Q\right)}{\partial X}\right|_{X=X^*} \tag{23}$$

We then have

$$\begin{aligned}
\frac{XQ\left(\alpha - Q\right)}{r - \mu} - \delta Q &= AX^{\beta} \\
\Rightarrow \frac{\beta}{X}AX^{\beta} &= \frac{Q\left(\alpha - Q\right)}{r - \mu} \\
\Rightarrow AX^{\beta} &= \frac{X}{\beta}\frac{Q\left(\alpha - Q\right)}{r - \mu} \\
\Rightarrow \frac{XQ\left(\alpha - Q\right)}{r - \mu} - \delta Q &= \frac{X}{\beta}\frac{Q\left(\alpha - Q\right)}{r - \mu} \\
\Rightarrow \left(1 - \frac{1}{\beta}\right)\frac{XQ\left(\alpha - Q\right)}{r - \mu} &= \delta Q \\
\Rightarrow \left(\frac{\beta - 1}{\beta}\right)X &= \frac{\left(r - \mu\right)\delta}{\alpha - Q}
\end{aligned}$$

So, then

$$X^* = \frac{\beta}{\beta - 1}\frac{\left(r - \mu\right)\delta}{\alpha - Q} \tag{24}$$

Filling these values into (21), we obtain

$$\begin{aligned}
Q^* &= \frac{1}{2}\left(\alpha - \frac{\delta\left(r - \mu\right)}{\frac{\beta}{\beta - 1}\frac{(r-\mu)\delta}{\alpha - Q}}\right) \\
Q^* &= \frac{1}{2}\left(\alpha - \frac{\beta - 1}{\beta}\left(\alpha - Q^*\right)\right) \\
2Q^* &= \alpha - \frac{\beta - 1}{\beta}\alpha + \frac{\beta - 1}{\beta}Q^* \\
\left(\frac{2\beta}{\beta} - \frac{\beta - 1}{\beta}\right)Q^* &= \alpha\left(\frac{\beta}{\beta} - \frac{\beta - 1}{\beta}\right) \\
\left(\frac{\beta + 1}{\beta}\right)Q^* &= \frac{\alpha}{\beta} \\
Q^* &= \frac{\alpha}{\beta + 1}
\end{aligned}$$

$$\begin{aligned}
X^* &= \frac{\beta}{\beta-1}\frac{(r-\mu)\,\delta}{\alpha-\frac{\alpha}{\beta+1}} \\
&= \frac{\beta}{\beta-1}\frac{(r-\mu)\,\delta}{\frac{\alpha\beta+\alpha-\alpha}{\beta+1}} \\
&= \frac{\beta}{\beta-1}\frac{(r-\mu)\,\delta}{\beta\frac{\alpha}{\beta+1}} \\
&= \frac{\beta+1}{\beta-1}\frac{(r-\mu)\,\delta}{\alpha}
\end{aligned}$$

6.1.2 Duopoly

We have that Q_1 and Q_2 are the investment size for the leader and the follower, respectively. We then have that the value is equal to:

$$V_2^*\left(X,Q_1,Q_2\right) = \frac{XQ_2\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu} - \delta Q_2 \tag{25}$$

Now, we want to maximize with respect to Q_2

$$\begin{aligned}
\frac{\partial}{\partial Q_2}V_2^*\left(X,Q_1,Q_2\right) &= \frac{X\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu} - \frac{XQ_2}{r-\mu} - \delta \\
\Rightarrow 0 &= \frac{\alpha X - Q_1X - Q_2^*X}{r-\mu} - \frac{XQ_2^*}{r-\mu} - \delta \\
&= \frac{-2XQ_2^* + \alpha X - Q_1X}{r-\mu} - \delta
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
2XQ_2^* &= \left(\alpha-Q_1\right)X - (r-\mu)\,\delta \\
Q_2^* &= \frac{1}{2}\left(\alpha-Q_1\right) - \frac{1}{2}\frac{(r-\mu)\,\delta}{X}
\end{aligned} \tag{26}$$

We have, just as in the monopoly case, that the net present value of the cash flow is of the form $F(X) = AX^\beta$. We, again, use value matching and smooth pasting, from (22) and (23). Therefore,

$$\begin{aligned}
A_2X^\beta &= \frac{XQ_2\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu} - \delta Q_2 \\
\Rightarrow \frac{\beta}{X}A_2X^\beta &= \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu} \\
\Rightarrow A_2X^\beta &= \frac{X}{\beta}\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu}
\end{aligned}$$

Combining these gives the following result

$$\frac{X^* Q_2 (\alpha - (Q_1 + Q_2))}{r - \mu} - \delta Q_2 = \frac{X^*}{\beta} \frac{Q_2 (\alpha - (Q_1 + Q_2))}{r - \mu}$$

$$\left(1 - \frac{1}{\beta}\right) \frac{X^* Q_2 (\alpha - (Q_1 + Q_2))}{r - \mu} = \delta Q_2$$

$$\frac{\beta - 1}{\beta} X^* = \frac{\delta (r - \mu)}{\alpha - (Q_1 + Q_2)}$$

$$X^* = \frac{\beta}{\beta - 1} \frac{\delta (r - \mu)}{\alpha - (Q_1 + Q_2)}$$

Now, plugging in Q_2^* from (26) gives

$$\begin{aligned} X^* &= \frac{\beta}{\beta - 1} \frac{\delta (r - \mu)}{\alpha - \left(\frac{1}{2} (\alpha - Q_1) - \frac{1}{2} \frac{(r-\mu)\delta}{X} + Q_1\right)} \\ &= \frac{\beta}{\beta - 1} \frac{\delta (r - \mu)}{\frac{1}{2} (\alpha - Q_1) + \frac{1}{2} \frac{(r-\mu)\delta}{X^*}} \end{aligned}$$

$$\frac{1}{2} (\alpha - Q_1) X^* + \frac{1}{2} (r - \mu) \delta = \frac{\beta}{\beta - 1} \delta (r - \mu)$$

$$\frac{1}{2} (\alpha - Q_1) X^* = \left(\frac{\beta}{\beta - 1} - \frac{\frac{1}{2} (\beta - 1)}{\beta - 1}\right) \delta (r - \mu)$$

$$\begin{aligned} X_2^* &= \frac{2}{\alpha - Q_1} \left(\frac{\frac{1}{2}\beta + \frac{1}{2}}{\beta - 1}\right) \delta (r - \mu) \\ &= \frac{\beta + 1}{\beta - 1} \delta \frac{r - \mu}{\alpha - Q_1} \end{aligned} \tag{27}$$

Plugging this back into (26) gives the following value for Q_2^*

$$\begin{aligned} Q_2^* &= \frac{1}{2} (\alpha - Q_1) - \frac{1}{2} \frac{(r - \mu) \delta}{\frac{\beta+1}{\beta-1} \delta \frac{r-\mu}{\alpha-Q_1}} \\ &= \frac{1}{2} (\alpha - Q_1) - \frac{1}{2} \frac{\beta - 1}{\beta + 1} (\alpha - Q_1) \\ &= \frac{1}{2} \left(\frac{\beta + 1}{\beta + 1} - \frac{\beta - 1}{\beta + 1}\right) (\alpha - Q_1) \end{aligned}$$

$$Q_2^* = \frac{\alpha - Q_1}{\beta + 1} \tag{28}$$

It follows that a deterrence strategy occurs whenever the leader chooses a capacity level

Q_1 larger than $\hat{Q}_1(X)$ such that

$$\alpha - \hat{Q}_1 = \frac{\beta+1}{\beta-1}\delta\left(r-\mu\right)\frac{1}{X}$$
$$\hat{Q}_1 = \alpha - \frac{\beta+1}{\beta-1}\delta\left(r-\mu\right)\frac{1}{X}$$

Hence, in the complementary case, i.e., $Q_1 \leq \hat{Q}_1$, the follower invests at the same time as the leader.

The value of the leader is defined as follows,

$$\begin{aligned}
&V_1\left(X, Q_1\right) \\
&= \mathbb{E}\left[\int_{t=0}^{T_2^*} Q_1 X\left(t\right)\left(\alpha - Q_1\right) e^{-rt} dt - \delta Q_1 + \int_{t=T_2^*}^{\infty} Q_1 X\left(t\right)\left(\alpha - Q_1 - Q_2^*\right) e^{-rt} dt \middle| X = X\left(0\right)\right] \\
&= \mathbb{E}\left[\int_{t=0}^{\infty} Q_1 X\left(t\right)\left(\alpha - Q_1\right) e^{-rt} dt - \delta Q_1 - \int_{t=T_2^*}^{\infty} Q_1 X\left(t\right) Q_2^* e^{-rt} dt \middle| X = X\left(0\right)\right] \\
&= \underbrace{\int_{t=0}^{\infty} Q_1 \mathbb{E}\left[X\left(t\right)|X = X\left(0\right)\right]\left(\alpha - Q_1\right) e^{-rt} dt - \delta Q_1}_{1} + \underbrace{\mathbb{E}\left[\int_{t=T_2^*}^{\infty} Q_1 X\left(t\right) Q_2^* e^{-rt} dt \middle| X = X\left(0\right)\right]}_{2}
\end{aligned}$$

We will solve these parts separately, however for both we need the solution to $\mathbb{E}\left[X\left(t\right)|X = X\left(0\right)\right]$. It can be proved that if $X \sim \text{GBM}\left(\mu, \sigma\right)$, we have that $\mathbb{E}\left[X\left(t\right)|X = X\left(0\right)\right] = Xe^{\mu t}$.

First we will take a look at the first part

$$\begin{aligned}
&\int_{t=0}^{\infty} Q_1 \mathbb{E}\left[X\left(t\right)|X = X\left(0\right)\right]\left(\alpha - Q_1\right) e^{-rt} dt - \delta Q_1 \\
&= \int_{t=0}^{\infty} Q_1 X\left(\alpha - Q_1\right) e^{(\mu - r)t} dt - \delta Q_1 \\
&= Q_1\left(\alpha - Q_1\right) X \int_{t=0}^{\infty} e^{(\mu - r)t} dt - \delta Q_1 \\
&= Q_1\left(\alpha - Q_1\right) X \left[\frac{1}{\mu - r} e^{(\mu - r)t}\right]_{t=0}^{t=\infty} - \delta Q_1 \\
&= \frac{Q_1\left(\alpha - Q_1\right) X}{r - \mu} - \delta Q_1
\end{aligned}$$

Next, we'll consider the second part

$$\mathbb{E}\left[\int_{t=T_2^*}^{\infty} Q_1 X\left(t\right) Q_2^*\left(Q_1\right) e^{-rt} dt \middle| X\left(0\right) = X\right]$$

We can prove that if $T = \inf(t|x_t \geq x^*, x_0 = x)$ then $\mathbb{E}\left[e^{-rT}\middle|x_0 = x\right] = \left(\frac{x}{x^*}\right)^{\beta_1}$, where $x_t \sim GBM(\mu, \sigma)$, using this we can rewrite,

$$\begin{aligned}
&\mathbb{E}\left[\int_{t=T_2^*}^{\infty} Q_1 X(t) Q_2^*(Q_1) e^{-rt} dt \middle| X(0) = X\right] \\
&= \mathbb{E}\left[e^{rT_2^*}\middle| X(0) = X\right] \mathbb{E}\left[\int_{t=0}^{\infty} Q_1 X(t) Q_2^* e^{-rt} dt \middle| X(0) = X_2^*\right] \\
&= \left(\frac{X}{X_2^*}\right)^{\beta_1} \frac{Q_1 Q_2^* X_2^*}{r-\mu}
\end{aligned}$$

combining these two results, we obtain the following,

$$V_1(X, Q_1) = \frac{Q_1(\alpha - Q_1) X}{r-\mu} - \delta Q_1 - \left(\frac{X}{X_2^*}\right)^{\beta_1} \frac{Q_1 Q_2^* X_2^*}{r-\mu} \tag{29}$$

