A Real Options Analysis of Regime Switching and Lead Time on the Investment Decision of a Monopoly

by

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Abstract

This thesis analyzes a real options investment model of a monopolist that encounters lead time to install the investment and regime switches allowing a variable, irreversible, one-time investment capacity. The objective is to investigate the effect of these extensions on the investment decision by studying their effects in two separate models and one combined model. Lead time is assumed to follow a known distribution and regime switches occur randomly based on constant rate parameters. Hence, the real options model includes stochastic installation time as well as stochastic regime switches and has led to the following conclusions.

First, an increase in the expectation of lead time for a no-switch model postpones the investment while the investment capacity remains constant. An increase in the volatility of lead time decreases the investment trigger. Yet, the effect of the volatility of lead time on the timing is inferior to the effect of the expectation of lead time.

Second, regime-switching affects the investment decision. The investment capacity depends on the long-term investment strategy of the firm as the firm can only invest once. However, the effect of regime switch rates on the investment trigger is contingent on the magnitude of the rates. Low rates prioritize the current regime in the investment strategy. Since the long-term perspective is captured by the investment capacity, the investment capacities are altered to the current regime at the expense of the timing. On the contrary, high switch rates result in the firm emphasizing the long-term. The net present value of the investment is now preeminent in both dimensions of the investment decision.

In the combination of both extensions, the investment capacity is adjusted to mitigate the risk of regime switches occurring in the installation period. Though, this effect is limited as the investment capacity emphasizes the long-term perspective of the investment. Similar to the no-regime switch model, the investment is postponed to compensate for the loss in net present value due to the lead time. Yet, an additional effect of lead time on the effect of the regime switch rates is observed. Namely, at large lead times, the net present value of the investment becomes more prevalent in the timing of the investment.

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Contents

1 Introduction

In the past, a mathematical understanding of the optimal strategic investment decision of firms in a dynamic framework has been studied extensively. One of the most prominent works in literature is Investment under Uncertainty by Dixit and Pindyck (1994). They explain the real options model on how firms mathematically decide on entering and expanding in markets by several dimensions such as new capital equipment, workforce, or the development of new products. Based on their work, many researchers further extend the models by Dixit and Pindyck to study the optimal investment strategy of firms in real-life situations. This thesis specifically builds on the model by Huisman and Kort (2015) in which the timing and capacity of the investment are studied for entry deterrence and accommodation in a duopoly framework. It extends the monopoly model of Huisman and Kort (2015) to include present-day challenges for firms that are not considered yet.

One of the most recent events in markets is geopolitical unrest as a result of the current tensions between nations such as the USA and China (see e.g. Nellis et al. (2023) on how the Biden administration aims to hinder the tech industry in China). However, the tech industry is not the only industry that is prone to market alterations. Another recent development on the international stage with economic consequences is the special operation of Russia in Ukraine that has been ongoing for over a year. In particular, the European Union (E.U.C., 2023) and USA (Treasury, 2023) impose sanctions on the Russian natural-resource-rich economy resulting in a shift in the equilibrium of markets compared to the situation before the Russian invasion. Hence, including additional market uncertainties in the investment decision is needed from a practical perspective.

Altered market dynamics may affect the investment decision of a firm and it has piqued the interest of researchers over the years. Several papers explore the effect of geopolitical unrest on industries, such as tourism (Webster and Ivanov, 2015) and water reserves (Al-Masri et al., 2021). Both papers conclude that geopolitical instability affects these industries. Therefore, it is significant to include these uncertainties in real options models. Thus, the first objective of this thesis is to derive the investment decision in a real options model that includes additional market uncertainties due to geopolitical unrest.

There are multiple ways to include extra uncertainties that are in line with geopolitical unrest in the investment decision. For example, Lin and Huang (2010) used an adjusted stochastic process for the occurrence of unexpected negative events, such as irreversible climate change, on shocks of the stochastic process. In their paper, future discounted benefits in the mining industry are uncertain. Their formula implies that the revenue is negatively affected by an impacted amount whenever an event occurs. Their model coincides with the effect of geopolitical factors on the revenue of the firm as geopolitical unrest may decrease the expected revenue by a certain amount. Therefore, the market shrinks because of geopolitical unrest, and the firm's margins decrease.

However, it may also be that geopolitical factors are positive. For example, the USA has implemented the Inflation Reduction Act to promote local investments in clean energy technology, manufacturing, and innovation (Podesta, 2023). Hence, the USA aims for more investments in the local economy. The effect of geopolitical unrest could thus be more in line with the model of Daming et al. (2014), who used a positive jump that corresponds with radical technological innovation for a duopoly. They also included a waiting time for the investment similar to an installation time of the investment. However, they did not find an expression for optimal capacity in their model as their demand function only depends on technological innovation and is not flexible like in the model of Huisman and Kort (2015). Furthermore, Daming et al. (2014) used an approximated diffusion process for the jump-diffusion process with adjusted parameters. This yields an analytical result, though, the precision of this analytical solution is questionable. Martzoukos and Trigeorgis (2002) have shown that this method undervalues the option if the jump mean of different classes of jumps varies significantly around their average value. Therefore, they proposed a new method that numerically estimates the option value.

Yet, a shortcoming of including a jump in the stochastic process is that the duration of the alteration is ephemeral whereas geopolitical unrest could persist over longer periods. Another way to include geopolitical unrest in a real options model is by switching the parameters of the stochastic process. Such a process is called a regime switch model (see for example Bensoussan et al. (2017) for a regime switch model for a duopoly with a follower and a leader, or Nishihara (2020) for a model where one regime is more favorable for the firm than the other regime). In the current literature, the continuous regime-switching model is studied by Guo et al. (2005). They derived a real options model that combines both optimal investment capacity and regime switching. However, they used an iso-elastic demand function whereas Huisman and Kort (2015) used a linear demand function. Furthermore, Guo et al. (2005) allowed for a lumpy and incremental investment capacity. Hence, they tolerated a correction of the investment capacity if it turns out to be sub-optimal. Therefore, a one-time, irreversible investment decision for a regime-switching real options model using a linear demand function is studied in this thesis. Such a model has not been studied yet and might be relevant to investigate further.

In Balter et al. (2023), a real options model with a single regime switch is derived recently with similar assumptions as in this thesis. In their paper, they studied the investment decision of a firm that produces a product that follows a product life cycle. A single regime switch model is used to model the life cycle of a product. This single-switch model arises as a simplification of a continuous regime-switching model. Namely, if one of the regime-switch parameters in the continuous regime-switch model is approximately zero, the regime-switch model becomes equivalent to a single regime-switch model. Therefore, this model is also studied in the section that derives the investment decision taking geopolitical unrest into account. Furthermore, it is studied how their single regime switch model relates to the continuous regime switch model that corresponds to the geopolitical unrest model.

Moreover, other factors than geopolitical unrest could alter market dynamics. A recent example of such an event is the COVID pandemic. During this pandemic, the demand for computer chips soared while lead times increased (King et al., 2021). Since firms in the semiconductor industry could not respond adequately to these changes in the investment environment, big shortages in computer chips followed (Sweney, 2021). Therefore, it is essential to expand the current understanding of the optimal investment strategy of firms by including lead time in combination with altered market dynamics.

In the current literature, several researchers have used real options models to include lead time in the investment decision. Balliauw (2021) looked into expanding the capacity of a port. Though, he used numerical approximation to include lead time in the investment decision whereas this thesis aims to find an explicit expression for the optimal investment decision that includes an installation time of the investment. A researcher who found an analytical answer to the investment problem was Grenadier (2000), although, they both analyzed the expansion capacity of a firm in an already existing market.

Similar to the monopoly model of Huisman and Kort (2015), the emphasis of this thesis is placed on determining the irreversible, one-time investment decision to enter a new market and include lead time and changing market dynamics which have been not considered yet simultaneously. Kauppinen et al. (2018) derived a model that included lead time and stochastic revenues and operating costs. Nevertheless, they did not include a flexible investment capacity. Furthermore, they could only derive a numerical solution to the investment problem, because they included both a variable operating cost and a variable price of the project. Therefore, in the literature, there are no studies that derive an explicit expression for a real options problem that includes only lead time in the one-time, irreversible investment decision in the dimensions of timing and capacity for a monopoly. The second aim of this thesis is thus to derive an explicit expression for the investment capacity and timing of the investment by including lead time.

Installation time and regime switching as extensions of the model by Huisman and Kort (2015) individually affect the investment decision as they change the kind of uncertainty in the real options model. However, these independent effects may not be the only effects present in the investment decision of the firm if these extensions are applied separately. A combined model is therefore derived and analyzed to find the additional relations between the extension parameters and the investment decision. This model including both geopolitical unrest and time to build is the main topic of this thesis as these extensions combined have not been considered yet in a real options model but are relevant from a practical and theoretical perspective.

In this thesis, it is found that if installation time is the only considered extension in the investment decision, the investment trigger changes while the investment capacity remains constant. If lead time is deterministic, an increase in the lead time postpones the investment. When lead time is stochastic, an increase in the expected lead time increases the investment trigger, similar to the effect of deterministic lead time. Yet, if the uncertainty of the building time increases, the investment trigger decreases as a result of the convexity of the payoff function. This increase in investment trigger coincides with the discount factor corrected for the drift rate over the lead time. Hence, the net present value of the revenue of the investment when the investment is installed does not change when lead time is included.

In the separate regime-switching model, rates that define a Poisson process are used to model the regime switches. These regime-switch rates affect both the investment trigger and the investment capacity of the firm. The investment capacity depends on the long-term perspective of the firm as it cannot adjust the investment capacity when it has invested. Furthermore, the effects of the regime switch rates on the investment decision decrease for higher rates because for very high regime switch rates, the investment decision converges to a no-regime-switch model.

Particularly, if one regime switch rate tends to infinity, the firm is almost always in one regime as the regime switches immediately back. Hence, the real options model is identical to a no-switch model. If both regime switch rates tend to infinity with a similar convergence rate, the regime switches constantly and the firm is practically in one regime that is an average of both regimes.

The effect of the regime switch rates on the timing of the investment is twofold. For a lower regime switch rate in a certain regime, that regime is prioritized in the investment decision. However, the investment capacity focuses on the long term and not on a particular regime. Hence, the firm alters the investment capacity at the expense of the timing. This results in an increase in the investment capacity by postponing the investment if an advantageous change in the switch parameters occurs and vice versa. This is the so-called capacity effect of the regime switch rates on the timing of the investment. For a higher regime switch rate in a certain regime, the investment capacity does not change majorly if the switch rates change. To capture the change in the net present value of the investment because of the change in switch rates nevertheless, the firm alters the timing in the regime. This is the net present value effect.

In the combined model with both lead time and regime switching, lead time minorly affects the investment capacity because a regime switch may occur during the installation period. Hence, the firm mitigates the risk of being in the other regime at the moment the production plant has been installed. Yet, the hedging effect due to the lead time is limited as the long term is more preeminent in the investment capacity.

Similar to the effect of lead time in the no-switch model, the investment is postponed for higher lead times in the combined model. Besides the postponement of the investment, lead time also affects the effect of the regime switch rates on the investment trigger. An increase in lead time causes the firm to emphasize the long-term perspective more over the current regime in the investment decision. Therefore, the net present value effect becomes more preeminent than the capacity effect in determining the timing of the investment if lead time increases.

This new model gives insight into the theoretically optimal investment decision of a monopoly that cannot directly apply previous real options models because the assumptions in these real options models do not apply to their business model. Furthermore, the recent developments on the international stage challenge the current assumptions in real options models. Therefore, firms may face situations in the future where they have to adjust their investment strategies, even though currently they may not face violations of assumptions in real options models. If the investment strategy needs to be adjusted, these firms can use the regime-switching real options model including lead time to correct their investment strategies to the new situation.

In the remainder of this thesis, the baseline monopoly model of Huisman and Kort (2015) is understood first to extend their model. Therefore, in Section 2, the investment trigger and investment capacity are determined in their monopoly model. Secondly, this model is extended by including time to build in Section 3. Next, the model including geopolitical unrest is studied in Section 4. When these models are derived separately in the previous sections, both these extensions are combined in Section 5. In Section 6, the effect of the parameters on the investment decision is analyzed. Furthermore, special cases of the combined model are also studied in this section. Like in the model of Huisman and Kort, this thesis uses a linear demand function. In Section 7, an iso-elastic demand function is used to compare the outcomes in the model using the linear demand function. Section 7 makes the findings for the model using a linear demand function more robust or finds anomalies based on the demand function. In Section 8, the model is applied to public data of ASML to analyze the investment decision of ASML and how this relates to the mathematically optimal investment decision. Lastly, in Section 9 and Section 10, the findings and shortcomings of the model are discussed, ending with future research.

2 Model of Huisman and Kort (2015)

To be able to extend the model of Huisman and Kort (2015), their monopoly model has to be understood first. In this section, the steps they took in their paper are derived. Hence, it is clear where to adjust their model to extend the model to include time to build and regime switching.

First, the initialization for the price of a product is defined. Like in their paper, this is

$$P(t) = X(t)\Big(1 - \eta Q(t)\Big)$$

Here, η is a constant, and X_t is an exogenous shock process with

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

with $dW(t)$ as the increment of a Wiener process. Furthermore, it is assumed, like Huisman and Kort do, that the firm is risk neutral and discounts against a rate r, with $r > \mu$. This inequality needs to hold, otherwise, it would be more profitable for the firm to always wait with investing and an optimum would not exist (Dixit and Pindyck, 1994).

Moreover, it is assumed that the firm is a monopoly and can only invest once with a flexible capacity in the production of a product. Hence, the investment capacity does not change over time. The value of a firm is given by

$$V(X,Q) = \max_{T\geq 0, Q(t)\geq 0} \mathbb{E}\Big[\int_{t=T}^{\infty} QP(t)exp(-rt)dt - I_T(Q)exp(-rT)|X(0) = X\Big]$$

2.1 Derivation of the Value of the Firm at the Moment of Investment

First, the optimal capacity the firm has to invest with given that the firm enters the market, i.e. $X(t) \geq X^*$, is derived. Therefore, the value of the firm at the moment of investment is maximized and the linear investment costs ($I_t = \delta Q$) are substituted:

$$V(X,Q) = \mathbb{E}\Big[\int_{t=0}^{\infty} QX(t)\Big(1 - \eta Q\Big)exp(-rt)dt - \delta Q|X(0) = X\Big]$$

In the other case that $X(t) < X^*$, the optimal capacity is 0, or in other words, the firm does not invest yet. Therefore, the value of the firm at the moment of investment is

$$\begin{aligned} V(X) &= \max_{Q\geq 0} \mathbb{E}\Big[\int_{t=0}^{\infty} QX(t)\Big(1 - \eta Q\Big)exp(-rt)dt - \delta Q|X(0) = X\Big] \\ &= \max_{Q\geq 0} \Big(Q(1-\eta Q)\int_{t=0}^{\infty} \mathbb{E}[X(t)|X(0) = X]exp(-rt)dt - \delta Q\Big) \end{aligned}$$

To find an analytical solution to this problem, the value for $\mathbb{E}[X(t)|X(0) = X]$ has to be found. First, a new variable is defined: $Y(t) = ln(X(t))$. From Ito's Lemma, $dY = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz_t$ is obtained, where $z_t \sim N(0,t)$ with $z_0 = 0$. Therefore, the following algebraic steps can be taken

$$
\begin{aligned}
Y_t - Y_0 &= \int_0^t (\mu - \frac{1}{2}\sigma^2)dt + \int_0^t \sigma dz \\
&= (\mu - \frac{1}{2}\sigma^2)t + \sigma z_t \\
&= Y_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma z_t \\
ln(X_t) &= ln(X_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma z_t \\
X_t &= X_0 exp((\mu - \frac{1}{2}\sigma^2)t + \sigma z_t)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X_t|X_0 = X] &= \int_{-\infty}^{\infty} X exp((\mu - \frac{1}{2}\sigma^2)t + \sigma z)\frac{1}{\sqrt{t}\sqrt{2\pi}} exp(-\frac{z^2}{2t})dz \\
&= X exp((\mu - \frac{1}{2}\sigma^2)t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}\sqrt{2\pi}} exp(-\frac{z^2}{2t} + \sigma z)dz \\
&= X exp((\mu - \frac{1}{2}\sigma^2)t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}\sqrt{2\pi}} exp(-\frac{(z-\sigma t)^2}{2t} - \frac{\sigma^2 t}{2})dz \\
&= X exp(\mu t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}\sqrt{2\pi}} exp(-\frac{(z-\sigma t)^2}{2t})dz \\
&= X exp(\mu t)
\end{aligned}
$$

Hence, the expected value of X_t depends on the current value of the stochastic process, the drift rate, and the amount of time X_t is in the future. Substituting this expression into the formula for the value of the firm gives

$$
\begin{aligned}
V(X) &= \max_{Q\geq 0} \Big(Q(1-\eta Q) \int_{t=0}^{\infty} \mathbb{E}[X(t)|X(0) = X] exp(-rt)dt - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(Q(1-\eta Q) \int_{t=0}^{\infty} X * exp(\mu t) exp(-rt)dt - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(XQ(1-\eta Q) \int_{t=0}^{\infty} exp((\mu - r)t)dt - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(XQ(1-\eta Q) \Big[\frac{1}{\mu - r} exp((\mu - r)t)\Big]_{t=0}^{t=\infty} - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(\frac{XQ(1-\eta Q)}{r - \mu} - \delta Q\Big)
\end{aligned}
$$

To find the optimal investment capacity, the first-order condition (FOC) with respect to Q is applied. This step yields

$$\begin{aligned}
\frac{X(1-2\eta Q)}{r-\mu} - \delta &= 0 \\
X(1-2\eta Q) &= \delta(r-\mu) \\
2\eta Q &= 1 - \frac{\delta(r-\mu)}{X} \\
Q^*(X) &= \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu)}{X}\Big)
\end{aligned}$$

To check whether this expression for the investment capacity is a maximum, the second-order condition of $V(X,Q)$ with respect to Q is determined:

$$\frac{\partial^2 V}{Q^2} = \frac{-2\eta X}{r-\mu} < 0$$

as $X > 0$ and $\eta > 0$, and $r > \mu$. Therefore, the FOC yields a global maximum.

This value for optimal investment capacity is substituted into the value of the firm if $X(t) \geq X^*$ to remove the dimension of capacity in the value of the firm at the moment of investment. This gives

$$\begin{aligned}
V(X,Q^*) &= \frac{XQ^*(1-\eta Q^*)}{r-\mu} - \delta Q^* \\
&= \frac{X\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big)(1-\eta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big)}{r-\mu} - \delta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big) \\
&= \frac{\Big(X-\delta(r-\mu)\Big)\Big(1+\frac{\delta(r-\mu)}{X}\Big)}{4\eta(r-\mu)} - \delta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big) \\
&= \frac{\Big(X-\delta(r-\mu)\Big)\Big(X+\delta(r-\mu)\Big)}{4X\eta(r-\mu)} - \frac{\delta X-\delta^2(r-\mu)}{2\eta X} \\
&= \frac{X^2-\delta^2(r-\mu)^2}{4X\eta(r-\mu)} - \frac{2\delta X(r-\mu)-2\delta^2(r-\mu)^2}{4X\eta(r-\mu)} \\
&= \frac{X^2-2\delta(r-\mu)X+\delta^2(r-\mu)^2}{4X\eta(r-\mu)} \\
&= \frac{(X-\delta(r-\mu))^2}{4X\eta(r-\mu)}
\end{aligned}$$

$V(X)$ is a polynomial of the form $aX+b+c/X$ and it has a similar polynomial with respect to μ. Therefore, the effects of X and μ on the value of the firm are not monotone. The reason μ is not monotone is that a higher drift increases the net present value of the revenue of the firm, but also increases the total cost of investment. The reason for X is that Huisman and Kort allow for a variable Q. It is found that for small values of X ($X < \delta(r-\mu)$), the optimal investment capacity becomes negative. However, at these small values for the stochastic process, the firm is not likely to invest and the investment capacity at these values for X_t is probably equal to zero. Yet, the investment trigger should never be lower than this value as the investment capacity should be bigger than zero.

2.2 Derivation of the Value of waiting

To determine the value of waiting to invest is determined by using the Bellman equation. This equation is defined as

$$rF(x) = \pi + \frac{1}{dt}\mathbb{E}[dF]$$

Using Ito's Lemma, dF is derived. This is

$$\begin{aligned} dF &= \frac{\partial F(X)}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 X^2 dt + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}\sigma dz_t \\ &= \frac{\partial F(X)}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 X^2 dt + \frac{\partial F}{\partial X}\sigma dz_t \end{aligned}$$

As in Dixit and Pindyck (1994), $F(X)$ is assumed to have the form AX^β. Therefore, $\frac{\partial F}{\partial t} = 0$. Using this expression and given that $\pi = 0$ as the firm has not invested yet, the Bellman equation is

$$\begin{aligned} rF(X) &= \frac{1}{dt}\mathbb{E}\Big[\frac{\partial F(X)}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 X^2 dt + \frac{\partial F}{\partial X}\sigma dz_t\Big] \\ &= \frac{1}{dt}\mathbb{E}\Big[\frac{\partial F(X)}{\partial X}\mu X dt + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 X^2 dt\Big] \\ &= \frac{\partial F(X)}{\partial X}\mu X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\sigma^2 X^2 \end{aligned}$$

as dz_t is a Wiener process and has an expected value of zero.

Since $F(X)$ has the form AX^β, $F'(X) = \beta AX^{\beta-1}$ and $F''(X) = \beta(\beta-1)AX^{\beta-2}$. Substituting these expressions into the Bellman equation yields the following:

$$\begin{aligned} rAX^\beta &= \beta AX^{\beta-1}\mu X + \frac{1}{2}\beta(\beta-1)AX^{\beta-2}\sigma^2 X^2 \\ r &= \beta\mu + \frac{1}{2}\beta(\beta-1)\sigma^2 \\ r &= \beta\mu + \frac{1}{2}\beta^2\sigma^2 - \frac{1}{2}\beta\sigma^2 \\ \frac{1}{2}\beta^2\sigma^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r &= 0 \\ \beta &= \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\Big(\frac{1}{2} - \frac{\mu}{\sigma^2}\Big)^2 + \frac{2r}{\sigma^2}} \end{aligned}$$

As Dixit and Pindyck (1994) have shown, only the positive β is a solution to the differential equation with $A \neq 0$, so $F(X) = AX^\beta$. They also show that

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\Big(\frac{1}{2} - \frac{\mu}{\sigma^2}\Big)^2 + \frac{2r}{\sigma^2}} > 1$$

2.3 Derivation of the Investment Decision

X^* is derived using the smooth pasting conditions from Dixit and Pindyck (1994) and the expressions above:

$$\begin{cases} V(X^*) = F(X^*) \\ \frac{\partial V(X)}{\partial X}\Big|_{X=X^*} = \frac{\partial F(X)}{\partial X}\Big|_{X=X^*} \end{cases}$$

Since $F(X) = AX^{\beta}$, $F'(X) = \beta AX^{\beta-1} = \frac{\beta}{X}F(X)$. Hence,

$$\begin{aligned}
V'(X^*) &= \frac{\beta}{X}V(X^*) \\
\frac{Q(1-\eta Q)}{r-\mu} &= \frac{\beta}{X^*}\Big(\frac{X^*Q(1-\eta Q)}{r-\mu} - \delta Q\Big) \\
\frac{(\beta-1)(1-\eta Q)X^*}{r-\mu} &= \beta\delta \\
(\beta-1)(1-\eta Q)X^* &= \beta\delta(r-\mu) \\
(\beta-1)(1-\eta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X^*}\Big))X^* &= \beta\delta(r-\mu) \\
\frac{1}{2}(\beta-1)\Big(1+\frac{\delta(r-\mu)}{X^*}\Big)X^* &= \beta\delta(r-\mu) \\
\frac{1}{2}(\beta-1)\Big(X^*+\delta(r-\mu)\Big) &= \beta\delta(r-\mu) \\
(\beta-1)X^* &= (2\beta-(\beta-1))\delta(r-\mu) \\
X^* &= \frac{\beta+1}{\beta-1}\delta(r-\mu)
\end{aligned}$$

Since $\beta > 1$, it holds that $X^* \geq \delta(r-\mu)$. Hence, the optimal investment capacity at the moment of investing is always bigger than zero. This is also shown if the investment capacity at the moment of investment is determined using $Q^* = Q(X^*)$ below

$$\begin{aligned}
Q^* = Q(X^*) &= \frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)}{X^*}\Big) \\
&= \frac{1}{2\eta}\Big(1-\frac{\beta-1}{\beta+1}\Big) \\
&= \frac{1}{2\eta}\Big(\frac{\beta+1-(\beta-1)}{\beta+1}\Big) \\
&= \frac{1}{2\eta}\Big(\frac{2}{\beta+1}\Big) \\
&= \frac{1}{(\beta+1)\eta}
\end{aligned}$$

With these findings, A is derived:

$$\begin{aligned}
AX^{*\beta} = V(X^*, Q^*) &= \frac{(X^*-\delta(r-\mu))^2}{4X^*\eta(r-\mu)} \\
&= \frac{\Big(\frac{\beta+1}{\beta-1}\delta(r-\mu)-\delta(r-\mu)\Big)^2}{4\frac{\beta+1}{\beta-1}\delta(r-\mu)\eta(r-\mu)} \\
&= \frac{\Big(\frac{2}{\beta-1}\delta(r-\mu)\Big)^2}{4\frac{\beta+1}{\beta-1}\delta(r-\mu)\eta(r-\mu)} \\
&= \frac{\delta}{(\beta^2-1)\eta} \\
A &= \frac{\delta}{(\beta^2-1)\eta}X^{*-\beta} \\
&= \frac{\delta\Big(\frac{\beta+1}{\beta-1}\delta(r-\mu)\Big)^{-\beta}}{(\beta^2-1)\eta}
\end{aligned}$$

To analyze the effect of the parameters that identify $X(t)$, the derivatives of Q^* and X^* with respect to β are derived below:

$$\begin{aligned}\frac{\partial X^*}{\partial \beta} &= \frac{\partial}{\partial \beta} X^* \\ &= \frac{\partial}{\partial \beta}\Big[\frac{\beta+1}{\beta-1}\delta(r-\mu)\Big] \\ &= \frac{\partial}{\partial \beta}\Big[(1+\frac{2}{\beta-1})\delta(r-\mu)\Big] \\ &= -\frac{2\delta(r-\mu)}{(\beta-1)^2} < 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Q^*}{\partial \beta} &= \frac{\partial}{\partial \beta} Q^* \\ &= \frac{\partial}{\partial \beta}\frac{1}{(\beta+1)\eta} \\ &= -\frac{1}{(\beta+1)^2\eta} < 0\end{aligned}$$

Hence, increases in β decrease both the investment trigger and investment capacity. These expressions of the investment decision are also found by Huisman and Kort (2015). [1] Thus, the monopoly model of Huisman and Kort (2015) is studied in depth and their analysis is understood. Now, it rests to loosen some of their assumptions to extend this model. In Section 3, the assumption of a negligible installation period is relaxed. In Section 4, the assumption that the stochastic process follows a geometric Brownian motion with constant drift and volatility is modified. A Brownian motion with a switching drift and volatility is considered in this section where the drift and volatility change simultaneously in a given state. These state switches are not given at certain moments but follow a Poisson distribution. In Section 5, both lead time and regime switching are included in the model. In this section, a model is derived that includes both extensions.

[1] The welfare analysis is also derived. Yet, this part from the monopoly model of Huisman and Kort (2015) is not used in the remainder of this thesis. One can find these derivations in Appendix A.

3 Extension by including Time to Build

In contrast with the model of Huisman and Kort (2015), it can take time to enter a market. For instance, it takes time to build a new facility that produces a new product, while the investment has already been made. Nevertheless, their model can be adjusted to include time to build. The duration to install the investment is denoted by θ. The lead time is assumed to be known before the investment and it is independent of the investment capacity.[2] Therefore, it is assumed that the investment capacity only affects the investment costs but not the duration of the installation of the investment.

First, the investment model with deterministic time to build is studied as the investment decision is altered in one dimension. However, it may also be that the firm has an expectation and a certain window for when the investment is installed. E.g. building the new production facility is expected to take half a year, but could take anywhere between three months up to a year. A stochastic time to install the investment is thus more representative of the investment decision of the firm. The model with a stochastic lead time is studied in the latter part of the section.

3.1 Deterministic Time to Build

The model by Huisman and Kort (2015) is altered to include lead time in the investment decision. Lead time delays the moment the investment generates revenue with respect to the moment the firm invests. The new value function of the monopolist becomes

$$V^{CLT}(X) = \max_{T\geq 0, Q\geq 0} \mathbb{E}\Big[\int_{t=T+\theta}^{\infty} QP(t)exp(-rt)dt - \delta Q exp(-rT)|X(0) = X\Big]$$

If a constant time to build is assumed, the exact delay is known beforehand. Hence, including lead time in the value of the firm at the moment of investment results in

$$\begin{aligned}
V^{CLT}(X) &= \max_{Q(t)\geq 0} \mathbb{E}\Big[\int_{t=\theta}^{\infty} Q(t)X(t)\Big(1-\eta Q(t)\Big)exp(-rt)dt - \delta Q(t)|X(0) = X\Big] \\
&= \max_{Q\geq 0} \Big(Q(1-\eta Q)\int_{t=\theta}^{\infty} \mathbb{E}[X(t)|X(0)=X]exp(-rt)dt - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(Q(1-\eta Q)\int_{t=\theta}^{\infty} Xexp((\mu-r)t)dt - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(Q(1-\eta Q)X\Big[\frac{1}{\mu-r}exp((\mu-r)t)\Big]_{t=\theta}^{t=\infty} - \delta Q\Big) \\
&= \max_{Q\geq 0} \Big(\frac{Q(1-\eta Q)X}{r-\mu}exp((\mu-r)\theta) - \delta Q\Big)
\end{aligned}$$

[2] If the lead time depends on the investment capacity, an explicit expression for the optimal investment capacity and threshold are hard to find in the linear demand model. Using this demand function, a numerical solution can still be found for a lead time that depends on the installation time. If the iso-elastic demand function is used, an explicit solution of the investment decision can be found for a lead time depending on the investment capacity. For this, see Appendix B.