Now, inserting (27) and (28) into (29) gives

$$\begin{aligned}
&V_1(X^*, Q_1) \\
&= \frac{Q_1(\alpha - Q_1) X}{r-\mu} - \delta Q_1 - \left(X \frac{\beta - 1}{\beta + 1} \frac{1}{\delta} \frac{\alpha - Q_1}{r-\mu}\right)^{\beta} * \frac{Q_1(\alpha - Q_1)}{(\beta+1)(r-\mu)} \frac{\beta+1}{\beta-1} \frac{(r-\mu)}{\alpha - Q_1} \delta \\
&= \frac{Q_1(\alpha - Q_1) X}{r-\mu} - \delta Q_1 - \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{(\alpha - Q_1)}{r-\mu}\right)^{\beta} \frac{\delta Q_1}{\beta - 1}
\end{aligned} \tag{30}$$

Taking the first order condition with respect to Q_1 gives the following result

$$\phi(X, Q_1) = \frac{(\alpha - Q_1) X}{r-\mu} - \frac{Q_1 X}{r-\mu} - \delta - f(X, Q_1) \tag{31}$$

where

$$
\begin{aligned}
&f(X,Q_1)\\
&= -\beta\frac{X(\beta-1)}{\delta(\beta+1)(r-\mu)}\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta-1}\frac{\delta Q_1}{\beta-1}+\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}\frac{\delta}{\beta-1}\\
&= -\beta\frac{X(\beta-1)}{\delta(\beta+1)(r-\mu)}\frac{\delta(\beta+1)(r-\mu)}{X(\beta-1)(\alpha-Q_1)}\frac{\delta Q_1}{\beta-1}\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}\\
&+\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}\frac{\delta}{\beta-1}\\
&= \left(\frac{\delta}{\beta-1}-\frac{\delta Q_1}{(\beta-1)(\alpha-Q_1)}\beta\right)\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}\\
&= \frac{(\alpha-(\beta+1)Q_1)\delta}{(\beta-1)(\alpha-Q_1)}\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}
\end{aligned}
$$

Inserting this result into (31)

$$
\phi(X,Q_1)=\frac{(\alpha-2Q_1)X}{(r-\mu)}-\delta-\frac{\delta(\alpha-(\beta+1)Q_1)}{(\beta-1)(\alpha-Q_1)}\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta}=0 \qquad (32)
$$

Before the leader has invested, so when $X < X_1$, the firm holds an option to invest. The option value is

$$
F_1(X)=A_1X^{\beta}
$$

We want to apply smooth pasting and value matching to

$$
V_1(X^*,Q_1)=\frac{Q_1(\alpha-Q_1)X}{r-\mu}-\delta Q_1-\left(\frac{X}{\delta}\frac{\beta-1}{\beta+1}\frac{(\alpha-Q_1)}{r-\mu}\right)^{\beta}\frac{\delta Q_1}{\beta-1} \qquad (33)
$$

First, note that we can extend the product rule further for function fgh,

$$
(fgh)'=(fg)'h+fgh'=f'gh+gf'h+fgh'
$$

Now, V_1 can be rewritten into

$$
\frac{Q_1(X)(\alpha-Q_1(X))}{r-\mu}X-\delta Q_1(X)-\left(\frac{\beta-1}{\delta(\beta+1)(r-\mu)}\right)^{\beta}\frac{\delta}{\beta-1}\underbrace{X^{\beta}Q_1(X)(\alpha-Q_1(X))^{\beta}}
$$

If we take the derivative of the last part, we get

$$
\beta X^{\beta-1}Q_1(X)(\alpha-Q_1(X))^{\beta}+X^{\beta}\frac{\partial Q_1(X)}{\partial X}(\alpha-Q_1(X))^{\beta}-\beta\frac{\partial Q_1(X)}{\partial X}X^{\beta}Q_1(X)(\alpha-Q_1(X))^{\beta-1}
$$

$$\Rightarrow \left(\frac{\beta}{X} + \frac{\frac{\partial Q_1(X)}{\partial X}}{Q_1(X)} - \beta \frac{\frac{\partial Q_1(X)}{\partial X}}{\alpha - Q_1(X)} \right) \left(X^\beta Q_1(X) (\alpha - Q_1(X))^\beta \right)$$

Now, taking the derivative

$$\begin{aligned} \frac{\partial V_1(X^*, Q_1)}{\partial X} =& \frac{Q_1(\alpha - Q_1)}{r - \mu} + \frac{X \frac{\partial Q_1}{\partial X} (\alpha - Q_1)}{r - \mu} - \frac{X \frac{\partial Q_1(X)}{\partial X} Q_1(X)}{r - \mu} - \delta \frac{\partial Q_1(X)}{\partial X} \\ & - \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{\alpha - Q_2(X)}{r - \mu} \right)^\beta \frac{\delta Q_1(X)}{\beta - 1} \left(\frac{\beta Q_1(\alpha - Q_1) + X \frac{\partial Q_1}{\partial X} (\alpha - Q_1(X)) - \beta X Q_1 \frac{\partial Q_1}{\partial X}}{X Q_1(\alpha - Q_1)} \right) \end{aligned}$$

Because we have (22) and (23), we have

$$V_1(X^*, Q_1) = \frac{X}{\beta} \frac{\partial}{\partial X} V_1(X^*, Q_1)$$

Using this, (33), and (34),

$$\begin{aligned} 0 =& \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \frac{Q_1(\alpha - Q_1) X}{\beta(r - \mu)} - \frac{X^2 \frac{\partial Q_1}{\partial X} (\alpha - 2Q_1)}{\beta(r - \mu)} - \delta Q_1 + \delta \frac{X}{\beta} \frac{\partial Q_1(X)}{\partial X} \\ & + \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{(\alpha - Q_1)}{r - \mu} \right)^\beta \frac{\delta}{\beta - 1} \frac{1}{\beta} \left(\frac{X \frac{\partial Q_1}{\partial X} (\alpha - Q_1) - \beta X Q_1 \frac{\partial Q_1}{\partial X}}{\alpha - Q_1} \right) \\ =& \left(\frac{\beta - 1}{\beta} \right) \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \frac{X^2 \frac{\partial Q_1}{\partial X} (\alpha - 2Q_1)}{\beta(r - \mu)} - \delta Q_1 + \delta \frac{X}{\beta} \frac{\partial Q_1(X)}{\partial X} \\ & + \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{(\alpha - Q_1)}{r - \mu} \right)^\beta \frac{\delta}{\beta - 1} \frac{1}{\beta} \left(\frac{X \frac{\partial Q_1}{\partial X} (\alpha - (1 + \beta) Q_1)}{\alpha - Q_1} \right) \end{aligned}$$

Using equation (32), we can write

$$\left(\frac{X(\beta - 1)(\alpha - Q_1)}{\delta(\beta + 1)(r - \mu)} \right)^\beta = \left(\frac{(\alpha - 2Q_1) X}{r - \mu} - \delta \right) \frac{(\beta - 1)(\alpha - Q_1)}{\delta(\alpha - (\beta + 1) Q_1)}$$

Plugging this in,

$$
\begin{aligned}
0 &= \left(\frac{\beta-1}{\beta}\right)\frac{Q_1\left(\alpha-Q_1\right)X}{r-\mu} - \frac{X^2\frac{\partial Q_1}{\partial X}\left(\alpha-2Q_1\right)}{\beta\left(r-\mu\right)} - \delta Q_1 + \delta\frac{X}{\beta}\frac{\partial Q_1\left(X\right)}{\partial X} \\
&+ \left(\frac{\left(\alpha-2Q_1\right)X}{r-\mu}-\delta\right)\frac{\left(\beta-1\right)\left(\alpha-Q_1\right)}{\delta\left(\alpha-\left(\beta+1\right)Q_1\right)}\frac{\delta}{\beta-1}\frac{1}{\beta}\left(\frac{X\frac{\partial Q_1}{\partial X}\left(\alpha-\left(1+\beta\right)Q_1\right)}{\alpha-Q_1}\right) \\
&= \left(\beta-1\right)\frac{Q_1\left(\alpha-Q_1\right)X}{r-\mu} - \frac{X^2\frac{\partial Q_1}{\partial X}\left(\alpha-2Q_1\right)}{r-\mu} - \beta\delta Q_1 - \delta X\frac{\partial Q_1\left(X\right)}{\partial X} \\
&+ \left(\frac{\left(\alpha-2Q_1\right)X}{r-\mu}-\delta\right)\left(X\frac{\partial Q_1}{\partial X}\right) \\
&= \left(\beta-1\right)\frac{Q_1\left(\alpha-Q_1\right)X}{r-\mu} - \beta\delta Q_1
\end{aligned}
$$

So the resulting equation gives

$$
0 = \left(\beta-1\right)\frac{\left(\alpha-Q_1\right)X}{r-\mu} - \beta\delta \tag{35}
$$

Now, to compute the value for X_1 and Q_1, we combine (32) and (35). First we'll rewrite (35):

$$
\begin{aligned}
\frac{\left(\alpha-Q_1\right)X}{r-\mu} &= \frac{\beta\delta}{\beta-1} \\
\Rightarrow X &= \frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\left(\alpha-Q_1\right)}
\end{aligned}
$$

Which subsequently gives us,

$$
\begin{aligned}
0 &= \frac{\left(\alpha-2Q_1\right)\beta\delta\left(r-\mu\right)}{\left(r-\mu\right)\left(\beta-1\right)\left(\alpha-Q_1\right)} - \delta - \frac{\delta\left(\alpha-\left(\beta+1\right)Q_1\right)}{\left(\beta-1\right)\left(\alpha-Q_1\right)}\left(\frac{\beta\delta\left(r-\mu\right)\left(\beta-1\right)\left(\alpha-Q_1\right)}{\left(\beta-1\right)\left(\alpha-Q_1\right)\left(\beta+1\right)\left(r-\mu\right)\delta}\right)^{\beta} \\
&= \left(\alpha-2Q_1\right)\beta\delta - \delta\left(\beta-1\right)\left(\alpha-Q_1\right) - \left(\frac{\beta}{\beta+1}\right)^{\beta}\delta\left(\alpha-\left(\beta+1\right)Q_1\right) \\
&= \delta\left(\alpha\beta-2\beta Q_1-\alpha\beta+\beta Q_1+\alpha-Q_1\right) - \left(\frac{\beta}{\beta+1}\right)^{\beta}\delta\left(\alpha-\left(\beta+1\right)Q_1\right) \\
&= \delta\left(\alpha-\left(\beta+1\right)Q_1\right) - \left(\frac{\beta}{\beta+1}\right)^{\beta}\delta\left(\alpha-\left(\beta+1\right)Q_1\right)
\end{aligned}
$$

$$
\Rightarrow 0 = \alpha-\left(\beta+1\right)Q_1
$$

From there we can conclude that

$$
Q_1 = \frac{\alpha}{\beta+1} \tag{36}
$$

Inserting this into (35) gives us

$$\begin{aligned}
\beta\delta &= (\beta-1)\frac{(\alpha-Q_1)X}{r-\mu} \\
&= (\beta-1)\frac{\left(\alpha-\frac{\alpha}{\beta+1}\right)X}{r-\mu} \\
&= (\beta-1)\frac{\beta\frac{\alpha}{\beta+1}X}{r-\mu}
\end{aligned}$$

From there we can determine the optimal investment timing in case of a deterrence strategy,

$$X_1 = \frac{\delta}{\alpha}(r-\mu)\frac{\beta+1}{\beta-1} \tag{37}$$

6.1.3 Triopoly

We have for the third entrant,V

$$\begin{aligned}
V_3(X,Q_3) &= \mathbb{E}\left[\int_{t=0}^{\infty} Q_3X(t)(\alpha-(Q_1+Q_2+Q_3))e^{-rt}dt \middle| X(0)=X\right] - \delta Q_3 \\
&= \int_{t=0}^{\infty} Q_3\mathbb{E}\left[X(t)|X(0)=X\right](\alpha-(Q_1+Q_2+Q_3))e^{-rt}dt - \delta Q_3 \\
&= \int_{t=0}^{\infty} Q_3Xe^{\mu t}(\alpha-(Q_1+Q_2+Q_3))e^{-rt}dt - \delta Q_3 \\
&= Q_3X(\alpha-(Q_1+Q_2+Q_3))\int_{t=0}^{\infty} e^{t(\mu-r)}dt - \delta Q_3 \\
&= Q_3X(\alpha-(Q_1+Q_2+Q_3))\left[\frac{1}{\mu-r}e^{t(\mu-r)}\right]_{t=0}^{t=\infty} \\
&= \frac{Q_3X(\alpha-(Q_1+Q_2+Q_3))}{r-\mu} - \delta Q_3
\end{aligned}$$

$$V_3(X,Q_3) = \frac{Q_3X(\alpha-(Q_1+Q_2+Q_3))}{r-\mu} - \delta Q_3 \tag{38}$$

Now, we want to maximize with respect to Q_3

$$\begin{aligned}
\frac{\partial}{\partial Q_3}V_3^*(X,Q_3) &= \frac{X(\alpha-(Q_1+Q_2+Q_3))}{r-\mu} - \frac{XQ_3}{r-\mu} - \delta \\
\Rightarrow 0 &= \frac{X(\alpha-(Q_1+Q_2))}{r-\mu} - 2\frac{XQ_3}{r-\mu} - \delta
\end{aligned}$$

$$\Downarrow$$

$$
\begin{aligned}
2XQ_3^* &= \left(\alpha - \left(Q_1 + Q_2\right)\right) X - \left(r - \mu\right) \delta \\
Q_3^* &= \frac{1}{2}\left(\alpha - \left(Q_1 + Q_2\right)\right) - \frac{1}{2}\frac{\left(r - \mu\right)\delta}{X}
\end{aligned} \tag{39}
$$

We have, just as in the monopoly case, that the net present value of the cash flow is of the form $F(X) = AX^{\beta}$. We, again, use value matching and smooth pasting, from (22) and (23). Therefore,

$$
\begin{aligned}
A_3 X^{\beta} &= \frac{XQ_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu} - \delta Q_3 \\
\Rightarrow \frac{\beta}{X} A_3 X^{\beta} &= \frac{Q_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu} \\
\Rightarrow A_3 X^{\beta} &= \frac{X}{\beta}\frac{Q_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu}
\end{aligned}
$$

Combining these gives the following result

$$
\frac{X^* Q_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu} - \delta Q_3 = \frac{X^*}{\beta}\frac{Q_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu}
$$

$$
\begin{aligned}
\left(1 - \frac{1}{\beta}\right)\frac{X^* Q_3\left(\alpha - \left(Q_1 + Q_2 + Q_3\right)\right)}{r - \mu} &= \delta Q_3 \\
\frac{\beta - 1}{\beta} X^* &= \frac{\delta\left(r - \mu\right)}{\alpha - \left(Q_1 + Q_2 + Q_3\right)} \\
X^* &= \frac{\beta}{\beta - 1}\frac{\delta\left(r - \mu\right)}{\alpha - \left(Q_1 + Q_2 + Q_3\right)}
\end{aligned}
$$

Now, plugging in Q_2^* from (39) gives

$$
\begin{aligned}
X^* &= \frac{\beta}{\beta - 1}\frac{\delta\left(r - \mu\right)}{\alpha - \left(\frac{1}{2}\left(\alpha - \left(Q_1 + Q_2\right)\right) - \frac{1}{2}\frac{\left(r - \mu\right)\delta}{X} + \left(Q_1 + Q_2\right)\right)} \\
&= \frac{\beta}{\beta - 1}\frac{\delta\left(r - \mu\right)}{\frac{1}{2}\left(\alpha - \left(Q_1 + Q_2\right)\right) + \frac{1}{2}\frac{\left(r - \mu\right)\delta}{X^*}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\left(\alpha - \left(Q_1 + Q_2\right)\right) X^* + \frac{1}{2}\left(r - \mu\right)\delta &= \frac{\beta}{\beta - 1}\delta\left(r - \mu\right) \\
\frac{1}{2}\left(\alpha - \left(Q_1 + Q_2\right)\right) X^* &= \left(\frac{\beta}{\beta - 1} - \frac{\frac{1}{2}\left(\beta - 1\right)}{\beta - 1}\right)\delta\left(r - \mu\right)
\end{aligned}
$$

$$
\begin{aligned}
X_3^* &= \frac{2}{\alpha - (Q_1 + Q_2)} \left(\frac{\frac{1}{2}\beta + \frac{1}{2}}{\beta - 1} \right) \delta (r - \mu) \\
&= \frac{\beta + 1}{\beta - 1} \delta \frac{r - \mu}{\alpha - (Q_1 + Q_2)}
\end{aligned}
\tag{40}
$$

Plugging this back into (26) gives the following value for Q_2^*

$$
\begin{aligned}
Q_3^* &= \frac{1}{2} (\alpha - (Q_1 + Q_2)) - \frac{1}{2} \frac{(r - \mu)\delta}{\frac{\beta+1}{\beta-1}\delta \frac{r-\mu}{\alpha-(Q_1+Q_2)}} \\
&= \frac{1}{2} (\alpha - Q_1) - \frac{1}{2} \frac{\beta - 1}{\beta + 1} (\alpha - (Q_1 + Q_2)) \\
&= \frac{1}{2} \left(\frac{\beta + 1}{\beta + 1} - \frac{\beta - 1}{\beta + 1} \right) (\alpha - (Q_1 + Q_2))
\end{aligned}
$$

$$
Q_3^* = \frac{\alpha - (Q_1 + Q_2)}{\beta + 1} \tag{41}
$$

The value of the second entrant is defined as follows,

$$
\begin{aligned}
&V_2 (X, Q_2) \\
&= \mathbb{E} \left[\int_{t=0}^{T_3^*} Q_2 X (t) (\alpha - (Q_1 + Q_2)) e^{-rt} dt - \delta Q_2 + \int_{t=T_3^*}^{\infty} Q_2 X (t) (\alpha - (Q_1 + Q_2 + Q_3)) e^{-rt} dt \middle| X = X (0) \right] \\
&= \mathbb{E} \left[\int_{t=0}^{\infty} Q_2 X (t) (\alpha - (Q_1 + Q_2)) e^{-rt} dt - \delta Q_1 - \int_{t=T_3^*}^{\infty} Q_2 X (t) Q_3^* e^{-rt} dt \middle| X = X (0) \right] \\
&= \underbrace{\int_{t=0}^{\infty} Q_2 \mathbb{E} [X (t) | X = X (0)] (\alpha - (Q_1 + Q_2)) e^{-rt} dt - \delta Q_2}_{1} + \underbrace{\mathbb{E} \left[\int_{t=T_3^*}^{\infty} Q_2 X (t) Q_3^* e^{-rt} dt \middle| X = X (0) \right]}_{2}
\end{aligned}
$$

We will solve these parts separately, however for both we need the solution to $\mathbb{E} [X (t) | X = X (0)]$. It can be proved that if $X \sim \text{GBM} (\mu, \sigma)$, we have that $\mathbb{E} [X (t) | X = X (0)] = X e^{\mu t}$.

First we will take a look at the first part

$$\begin{aligned}
&\int_{t=0}^{\infty} Q_2 \mathbb{E}\left[X\left(t\right)|X=X\left(0\right)\right]\left(\alpha-\left(Q_1+Q_2\right)\right)e^{-rt}dt-\delta Q_1 \\
&=\int_{t=0}^{\infty} Q_2 X\left(\alpha-\left(Q_1+Q_2\right)\right)e^{(\mu-r)t}dt-\delta Q_2 \\
&=Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X\int_{t=0}^{\infty} e^{(\mu-r)t}dt-\delta Q_2 \\
&=Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X\left[\frac{1}{\mu-r}e^{(\mu-r)t}\right]_{t=0}^{t=\infty}-\delta Q_2 \\
&=\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu}-\delta Q_2
\end{aligned}$$

Next, we'll consider the second part

$$\mathbb{E}\left[\int_{t=T_3^*}^{\infty} Q_2 X\left(t\right)Q_3^*\left(Q_2\right)e^{-rt}dt\middle|X\left(0\right)=X\right]$$

We can prove that if $T=\inf\left(t|x_t\geq x^*,x_0=x\right)$ then $\mathbb{E}\left[e^{-rT}\middle|x_0=x\right]=\left(\frac{x}{x^*}\right)^{\beta_1}$, where $x_t\sim GBM\left(\mu,\sigma\right)$, using this we can rewrite,

$$\begin{aligned}
&\mathbb{E}\left[\int_{t=T_2^*}^{\infty} Q_2 X\left(t\right)Q_3^*\left(Q_2\right)e^{-rt}dt\middle|X\left(0\right)=X\right] \\
&=\mathbb{E}\left[e^{rT_3^*}\middle|X\left(0\right)=X\right]\mathbb{E}\left[\int_{t=0}^{\infty} Q_2\left(t\right)Q_3^* e^{-rt}dt\middle|X\left(0\right)=X_3^*\right] \\
&=\left(\frac{X}{X_3^*}\right)^{\beta_1}\frac{Q_2 Q_3^* X_3^*}{r-\mu}
\end{aligned}$$

Combining these two results, we obtain the following,

$$V_2\left(X,Q_2\right)=\frac{Q_2\left(\alpha-\left(Q_1-Q_2\right)\right)X}{r-\mu}-\delta Q_2-\left(\frac{X}{X_3^*}\right)^{\beta_1}\frac{Q_2 Q_3^* X_3^*}{r-\mu} \tag{42}$$

Now, inserting (40) and (41) into (42) gives

$$\begin{aligned}
&V_2\left(X^*, Q_2\right) \\
&= \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right) X}{r-\mu}-\delta Q_2 \\
&- \left(X \frac{\beta-1}{\beta+1} \frac{1}{\delta} \frac{\alpha-\left(Q_1+Q_2\right)}{r-\mu}\right)^\beta * \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)}{(\beta+1)(r-\mu)} \frac{\beta+1}{\beta-1} \frac{(r-\mu)}{\alpha-\left(Q_1+Q_2\right)} \delta \\
&= \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right) X}{r-\mu}-\delta Q_2-\left(\frac{X}{\delta} \frac{\beta-1}{\beta+1} \frac{\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu}\right)^\beta \frac{\delta Q_2}{\beta-1}
\end{aligned} \tag{43}$$

Taking the first order condition with respect to Q_2 gives the following result

$$\phi_2\left(X, Q_2\right)=\frac{\left(\alpha-\left(Q_1+Q_2\right)\right) X}{r-\mu}-\frac{Q_2 X}{r-\mu}-\delta-f\left(X, Q_2\right) \tag{44}$$

where

$$\begin{aligned}
&f\left(X, Q_2\right) \\
&= -\beta \frac{X(\beta-1)}{\delta(\beta+1)(r-\mu)}\left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^{\beta-1} \frac{\delta Q_2}{\beta-1} \\
&+ \left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^\beta \frac{\delta}{\beta-1} \\
&= -\beta \frac{X(\beta-1)}{\delta(\beta+1)(r-\mu)} \frac{\delta(\beta+1)(r-\mu)}{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)} \frac{\delta Q_2}{\beta-1}\left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^\beta \\
&+ \left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^\beta \frac{\delta}{\beta-1} \\
&= \left(\frac{\delta}{\beta-1}-\frac{\delta Q_2}{(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)} \beta\right)\left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^\beta \\
&= \frac{\left(\alpha-\left(Q_1+(\beta+1) Q_2\right)\right) \delta}{(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}\left(\frac{X(\beta-1)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta(\beta+1)(r-\mu)}\right)^\beta
\end{aligned}$$

The option value is

$$F_2(X)=A_2 X^\beta$$

We want to apply smooth pasting and value matching to

$$V_2\left(X^*, Q_2\right)=\frac{Q_2\left(\alpha-Q_2\right) X}{r-\mu}-\delta Q_2-\left(\frac{X}{\delta} \frac{\beta-1}{\beta+1} \frac{\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu}\right)^\beta \frac{\delta Q_2}{\beta-1} \tag{45}$$

First, note that we can extend the product rule further for function fgh,

$$(fgh)' = (fg)' h + fgh' = f'gh + gf'h + fgh'$$

Now, V_2 can be rewritten into

$$\begin{aligned} V_2(X^*, Q_2) = & \frac{Q_2(X)(\alpha - (Q_1 + Q_2)(X))}{r - \mu} X - \delta Q_2(X) \\ & - \left(\frac{\beta - 1}{\delta(\beta + 1)(r - \mu)} \right)^{\beta} \frac{\delta}{\beta - 1} \underbrace{X^{\beta} Q_2(X)(\alpha - (Q_1 + Q_2)(X))^{\beta}} \end{aligned}$$