Again, the first-order condition is applied to find the optimal investment capacity:

$$\begin{aligned}\frac{\partial}{\partial Q}V^{CLT}(X) &= 0\\ \frac{(1-2\eta Q)X}{r-\mu}exp((\mu-r)\theta)-\delta &= 0\\ (1-2\eta Q)X &= \delta(r-\mu)exp((r-\mu)\theta)\\ Q^*_{CLT}(X) &= \frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)\end{aligned} \tag{1}$$

The second-order condition of the value of the firm with respect to the investment capacity is

$$\frac{-2\eta X}{(r-\mu)exp((r-\mu)\theta)} < 0$$

Hence, FOC yields a global maximum and the expression for the investment capacity maximizes the value of the firm. Substituting this optimal investment capacity into the value of the firm at the moment of investment gives

$$\begin{aligned}V^{CLT}(X) =& \frac{\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)(1-\eta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big))X}{r-\mu}exp((\mu-r)\theta)\\ &-\delta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)\\ =&\frac{Xexp((\mu-r)\theta)}{4\eta(r-\mu)}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)\Big(1+\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)\\ &-\frac{\delta}{2\eta}\Big(\frac{X-\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)\\ =&\frac{Xexp((\mu-r)\theta)}{4\eta(r-\mu)}\Big(1-\Big(\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big)^2\Big)-\frac{2\delta X-2\delta^2(r-\mu)exp((r-\mu)\theta)}{4\eta X}\\ =&\frac{X^2-(\delta(r-\mu)exp((r-\mu)\theta))^2}{4\eta(r-\mu)Xexp((r-\mu)\theta)}-\frac{2\delta X(r-\mu)exp((r-\mu)\theta)-2\delta^2((r-\mu)exp((r-\mu)\theta))^2}{4\eta X(r-\mu)exp((r-\mu)\theta)}\\ =&\frac{X^2-2\delta X(r-\mu)exp((r-\mu)\theta)+(\delta(r-\mu)exp((r-\mu)\theta))^2}{4\eta(r-\mu)Xexp((r-\mu)\theta)}\\ =&\frac{(X-\delta(r-\mu)exp((r-\mu)\theta))^2}{4\eta(r-\mu)Xexp((r-\mu)\theta)}\end{aligned}$$

To find the threshold X_t has to pass to make the investment worthwhile, the same value of waiting as Huisman and Kort is used because the form of the value of waiting does not change if lead time is included. The form of the value of waiting thus remains $F(X) = AX^{\beta}$. The smooth-pasting and value-matching conditions used previously are applied. Applying these conditions yields

$$
\begin{aligned}
\frac{\beta}{X}\Big(\frac{Q(1-\eta Q)X}{r-\mu}exp((\mu-r)\theta)-\delta Q\Big) &= \frac{Q(1-\eta Q)}{r-\mu}exp((\mu-r)\theta)\\
(\beta-1)X(1-\eta Q)exp((\mu-r)\theta) &= \beta\delta(r-\mu)\\
(\beta-1)X(1-\eta\frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X}\Big))exp((\mu-r)\theta) &= \beta\delta(r-\mu)\\
(\beta-1)X(\frac{1}{2}+\frac{1}{2}\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X})exp((\mu-r)\theta) &= \beta\delta(r-\mu)\\
\frac{1}{2}(\beta-1)\Big(X+\delta(r-\mu)exp((r-\mu)\theta)\Big)exp((\mu-r)\theta) &= \beta\delta(r-\mu)\\
(\beta-1)Xexp((\mu-r)\theta) &= (2\beta-(\beta-1))\delta(r-\mu)\\
X^*_{CLT} &= \frac{\beta+1}{\beta-1}\delta(r-\mu)exp((r-\mu)\theta)\\
&= X^*exp((r-\mu)\theta)
\end{aligned}
\tag{2}
$$

where X^* is the investment trigger without lead time from Section 2. Hence, the investment trigger is multiplied by a factor $exp((r-\mu)\theta)$. This factor is greater or equal to one because $\mu < r$ and $\theta > 0$.

Substituting the investment trigger into the equation for $V(X)$ to find the value of investing and the value of waiting gives

$$
\begin{aligned}
A(X^*_{CLT})^\beta = V^{CLT}(X^*_{CLT}) &= \frac{(X^*_{CLT}-\delta(r-\mu)exp((r-\mu)\theta))^2}{4\eta(r-\mu)X^*_{CLT}exp((r-\mu)\theta)}\\
&= \frac{(X^*exp((r-\mu)\theta)-\delta(r-\mu)exp((r-\mu)\theta))^2}{4\eta(r-\mu)X^*exp((r-\mu)\theta)exp((r-\mu)\theta)}\\
&= \frac{exp(2(r-\mu)\theta)(X^*-\delta(r-\mu))^2}{4\eta(r-\mu)X^*exp(2(r-\mu)\theta)}\\
&= \frac{(X^*-\delta(r-\mu))^2}{4\eta(r-\mu)X^*}\\
&= \frac{\delta}{(\beta^2-1)\eta}\\
A &= \frac{\delta(X^*_{CLT})^{-\beta}}{(\beta^2-1)\eta}\\
&= \frac{\delta\Big(\frac{\beta+1}{\beta-1}\delta(r-\mu)exp((r-\mu)\theta)\Big)^{-\beta}}{(\beta^2-1)\eta}
\end{aligned}
$$

Likewise, the investment capacity is

$$
\begin{aligned}
Q^*_{CLT} = Q^*_{CLT}(X^*_{CLT}) &= \frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{X^*_{CLT}}\Big)\\
&= \frac{1}{2\eta}\Big(1-\frac{\delta(r-\mu)exp((r-\mu)\theta)}{\frac{\beta+1}{\beta-1}\delta(r-\mu)exp((r-\mu)\theta)}\Big)\\
&= \frac{1}{2\eta}\Big(1-\frac{\beta-1}{\beta+1}\Big)\\
&= \frac{1}{(\beta+1)\eta}
\end{aligned}
\tag{3}
$$

Compared to the findings of Huisman and Kort (2015), the investment capacity of the monopoly at the trigger does not change if time to build is included. On the other hand, the threshold, X^*_{CLT}, increases by a factor $exp((r-\mu)\theta)$ with respect to X^*. This increase in investment trigger results in a similar expectation of the revenue at the moment of investment as the expected revenue in the model without lead time:

$$\begin{aligned} R(X^*_{CLT}) &= \frac{Q^*_{CLT}(1-\eta Q^*_{CLT})X^*_{CLT}}{r-\mu} exp((\mu-r)\theta) \\ &= \frac{Q^*(1-\eta Q^*)X^* exp((r-\mu)\theta)}{r-\mu} exp((\mu-r)\theta) \\ &= \frac{Q^*(1-\eta Q^*)X^*}{r-\mu} = R(X^*) \end{aligned}$$

Intuitively, as the revenue is discounted over the lead time corrected for the drift rate of the investment and the threshold is altered to compensate for this loss in the NPV of the investment, the expected revenue at the moment of investment does not change.

Furthermore, the investment trigger including lead time tends to the investment trigger without lead time if the lead time tends to zero. Mathematically, $X^*_{CLT} \to X^*$ as $\theta \to 0$. Therefore, if the duration of the placement of the investment becomes negligible, the threshold including time to build becomes equal to the threshold of a similar investment without time to build.

To analyze the effect of lead time on the threshold and optimal capacity, X^*_{CLT} and Q^*_{CLT} are differentiated with respect to θ. This gives:

$$\frac{\partial X^*_{CLT}}{\partial \theta} = \frac{\beta+1}{\beta-1}\delta(r-\mu)^2 exp((r-\mu)\theta) > 0 \tag{4}$$

and

$$\frac{\partial Q^*}{\partial \theta} = 0 \tag{5}$$

Hence, an increase in installation time causes the threshold to increase and the investment is postponed. The firm thus expects to invest later for increases in the installation time of the investment. This is in line with the results of Balliauw (2021). However, he also found that an increase in lead time may decrease the investment capacity. This decrease in investment capacity could result from the difference in demand function as he used $P_t = X_t - BQ_t$. Furthermore, he analyses the optimal expansion capacity of an already existing port, not the entry of a monopolist into a new market. Though, whether this contrary finding is because of the difference in price function or because of the expansive investment strategy is hard to tell as he used numerical approximation to determine the effect of lead time on the investment decision.

Yet, the optimal investment capacity is not affected by including lead time. Interestingly, the formula for the optimal capacity with any level of the stochastic process X_t does depend on lead time (see Equation 1). Though, this term cancels if the optimal investment threshold is substituted. These findings also hold if an iso-elastic demand function is used. For this, see Appendix B.

3.2 Stochastic Time to Build

A deterministic time to build may not be realistic because many unforeseen factors may affect the building process of the production plant. Nevertheless, the firm may have other information about the installation time which could indicate the distribution of the lead time. A stochastic installation time of the investment may thus be more representative of the real-world investment decision of a firm. This section derives and studies this extension of a stochastic lead time. The assumptions for the installation time are that it is a variable with a known distribution, say F_θ, and it is still independent of the investment capacity.

As the monopolist optimizes over the expectation of their payoff function, the value of the investment becomes

$$V^{VLT}(X) = \max_{T\geq 0, Q(t)\geq 0} \mathbb{E}\Big[\int_{t=T+\theta}^{\infty} Q(t)P(t)exp(-rt)dt - \delta Q(t)exp(-rT)|X(0) = X, \theta \sim F_\theta \Big]$$

The value of the firm at the moment of investment is now

$$\begin{aligned}
V^{VLT}(X) &= \max_{Q\geq 0} \mathbb{E}\Big[\int_{t=\theta}^{\infty} Q(t)P(t)exp(-rt)dt - \delta Q(t)exp(-rT)\Big| X(0) = X, \theta \sim F_\theta \Big] \\
&= \max_{Q\geq 0} \mathbb{E}_\theta \Bigg[\mathbb{E}_X \Big[\int_{t=\theta}^{\infty} QX(t)\Big(1-\eta Q\Big)exp(-rt)dt - \delta Q|X(0) = X \Big] \Bigg| \theta \sim F_\theta \Bigg] \\
&= \max_{Q\geq 0} \mathbb{E}_\theta \Bigg[Q\Big(1-\eta Q\Big) \int_{t=\theta}^{\infty} Xexp((\mu - r)t)dt - \delta Q \Big| \theta \sim F_\theta \Bigg] \\
&= \max_{Q\geq 0} \mathbb{E}_\theta \Bigg[\frac{Q(1-\eta Q)X}{r-\mu} exp((\mu - r)\theta) - \delta Q \Big| \theta \sim F_\theta \Bigg] \\
&= \max_{Q\geq 0} \int_{\Theta} \Bigg(\frac{Q(1-\eta Q)X}{r-\mu} exp((\mu - r)\theta) - \delta Q \Bigg) dF_\theta \\
&= \max_{Q\geq 0} \Big(\frac{Q(1-\eta Q)X}{r-\mu} \int_{\Theta} exp((\mu - r)\theta)dF_\theta - \delta Q \Big) \\
&= \max_{Q\geq 0} \Big(\frac{Q(1-\eta Q)X}{r-\mu} M_\theta(\mu - r) - \delta Q \Big)
\end{aligned}$$

where $M_\theta(t)$ is the moment generating function (MGF) of θ defined as

$$M_\theta(t) = \int_{\Theta} e^{t\theta} dF_\theta = \mathbb{E}_\theta[e^{t\theta}]$$

With a similar approach to find optimal capacity depending on X_t ($Q^*_{VLT}(X)$) and the threshold (X^*_{VLT}), the optimal investment capacity (Q^*_{VLT}) is derived, like in the model for constant lead time. FOC of the value of the firm at the moment of investment with respect to capacity is applied to find the optimal investment capacity. This step yields

$$\begin{aligned}
\frac{\partial}{\partial Q} V^{VLT}(X) &= 0 \\
\frac{(1-2\eta Q)X}{r-\mu} M_\theta(\mu - r) - \delta &= 0 \\
(1-2\eta Q)X &= \frac{\delta(r-\mu)}{M_\theta(\mu - r)} \\
Q^*_{VLT}(X) &= \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu)}{XM_\theta(\mu - r)}\Big)
\end{aligned}$$

The second-order condition of the value of the firm at the moment of investment with respect to the investment capacity for a stochastic time to build is

$$\frac{-2\eta X M_\theta(\mu - r)}{r - \mu} < 0$$

This inequality holds because the discount rate (r) is bigger than the drift rate (μ) and the moment generating function ($M_\theta(t)$), the elasticity parameter (η), and the stochastic process (X) are always bigger than zero.

Applying the smooth-pasting and value-matching conditions gives

$$\begin{aligned}
\frac{\beta}{X}\Big(\frac{Q(1-\eta Q)X}{r-\mu}M_\theta(\mu - r) - \delta Q\Big) &= \frac{Q(1-\eta Q)}{r-\mu}M_\theta(\mu - r) \\
(\beta - 1)X(1-\eta Q)M_\theta(\mu - r) &= \beta\delta(r-\mu) \\
(\beta - 1)X(1-\eta\frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu)}{XM_\theta(\mu - r)}\Big))M_\theta(\mu - r) &= \beta\delta(r-\mu) \\
(\beta - 1)X(\frac{1}{2} + \frac{1}{2}\frac{\delta(r-\mu)}{XM_\theta(\mu - r)})M_\theta(\mu - r) &= \beta\delta(r-\mu) \\
\frac{1}{2}(\beta - 1)\Big(X + \frac{\delta(r-\mu)}{M_\theta(\mu - r)}\Big)M_\theta(\mu - r) &= \beta\delta(r-\mu) \\
(\beta - 1)XM_\theta(\mu - r) &= (2\beta - (\beta - 1))\delta(r-\mu) \\
X^*_{VLT} &= \frac{\beta+1}{\beta-1}\frac{\delta(r-\mu)}{M_\theta(\mu - r)}
\end{aligned} \tag{6}$$

and substituting the threshold above into the formula for optimal capacity gives

$$\begin{aligned}
Q^*_{VLT} = Q^*(X^*) &= \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu)}{X^*_{VLT}M_\theta(\mu - r)}\Big) \\
&= \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu)}{\frac{\beta+1}{\beta-1}\frac{\delta(r-\mu)}{M_\theta(\mu-r)}M_\theta(\mu - r)}\right) \\
&= \frac{1}{2\eta}\Big(1 - \frac{\beta-1}{\beta+1}\Big) \\
&= \frac{1}{(\beta+1)\eta}
\end{aligned} \tag{7}$$

Firstly, the optimal capacity remains the same over all the three different models as $Q^*_{VLT} = Q^*_{CLT} = Q^* = \frac{1}{(\beta+1)\eta}$. Hence, including lead time in the investment decision does not affect the investment capacity for a linear demand function, though the investment trigger changes with respect to the distribution of θ. Differentiating the expressions for X^* and Q^* with respect to the moment-generating function gives insight into the behavior of the model for different distributions of lead time. Doing this yields

$$\frac{\partial X^*_{VLT}}{\partial M_\theta(\mu - r)} = -\frac{\beta+1}{\beta-1}\frac{\delta(r-\mu)}{(M_\theta(\mu - r))^2} < 0 \tag{8}$$

and

$$\frac{\partial Q^*_{VLT}}{\partial M_\theta(t)} = 0 \tag{9}$$

Therefore, a higher MGF for θ gives a lower threshold, whereas the investment capacity is not affected by the moment generating function and thus the distribution of θ. To make it more insightful, $\theta \sim N(\mu_\theta, \sigma^2_\theta)$ is assumed. This assumption can be validated as the building speed is analyzed. Namely, if the building speed on a very small time interval is independent and identically distributed, the total building time tends to a normal distribution by the central limit theorem.