If we take the derivative of the last part, we get

$$\beta X^{\beta - 1} Q_2 (\alpha - (Q_1 + Q_2))^{\beta} + X^{\beta} \frac{\partial Q_2}{\partial X} (\alpha - Q_2)^{\beta} - \beta \frac{\partial Q_2}{\partial X} X^{\beta} Q_2 (\alpha - (Q_1 + Q_2))^{\beta - 1}$$

$$\Rightarrow \left(\frac{\beta}{X} + \frac{\frac{\partial Q_2}{\partial X}}{Q_2} - \beta \frac{\frac{\partial Q_2}{\partial X}}{\alpha - (Q_1 + Q_2)} \right) \left(X^{\beta} Q_2 (\alpha - (Q_1 + Q_2))^{\beta} \right)$$

Now, taking the derivative

$$\begin{aligned} \frac{\partial V_2(X^*, Q_1)}{\partial X} = & \frac{Q_2(\alpha - (Q_1 + Q_2))}{r - \mu} + \frac{X \frac{\partial Q_2}{\partial X} (\alpha - (Q_1 + Q_2))}{r - \mu} - \frac{X \frac{\partial Q_2(X)}{\partial X} Q_2(X)}{r - \mu} - \delta \frac{\partial Q_2(X)}{\partial X} \\ & - \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{\alpha - (Q_1 + Q_2)(X)}{r - \mu} \right)^{\beta} \frac{\delta Q_2(X)}{\beta - 1} \\ & * \left(\frac{\beta Q_2 (\alpha - (Q_1 + Q_2)) + X \frac{\partial Q_2}{\partial X} (\alpha - (Q_1 + Q_2)(X)) - \beta X Q_2 \frac{\partial Q_2}{\partial X}}{X Q_2 (\alpha - (Q_1 + Q_2))} \right) \end{aligned} \tag{46}$$

Because we have (22) and (23), we have

$$V_1(X^*, Q_1) = \frac{X}{\beta} \frac{\partial}{\partial X} V_1(X^*, Q_1)$$

Using this, (45), and (46),

$$
\begin{aligned}
0 =& \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{\beta\left(r-\mu\right)} - \frac{X^2\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+2Q_2\right)\right)}{\beta\left(r-\mu\right)} - \delta Q_2 \\
&+ \delta\frac{X}{\beta}\frac{\partial Q_2\left(X\right)}{\partial X} + \left(\frac{X}{\delta}\frac{\beta-1}{\beta+1}\frac{\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu}\right)^{\beta}\frac{\delta}{\beta-1}\frac{1}{\beta}\left(\frac{X\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+Q_2\right)\right)-\beta XQ_2\frac{\partial Q_2}{\partial X}}{\alpha-\left(Q_1+Q_2\right)}\right) \\
=& \left(\frac{\beta-1}{\beta}\right)\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \frac{X^2\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+2Q_2\right)\right)}{\beta\left(r-\mu\right)} - \delta Q_2 + \delta\frac{X}{\beta}\frac{\partial Q_2\left(X\right)}{\partial X} \\
&+ \left(\frac{X}{\delta}\frac{\beta-1}{\beta+1}\frac{\left(\alpha-\left(Q_1+Q_2\right)\right)}{r-\mu}\right)^{\beta}\frac{\delta}{\beta-1}\frac{1}{\beta}\left(\frac{X\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+\left(1+\beta\right)Q_2\right)\right)}{\alpha-\left(Q_1+Q_2\right)}\right)
\end{aligned}
$$

Using equation (44), we can write

$$
\left(\frac{X\left(\beta-1\right)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta\left(\beta+1\right)\left(r-\mu\right)}\right)^{\beta} = \left(\frac{\left(\alpha-\left(Q_1+2Q_2\right)\right)X}{r-\mu}-\delta\right)\frac{\left(\beta-1\right)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta\left(\alpha-\left(Q_1+\left(\beta+1\right)Q_2\right)\right)}
$$

Plugging this in,

$$
\begin{aligned}
0 =& \left(\frac{\beta-1}{\beta}\right)\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \frac{X^2\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+2Q_2\right)\right)}{\beta\left(r-\mu\right)} - \delta Q_2 + \delta\frac{X}{\beta}\frac{\partial Q_2\left(X\right)}{\partial X} \\
&+ \left(\frac{\left(\alpha-\left(Q_1+2Q_2\right)\right)X}{r-\mu}-\delta\right)\frac{\left(\beta-1\right)\left(\alpha-\left(Q_1+Q_2\right)\right)}{\delta\left(\alpha-\left(Q_1+\left(\beta+1\right)Q_2\right)\right)}\frac{\delta}{\beta-1}\frac{1}{\beta}\left(\frac{X\frac{\partial Q_2}{\partial X}\left(\alpha-\left(Q_1+\left(\beta+1\right)Q_2\right)\right)}{\alpha-\left(Q_1+Q_2\right)}\right) \\
=& \left(\beta-1\right)\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \frac{X^2\frac{\partial Q_1}{\partial X}\left(\alpha-\left(Q_1+2Q_2\right)\right)}{r-\mu} - \beta\delta Q_2 - \delta X\frac{\partial Q_2\left(X\right)}{\partial X} \\
&+ \left(\frac{\left(\alpha-\left(Q_1+2Q_2\right)\right)X}{r-\mu}-\delta\right)\left(X\frac{\partial Q_2}{\partial X}\right) \\
=& \left(\beta-1\right)\frac{Q_2\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \beta\delta Q_2
\end{aligned}
$$

So the resulting equation gives

$$
0 = \left(\beta-1\right)\frac{\left(\alpha-\left(Q_1+Q_2\right)\right)X}{r-\mu} - \beta\delta \tag{47}
$$

Now, to compute the value for X_1 and Q_1, we combine (44) and (47). First we'll rewrite

(47):

$$\frac{(\alpha - (Q_1 + Q_2))\, X}{r - \mu} = \frac{\beta\delta}{\beta - 1}$$
$$\Rightarrow X = \frac{\beta\delta\,(r-\mu)}{(\beta-1)\,(\alpha - (Q_1+Q_2))}$$

Which subsequently gives us,

$$\begin{aligned}
0 &= \frac{(\alpha - (Q_1 + 2Q_2))\,\beta\delta\,(r-\mu)}{(r-\mu)\,(\beta-1)\,(\alpha-(Q_1+Q_2))} - \delta \\
&\quad - \frac{\delta\,(\alpha - (Q_1 + (\beta+1)\,Q_2))}{(\beta-1)\,(\alpha-(Q_1+Q_2))}\left(\frac{\beta\delta\,(r-\mu)\,(\beta-1)\,(\alpha-(Q_1+Q_2))}{(\beta-1)\,(\alpha-(Q_1+Q_2))\,(\beta+1)\,(r-\mu)\,\delta}\right)^{\beta} \\
&= (\alpha - (Q_1 + 2Q_2))\,\beta\delta - \delta\,(\beta-1)\,(\alpha-(Q_1+Q_2)) - \left(\frac{\beta}{\beta+1}\right)^{\beta}\delta\,(\alpha-(Q_1+(\beta+1)\,Q_2)) \\
&= \alpha\beta - \beta Q_1 - 2\beta Q_2 - \alpha\beta + \beta Q_1 + \beta Q_2 + \alpha - Q_1 - Q_2 - \left(\frac{\beta}{\beta+1}\right)^{\beta}(\alpha-(Q_1+(\beta+1)\,Q_2)) \\
&= \alpha - (Q_1 + (\beta+1)\,Q_2) - \left(\frac{\beta}{\beta+1}\right)^{\beta}(\alpha-(Q_1+(\beta+1)\,Q_2))
\end{aligned}$$
$$\Rightarrow 0 = \alpha - (Q_1 + (\beta+1)\,Q_2)$$

From there we can conclude that

$$Q_2^* = \frac{\alpha - Q_1}{\beta + 1} \tag{48}$$

Inserting this into (47) gives us

$$\begin{aligned}
X &= \frac{(r-\mu)\,\beta\delta}{(\beta-1)\left(\alpha - Q_1 - \frac{\alpha-Q_1}{\beta+1}\right)} \\
&= \frac{(r-\mu)\,\beta\delta}{(\beta-1)\,\frac{\beta}{\beta+1}\,(\alpha-Q_1)}
\end{aligned}$$

Hence, we have

$$X_2^* = (r-\mu)\,\delta\frac{\beta+1}{\beta-1}\frac{1}{\alpha-Q_1} \tag{49}$$

The value of the third entrant is defined as follows,

$$
\begin{aligned}
&V_1(X, Q_1) \\
&= \mathbb{E}\left[\int_{t=0}^{T_2^*} Q_1 X(t)(\alpha - Q_1) e^{-rt} dt - \delta Q_1 + \int_{t=T_2^*}^{T_3^*} Q_1 X(t)(\alpha - (Q_1 + Q_2)) e^{-rt} dt \right.\\
&\qquad \left. + \int_{t=T_3^*}^{\infty} Q_1 X(t)(\alpha - (Q_1 + Q_2 + Q_3)) e^{-rt} \middle| X = X(0) \right] \\
&= \mathbb{E}\left[\int_{t=0}^{\infty} Q_1 X(t)(\alpha - Q_1) e^{-rt} dt - \delta Q_1 - \int_{t=T_2^*}^{\infty} Q_1 X(t) Q_2^* e^{-rt} dt \right.\\
&\qquad \left. - \int_{t=T_3^*}^{\infty} Q_1 X(t) Q_3^* e^{-rt} dt \middle| X(0) = X \right]
\end{aligned}
$$

As we have done before, we can usually compute the expected values and integrals. The value function will become

$$
V_1(X, Q_1) = \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \delta Q_1 - \left(\frac{X}{X_2^*}\right)^{\beta_1} \frac{Q_1 Q_2^* X_2^*}{r - \mu} - \left(\frac{X}{X_3^*}\right)^{\beta_1} \frac{Q_1 Q_3^* X_3^*}{r - \mu} \tag{50}
$$

Now, inserting (40), (41), (48), and (49) into (50) gives

$$
\begin{aligned}
&V_1(X, Q_1) \\
&= \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \delta Q_1 \\
&- \left(X \frac{\beta - 1}{\beta + 1} \frac{1}{\delta} \frac{\alpha - Q_1}{r - \mu}\right)^{\beta} * \frac{Q_1(\alpha - Q_1)}{(\beta + 1)(r - \mu)} \frac{\beta + 1}{\beta - 1} \frac{(r - \mu)}{\alpha - Q_1} \delta \\
&- \left(X \frac{\beta - 1}{\beta + 1} \frac{1}{\delta} \frac{\alpha - (Q_1 + Q_2)}{r - \mu}\right)^{\beta} * \frac{Q_1(\alpha - (Q_1 + Q_2))}{(\beta + 1)(r - \mu)} \frac{\beta + 1}{\beta - 1} \frac{(r - \mu)}{\alpha - (Q_1 + Q_2)} \delta \\
&= \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \delta Q_1 - \left(X \frac{\beta - 1}{\beta + 1} \frac{\alpha - Q_1}{\delta(r - \mu)}\right)^{\beta} \frac{\delta Q_1}{\beta - 1} - \left(X \frac{\beta - 1}{\beta + 1} \frac{\alpha - Q_1 - \frac{\alpha - Q_1}{\beta + 1}}{\delta(r - \mu)}\right)^{\beta} \frac{\delta Q_1}{\beta - 1} \\
&= \frac{Q_1(\alpha - Q_1) X}{r - \mu} - \delta Q_1 - \left(X \frac{\beta - 1}{\beta + 1} \frac{\alpha - Q_1}{\delta(r - \mu)}\right)^{\beta} \frac{\delta Q_1}{\beta - 1} - \left(X \frac{\beta - 1}{(\beta + 1)^2} \frac{\beta(\alpha - Q_1)}{\delta(r - \mu)}\right)^{\beta} \frac{\delta Q_1}{\beta - 1}
\end{aligned}
\tag{51}
$$

Taking the first order condition with respect to Q_1 gives the following result

$$
\phi_1(X, Q_1) = \frac{(\alpha - Q_1) X}{r - \mu} - \frac{Q_1 X}{r - \mu} - \delta - f(X, Q_1) - g(X, Q_1) \tag{52}
$$

We have computed $f(X,Q_1)$ before and from here, $g(X,Q_1)$ can be easily derived as well. So we now have

$$\begin{aligned}\phi_1 =& \frac{X(\alpha-2Q_1)}{r-\mu}-\delta \\ &-\frac{(\alpha-(\beta+1)Q_1)\delta}{(\beta-1)(\alpha-Q_1)}\left(\frac{X(\beta-1)(\alpha-Q_1)}{\delta(\beta+1)(r-\mu)}\right)^{\beta} \\ &-\frac{(\alpha-(\beta+1)Q_1)\delta}{(\beta-1)(\alpha-Q_1)}\left(\frac{X(\beta-1)}{(\beta+1)^2}\frac{\beta(\alpha-Q_1)}{\delta(r-\mu)}\right)^{\beta} \\ &=0\end{aligned} \tag{53}$$

The option value is

$$F_1(X)=A_1X^{\beta}$$

We want to apply smooth pasting and value matching to

$$\begin{aligned}V_1(X,Q_1)=&\frac{Q_1(\alpha-Q_1)X}{r-\mu}-\delta Q_1 \\ &-\left(X\frac{\beta-1}{\beta+1}\frac{\alpha-Q_1}{\delta(r-\mu)}\right)^{\beta}\frac{\delta Q_1}{\beta-1} \\ &-\left(X\frac{\beta-1}{(\beta+1)^2}\frac{\beta(\alpha-Q_1)}{\delta(r-\mu)}\right)^{\beta}\frac{\delta Q_1}{\beta-1}\end{aligned} \tag{54}$$

Now, taking the derivative

$$\begin{aligned}&\frac{\partial V_1(X,Q_1)}{\partial X} \\ &=\frac{Q_1(\alpha-Q_1)}{r-\mu}+\frac{X\frac{\partial Q_1}{\partial X}(\alpha-Q_1)}{r-\mu}-\frac{X\frac{\partial Q_1}{\partial X}Q_1}{r-\mu}-\delta\frac{\partial Q_1}{\partial X} \\ &-\left(\frac{X}{\delta}\frac{\beta-1}{\beta+1}\frac{\alpha-Q_1}{r-\mu}\right)^{\beta}\frac{\delta Q_1}{\beta-1}\left(\frac{\beta Q_1(\alpha-Q_1)+X\frac{\partial Q_1}{\partial X}(\alpha-Q_1)-\beta XQ_1\frac{\partial Q_1}{\partial X}}{XQ_1(\alpha-Q_1)}\right) \\ &-\left(\frac{X}{\delta}\frac{\beta(\beta-1)}{(\beta+1)^2}\frac{\alpha-Q_1}{r-\mu}\right)^{\beta}\frac{\delta Q_1}{\beta-1}\left(\frac{\beta Q_1(\alpha-Q_1)+X\frac{\partial Q_1}{\partial X}(\alpha-Q_1)-\beta XQ_1\frac{\partial Q_1}{\partial X}}{XQ_1(\alpha-Q_1)}\right)\end{aligned} \tag{55}$$

Applying smooth pasting and value matching gives us:

$$
\begin{aligned}
0 =& \frac{Q_1 (\alpha - Q_1) X}{r-\mu} - \frac{1}{\beta} \frac{Q_1 (\alpha - Q_1) X}{(r-\mu)} - \frac{X^2 \frac{\partial Q_1}{\partial X} (\alpha - 2Q_1)}{\beta (r-\mu)} - \delta Q_1 + \delta \frac{X}{\beta} \frac{\partial Q_1 (X)}{\partial X} \\
&+ \left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{(\alpha - Q_1)}{r-\mu} \right)^{\beta} \frac{\delta}{\beta - 1} \frac{1}{\beta} \left(\frac{X \frac{\partial Q_1}{\partial X} (\alpha - (\beta+1) Q_1)}{\alpha - Q_1} \right) \\
&+ \left(\frac{X}{\delta} \frac{\beta (\beta - 1)}{(\beta + 1)^2} \frac{(\alpha - Q_1)}{r-\mu} \right)^{\beta} \frac{\delta}{\beta - 1} \frac{1}{\beta} \left(\frac{X \frac{\partial Q_1}{\partial X} (\alpha - (\beta+1) Q_1)}{\alpha - Q_1} \right) \\
=& \left(\frac{\beta - 1}{\beta} \right) \frac{Q_1 (\alpha - Q_1) X}{r-\mu} - \frac{X^2 \frac{\partial Q_1}{\partial X} (\alpha - 2Q_1)}{\beta (r-\mu)} - \delta Q_1 + \delta \frac{X}{\beta} \frac{\partial Q_1 (X)}{\partial X} \\
&+ \frac{X}{\beta} \frac{\partial Q_1}{\partial X} \left(\left(\frac{X}{\delta} \frac{\beta - 1}{\beta + 1} \frac{(\alpha - Q_1)}{r-\mu} \right)^{\beta} \frac{\delta}{\beta - 1} \left(\frac{(\alpha - (1+\beta) Q_1)}{\alpha - Q_1} \right) \right. \\
&+ \left. \left(\frac{X}{\delta} \frac{\beta (\beta - 1)}{(\beta + 1)^2} \frac{(\alpha - Q_1)}{r-\mu} \right)^{\beta} \frac{\delta}{\beta - 1} \left(\frac{(\alpha - (1+\beta) Q_1)}{\alpha - Q_1} \right) \right)
\end{aligned}
$$

Using equation (52), we can write

$$
\begin{aligned}
\frac{(\alpha - (\beta+1) Q_1) \delta}{(\beta - 1)(\alpha - Q_1)} \left(\frac{X (\beta - 1)(\alpha - Q_1)}{\delta (\beta + 1)(r-\mu)} \right)^{\beta} - \frac{(\alpha - (\beta+1) Q_1) \delta}{(\beta - 1)(\alpha - Q_1)} \left(\frac{X (\beta - 1)}{(\beta+1)^2} \frac{\beta (\alpha - Q_1)}{\delta (r-\mu)} \right)^{\beta} \\
= \frac{X (\alpha - 2Q_1)}{r-\mu} - \delta
\end{aligned}
$$

Plugging this in,

$$
0 = \frac{\beta - 1}{\beta} \frac{X Q_1 (\alpha - Q_1)}{r-\mu} - \frac{X^2 \frac{\partial Q_1}{\partial X} (\alpha - 2Q_1)}{\beta (r-\mu)} - \delta Q_1 + \delta \frac{X}{\beta} \frac{\partial Q_1}{\partial X} + \frac{X}{\beta} \frac{\partial Q_1}{\partial X} \left(\frac{X (\alpha - 2Q_1)}{r-\mu} - \delta \right)
$$

$$
\frac{\beta - 1}{\beta} \frac{X Q_1 (\alpha - Q_1)}{r-\mu} = \delta Q_1
$$

$$
\Rightarrow X = \frac{\delta \beta (r-\mu)}{(\beta - 1)(r-\mu)(\alpha - Q_1)} \tag{56}
$$

Which subsequently gives us,

$$\begin{aligned}
0 =& \frac{(\alpha - 2Q_1)\,\beta\delta\,(r-\mu)}{(r-\mu)\,(\beta-1)\,(\alpha - Q_1)} - \delta \\
&- \frac{\delta\,(\alpha - (\beta+1)\,Q_1)}{(\beta-1)\,(\alpha-Q_1)} \left(\frac{\beta\delta\,(r-\mu)\,(\beta-1)\,(\alpha-Q_1)}{(\beta-1)\,(\alpha-Q_1)\,(\beta+1)\,(r-\mu)\,\delta} \right)^{\beta} \\
&- \frac{\delta\,(\alpha - (\beta+1)\,Q_1)}{(\beta-1)\,(\alpha-Q_1)} \left(\frac{\beta\delta\,(r-\mu)\,(\beta\,(\beta-1))\,(\alpha-Q_1)}{(\beta-1)\,(\alpha-Q_1)\left((\beta+1)^2\right)(r-\mu)\,\delta} \right)^{\beta} \\
=& (\alpha - 2Q_1)\,\beta\delta - \delta\,(\beta-1)\,(\alpha-Q_1) - \left(\frac{\beta}{\beta+1}\right)^{\beta} \delta\,(\alpha - (\beta+1)\,Q_1) \\
&- \left(\frac{\beta}{\beta+1}\right)^{2\beta} \delta\,(\alpha - (\beta+1)\,Q_1) \\
=& \alpha\beta - 2\beta Q_1 - \alpha\beta + \beta Q_1 + \alpha - Q_1 \\
&- \left(\frac{\beta}{\beta+1}\right)^{\beta} (\alpha - (\beta+1)\,Q_1) - \left(\frac{\beta}{\beta+1}\right)^{2\beta} (\alpha - (\beta+1)\,Q_1) \\
&= \alpha - ((\beta+1)\,Q_1) - \left(\frac{\beta}{\beta+1}\right)^{\beta} (\alpha - (\beta+1)\,Q_1) - \left(\frac{\beta}{\beta+1}\right)^{2\beta} (\alpha - (\beta+1)\,Q_1) \\
\Rightarrow& \; 0 = \alpha - (\beta+1)\,Q_1
\end{aligned}$$

From there we can conclude that

$$Q_1^* = \frac{\alpha}{\beta+1} \tag{57}$$

Inserting this into (56) gives us

$$\begin{aligned}
X &= \frac{(r-\mu)\,\beta\delta}{(\beta-1)\left(\alpha - \frac{\alpha}{\beta+1}\right)} \\
&= \frac{(r-\mu)\,\beta\delta}{(\beta-1)\,\frac{\alpha\beta}{\beta+1}}
\end{aligned}$$

Hence, we have

$$X_1^* = (r-\mu)\,\frac{\beta+1}{\beta-1}\frac{\delta}{\alpha}$$

6.1.4 Oligopoly

We will prove the statement by strong induction. Firstly, we will prove that the statement holds for the last entrant n. Next, we will consider firm i, where $1 \le i < n$, assuming that the statement holds for all j such that $i < j \le n$.