For a normally distributed variable, its moment generating function is $M_\theta(t) = exp(\mu_\theta t + \frac{1}{2}\sigma_\theta^2 t^2)$. Hence,

$$\frac{\partial M_\theta(t)}{\partial \mu_\theta} = texp(\mu_\theta t + \frac{1}{2}\sigma_\theta^2 t^2)$$

and

$$\frac{\partial M_\theta(t)}{\partial \sigma_\theta^2} = \frac{1}{2}t^2 exp(\mu_\theta t + \frac{1}{2}\sigma_\theta^2 t^2)$$

Thus, if the lead time is normally distributed, an increase in the expected lead time, μ_θ, decreases the MGF as $\mu - r < 0$, increasing the investment trigger, X^*. An increase in uncertainty of the lead time, σ_θ^2, increases the moment generating function as $(\mu - r)^2 > 0$ and decreases the threshold. Intuitively, this can be explained by the convexity of $exp(x)$. An increase in the variance of the lead time results in a higher potential gain compared to the potential risk increasing the expected value of investing.

The effect of the standard deviation of lead time on the investment triggers is visualized in Figure 1.

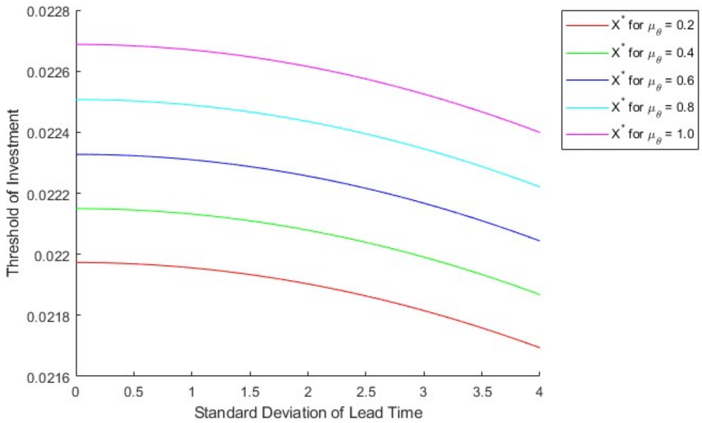


Figure 1: Plot of X^*_{VLT} as a function of σ_θ with lead time having a normal distribution according to the legend and parameter values: $\mu_X = 0.06$, $\sigma_X = 0.2$, $r = 0.1$, $\delta = 0.1$, $\eta = 0.05$

Figure 1 illustrates that a higher expected lead time increases the investment trigger and an increase in the standard deviation of lead time decreases the investment trigger. Furthermore, it shows that for a sufficiently high value of the standard deviation of lead time, the investment trigger is smaller than for smaller values of expected lead time. Namely, the investment trigger for an expected lead time of 0.4 and no uncertainty for the building time is bigger than for a lead time with an expected lead time of 0.6 and a standard deviation of 4. However, the latter instance is not realistic as the probability that the lead time is negative is not negligible. Hence, a high variance in the lead time is not realistic even though it has a big effect on the investment trigger.

The optimal capacity, on the contrary, is not affected by the expected lead time nor its uncertainty in this model because a different MGF does not affect the investment capacity.

The effect of μ_X and r on the investment trigger are also studied for a variable uncertainty of time to build. Below are graphs of different values of μ_X and r and what their effect is on the investment trigger for a variable uncertainty of lead time.

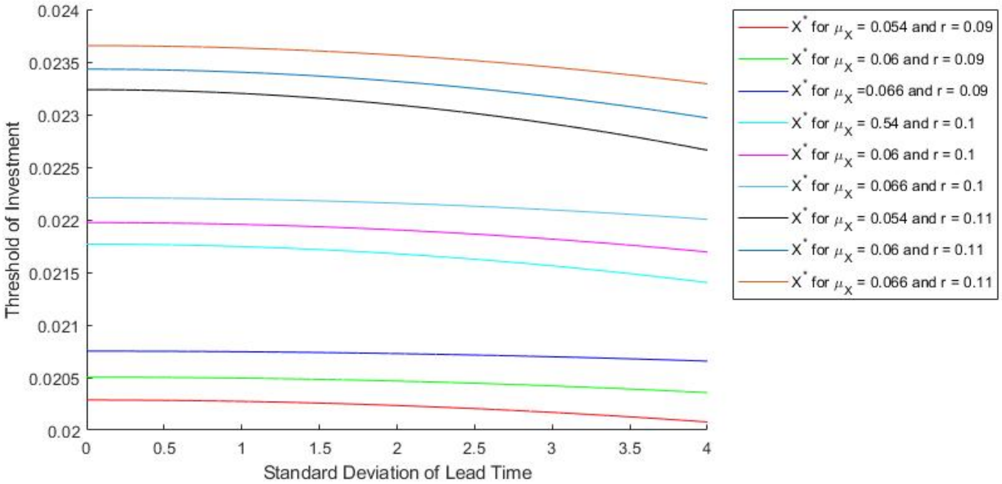


Figure 2: Plot of X^*_{VLT} as a function of σ_θ with lead time having a normal distribution with an expected duration of $\mu_\theta = 0.2$, parameter values: $\sigma_X = 0.2$, $\delta = 0.1$, $\eta = 0.05$ and other parameters according to the legend

Figure 2 illustrates that the threshold again decreases with an increase in the standard deviation of lead time. This change is small because the difference in the threshold for $\sigma_\theta = 0$ (i.e. constant lead time) and $\sigma_\theta = 4$ (very high uncertainty compared to the expected duration of the project, which is not realistic) is small. Therefore, it seems that the effect of the variance of lead time on the investment trigger is almost negligible compared to the effect of the drift of the stochastic process and discount rate.

This is also supported if the percentage difference of the moment generating function with respect to the volatility of lead time is analyzed, σ_θ^2. A percentage difference in the drift and discount rate is

$$\frac{\partial M_\theta(\mu - r)}{\partial \sigma_\theta^2} / M_\theta(\mu - r) = \frac{1}{2}(\mu - r)^2 > 0$$

This is a very small number as realistic values for μ_X and r lay mostly between -0.1 and 0.1 for annual data and $\mu - r \in (-1, 0)$ is likely to hold. Hence, the percentage difference in threshold level due to an increase in uncertainty in lead time is likely to be small, even for high values for this uncertainty.

Moreover, the percentage difference by increasing the expected lead time is $\frac{\partial M_\theta(\mu - r)}{\partial \mu_\theta} / M_\theta(\mu - r) = \mu - r < 0$. This change has a bigger magnitude than the percentage difference in expected lead time. Therefore, the expectation of lead time has a bigger effect on the investment trigger than the standard deviation of lead time.

4 Extension by including Geopolitical Unrest

In this section, the real options model that includes geopolitical unrest is derived by allowing for regime switching. Additionally, the relationship between the investment decision and its parameters is studied. The distinction is made between a single regime switch model and a continuous regime switch model. It starts with the single regime switch model as this model is intuitively easier. The part about the single regime switch model is divided further into two segments. One where a regime switch is made to an investment discouraging regime and another where the new regime stimulates investment. In the next part, the continuous regime-switching model is derived.

The applications of the regime switch models are based on economic theories. For instance, the model for a discouraging new regime is in line with the product life cycle theory (Balter et al., 2023), and the continuous regime switch model is based on a model that includes geopolitical unrest in the investment decision (Guo et al., 2005).[3] Therefore, the emphasis of this thesis is the continuous regime switch model as it incorporates geopolitical unrest in the investment decision. Yet, the single regime switch model is also important as it helps elucidate the approach to derive the continuous regime switch model.

The stochastic process that allows for regime switching is defined as follows:

$$dX_t = \mu_R X_t dt + \sigma_R X_t dZ_t$$

here, dZ_t is a Wiener process and μ_R and σ_R are the parameter values in a certain regime $R \in \{1, 2\}$.

4.1 Single Regime Switch Model

In this section, the simple model is studied, where a switch between regimes is only made once with rate λ. The firm starts in state 1. State 2 is thus the absorbing state. Since this is the final state, the regime cannot switch back to state 1. The value of the firm at the moment of investment in state 2 is, therefore, known. Hence, the same model as in Section 2 can be applied in state 2 resulting in

$$V_2(X) = \max_{T \geq 0, Q(t) \geq 0} \mathbb{E}\Big[\int_{t=T}^{\infty} Q(t)P(t)exp(-rt)dt - \delta Q(t)exp(-rT) | X(0) = X \Big]$$

In state 2, $dX_t = \mu_2 X_t dt + \sigma_2 X_t dZ_t$. Following the same derivations as in Section 2, the expressions for the value of waiting are obtained. Here for the sake of notation denoted by $F_2(X) = BX^{\alpha}$, the investment trigger X_2^* and optimal investment capacity Q_2^*. The value of the firm at the moment of investment in the second regime is

$$V_2(X, Q) = \frac{Q(1 - \eta Q)X}{r - \mu_2} - \delta Q$$

Finding the value of the firm in state 1 is different from the derivation for $V_2(X)$.[4] The NPV of the value of investing in state 1 is found by applying the Bellman equation (Guo et al., 2005). The revenue of the investment in state 1, denoted by $R_1(X)$, is defined by the Bellman equation

[3] A jump-diffusion process can also be used to model geopolitical unrest. Appendix C contains the derivation of the approximated jump-diffusion process to apply in this stochastic process to the model of Huisman and Kort (2015).

[4] This will be adjusted if a continuous regime-switching model is used. Then both $V_1(X)$ and $V_2(X)$ are defined by the Bellman equation for the net present value of the investment. See Section 4.2 for a full derivation if the regime can also jump from state 2 to state 1.

as

$$rR_1(X) = Q(1-\eta Q)X + \mu_1 X R_1'(X) + \frac{1}{2}\sigma_1^2 X^2 R_1''(X) + \lambda\Big(R_2(X) - R_1(X)\Big)$$

Furthermore, the NPV of the revenue at any point in time when the regime is in state 2 for a given Q ($R_2(X)$) is $\frac{Q(1-\eta Q)X}{r-\mu_2}$. In the no-switch model, the form of the revenue has a constant return to scale with respect to X. Therefore, the guess of the form of the revenue in regime 1 has this form as well: $R_1(X) = cX$, $R_1'(X) = c$, and $R_1''(X) = 0$. Thus, the Bellman equation can be written as

$$\begin{aligned}
(r+\lambda)cX &= Q(1-\eta Q)X + \mu_1 X c + \lambda \frac{Q(1-\eta Q)X}{r-\mu_2} \\
(r+\lambda)cX &= Q(1-\eta Q)X\Big(1 + \frac{\lambda}{r-\mu_2}\Big) + \mu_1 X c \\
(r+\lambda-\mu_1)c &= Q(1-\eta Q)\Big(\frac{r+\lambda-\mu_2}{r-\mu_2}\Big) \\
c &= \frac{Q(1-\eta Q)(r+\lambda-\mu_2)}{(r-\mu_2)(r+\lambda-\mu_1)} = Q(1-\eta Q)\Lambda
\end{aligned}$$

where $\Lambda = \frac{r+\lambda-\mu_2}{(r-\mu_2)(r+\lambda-\mu_1)}$.

By using this expression, the value of the firm is derived if the regime is in state 1. This is given by

$$V_1(X) = \max_{Q\geq 0}\Big(Q(1-\eta Q)\Lambda X - \delta Q\Big)$$

Applying FOC to obtain the optimal investment quantity yields

$$\begin{aligned}
\frac{\partial V_1(X,Q)}{\partial Q} &= 0 \\
(1-2\eta Q)\Lambda X - \delta &= 0 \\
Q_1^*(X) &= \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda X}\Big)
\end{aligned}$$

Furthermore, the SOC is

$$-2\eta\Lambda X < 0$$

as η, Λ and X_t are bigger than zero. Hence, the FOC yields a global maximum.

The value of the firm at the moment of investment given that the regime is in state 1 and that the firm optimizes its profits is

$$
\begin{aligned}
V_1(X) &= Q_1^*(X)(1-\eta Q_1^*(X))\Lambda X - \delta Q_1^*(X) \\
&= \frac{1}{2\eta}\Big(1-\frac{\delta}{\Lambda X}\Big)\bigg(1-\eta\frac{1}{2\eta}\Big(1-\frac{\delta}{\Lambda X}\Big)\bigg)\Lambda X - \delta\frac{1}{2\eta}\Big(1-\frac{\delta}{\Lambda X}\Big) \\
&= \frac{\Lambda X}{2\eta}\Big(1-\frac{\delta}{\Lambda X}\Big)\Big(\frac{1}{2}+\frac{1}{2}\frac{\delta}{\Lambda X}\Big) - \frac{\delta}{2\eta}\Big(1-\frac{\delta}{\Lambda X}\Big) \\
&= \frac{1}{4\eta}(\Lambda X-\delta)\Big(1+\frac{\delta}{\Lambda X}\Big) - \frac{\delta}{2\eta} + \frac{\delta^2}{2\eta\Lambda X} \\
&= \frac{1}{4\eta\Lambda X}(\Lambda X-\delta)(\Lambda X+\delta) - \frac{2\delta\Lambda X}{4\eta\Lambda X} + \frac{2\delta^2}{4\eta\Lambda X} \\
&= \frac{(\Lambda X)^2-\delta^2}{4\eta\Lambda X} - \frac{2\delta\Lambda X}{4\eta\Lambda X} + \frac{2\delta^2}{4\eta\Lambda X} \\
&= \frac{(\Lambda X)^2-2\delta\Lambda X+\delta^2}{4\eta\Lambda X} \\
&= \frac{(\Lambda X-\delta)^2}{4\eta\Lambda X}
\end{aligned} \tag{10}
$$

Like the value of the firm at the moment of investment, the value of waiting is different in the two regimes. Hence, two distinct investment triggers corresponding to the regimes are part of the investment decision instead of the investment decision consisting of one investment trigger and one investment capacity in the no-switch model.