For the n-th entrant, we have the following value function

$$
\begin{aligned}
V_n(X, Q_n) &= \mathbb{E}\left[\int_{t=0}^{\infty} Q_n X(t) \left(\alpha - \sum_{j=1}^{n} Q_j\right) e^{-rt} dt \middle| X(0) = X\right] - \delta Q_n \\
&= \int_{t=0}^{\infty} Q_n \mathbb{E}\left[X(t) | X(0) = X\right] \left(\alpha - \sum_{j=1}^{n} Q_j\right) e^{-rt} dt - \delta Q_n \\
&= \int_{t=0}^{\infty} Q_n X e^{\mu t} \left(\alpha - \sum_{j=1}^{n} Q_j\right) e^{-rt} dt - \delta Q_n \\
&= Q_n X \left(\alpha - \sum_{j=1}^{n} Q_j\right) \int_{t=0}^{\infty} e^{(\mu - r)t} dt - \delta Q_n \\
&= Q_n X \left(\alpha - \sum_{j=1}^{n} Q_j\right) \left[\frac{1}{\mu - r} e^{(\mu - r)t}\right]_{t=0}^{t=\infty} - \delta Q_n
\end{aligned}
$$

$$
V_n(X, Q_n) = \frac{Q_n X \left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r - \mu} - \delta Q_n \tag{58}
$$

If we now take the derivative of $V_n(X, Q_n)$ with respect to Q_n we obtain the following,

$$
\begin{aligned}
\frac{\partial V_n(X, Q_n)}{\partial Q_n} &= \frac{X \left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r - \mu} - \frac{Q_n X}{r - \mu} - \delta \\
0 &= \frac{X \left(\alpha - \sum_{j=1}^{n-1} Q_j - 2Q_n\right)}{r - \mu} - \delta \\
\delta(r - \mu) &= X \left(\alpha - \sum_{j=1}^{n-1} Q_j\right) - 2Q_n X
\end{aligned}
$$

$$
Q_n = \frac{1}{2} \left(\alpha - \sum_{j=1}^{n-1} Q_j\right) - \frac{1}{2} \frac{\delta(r - \mu)}{X} \tag{59}
$$

If we again apply smooth pasting and value matching to (58), we obtain

$$\begin{aligned}
AX^{\beta} &= \frac{Q_n X\left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r-\mu} - \delta Q_n \\
\frac{\beta}{X} AX^{\beta} &= \frac{Q_n\left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r-\mu} \\
\frac{1}{\beta}\frac{Q_n X\left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r-\mu} &= \frac{Q_n X\left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r-\mu} - \delta Q_n \\
\delta &= \frac{\beta-1}{\beta}\frac{X\left(\alpha - \sum_{j=1}^{n} Q_j\right)}{r-\mu}
\end{aligned}$$

$$X = \frac{\beta}{\beta-1}\frac{\delta\left(r-\mu\right)}{\alpha - \sum_{j=1}^{n} Q_j} \tag{60}$$

Now, if we insert (59) into (60), we obtain

$$\begin{aligned}
X &= \frac{\beta}{\beta-1}\frac{\delta\left(r-\mu\right)}{\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) - \frac{1}{2}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) + \frac{1}{2}\frac{\delta(r-\mu)}{X}} \\
&= \frac{\beta}{\beta-1}\frac{\delta\left(r-\mu\right)}{\frac{1}{2}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) + \frac{1}{2}\frac{\delta(r-\mu)}{X}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{1}{2}X\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) + \frac{1}{2}\delta\left(r-\mu\right) &= \frac{\beta}{\beta-1}\delta\left(r-\mu\right) \\
X\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) &= \frac{2\beta-\beta+1}{\beta-1}\delta\left(r-\mu\right)
\end{aligned}$$

$$X_n^* = \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha - \sum_{j=1}^{n-1} Q_j} \tag{61}$$

Plugging (61) back into (59) gives us

$$\begin{aligned}
Q_n^* &= \frac{1}{2}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) - \frac{1}{2}\frac{\delta\left(r-\mu\right)}{\frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha-\sum_{j=1}^{n-1}Q_j}} \\
&= \frac{1}{2}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) - \frac{1}{2}\frac{\beta-1}{\beta+1}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) \\
&= \frac{1}{2}\frac{\beta+1-\beta+1}{\beta+1}\left(\alpha - \sum_{j=1}^{n-1} Q_j\right) \\
&= \frac{\alpha - \sum_{j=1}^{n-1} Q_j}{\beta+1}
\end{aligned}$$

So, both Q_n^* and X_n^* coincide with the equations mentioned before. Therefore, we have proven that for $i = n$, the equations holds. So the base case holds.

Now, we will look at i, where $1 \leq i < n$. Since we are using strong induction, we will assume that the equation holds for all k such that $i < k \leq n$. So, for these k, we have

$$X_k^* = \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha - \sum_{j=1}^{k-1} Q_j} \tag{62}$$

$$Q_k^* = \frac{\alpha - \sum_{j=1}^{k-1} Q_j}{\beta+1} \tag{63}$$

The value for the i-th entrant is defined by

$$\begin{aligned}
V_i\left(X, Q_i\right) =& \mathbb{E}\Bigg[\int_{t=0}^{T_{i+1}^*} Q_i X\left(t\right)\left(\alpha - \sum_{k=1}^{i} Q_j\right) e^{-rt}dt - \delta Q_i \\
&+ \sum_{k=i+1}^{n-1}\int_{t=T_k^*}^{T_{k+1}^*} Q_i X\left(t\right)\left(\alpha - \sum_{j=1}^{k} Q_j\right) e^{-rt}dt \\
&+ \int_{t=T_n^*}^{\infty} Q_i X\left(t\right)\left(\alpha - \sum_{j=1}^{n} Q_j\right) e^{-rt}dt \Bigg| X\left(0\right) = X\Bigg] \\
=& \mathbb{E}\Bigg[\int_{t=0}^{\infty} Q_i X\left(t\right)\left(\alpha - \sum_{j=1}^{i} Q_j\right) e^{-rt} - \delta Q_i - \sum_{k=i+1}^{n-1}\int_{t=T_k^*}^{\infty} Q_i X\left(t\right) Q_k e^{-rt}dt \\
&- \int_{t=T_n^*}^{\infty} Q_i X\left(t\right) Q_n e^{-rt}dt \Bigg| X\left(0\right) = X\Bigg]
\end{aligned}$$

$$V_i(X,Q_i)=\mathbb{E}\left[\int_{t=0}^{\infty}Q_iX(t)\left(\alpha-\sum_{j=1}^{i}Q_j\right)e^{-rt}-\delta Q_i\right.$$
$$\left.-\sum_{k=i+1}^{n}\int_{t=T_k^*}^{\infty}Q_iX(t)Q_ke^{-rt}dt\right|X(0)=X\Bigg] \tag{64}$$

We know, from previous calculations, that

$$\mathbb{E}\left[\int_{t=T_k^*}^{\infty}Q_iX(t)Q_ke^{-rt}dt\middle|X(0)=X\right]=\left(\frac{X}{X_k^*}\right)^{\beta}\frac{Q_iQ_k^*X_k^*}{r-\mu}$$

We also have calculated before the following

$$\mathbb{E}\left[\int_{t=0}^{\infty}Q_iX(t)\left(\alpha-\sum_{j=1}^{i}Q_j\right)e^{-rt}dt\middle|X(0)=X\right]=\frac{Q_iX\left(\alpha-\sum_{j=1}^{i}Q_j\right)}{r-\mu}$$

Combining these two results, (62), (63) and (64) gives

$$\begin{aligned}V_i(X,Q_i)=&\frac{Q_iX\left(\alpha-\sum_{j=1}^{i}Q_j\right)}{r-\mu}-\delta Q_i\\&-\sum_{k=i+1}^{n}\left(X\frac{\beta-1}{\beta+1}\frac{\alpha-\sum_{j=1}^{k-1}Q_j}{\delta(r-\mu)}\right)^{\beta}*\frac{Q_i\left(\alpha-\sum_{j=1}^{k-1}Q_j\right)}{(\beta+1)(r-\mu)}\frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha-\sum_{j=1}^{k-1}Q_j}\\=&\frac{Q_iX\left(\alpha-\sum_{j=1}^{i}Q_j\right)}{r-\mu}-\delta Q_i\\&-\sum_{k=i+1}^{n}\left(X\frac{\beta-1}{\beta+1}\frac{\alpha-\sum_{j=1}^{k-1}Q_j}{\delta(r-\mu)}\right)^{\beta}*\frac{\delta Q_i}{\beta-1}\end{aligned} \tag{65}$$

We now want to prove that for all values of $\ell\in N^+$,

$$\alpha-\sum_{j=1}^{i+\ell}Q_j=f_\ell(\beta)\left(\alpha-\sum_{j=1}^{i}Q_j\right) \tag{66}$$

where $f_\ell(\beta)$ is some function of β.

Base case ($\ell=0$): $\alpha-\sum_{j=1}^{i}Q_i=\alpha-\sum_{j=1}^{i}Q_i$

Induction step: Assume now the function holds for ℓ, so that $\alpha-\sum_{j=1}^{i+\ell}Q_j=f_\ell(\beta)(\alpha-$

$\sum_{j=1}^{i+\ell} Q_j)$ holds. We will now consider $\ell + 1$.

$$
\begin{aligned}
\alpha - \sum_{j=1}^{i+\ell+1} Q_j &= \alpha - \sum_{j=1}^{i+\ell} Q_j - Q_{\ell+1} \\
&= f_\ell(\beta)\left(\alpha - \sum_{j=1}^{i+\ell} Q_j\right) - \frac{\alpha - \sum_{j=1}^{i+\ell} Q_j}{\beta+1} \\
&= \frac{f_\ell(\beta)(\beta+1)\left(\alpha - \sum_{j=1}^{i+\ell} Q_j\right) - \left(\alpha - \sum_{j=1}^{i+\ell} Q_j\right)}{\beta+1} \\
&= f_{\ell+1}(\beta)\left(\alpha - \sum_{j=1}^{i+\ell} Q_j\right)
\end{aligned}
$$

The second step is due to our original assumption (63). Therefore, we have now proven by induction that (66) holds.

Inserting this result into (65) gives the following equation for $V_i(X, Q_i)$

$$
\begin{aligned}
V_i(X, Q_i) = &\frac{Q_i X \left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \delta Q_i \\
&- \sum_{k=i+1}^{n} \left(X \frac{\beta-1}{\beta+1} f_{k-i}(\beta) \frac{\alpha - \sum_{j=1}^{i} Q_j}{\delta(r-\mu)} \right)^{\beta} * \frac{\delta Q_i}{\beta-1}
\end{aligned}
$$

(67)

If we now take the derivative with respect to Q_i

$$
\phi_i(X, Q_i) = \frac{X\left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \frac{Q_i X}{r-\mu} - \delta - \sum_{k=i+1}^{n} g(X, Q_i)
$$

where

$$
\begin{aligned}
g\left(X,Q_i\right) = & \left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta} * \frac{\delta}{\beta-1} \\
& -\beta X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{1}{\delta\left(r-\mu\right)}\left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta-1} * \frac{\delta Q_i}{\beta-1} \\
= & \left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta} * \frac{\delta}{\beta-1} \\
& -\frac{\beta}{\alpha-\sum_{j=1}^{i}Q_j}\left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta} * \frac{\delta Q_i}{\beta-1} \\
= & \frac{\delta}{\beta-1}\left(\frac{\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i}{\alpha-\sum_{j=1}^{i}Q_j}\right)\left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta}
\end{aligned}
$$

With this result we get

$$
\begin{aligned}
\phi_i\left(X,Q_i\right) = & \frac{X\left(\alpha-\sum_{j=1}^{i}Q_j-Q_i\right)}{r-\mu}-\delta \\
& -\sum_{k=i+1}^{n}\frac{\delta\left(\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i\right)}{\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i}Q_j\right)}\left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta}=0
\end{aligned}
\tag{68}
$$

Applying value matching and smooth pasting,

$$
V_i\left(X^*,Q_i\right)=\frac{X}{\beta}\frac{\partial}{\partial X}V_i\left(X^*,Q_i\right)
$$

We first determine

$$
\begin{aligned}
\frac{\partial}{\partial X}V_i\left(X,Q_i\right) = & \frac{Q_i\left(\alpha-\sum_{j=1}^{i}Q_j\right)}{r-\mu}+\frac{\frac{\partial Q_i}{\partial X}\left(\alpha-\sum_{j=1}^{i}Q_j\right)X}{r-\mu}-\frac{\frac{\partial Q_i}{\partial X}Q_iX}{r-\mu}-\delta\frac{\partial Q_i}{\partial X} \\
& -\sum_{k=i+1}^{n}\left(\frac{\beta}{X}+\frac{\frac{\partial Q_i}{\partial X}}{Q_i}-\beta\frac{\frac{\partial Q_i}{\partial X}}{\alpha-\sum_{j=1}^{i}Q_j}\right)\frac{\delta Q_i}{\beta-1}\left(X\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta}
\end{aligned}
$$