Furthermore, the value of waiting cannot be derived immediately as there are two cases: a new state that stimulates investment and a new regime that discourages investment because the Bellman equation changes at some values of the stochastic process. These instances are studied in the upcoming segments and determine the investment decision.

4.1.1 Investment Discouraging New Regime

The case where the new regime discourages investment has a region where the firm waits with investing and a region where it invests. The value of investing is given above, whereas the value of waiting is different from the previous models. The case that the new regime discourages investment is studied. This condition results in the investment trigger in state 2 being bigger than the investment trigger in regime 1: $X_2^* > X_1^*$. The subscript denotes the state the firm is in.

In this case, the regime may switch with rate λ in the waiting domain and the value of waiting becomes $F_2(X)$. Hence, the Bellman equation is now

$$
\begin{aligned}
rF_1 &= \mu_1 X F_1' + \frac{1}{2}\sigma_1^2 X^2 F_1'' + \lambda(F_2 - F_1) \\
(r+\lambda)F_1 &= \mu_1 X F_1' + \frac{1}{2}\sigma_1^2 X^2 F_1'' + \lambda B X^\alpha
\end{aligned}
$$

Since $X_2^* > X_1^*$, the above-defined expression always holds. The value of waiting in regime 1 does not change as X_t passes X_2^* because the firm has already invested if it would still be in regime 1. Hence, the jump in regime while the firm waits with investing can only be from waiting to invest to waiting to invest. The firm never immediately invests if a regime switch occurs in this case.

To find the solution to this differential equation, the homogeneous solution to the problem is derived first:

$$(r+\lambda)F_1 = \mu_1 X F_1' + \frac{1}{2}\sigma_1^2 X^2 F_1''$$

Let $F_1(X)$ have the form AX^β. Substituting this form into the Bellman equation results in

$$\begin{aligned}(r+\lambda)AX^\beta &= \mu_1 X\beta AX^{\beta-1} + \frac{1}{2}\sigma_1^2 X^2 \beta(\beta-1)A_1 X^{\beta-2}\\ r+\lambda &= \mu_1\beta + \frac{1}{2}\sigma_1^2\beta(\beta-1)\\ r+\lambda-\mu_1\beta-\frac{1}{2}\sigma_1^2\beta(\beta-1) &= 0\end{aligned}$$

Hence, β solves for $g(\beta)=0$, where $g(\beta) = r+\lambda-\mu_1\beta-\frac{1}{2}\sigma_1^2\beta(\beta-1)$. Note that there are 2 possible β's: a positive and a negative. So the homogeneous solution to this differential equation has the form $F_1(X) = A_1X^{\beta_1} + A_2X^{\beta_2}$, where β_1 and β_2 are the positive and negative solution to $g(\beta)=0$.

To find the particular solution to the differential equation, it is assumed that the particular solution has the form A_3X^α. Working this out yields

$$\begin{aligned}(r+\lambda)A_3X^\alpha &= \mu_1\alpha A_3X^\alpha + \frac{1}{2}\sigma_1^2\alpha(\alpha-1)A_1X^\alpha + \lambda BX^\alpha\\ (r+\lambda) &= \mu_1\alpha + \frac{1}{2}\sigma_1^2\alpha(\alpha-1) + \lambda B/A_3\\ \lambda B/A_3 &= (r+\lambda) - \mu_1\alpha - \frac{1}{2}\sigma_1^2\alpha(\alpha-1)\\ A_3 &= \frac{\lambda B}{r+\lambda-\mu_1\alpha-\frac{1}{2}\sigma_1^2\alpha(\alpha-1)} = \frac{\lambda}{g(\alpha)}B\end{aligned}$$

Where α and B are derived using the parameters in the second regime. Therefore, $F_1(X) = A_1X^{\beta_1}+A_2X^{\beta_2}+A_3X^\alpha$. Furthermore, the boundary condition $F_1(0)=0$ has to hold. Therefore, $A_2 = 0$ and $F_1(X) = A_1X^{\beta_1} + A_3X^\alpha$. As the smooth-pasting and value-matching conditions are applied, the following system of equations is obtained

$$\begin{aligned}&\begin{cases} F_1(X_1^*) = V_1(X)\\ \frac{\partial F_1(X)}{\partial X}\Big|_{X=X_1^*} = \frac{\partial V_1(X)}{\partial X}\Big|_{X=X_1^*}\end{cases}\\ &\begin{cases} A_1(X_1^*)^{\beta_1} + \frac{\lambda}{g(\alpha)}B(X_1^*)^\alpha = \frac{(\Lambda X_1^*-\delta)^2}{4\eta\Lambda X_1^*}\\ \beta_1A_1(X_1^*)^{\beta_1-1} + \frac{\lambda}{g(\alpha)}B(X_1^*)^{\alpha-1} = \frac{\Lambda}{4\eta} - \frac{\delta^2}{4\eta\Lambda(X_1^*)^2}\end{cases}\\ &\begin{cases} A_1(X_1^*)^{\beta_1} + \frac{\lambda}{g(\alpha)}B(X_1^*)^\alpha = \frac{(\Lambda X_1^*-\delta)^2}{4\eta\Lambda X_1^*}\\ \beta_1A_1(X_1^*)^{\beta_1} + \frac{\lambda}{g(\alpha)}B(X_1^*)^{\alpha} = \frac{\Lambda X_1^*}{4\eta} - \frac{\delta^2}{4\eta\Lambda X_1^*}\end{cases}\end{aligned} \tag{11}$$

The solutions for A_1 and X_1^* are obtained by solving the system numerically as the system has 5 different powers of X in its terms and applying substitution does not result in an explicit expression. Q_1^* is found by substituting the investment trigger into the optimal investment capacity function. Below is a plot of $V_1(X)$, $F_1(X)$, and the investment trigger X_1^* for an investment discouraging new regime. Here, $X_1^* = 0.0170$ with $Q_1^* = 7.9813$ and $X_2^* = 0.0176$ with $Q_2^* = 7.7257$ for the parameter values described below the figure.

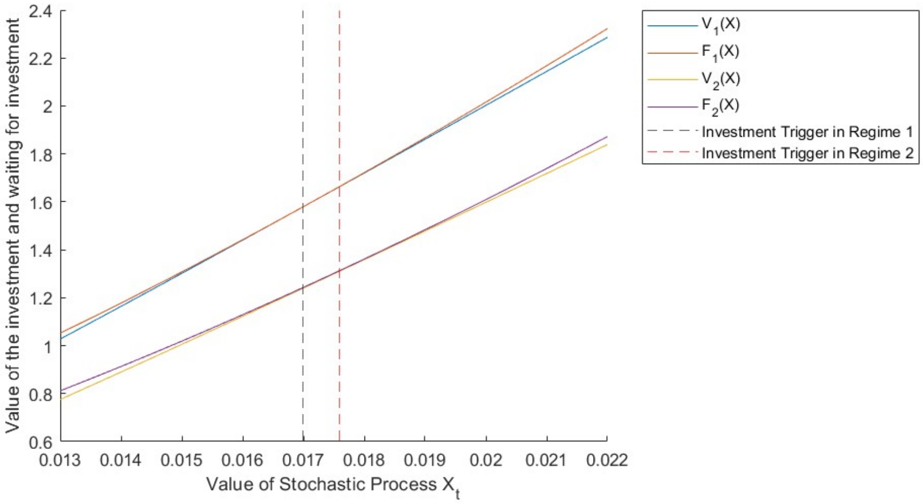


Figure 3: Plot of the value of waiting and the value of the firm at the moment of investment in both regimes as a function of X with parameter values: $\mu_1 = 0.08$, $\mu_2 = 0.06$, $\sigma_1 = 0.05$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda = 0.1$, $\delta = 0.1$ and $\eta = 0.05$

4.1.2 Investment Stimulating New Regime

It may also occur that the investment trigger in the second state is lower than in the initial state, i.e. $X_2^* < X_1^*$. This results in a potential situation where $X_2^* \geq X_t < X_1^*$ and the firm waits with investing because it is still in state 1. A state switch in this region will result in an immediate investment by the firm with a certain optimal capacity Q_2^* corresponding to the level of X_t. Therefore, the value of waiting has two different differential equations: one where the state switch occurs when $X_t < X_2^*$ and the firm waits with investing, F_1 thus jumps to F_2, and one where a switch causes the firm to invest immediately, so F_1 jumps to V_2. The latter takes place if $X_2^* < X_t < X_1^*$. This region is called the transient region, like in the paper by Guo et al. (2005).

With a similar derivation as in the section for a discouraging new state, $F_1(X) = A_1 X^{\beta_1} + A_3 X^{\alpha}$ if $X \in [0, X_2)$ is found, where β_1, A_3, and α are the same as in the case for a discouraging new regime because the differential equation remains the same in this region for X. However, A_1 is calculated differently.

To find A_1, the value of waiting in the initial regime for $X_t > X_2^*$ has to be defined. As stated above, a regime switch in the transient region results in an immediate investment by the firm. Therefore, the Bellman equation is adjusted to include a regime switch in the transient region:

$$rF_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda(V_2(X) - F_1(X))$$
$$(r + \lambda)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda V_2(X)$$

To solve this differential equation, the homogeneous solution is found first. It is derived by solving

$$(r + \lambda)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X)$$

Hence, it is assumed that the homogeneous solution has the form HX^{ϵ}. This results in the following equation

$$(r+\lambda)HX^{\epsilon} = \mu_1 X\epsilon HX^{\epsilon-1} + \frac{1}{2}\sigma_1^2 X^2\epsilon(\epsilon-1)HX^{\epsilon-2}$$
$$(r+\lambda)HX^{\epsilon} = \mu_1\epsilon HX^{\epsilon} + \frac{1}{2}\sigma_1^2\epsilon(\epsilon-1)HX^{\epsilon}$$
$$(r+\lambda) = \mu_1\epsilon + \frac{1}{2}\sigma_1^2\epsilon(\epsilon-1)$$

Therefore, ϵ solves $g(\epsilon) = 0$ for g defined like in the case for the discouraging new regime. Two ϵs are obtained that are equal to β_1 and β_2 from the previous section and the homogeneous solution has the form $H_1X^{\beta_1} + H_2X^{\beta_2}$.

Now the particular solution to this differential equation has to be found. The case for a fixed capacity, say K, is studied first to understand the intuition behind finding the value of waiting in the transient region. Since the value of the firm at the moment of investment is now a linear function with respect to X, the assumed form of the particular solution is also linear in X: $aX + b$. In this instance, the differential equation would be

$$(r+\lambda)(aX+b) = \mu_1 Xa + \frac{1}{2}\sigma_1^2X^2 * 0 + \lambda\Big(\frac{K(1-\eta K)X}{r-\mu_2} - \delta K\Big)$$
$$(r+\lambda-\mu_1)aX + (r+\lambda)b = \lambda\Big(\frac{K(1-\eta K)X}{r-\mu_2} - \delta K\Big)$$

This step results in $a = \frac{\lambda}{(r+\lambda-\mu_1)(r-\mu_2)}K(1-\eta K)$ and $b = -\frac{\lambda}{r+\lambda}\delta K$ for the particular solution $(aX+b)$ of the differential equation.

However, the investment capacity is not fixed but flexible. It optimizes the value of the firm at the moment of investment. As shown at the beginning of this segment, the optimal investment capacity in state 2 is a function of X. Substituting this investment capacity results in $V_2(X)$ being a polynomial in X of the form $aX + b + \frac{c}{X}$. The relation between $V_2(X)$ and X is thus not linear, like in the constant capacity case.

Moreover, the firm does not only have 1 or 2 investment triggers but also an investment range in the transient region. Consequently, the firm has a region of optimal investment capacities that is mapped by the transient region. Therefore, the expression of the value of the firm at the moment of investment in regime 2 for different X_t has to be substituted into the differential equation to include this flexibility.

The particular solution is expected to have a similar polynomial form as $V_2(X)$ with respect to X. Hence, the particular solution to the differential equation is assumed to have the form $aX + b + c/X$. Substituting this expression into the differential equation yields

$$(r+\lambda)(aX+b+cX^{-1}) = \mu_1X(a-cX^{-2}) + \frac{1}{2}\sigma_1^2X^2 * 2 * cX^{-3} + \lambda\frac{(X-\delta(r-\mu_2)^2}{4X\eta(r-\mu_2)}$$
$$(r+\lambda-\mu_1)aX + (r+\lambda)b + (r+\lambda+\mu_1-\sigma_1^2)cX^{-1} = \lambda\Big(\frac{X^2-2\delta(r-\mu_2)X+\delta^2(r-\mu_2))^2}{4X\eta(r-\mu_2)}\Big)$$

Hence, $a = \frac{\lambda}{4\eta(r+\lambda-\mu_1)(r-\mu_2)}$, $b = -\frac{\lambda\delta}{2\eta(r+\lambda)}$ and $c = \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\lambda+\mu_1-\sigma_1^2)}$ are found. Therefore, the value of waiting is

$$F_1(X) = \begin{cases} A_1X^{\beta_1} + A_3X^{\alpha} \text{ if } X \in [0, X_2^*) \\ H_1X^{\beta_1} + H_2X^{\beta_2} + aX + b + c/X \text{ if } X \in [X_2^*, X_1^*) \end{cases} \tag{12}$$

Since $F_1(X)$ is continuous and continuously differentiable, the continuity conditions for the value of waiting in state 1 at X_2^* have to hold:

$$\begin{cases} \lim_{X\uparrow X_2^*} F_1(X) = \lim_{X\downarrow X_2^*} F_1(X) \\ \lim_{X\uparrow X_2^*} \frac{\partial F_1(X)}{\partial X} = \lim_{X\downarrow X_2^*} \frac{\partial F_1(X)}{\partial X} \end{cases}$$

$$\begin{cases} A_1 X_2^{*\beta_1} + A_3(X_2^*)^\alpha = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \beta_1 A_1(X_2^*)^{\beta_1 - 1} + \alpha A_3(X_2^*)^{\alpha-1} = \beta_1 H_1(X_2^*)^{\beta_1-1} + \beta_2 H_2(X_2^*)^{\beta_2-1} + a - c/(X_2^*)^2 \end{cases} \tag{13}$$

$$\begin{cases} A_1 X_2^{*\beta_1} + A_3(X_2^*)^\alpha = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \beta_1 A_1(X_2^*)^{\beta_1} + \alpha A_3(X_2^*)^{\alpha} = \beta_1 H_1(X_2^*)^{\beta_1} + \beta_2 H_2(X_2^*)^{\beta_2} + aX_2^* - c/X_2^* \end{cases}$$

However, this system cannot be solved as of yet, because the system has 3 unknown variables and 2 equations. Therefore, to find the investment trigger, the value-matching and smooth-pasting conditions at the moment of investment in regime 1 are included. These conditions are

$$\begin{cases} F_1(X_1^*) = V_1(X_1^*) \\ \frac{\partial F_1(X)}{\partial X}\Big|_{X=X_1^*} = \frac{\partial V_1(X)}{\partial X}\Big|_{X=X_1^*} \end{cases}$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(\Lambda X_1^* - \delta)^2}{4\eta\Lambda X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1-1} + \beta_2 H_2(X_1^*)^{\beta_2-1} + a - c/(X_1^*)^2 = \frac{\Lambda}{4\eta} - \frac{\delta^2}{4\eta\Lambda(X_1^*)^2} \end{cases} \tag{14}$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(\Lambda X_1^* - \delta)^2}{4\eta\Lambda X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} + aX_1^* - c/X_1^* = \frac{\Lambda X_1^*}{4\eta} - \frac{\delta}{4\eta\Lambda X_1^*} \end{cases}$$

Since the only additional unknown variable is X_1^*, the system is perfectly identified as the continuity conditions are included. It can thus be solved numerically to find X_1^*. Hence, the investment trigger in regime 1 can be determined. Below is a plot of a numerical solution to this system of equations to find the investment decision.