So then we have

$$
\begin{aligned}
0 = & \frac{Q_i X \left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \frac{1}{\beta}\frac{Q_i X \left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \frac{X^2 \frac{\partial Q_i}{\partial X}\left(\alpha - \sum_{j=1}^{i} Q_j - Q_i\right)}{\beta\left(r-\mu\right)} \\
& - \delta Q_i + \delta \frac{\partial Q_i}{\partial X}\frac{X}{\beta} \\
& + \sum_{k=i+1}^{n}\left(X\frac{\beta-1}{\beta+1} f_{k-i}\left(\beta\right)\frac{\alpha - \sum_{j=1}^{i} Q_j}{\delta\left(r-\mu\right)}\right)^{\beta}\left(\frac{\delta Q_i}{\beta-1}\frac{X}{\beta}\right)\left(\frac{\left(\alpha - \sum_{j=1}^{i} Q_j - \beta Q_i\right)\frac{\partial Q_i}{\partial X}}{Q_i\left(\alpha - \sum_{j=1}^{i} Q_j\right)}\right) \\
= & \left(\frac{\beta-1}{\beta}\right)\frac{Q_i X \left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \frac{X^2 \frac{\partial Q_i}{\partial X}\left(\alpha - \sum_{j=1}^{i} Q_j - Q_i\right)}{\beta\left(r-\mu\right)} - \delta Q_i + \delta\frac{\partial Q_i}{\partial X}\frac{X}{\beta} \\
& + \frac{X}{\beta}\frac{\partial Q_i}{\partial X}\sum_{k=i+1}^{n}\left(X\frac{\beta-1}{\beta+1} f_{k-i}\left(\beta\right)\frac{\alpha - \sum_{j=i}^{i} Q_j}{\delta\left(r-\mu\right)}\right)^{\beta}\left(\frac{\delta\left(\alpha - \sum_{j=1}^{i} Q_j - \beta Q_i\right)}{\left(\beta-1\right)\left(\alpha - \sum_{j=1}^{i} Q_j\right)}\right)
\end{aligned}
$$

Using ϕ_i we can write

$$
\begin{aligned}
0 = \left(\frac{\beta-1}{\beta}\right)\frac{Q_i X \left(\alpha - \sum_{j=1}^{i} Q_j\right)}{r-\mu} - \frac{X^2 \frac{\partial Q_i}{\partial X}\left(\alpha - \sum_{j=1}^{i} Q_j - Q_i\right)}{\beta\left(r-\mu\right)} - \delta Q_i + \delta\frac{\partial Q_i}{\partial X}\frac{X}{\beta} \\
+ \frac{X}{\beta}\frac{\partial Q_i}{\partial X}\left(\frac{X\left(\alpha - \sum_{j=1}^{i} Q_j - Q_i\right)}{r-\mu} - \delta\right) \\
\Rightarrow \left(\frac{\beta-1}{\beta}\right)\frac{X Q_i\left(\alpha - \sum_{j=1}^{i} Q_j\right)}{\delta\left(r-\mu\right)} = \delta Q_i
\end{aligned}
$$

$$
X = \frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\left(\alpha - \sum_{j=1}^{i} Q_j\right)} \tag{69}
$$

Plugging this back into ϕ_i

$$\begin{aligned}
0 =& \frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i}Q_j\right)}\frac{\left(\alpha-\sum_{j=1}^{i}Q_j-Q_i\right)}{r-\mu}-\delta \\
&-\sum_{k=i+1}^{n}\frac{\delta\left(\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i\right)}{\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i}Q_j\right)}\left(\frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i}Q_j\right)}\frac{\beta-1}{\beta+1}f_{k-i}\left(\beta\right)\frac{\alpha-\sum_{j=1}^{i}Q_j}{\delta\left(r-\mu\right)}\right)^{\beta} \\
=& \beta\left(\alpha-\sum_{j=1}^{i}Q_j-Q_i\right)-\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i}Q_j\right) \\
&-\sum_{k=i+1}^{n}\left(\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i\right)\left(\frac{\beta}{\beta+1}f_{k-i}\left(\beta\right)\right)^{\beta} \\
=& \alpha\beta-\sum_{j=1}^{i}\beta Q_j-\beta Q_i-\alpha\beta+\sum_{j=1}^{i}\beta Q_j+\alpha-\sum_{j=1}^{i}Q_j \\
&-\sum_{k=i+1}^{n}\left(\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i\right)\left(\frac{\beta}{\beta+1}f_{k-i}\left(\beta\right)\right)^{\beta} \\
=& \alpha-\sum_{j=1}^{i}Q_j-\beta Q_i-\left(\alpha-\sum_{j=1}^{i}Q_j-\beta Q_i\right)\sum_{k=i+1}^{n}\left(\frac{\beta}{\beta-1}f_{k-i}\left(\beta\right)\right)^{\beta}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \alpha-\sum_{j=1}^{i}Q_j-\beta Q_i=0 \\
&\Rightarrow \alpha-\sum_{j=1}^{i-1}Q_j-\left(1+\beta\right)Q_i=0 \\
&\Rightarrow Q_i^{*}=\frac{\alpha-\sum_{j=1}^{i-1}Q_j}{\beta+1} \qquad (70)
\end{aligned}$$

If we insert (70) into (69), we get

$$\begin{aligned}
X &= \frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\left(\alpha-\sum_{j=1}^{i-1}Q_j-\frac{\alpha-\sum_{j=1}^{i-1}Q_j}{\beta+1}\right)} \\
&= \frac{\beta\delta\left(r-\mu\right)}{\left(\beta-1\right)\frac{\beta\left(\alpha-\sum_{j=1}^{i-1}Q_j\right)}{\beta+1}}
\end{aligned}$$

$$X_i^* = \frac{\beta+1}{\beta-1}\frac{\delta\,(r-\mu)}{\alpha-\sum_{j=1}^{i-1}Q_j} \tag{71}$$

We will now try to prove by induction that the following holds for Q_i^*, based on the recursive function (70),

$$Q_i^* = (\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{\beta+1} \tag{72}$$

First, we will observe the base case, so $i=1$:

$$(\frac{\beta}{\beta+1})^0\frac{\alpha}{\beta+1} = \frac{\alpha}{\beta+1} = Q_1^*$$

Next, we will assume, by induction, that the statement holds for i, hence we have

$$Q_i^* = \frac{\alpha-\sum_{j=1}^{i-1}Q_j}{\beta+1} = (\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{\beta+1}$$

Now, we will observe for $i+1$,

$$\begin{aligned}
Q_{i+1}^* &= \frac{\alpha-\sum_{j=1}^{i}Q_j}{\beta+1} = \frac{\alpha-\sum_{j=1}^{i-1}Q_j}{\beta+1} - \frac{Q_i}{\beta+1} \\
&= (\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{\beta+1} - (\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{(\beta+1)^2} \\
&= (\beta+1-1)(\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{(\beta+1)^2} \\
&= (\frac{\beta}{\beta+1})(\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{\beta+1} \\
&= (\frac{\beta}{\beta+1})^{i}\frac{\alpha}{\beta+1}
\end{aligned}$$

Hence, by induction, we have proven that (72) holds. Combining (71) and (72) gives the following,

$$\begin{aligned}
X_i^* &= \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha-\sum_{j=1}^{i-1}Q_j} \\
&= \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{(\beta+1)(\frac{\alpha-\sum_{j=1}^{i-1}Q_j}{\beta+1})} \\
&= \frac{\delta}{\beta-1}\frac{r-\mu}{(\frac{\beta}{\beta+1})^{i-1}\frac{\alpha}{\beta+1}}
\end{aligned}$$

$$\Rightarrow X_i^* = \frac{\beta+1}{\beta-1}\delta\frac{r-\mu}{\alpha}(\frac{\beta+1}{\beta})^{i-1} \tag{73}$$

6.2 General inverse demand function

In this section, we will consider the following price function

$$P = Xh(Q)$$

Here, $h(Q)$ is an unknown function.

6.2.1 Monopoly

The value function in case of a monopoly is as follows

$$V(X,Q) = \frac{Qh(Q)X}{r-\mu} - \delta Q$$

Firstly, we take the derivative with respect to Q

$$\frac{\partial V(X,Q)}{\partial Q} = \frac{h'(Q)XQ}{r-\mu} + \frac{Xh(Q)}{r-\mu} - \delta = 0 \tag{74}$$

Next, the derivative with respect to X is

$$\frac{\partial V(X,Q)}{\partial X} = \frac{Qh(Q)}{r-\mu}$$

Applying value matching and smooth pasting

$$\begin{aligned}
0 &= \frac{Xh(Q)Q}{r-\mu} - \frac{X}{\beta}\frac{Qh(Q)}{r-\mu} - \delta Q \\
\left(\frac{\beta-1}{\beta}\right)\frac{h(Q)X}{r-\mu} &= \delta \\
X &= \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q)} \qquad (75)
\end{aligned}$$

Now, inserting (75) into (74) gives us

$$\begin{aligned}
\frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{h(Q)}\frac{h'(Q)Q+h(Q)}{r-\mu} - \delta = 0 \\
\frac{\beta}{\beta-1}\frac{h'(Q)Q+h(Q)}{h(Q)} = 1 \\
\beta\left(Qh'(Q)+h(Q)\right) = (\beta-1)h(Q) \\
\beta Qh'(Q) + h(Q) = 0 \qquad (76)
\end{aligned}$$

The equations (76) together with (75) make up the investment decision in the monopoly case.

6.2.2 Duopoly

We will now consider a duopoly. The value function is defined as

$$V_2(X, Q) = \frac{XQ_2h(Q_1 + Q_2)}{r - \mu} - \delta Q_2$$

taking the derivative with respect to Q_2 we obtain

$$\begin{aligned}\frac{\partial V_2(X, Q)}{\partial Q_2} &= \frac{X_2h(Q_2 + Q_2)}{r - \mu} + \frac{X_2Q_2h'(Q_1 + Q_2)}{r - \mu} - \delta \\ &= X\frac{h(Q_1 + Q_2) + Q_2h'(Q_1 + Q_2)}{r - \mu} - \delta \\ &= 0\end{aligned}$$

Applying value matching and smooth pasting

$$\begin{aligned}0 &= \frac{X_2Q_2h(Q_1 + Q_2)}{r - \mu} - \frac{X_2}{\beta}\frac{Q_2h(Q_1 + Q_2)}{r - \mu} - \delta Q_2 \\ &= \frac{\beta - 1}{\beta}\frac{X_2h(Q_1 + Q_2)}{r - \mu} - \delta\end{aligned}$$

$$X_2 = \frac{\beta}{\beta - 1}\frac{\delta(r - \mu)}{h(Q_1 + Q_2)} \tag{77}$$

We can simplify this further by plugging in X_2

$$\begin{aligned}\frac{\beta}{\beta - 1}\frac{\delta(r - \mu)}{h(Q_1 + Q_2)}\frac{h(Q_1 + Q_2) + Q_2h'(Q_1 + Q_2)}{r - \mu} - \delta = 0 \\ \frac{\beta}{\beta - 1}\frac{h(Q_1 + Q_2) + Q_2h'(Q_1 + Q_2)}{h(Q_1 + Q_2)} - 1 = 0 \\ \beta(h(Q_1 + Q_2) + Q_2h'(Q_1 + Q_2)) = (\beta - 1)h(Q_1 + Q_2) \\ \beta Q_2h'(Q_1 + Q_2) + h(Q_1 + Q_2) = 0\end{aligned} \tag{78}$$

Using (78) and (77), we occasionally are able to find an explicit solution for Q_2, which is actually necessary for us to find the optimal decision for the first entrant.

Let us assume that we have found an explicit formula for Q_2 which only depends on Q_1. We then have the following value function for the first entrant

$$
\begin{aligned}
V_1(Q,X) =& \mathbb{E}\Bigg[\int_{t=0}^{T_2^*} Q_1 X(t) h(Q_1) e^{-rt} dt - \delta Q_1 \\
& + \int_{t=T_2^*}^{\infty} Q_1 X(t) h(Q_1+Q_2) e^{-rt} dt \Bigg| X(0)=X\Bigg] \\
=& \mathbb{E}\Bigg[\int_{t=0}^{\infty} Q_1 X(t) h(Q_1) e^{-rt} dt - \delta Q_1 \\
& - \int_{t=T_2^*}^{\infty} Q_1 X(t) \left(h(Q_1+Q_2) - h(Q_1)\right) e^{-rt} dt \Bigg| X(0)=X\Bigg] \\
=& \frac{Q_1 h(Q_1) X_1}{r-\mu} - \delta Q_1 - \left(\frac{X_1}{X_2}\right)^{\beta} X_2 \frac{Q_1 \left(h(Q_1+Q_2) - h(Q_1)\right)}{r-\mu}
\end{aligned}
$$

Now, by (78), we know that Q_2 only depends on Q_1. Similarly, since X_2 only depends on Q_1 and Q_2, which in turn only depends on Q_1, we know that X_2 only depends on Q_1.

Therefore, we will replace the last part of the equation by $g(Q_1)$. Hence, we will remain with the following value function.

$$
V_1(Q,X) = \frac{Q_1 h(Q_1) X_1}{r-\mu} - \delta Q_1 - X_1^{\beta} g(Q_1) \tag{79}
$$

where

$$
g\left(Q_1\right) = X_2^{1-\beta} \frac{Q_1 \left(h(Q_1+Q_2) - h(Q_1)\right)}{r-\mu} \tag{80}
$$

First, we will take the derivative with respect to Q_1

$$
\begin{aligned}
\frac{\partial V_1}{\partial Q_1} &= \frac{h\left(Q_1\right) X_1}{r-\mu} + \frac{h'\left(Q_1\right) Q_1 X_1}{r-\mu} - \delta + X_1^{\beta} \frac{\partial}{\partial Q_1} g\left(Q_1\right) \\
&= \frac{\left(h\left(Q_1\right) + h'\left(Q_1\right) Q_1\right)}{r-\mu} X_1 - \delta + X_1^{\beta} \frac{\partial}{\partial Q_1} g\left(Q_1\right) = 0
\end{aligned} \tag{81}
$$

Next, we will apply value matching and smooth pasting. In order to do that, we will need the derivative with respect to X_1

$$\begin{aligned}\frac{\partial V}{\partial X_1} =&\frac{Q_1 h\left(Q_1\right)}{r-\mu}+\frac{\frac{\partial Q_1}{\partial X_1} h\left(Q_1\right) X_1}{r-\mu}+\frac{\frac{\partial Q_1}{\partial X_1} h_1^{\prime}\left(Q_1\right) Q_1 X_1}{r-\mu} \\ & -\delta \frac{\partial Q_1}{\partial X_1}+\beta X_1^{\beta-1} g\left(Q_1\right)+\frac{\partial Q_1}{\partial X_1} \frac{\partial}{\partial Q_1} g\left(Q_1\right) X_1^\beta \\ =&\frac{Q_1 h\left(Q_1\right)}{r-\mu}+\frac{\partial Q_1}{\partial X_1} X_1 \frac{h\left(Q_1\right)+Q_1 h^{\prime}\left(Q_1\right)}{r-\mu}-\delta \frac{\partial Q_1}{\partial X_1} \\ & +X_1^\beta\left(\frac{\beta}{X_1} g(Q_1)+\frac{\partial Q_1}{\partial X_1} \frac{\partial}{\partial Q_1} g\left(Q_1\right)\right)\end{aligned}$$

Now, applying it gives us

$$\begin{aligned}0 =&\frac{Q_1 h(Q_1) X_1}{r-\mu}-\frac{X_1}{\beta} \frac{Q_1 h(Q_1)}{r-\mu}-\frac{X_1^2}{\beta} \frac{\partial Q_1}{\partial X_1} \frac{h(Q_1)+Q_1 h^{\prime}(Q_1)}{r-\mu} \\ & -\delta Q_1+\delta \frac{\partial Q_1}{\partial X_1} \frac{X_1}{\beta}+X_1^\beta g\left(Q_1\right)-X_1^\beta\left(g\left(Q_1\right)+\frac{X_1}{\beta} \frac{\partial Q_1}{\partial X_1} \frac{\partial}{\partial Q_1} g\left(Q_1\right)\right)\end{aligned} \tag{82}$$

We can rewrite (81) as

$$X_1^\beta=\frac{-\frac{h(Q_1)+h^{\prime}(Q_1) Q_1}{r-\mu} X_1+\delta}{\frac{\partial}{\partial Q_1} g\left(Q_1\right)}$$

Inserting this result into (82) yields

$$\begin{aligned}0 =&\left(\frac{\beta-1}{\beta}\right) \frac{Q_1 h(Q_1) X_1}{r-\mu}-\frac{X_1^2}{\beta} \frac{\partial Q_1}{\partial X_1} \frac{h(Q_1)+Q_1 h^{\prime}(Q_1)}{r-\mu}-\delta Q_1+\delta \frac{\partial Q_1}{\partial X_1} \frac{X_1}{\beta} \\ & -\left(\frac{-\frac{h(Q_1)+h^{\prime}(Q_1) Q_1}{r-\mu} X_1+\delta}{\frac{\partial}{\partial Q_1} g(Q_1)}\right) \frac{X_1}{\beta} \frac{\partial Q_1}{\partial X_1} \frac{\partial}{\partial Q_1} g(Q_1) \\ =&\frac{\beta-1}{\beta} \frac{Q_1 h(Q_1) X_1}{r-\mu}-\delta Q_1\end{aligned}$$

$$\begin{aligned}\delta &=\frac{\beta-1}{\beta} \frac{h(Q_1) Q_1}{r-\mu} \\ X_1 &=\frac{\beta \delta(r-\mu)}{(\beta-1) h(Q_1)}\end{aligned} \tag{83}$$

Inserting (83), (80), (77), and (78) into (81) gives us an equation that only depends on Q_1. Now, we can compute the optimal value for Q_1, and using this result, we can compute X_1, X_2, and Q_2.

6.3 Convex vs. Concave

6.3.1 Quadratic price inverse demand function

Let's assume a price function

$$P = X(\alpha - Q^2)$$

6.3.1.1 Monopoly

From the price function we can deduct that $h(Q) = \alpha - Q^2$. We can now easily compute the monopoly value by inserting this function into (76). This gives us the following

$$\begin{aligned} -2\beta Q^2 + \alpha - Q^2 &= 0 \\ \alpha &= (1+2\beta)Q^2 \\ Q &= \sqrt{\frac{\alpha}{1+2\beta}} \end{aligned} \tag{84}$$

We will ignore the negative root, since the investment quantity must be positive. Now, applying this to (75)

$$\begin{aligned} X &= \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha - \frac{\alpha}{1+2\beta}} \\ &= \frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\frac{2\alpha\beta}{1+2\beta}} \\ &= \frac{1+2\beta}{\beta-1}\frac{\delta(r-\mu)}{2\alpha} \end{aligned} \tag{85}$$

6.3.1.2 Duopoly

This this end, we first want to compute an equation for Q_2 and X_2. We can do this by inserting our function for $h(Q)$ into (78). This results in

$$
\begin{aligned}
0 &= \beta Q_2 \left(-2\left(Q_1+Q_2\right)\right) + \alpha - \left(Q_1+Q_2\right)^2 \\
&= -2\beta Q_1 Q_2 - 2\beta Q_2^2 + \alpha - Q_1^2 - 2Q_1Q_2 - Q_2^2 \\
&= (2\beta+1)Q_2^2 + (2\beta+2)Q_1Q_2 - \alpha + Q_1^2 \\
&= Q_2^2 + \frac{2\beta+2}{2\beta+1}Q_1Q_2 - \frac{\alpha - Q_1^2}{2\beta+1} \\
&= \left(Q_2^2 + \frac{\beta+1}{2\beta+1}Q_1\right)^2 - \frac{\left(\beta+1\right)^2}{(2\beta+1)^2}Q_1^2 - \frac{\alpha-Q_1^2}{2\beta+1} \\
&= \left(Q_2 + \frac{\beta+1}{2\beta+1}Q_1\right)^2 - \frac{\left(\beta^2+2\beta+1\right)Q_1^2}{(2\beta+1)^2} - \frac{(2\beta+1)\alpha-(2\beta+1)Q_1^2}{(2\beta+1)^2} \\
&= \left(Q_2 + \frac{\beta+1}{2\beta+1}Q_1\right)^2 - \frac{\beta^2 Q_1^2 + (2\beta+1)\,\alpha}{(2\beta+1)^2}
\end{aligned}
$$

We will again ignore the negative root, since the Q_1 and Q_2 both must be positive.

$$
\begin{aligned}
Q_2 + \frac{\beta+1}{2\beta+1}Q_1 &= \frac{\sqrt{\beta^2 Q_1^2 + (2\beta+1)\,\alpha}}{2\beta+1} \\
Q_2 &= \frac{\sqrt{\beta^2 Q_1^2 + (2\beta+1)\alpha}}{2\beta+1} - \frac{\beta+1}{2\beta+1}Q_1 \qquad (86)
\end{aligned}
$$

Now, using (83), (77), (86), (81), and (80), we can compute the optimal value for Q_1, and therefore for all decision variables.

6.3.2 Square-root inverse demand function

Next, we will consider the case where $h(Q) = \alpha - \sqrt{Q}$, hence the price function is as follows

$$P = X(\alpha - \sqrt{Q})$$

6.3.2.1 Monopoly

We can now easily compute the monopoly value by inserting this function into (76). This gives us the following

$$\begin{aligned} \beta Q\left(-\frac{1}{2}Q^{-\frac{1}{2}}\right)+\alpha-\sqrt{Q}&=0 \\ \alpha&=\left(1+\frac{1}{2}\beta\right)\sqrt{Q} \\ Q&=\left(\frac{\alpha}{1+\frac{1}{2}\beta}\right)^2 \end{aligned} \tag{87}$$

Now, applying this to (75)

$$\begin{aligned} X&=\frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha-\sqrt{Q}} \\ &=\frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha-\frac{\alpha}{1+\frac{1}{2}\beta}} \\ &=\frac{\beta}{\beta-1}\frac{\delta(r-\mu)}{\frac{\frac{1}{2}\alpha\beta}{1+\frac{1}{2}\beta}} \\ &=\frac{2+\beta}{\beta-1}\frac{\delta(r-\mu)}{\alpha} \end{aligned} \tag{88}$$

6.3.2.2 Duopoly

This this end, we first want to compute an equation for Q_2 and X_2. We can do this by inserting our function for $h(Q)$ into (78). This results in

$$
\begin{aligned}
-\frac{1}{2}\frac{\beta Q_2}{(Q_1+Q_2)^{\frac{1}{2}}} + \alpha - (Q_1+Q_2)^{\frac{1}{2}} = 0 \\
-\frac{1}{2}\beta Q_2 + \alpha(Q_1+Q_2)^{\frac{1}{2}} - (Q_1+Q_2) = 0 \\
-\frac{1}{2}\beta(Q_1+Q_2) + \alpha(Q_1+Q_2)^{\frac{1}{2}} - (Q_1+Q_2) \\
-(\frac{1}{2}\beta+1)(Q_1+Q_2) + \alpha(Q_1+Q_2)^{\frac{1}{2}} = -\frac{1}{2}\beta Q_1 \\
(Q_1+Q_2) - \frac{\alpha}{\frac{1}{2}\beta+1}(Q_1+Q_2)^{\frac{1}{2}} = \frac{\frac{1}{2}\beta Q_1}{\frac{1}{2}\beta+1} \\
(Q_1+Q_2) - \frac{2\alpha}{\beta+2}(Q_1+Q_2)^{\frac{1}{2}} = \frac{\beta Q_1}{\beta+2} \\
\left(\sqrt{Q_1+Q_2} - \frac{\alpha}{\beta+2}\right)^2 - \frac{\alpha^2}{(\beta+2)^2} = \frac{\beta Q_1}{\beta+2} \\
\left(\sqrt{Q_1+Q_2} - \frac{\alpha}{\beta+2}\right)^2 = \frac{\alpha^2+\beta^2 Q_1 + 2\beta Q_1}{(\beta+2)^2} \\
\sqrt{Q_1+Q_2} - \frac{\alpha}{\beta+2} = \frac{\sqrt{\alpha^2+\beta^2 Q_1+2\beta Q_1}}{\beta+2} \\
Q_1+Q_2 = \left(\frac{\sqrt{\alpha^2+(\beta^2+2\beta)Q_1}+\alpha}{\beta+2}\right)^2
\end{aligned}
$$

$$
Q_2 = \frac{2\alpha^2 + 2\alpha\sqrt{\alpha^2+(\beta^2+2\beta)Q_1} + (2\beta+4)Q_1}{(\beta+2)^2} \tag{89}
$$

Now, using (83), (77), (89), (81), and (80), we can compute the optimal value for Q_1, and therefore for all decision variables.