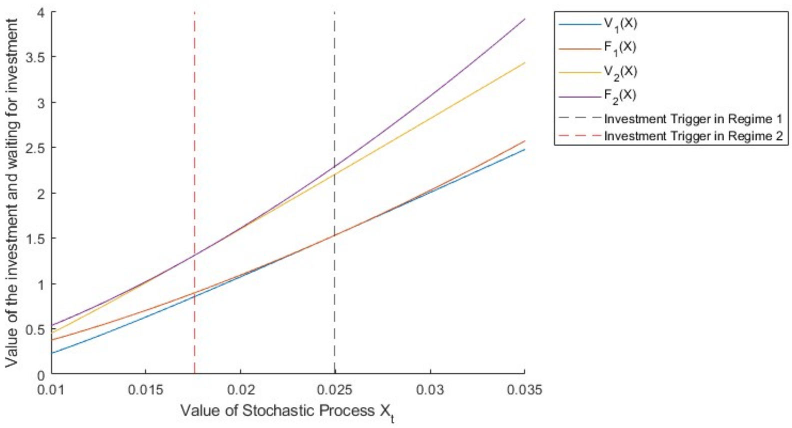


Figure 4: A plot of the value of waiting and the value of the firm at the moment of investment in both regimes as a function of X with parameter values: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.25$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda = 0.1$, $\delta = 0.1$, $\eta = 0.05$

For these parameter values, $X_1^* = 0.0250$ and $X_2^* = 0.0176$ are found. However, determining the investment capacity is different as the firm has two investment triggers and an investment range. Hence, in the next section, the different investment capacities are studied.

4.1.3 Analysis of the Investment Capacity

Next to these investment triggers, the firm also invests if the stochastic process, X_t, is between X_2^* and X_1^* and a switch occurs from regime 1 to regime 2 if the new regime stimulates investment and the initial regime discourages investment. Hence, the firm invests with an investment capacity given by $Q_2^*(X_t)$ for a level of X_t if a switch occurs in the transient region.

Since $X_t \in [X_2^*, X_1^*)$, and $Q_2^*(X)$ is strictly increasing in X, the investment capacity has a range bijectively mapped by the investment triggers. Therefore, there are three interesting investment capacities to analyze: $Q_1^* = Q_1^*(X_1^*)$, $Q_2^* = Q_2^*(X_2^*)$, and $Q_2^{'} = Q_2^*(X_1^*)$. The investment range is determined as follows $Q_S^* \in [Q_2^*, Q_2^{'})$. For the example for the parameters in Figure 4, $Q_1^* = 7.9393$, $Q_2^* = 7.7257$, and $Q^{'} = 8.3972$.

The investment capacities, Q_1^*, Q_2^* and $Q_2^{'}$, are studied below. From basic analysis, it can be shown that

$$\begin{aligned} Q_2^{'} &= Q_2^*(X_1^*) \\ &= \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu_2)}{X_1^*}\Big) \\ &> \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu_2)}{X_2^*}\Big) = Q_2^* \end{aligned}$$

and if $\mu_1 < \mu_2$

$$\begin{aligned} \Lambda &= \frac{r+\lambda-\mu_2}{(r-\mu_2)(r+\lambda-\mu_1)} \\ &< \frac{r+\lambda-\mu_1}{(r-\mu_2)(r+\lambda-\mu_1)} \\ &= \frac{1}{r-\mu_2} \end{aligned}$$

or if $\mu_1 > \mu_2$

$$\begin{aligned} \Lambda &= \frac{r+\lambda-\mu_2}{(r-\mu_2)(r+\lambda-\mu_1)} \\ &> \frac{r+\lambda-\mu_1}{(r-\mu_2)(r+\lambda-\mu_1)} \\ &= \frac{1}{r-\mu_2} \end{aligned}$$

Thus,

$$\begin{aligned} Q_2^{'} &= Q_2^*(X_1^*) \\ &= \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\mu_2)}{X_1^*}\Big) \\ &\begin{cases} > \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda X_1^*}\Big) = Q_1^* \text{ if } \mu_2 > \mu_1 \\ < \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda X_1^*}\Big) = Q_1^* \text{ if } \mu_2 < \mu_1 \end{cases} \end{aligned}$$

Therefore, it always holds that $Q_2^* < Q_2^{'}$, and $Q_2^{'} > Q_1^*$ if $\mu_2 > \mu_1$ or $Q_2^{'} < Q_1^*$ if $\mu_2 < \mu_1$.

Now rests the comparison between Q_1^* and Q_2^*.

Yet, the analysis of the investment trigger is done first as the thresholds imply the investment capacities. In Figure 5, whether the investment model follows a stimulating or discouraging new regime is computed. Computation is applied because of the absence of an explicit expression between the investment triggers and parameters of the model. Nevertheless, the parameters do affect the value of investing. Namely, an increase in (one of) the drift parameters increases Λ. Hence, an increase in $\Delta\mu$ does not universally increase Λ. However, since μ_1 is constant in the example, an increase in $\Delta\mu$ corresponds to an increase in μ_2 and thus an increase in Λ. An increase in Λ results in a higher value of the firm at the moment of investment in regime 1 as

$$\begin{aligned}\frac{\partial V_1(X)}{\partial \Lambda} &= \frac{\partial}{\partial \Lambda}\left(\frac{X\Lambda}{4\eta} - \frac{\delta}{2\eta} + \frac{\delta^2}{4\eta\Lambda X}\right) \\ &= \frac{X}{4\eta} - \frac{\delta^2}{4\eta\Lambda^2 X} \\ &> \frac{X}{4\eta} - \frac{X^2}{4\eta X} = 0\end{aligned}$$

This inequality holds because the investment trigger has to be larger than δ/Λ as otherwise, the investment capacity is negative. Hence, an increase of Λ increases the value of the firm at the moment of investment. However, like in the no-switch models, the value of waiting is affected by the volatility and the drift of the stochastic process. Therefore, the investment triggers are also affected by these parameters, like in the model of Huisman and Kort.

From an intuitive view, an increase in volatility increases the value of waiting and therefore, the investment trigger, whereas the value of the firm is not affected. Yet, the effect of the drift on the investment trigger is unknown as both the value of waiting and the value of the firm increase as the drift increases. Only an implicit function is derived for X_1^* from σ and μ contrary to the model from Section 2. Whether the investment trigger in regime 1 is smaller or bigger than in regime 2 is thus computed numerically using the system of equations described above. Doing this yields Figure 5 below.

Figure 5 illustrates that a $\Delta\sigma < -0.005$ always results in an investment stimulating new regime. Therefore, lower volatility in the relationship between price and capacity in the second regime causes the investment trigger to decrease with respect to the investment trigger in regime 1 as the value of waiting decreases.

The effect of $\Delta\mu$ is not monotone as is expected. An increase in $\Delta\mu$ for $\Delta\mu < 0$ and $\Delta\sigma \in (-0.1, 0.3)$ results in more investment-discouraging new regimes. However, for higher values of $\Delta\mu$, an increase in $\Delta\mu$ results in more investment stimulating new regimes. This result suggests that the relationship between $\Delta\mu$ and $\Delta\sigma$ to determine whether the new regime is investment stimulating or discouraging is not monotone.

Similar to the investment triggers, an explicit expression for the investment capacities is absent. Therefore, to determine whether Q_1 is bigger than Q_2 or vice versa, the investment capacities are computed numerically. In Figure 6, whether the investment capacity in the initial regime (Q_1^*) is bigger or smaller than in the second regime (Q_2^*) is computed for the same values for the parameters as in Figure 5.

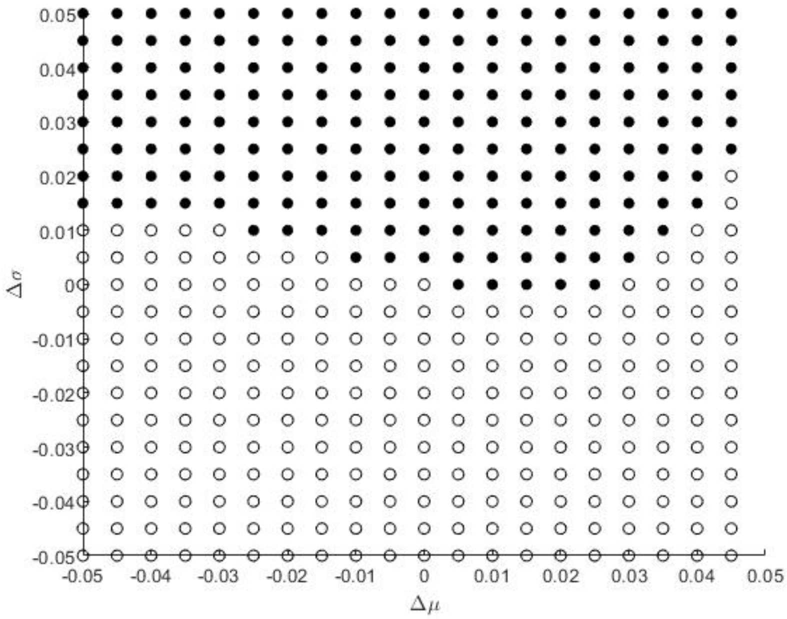


Figure 5: Plot of where X_1^* is bigger (here the open dots and denotes the investment stimulating new regime) or smaller (here the filled dots and denotes the investment discouraging new regime) than X_2^* for different values of $\Delta\mu$ and $\Delta\sigma$ with parameter values: $\mu_1 = 0.05$, $\mu_2 = \mu_1 + \Delta\mu$, $\sigma_1 = 0.1$, $\sigma_2 = \sigma_1 + \Delta\sigma$, $r = 0.1$, $\lambda = 0.1$, $\delta = 0.1$, $\eta = 0.05$

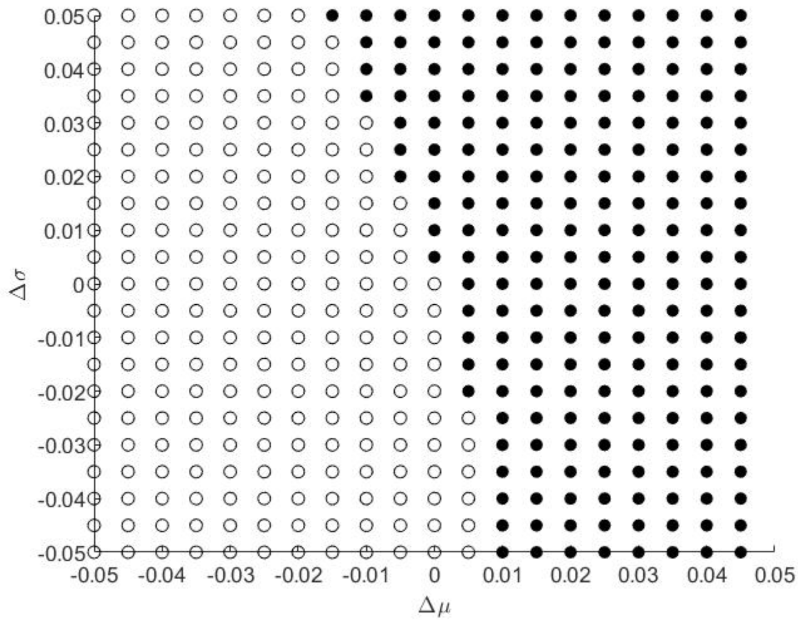


Figure 6: Plot of where Q_1^* is bigger (open dots) or smaller (filled dots) than Q_2^* for different values of $\Delta\mu$ and $\Delta\sigma$ with parameter values similar to the example in Figure 5.

Analyzing the region where Q_2^* is bigger than Q_1^*, a higher value of $\Delta\sigma$ allows for lower values of $\Delta\mu$. This coincides with a higher value of σ_2 with respect to σ_1 causing the value of waiting to increase in regime 2. Hence, the firm in regime 2 now invests later and with a relatively higher investment capacity than in regime 1. This results in a higher investment capacity for regime 2 compared to a similar model but with lower σ_2. Similarly, increases in $\Delta\mu$ (and thus increases in μ_2) increase the investment capacity in regime 2.

Intuitively, an increase in the drift of the stochastic process increases the NPV of the investment. The firm profits from this increase in NPV by increasing its investment capacity. Furthermore, from basic option theory, an increase in the volatility of the investment increases the value of waiting, while the value of investing is not affected by this change in volatility. This causes the firm to invest later and therefore with a higher investment capacity as the firm is in a better position if it invests. Both these effects are present in Figure 6 where the relationship of $\Delta\mu$ and $\Delta\sigma$ on which investment capacity is bigger is monotone.

In conclusion, a higher drift of the stochastic process in regime 2 compared to the drift in regime 1 increases both the value of investing and the value of waiting. The drift rate thus affects the investment trigger non-monotonically for different values of the volatility of the investment as whether the effect of the drift rate on the value of waiting is bigger than on the value of investing is unknown. This is in line with the results of Balter et al. (2023), who found that the drift has a twofold effect on the timing of the investment in a single-switch model.

An increase in the volatility of the investment postpones the investment as only the value of waiting increases. Moreover, the net effect of a higher drift in a certain regime results in a higher investment capacity in that regime 2 compared to the investment capacity in the other regime and an increase in volatility also increases the investment capacity. These conclusions are in line with the findings of Huisman and Kort (2015) in the model without switching but without regime switches of the stochastic process.

4.2 Continuous Regime Switch Model

The previous model can be applied in several fields. The single-switch models can for instance allow for the analysis of the life cycle of products (Balter et al., 2023), where the change in the drift of the Brownian motion is from a positive drift rate to a negative drift rate. Balter et al. also studied the case where the investment can only take place before the regime switch. Therefore, they have a different value of waiting in this instance than the one derived in Section 4.1 and they could find a closed-form solution for the investment trigger and investment capacity.

However, the single-switch model also has shortcomings as it does not allow for a switch back to the initial state. A real options model that allows for continuous switches between states can be used to analyze the effect of geopolitical unrest as the tensions between two countries rise and subside continuously. To model geopolitical unrest, the firm has a state in which the investment has a higher volatility and lower drift during periods of turmoil and another state in which the two countries are on friendly terms and the stochastic process has a higher drift and lower volatility. Therefore, a model that allows for continuous switching between two states is derived in the upcoming part.

The changes in state are defined as a Markov process with two states with a rate matrix $Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$. Hence, the probability matrix $P = I + Qdt$ for an infinitely small increment dt.

4.2.1 Derivation of the Value of the Firm at the Moment of Investment

From here the value of the firm in regime 1 and regime 2 is derived using the same assumptions as in Section 4.1 and that the firm knows which regime it is in. Similar to the NPV of the investment of the firm in regime 1 in the single switch model, a system of equations is obtained using the Bellman equation for $R_1(X)$ and $R_2(X)$ respectively. These are

$$\begin{cases} rR_1(X) & = Q(1-\eta Q)X + \mu_1 X R_1'(X) + \frac{1}{2}\sigma_1^2 R_1''(X) + \lambda_1(R_2(X) - R_1(X)) \\ rR_2(X) & = Q(1-\eta Q)X + \mu_2 X R_2'(X) + \frac{1}{2}\sigma_2^2 R_2''(X) + \lambda_2(R_1(X) - R_2(X)) \end{cases}$$

Since the revenue is linear with respect to X in the model of Huisman and Kort and $X = 0$ results in $R_i(X) = 0$, the revenue at the moment of investment is assumed to be $R_i(X) = b_i X$. Therefore, substituting this expression for $R_1(X)$ into the system of equations yields

$$\begin{aligned} (r+\lambda_1)b_1 X &= Q(1-\eta Q)X + \mu_1 X b + \frac{1}{2}\sigma_1^2 * 0 + \lambda_1 b_2 X \\ (r+\lambda_1-\mu_1)b_1 X &= Q(1-\eta Q)X + \lambda_1 b_2 X \\ b_1 &= \frac{Q(1-\eta Q) + \lambda_1 b_2}{r+\lambda_1-\mu_1} \end{aligned}$$

Similarly,

$$b_2 = \frac{Q(1-\eta Q) + \lambda_2 b_1}{r+\lambda_2-\mu_2}$$

Substituting this into the formula for b_1 gives

$$\begin{aligned} b_1 &= \frac{Q(1-\eta Q) + \lambda_1 \frac{Q(1-\eta Q)+\lambda_2 b_1}{r+\lambda_2-\mu_2}}{r+\lambda_1-\mu_1} \\ b_1 &= \frac{(r+\lambda_2-\mu_2)Q(1-\eta Q) + \lambda_1(Q(1-\eta Q)+\lambda_2 b_1)}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)} \\ (r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)b_1 &= Q(1-\eta Q)(r+\lambda_1+\lambda_2-\mu_2) + \lambda_1\lambda_2 b_1 \\ b_1 &= \frac{Q(1-\eta Q)(r+\lambda_1+\lambda_2-\mu_2)}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2} \\ &= \Lambda_1 Q(1-\eta Q) \end{aligned}$$

where $\Lambda_1 = \frac{r+\lambda_1+\lambda_2-\mu_2}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2}$

Similarly, b_2 is derived with $b_2 = \Lambda_2 Q(1-\eta Q)$ and $\Lambda_2 = \frac{r+\lambda_1+\lambda_2-\mu_1}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2}$. Thus,

$$R_i(X) = \Lambda_i Q(1-\eta Q)X$$

with $\Lambda_i = \frac{r+\lambda_1+\lambda_2-\mu_1-\mu_2+\mu_i}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2}$

Therefore, the value of investing in the two regimes is

$$\begin{aligned} V_1(X) &= R_1(X) - \delta Q = \Lambda_1 Q(1-\eta Q)X - \delta Q \\ V_2(X) &= R_2(X) - \delta Q = \Lambda_2 Q(1-\eta Q)X - \delta Q \end{aligned}$$

Using FOC yields the optimal investment capacity for any level of the stochastic process, X. This step yields

$$\begin{aligned} \frac{\partial V_i(X)}{\partial Q} &= 0 \\ \Lambda_i(1-2\eta Q)X - \delta &= 0 \\ 1 - 2\eta Q &= \frac{\delta}{\Lambda_i X} \\ Q_i^*(X) &= \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_i X}\Big) \end{aligned}$$

The SOC is

$$-2\eta\Lambda_i X < 0$$

because η, Λ_i, and X are greater than zero. Hence, the FOC yields a global maximum.

The derivation for the value of the firm at the moment of investment given X is similar to the one in the single switch model, but now with Λ_i instead of Λ. This is

$$V_i(X) = \frac{(\Lambda_i X - \delta)^2}{4\eta\Lambda_i X} \tag{15}$$

4.2.2 Derivation of the Value of Waiting

Using the Bellman equation, a system of differential equations is derived for the value of waiting in regime 1 and regime 2. For the sake of notation, regime 1 is the investment-discouraging regime, and regime 2 is the investment-stimulating regime. I.e. the investment trigger in regime 1 is bigger than in regime 2 ($X_2^* < X_1^*$). The starting regime does not depend on whether it is in an investment-stimulating or investment-discouraging regime. The starting regime can thus be any of the two regimes. Moreover, it is assumed that the current regime is known to the firm. Therefore, similar to the single switch model, in regime 1, the value of waiting is

$$\begin{cases} rF_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1(F_2(X) - F_1(X)) \text{ if } X \in [0, X_2^*) \\ rF_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1(V_2(X) - F_1(X)) \text{ if } X \in [X_2^*, X_1^*) \end{cases}$$

and in regime 2

$$rF_2(X) = \mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X) + \lambda_2(F_1(X) - F_2(X)) \text{ if } X \in [0, X_2^*)$$

Hence, if $X \in [0, X_2^*)$, the value of waiting should satisfy the following system of equations

$$\begin{cases} (r+\lambda_1)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 F_2(X) \\ (r+\lambda_2)F_2(X) = \mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X) + \lambda_2 F_1(X) \end{cases}$$

Finding a solution to this system of differential equations is different from the single-switch model as the regime can also switch back to the previous regime. Therefore, a different approach is used.

Say a homogeneous solution of the differential equation for F_1 has the form $A_1 X^{\gamma_1}$ and for F_2 has the form $A_2 X^{\gamma_2}$, where $\gamma_1 \neq \gamma_2$. The corresponding particular solution for F_2 has to be of the form $B_2 X^{\gamma_1}$ and the corresponding particular solution for F_1 has to be of the form $B_1 X^{\gamma_2}$. Therefore, the particular solution of the differential equation for F_2 solves for

$$\begin{aligned} (r+\lambda_2)B_2 X^{\gamma_1} &= \mu_2\gamma_1 B_2 X^{\gamma_1} + \frac{1}{2}\sigma_2^2 B_2 X^{\gamma_1}\gamma_1(\gamma_1 - 1) + \lambda_2 A_1 X^{\gamma_1} \\ r+\lambda_2 &= \mu_2\gamma_1 + \frac{1}{2}\sigma_2^2\gamma_1(\gamma_1 - 1) + \lambda_2 A_1/B_2 \\ r+\lambda_2 - \mu_2\gamma_1 - \frac{1}{2}\sigma_2^2\gamma_1(\gamma_1 - 1) &= \lambda_2 A_1/B_2 \\ B_2 &= \frac{\lambda_2}{g_2(\gamma_1)} A_1 \end{aligned}$$

where $g_i(\beta) = r + \lambda_i - \mu_i\beta - \frac{1}{2}\sigma_i^2\beta(\beta - 1)$. Using this result to find the homogeneous solution to the differential equation of F_1 yields

$$\begin{aligned}(r+\lambda_1)A_1X^{\gamma_1} &= \mu_1\gamma_1A_1X^{\gamma_1} + \frac{1}{2}\sigma_1^2\gamma_1(\gamma_1-1)A_1X^{\gamma_1} + \lambda_1\frac{\lambda_2}{g_2(\gamma_1)}A_1X^{\gamma_1}\\ r+\lambda_1 &= \mu_1\gamma_1 + \frac{1}{2}\sigma_1^2\gamma_1(\gamma-1) + \frac{\lambda_1\lambda_2}{g_2(\gamma_1)}\\ g_1(\gamma_1) &= \frac{\lambda_1\lambda_2}{g_2(\gamma_1)}\\ g_1(\gamma_1)g_2(\gamma_1) &= \lambda_1\lambda_2\end{aligned}$$

Similarly, $g_1(\gamma_2)g_2(\gamma_2) = \lambda_1\lambda_2$ and $B_1 = \frac{\lambda_1}{g_1(\gamma_2)}A_2$. Since $\frac{g_1(\gamma_2)}{\lambda_1} = \frac{\lambda_2}{g_2(\gamma_2)}$, $A_2 = \frac{\lambda_2}{g_2(\gamma_2)}B_1$ (Note that this only holds if $\lambda_i \neq 0$ for $i \in \{1,2\}$). Therefore, the solutions to the differential equations are found in the region $X_t \in [0, X_2^*)$, with γ_1 and γ_2 being solutions to $g_1(\gamma)g_2(\gamma) = \lambda_1\lambda_2$.

Furthermore, as the boundary condition, $F_i(X) = 0$ for $X = 0$, has to hold, γ_1 and γ_2 are positive and different solutions to the polynomial. Thus, in the region $X_t \in [0, X_2^*)$, the following value of waiting in the two regimes are defined as

$$\begin{cases} F_1(X) = A_1X^{\gamma_1} + B_1X^{\gamma_2} \\ F_2(X) = \dfrac{\lambda_2}{g_2(\gamma_1)}A_1X^{\gamma_1} + \dfrac{\lambda_2}{g_2(\gamma_2)}B_1X^{\gamma_2} \end{cases} \tag{16}$$

This result is in line with the findings of Luo and Yang (2017) and their continuous regime switch model for contingent claims.

Now the value of waiting has been defined in the region for $X_t \in [0, X_2^*)$ for both regimes. The value of waiting for $X_t \in [X_2^*, X_1^*)$ has still to be defined. In this region, only the value of waiting in regime 1 is analyzed because the firm has already invested if the firm is in regime 2 as $X_t > X_2^*$. F_1 has the following differential equation:

$$(r+\lambda_1)F_1(X) = \mu_1XF_1'(X) + \frac{1}{2}\sigma_1^2X^2F_1''(X) + \lambda_1V_2(X)$$

The homogeneous part of the solution for F_1 solves

$$(r+\lambda_1)F_1(X) = \mu_1XF_1'(X) + \frac{1}{2}\sigma_1^2X^2F_1''(X)$$

Hence, the guess for the homogeneous solution of F_1 is of the form HX^{β}. This gives

$$\begin{aligned}(r+\lambda_1)HX^{\beta} &= \mu_1\beta HX^{\beta} + \frac{1}{2}\sigma_1^2\beta(\beta-1)HX^{\beta}\\ (r+\lambda_1) &= \mu_1\beta + \frac{1}{2}\sigma_1^2\beta(\beta-1)\end{aligned}$$

Therefore, β is the solution to $g_1(\beta) = 0$. This yields 2 solutions for β: β_1 and β_2. The homogeneous solution to this differential equation is thus $H_1X^{\beta_1} + H_2X^{\beta_2}$.

To find the particular solution to this differential equation, the same approach is used as in the single-switch model with an investment-stimulating new regime. The firm has a transient region where it would invest in regime 2 but has not invested yet in regime 1. Hence, the investment capacity does not only correspond to one investment trigger but is flexible to include investing in the transient region at the moment of a regime switch. By removing the dimension of the investment capacity in the value of investing, the value of investing becomes a polynomial of the form $aX + b + c/X$. The guess of the particular solution of the differential equation has thus the form $aX + b + c/X$. Substituting this expression into the differential equation gives

$$(r+\lambda_1)(aX+b+cX^{-1}) = \mu_1 X(a - cX^{-1}) + \frac{1}{2}\sigma_1^2 X^2 * 2cX^{-3} + \lambda_1 \frac{(\Lambda_2 X - \delta)^2}{4\eta\Lambda_2 X}$$

$$\begin{cases} (r+\lambda_1-\mu_1)a = \lambda_1 \frac{\Lambda_2}{4\eta} \\ (r+\lambda_1)b = -\lambda_1 \frac{\delta}{2\eta} \\ (r+\lambda_1+\mu_1-\sigma_1^2)c = \lambda_1 \frac{\delta^2}{4\eta\Lambda_2} \end{cases}$$

$$\begin{cases} a = \frac{\lambda_1\Lambda_2}{4\eta(r+\lambda_1-\mu_1)} = \frac{\lambda_1}{g_1(1)}\frac{\Lambda_2}{4\eta} \\ b = -\frac{\lambda_1\delta}{2\eta(r+\lambda_1)} = -\frac{\lambda_1}{g_1(0)}\frac{\delta}{2\eta} \\ c = \frac{\lambda_1\delta^2}{4\eta\Lambda_2(r+\lambda_1+\mu_1-\sigma_1^2)} = \frac{\lambda_1}{g_1(-1)}\frac{\delta^2}{4\eta\Lambda_2} \end{cases}$$

Thus the particular solution to the differential equation is also found. $F_1(X)$ on the domain $X_t \in [X_2^*, X_1^*)$ is defined as

$$F_1(X) = H_1 X^{\beta_1} + H_2 X^{\beta_2} + aX + b + c/X \tag{17}$$

$F_1(X)$ is continuous and continuously differentiable. Hence, the same continuity conditions for $F_1(X)$ as in an investment-stimulating new regime in the single switch model have to hold:

$$\begin{cases} \lim_{X\uparrow X_2^*} F_1(X) = \lim_{X\downarrow X_2^*} F_1(X) \\ \lim_{X\uparrow X_2^*} \frac{\partial F_1(X)}{\partial X} = \lim_{X\downarrow X_2^*} \frac{\partial F_1(X)}{\partial X} \end{cases}$$

$$\begin{cases} A_1(X_2^*)^{\gamma_1} + B_1(X_2^*)^{\gamma_2} = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \gamma_1 A_1(X_2^*)^{\gamma_1-1} + \gamma_2 B_1(X_2^*)^{\gamma_2-1} = \beta_1 H_1(X_2^*)^{\beta_1-1} + \beta_2 H_2(X_2^*)^{\beta_2-1} + a - c/(X_2^*)^2 \end{cases} \tag{18}$$

$$\begin{cases} A_1(X_2^*)^{\gamma_1} + B_1(X_2^*)^{\gamma_2} = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \gamma_1 A_1(X_2^*)^{\gamma_1} + \gamma_2 B_1(X_2^*)^{\gamma_2} = \beta_1 H_1(X_2^*)^{\beta_1} + \beta_2 H_2(X_2^*)^{\beta_2} + aX_2^* - c/X_2^* \end{cases}$$

4.2.3 Finding the Investment Trigger and Optimal Investment Capacity

Implicit expressions for the investment triggers in the two regimes are derived together with two equations (the continuity conditions for $F_1(X)$ in X_2^*) with six unknowns (A_1, B_1, H_1, H_2, X_1^*, X_2^*). Therefore, if the smooth-pasting and value-matching conditions in the two regimes are included, four extra equations are obtained and the system is perfectly identified. Therefore, the smooth-pasting and value-matching conditions in regime 1 and regime 2 are used. These conditions in regime 2 are

$$\begin{cases} F_2(X_2^*) = V_2(X_2^*) \\ \frac{\partial F_2(X)}{\partial X}\Big|_{X=X_2^*} = \frac{\partial V_2(X)}{\partial X}\Big|_{X=X_2^*} \end{cases}$$

$$\begin{cases} \frac{\lambda_2}{g_2(\gamma_1)} A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1(X_2^*)^{\gamma_2} = \frac{(\Lambda_2 X_2^* - \delta)^2}{4\eta\Lambda_2 X_2^*} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1(X_2^*)^{\gamma_1 - 1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1(X_2^*)^{\gamma_2 - 1} = \frac{\Lambda_2}{4\eta} - \frac{\delta^2}{4\eta\Lambda_2 (X_2^*)^2} \end{cases} \tag{19}$$

$$\begin{cases} \frac{\lambda_2}{g_2(\gamma_1)} A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1(X_2^*)^{\gamma_2} = \frac{(\Lambda_2 X_2^* - \delta)^2}{4\eta\Lambda_2 X_2^*} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1(X_2^*)^{\gamma_2} = \frac{\Lambda_2 X_2^*}{4\eta} - \frac{\delta^2}{4\eta\Lambda_2 X_2^*} \end{cases}$$

These conditions in regime 1 are

$$\begin{cases} F_1(X_1^*) = V_1(X_1^*) \\ \frac{\partial F_1(X)}{\partial X}\Big|_{X=X_1^*} = \frac{\partial V_1(X)}{\partial X}\Big|_{X=X_1^*} \end{cases}$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(\Lambda_1 X_1^* - \delta)^2}{4\eta\Lambda_1 X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1 - 1} + \beta_2 H_2(X_1^*)^{\beta_2 - 1} + a - c/(X_1^*)^{-2} = \frac{\Lambda_1}{4\eta} - \frac{\delta^2}{4\eta\Lambda_1 (X_1^*)^2} \end{cases} \tag{20}$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(\Lambda_1 X_1^* - \delta)^2}{4\eta\Lambda_1 X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} + aX_1^* - c/X_1^* = \frac{\Lambda_1 X_1^*}{4\eta} - \frac{\delta^2}{4\eta\Lambda_1 X_1^*} \end{cases}$$

Hence, the full system of equations is

$$\begin{cases} A_1(X_2^*)^{\gamma_1} + B_1(X_2^*)^{\gamma_2} = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \gamma_1 A_1(X_2^*)^{\gamma_1} + \gamma_2 B_1(X_2^*)^{\gamma_2} = \beta_1 H_1(X_2^*)^{\beta_1} + \beta_2 H_2(X_2^*)^{\beta_2} + aX_2^* - c/X_2^* \\ \\ \frac{\lambda_2}{g_2(\gamma_1)} A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1(X_2^*)^{\gamma_2} = \frac{(\Lambda_2 X_2^* - \delta)^2}{4\eta\Lambda_2 X_2^*} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1(X_2^*)^{\gamma_2} = \frac{\Lambda_2 X_2^*}{4\eta} - \frac{\delta^2}{4\eta\Lambda_2 X_2^*} \\ \\ H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(\Lambda_1 X_1^* - \delta)^2}{4\eta\Lambda_1 X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} + aX_1^* - c/X_1^* = \frac{\Lambda_1 X_1^*}{4\eta} - \frac{\delta^2}{4\eta\Lambda_1 X_1^*} \end{cases} \tag{21}$$

This system of equations for certain parameter values is computed using Matlab. The following graph is obtained.

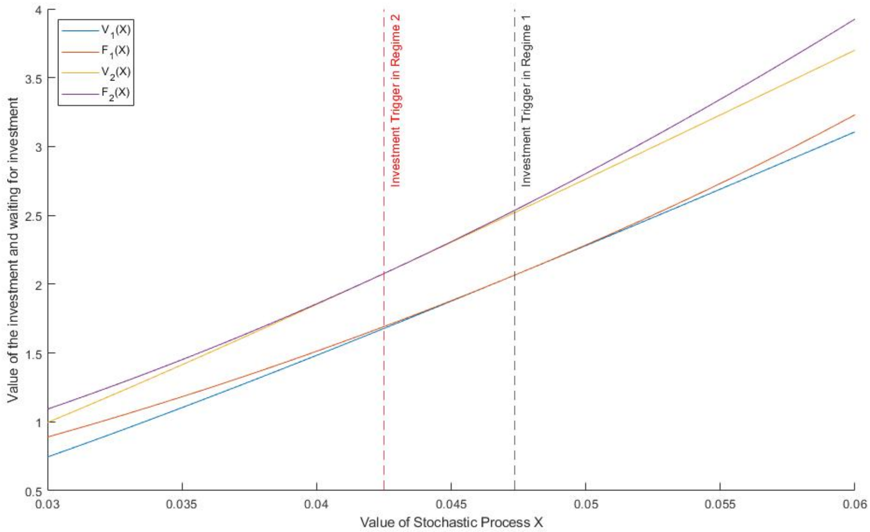


Figure 7: Plot of the value of waiting and the value of the investment in both regimes for a continuous regime switch model as a function of X with parameter values: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\delta = 0.25$ and $\eta = 0.05$

Using the investment triggers, the optimal investment capacity is obtained using the formula for $Q_i^*(X)$. For this example, $X_1^* = 0.0474$ with $Q_1^* = 7.0189$ and $X_2^* = 0.0425$ with $Q_2^* = 7.0277$, and the upper bound of the investment capacity in the transient region: $Q_2^{'} = 7.3327$. Similar to the single regime switch model, the investment decision is studied for different parameters of the stochastic process. For the relation with Q_2^*, the definition of the investment triggers is used: $X_1^* > X_2^*$. Hence,

$$\begin{aligned} Q_2^{'} = Q_2(X_1^*) &= \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_2 X_1^*}\Big) \\ &> \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_2 X_2^*}\Big) = Q_2^* \end{aligned}$$

Therefore, the upper bound of the investment capacity in the transient region is always bigger than the investment capacity at the trigger in regime 2. Furthermore, to determine the relation between Q_1^* and $Q_2^{'}$, the relation between Λ_1 and Λ_2 is analyzed first. This relation is

$$\begin{aligned} \Lambda_1 &= \frac{r + \lambda_1 + \lambda_2 - \mu_2}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} \\ \Lambda_1 &< \frac{r + \lambda_1 + \lambda_2 - \mu_1}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} = \Lambda_2 \text{ if } \mu_2 > \mu_1 \\ \Lambda_1 &> \frac{r + \lambda_1 + \lambda_2 - \mu_1}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} = \Lambda_2 \text{ if } \mu_2 < \mu_1 \end{aligned}$$

Consequently, the upper bound of the investment range is

$$\begin{aligned} Q_2^{'} = Q_2(X_1^*) &= \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_2 X_1^*}\Big) \\ Q_2^{'} &> \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_1 X_1^*}\Big) = Q_1^* \text{ if } \Lambda_1 < \Lambda_2 \iff \mu_2 > \mu_1 \\ Q_2^{'} &< \frac{1}{2\eta}\Big(1 - \frac{\delta}{\Lambda_1 X_1^*}\Big) = Q_1^* \text{ if } \Lambda_1 > \Lambda_2 \iff \mu_2 < \mu_1 \end{aligned}$$

Hence, it depends on the drift of the stochastic process whether the upper bound of the investment capacity range is bigger or smaller than the investment capacity at the investment trigger in regime 1. Now it rests to analyze which sets of parameters the regimes are investment-stimulating or discouraging and when the investment capacity in one regime is bigger than the investment capacity in the other. Below is a scatter plot of which regime is investment-discouraging and which regime is investment-stimulating. In this plot, the subscripts denote the parameters of the particular regime. This does not imply that regime 1 is always the investment-discouraging regime and regime 2 is the investment-stimulating regime.

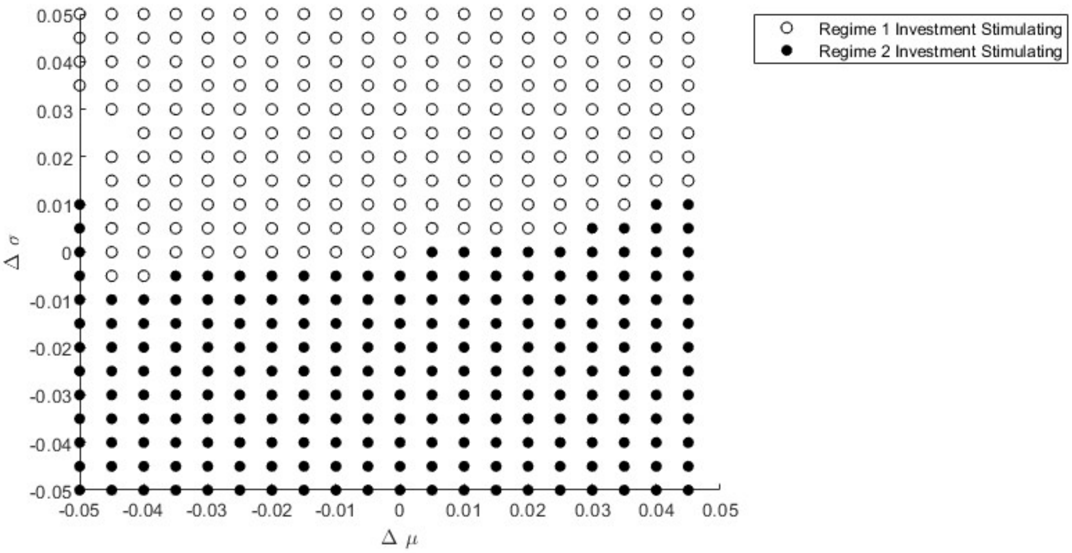


Figure 8: Scatter plot of which regime is investment-discouraging or investment-stimulating for parameter values: $\mu_1 = 0.05$, $\mu_2 = \mu_1 + \Delta\mu$, $\sigma_1 = 0.1$, $\sigma_2 = \sigma_1 + \Delta\sigma$, $r = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\delta = 0.25$ and $\eta = 0.05$

First, it was not possible to find solutions for $\mu_2 = 0$ and $\sigma_2 \in [0.015, 0.03]$ in Figure 8. Because the system consists of nonlinear equations for which guesses are used to determine which regime is investment-stimulating, the software may not always yield a viable answer. Therefore, this omission of solutions is treated as an area where definitive conclusions about the model could not be stated and the remainder of the plot is emphasized.

In the remainder of the figure, a non-monotone relationship between the parameters is detected in determining which regime stimulates investment (i.e. $X_i^* > X_j^*$ for $i \neq j$). There is a lower bound for the value of $\Delta\sigma$ and thus σ_2 for which Regime 1 is investment-stimulating. This implies that an increase in volatility of the stochastic process in regime 2 causes the investment trigger in regime 1 to decrease with respect to the investment trigger in regime 2.

Furthermore, an increase in μ_2 does not lead to regime 2 becoming the investment-stimulating regime as for $\mu_2 = 0$ and $\sigma_2 = 0.105$ regime 2 is the investment-stimulating regime, but for $\mu_2 = 0.04$ and $\sigma_2 = 0.105$ regime 1 is investment-stimulating. This can be explained by the twofold effect of the drift of the stochastic process as a lower drift decreases the NPV of the investment but also the value of waiting. Therefore, it depends on which effect of the drift is bigger.

Apart from the analysis of the investment trigger, whether the investment capacity at the trigger in regime 1 is bigger or smaller than the investment capacity of the trigger in regime 2 is also studied. Figure 9 is a scatter plot of this with similar parameters as in Figure 8.

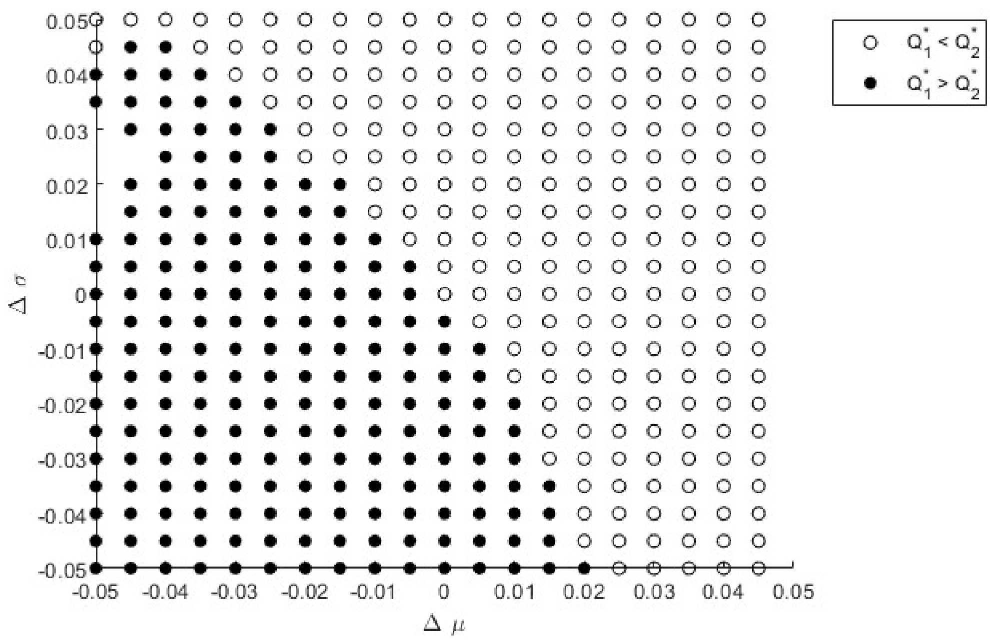


Figure 9: Scatter plot of whether the investment capacity in one regime is bigger or smaller than the other for parameter values: $\mu_1 = 0.05$, $\mu_2 = \mu_1 + \Delta\mu$, $\sigma_1 = 0.1$, $\sigma_2 = \sigma_1 + \Delta\sigma$, $r = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\delta = 0.25$ and $\eta = 0.05$

Similar to Figure 8, there is a gap in the figure for the same set of parameters because the software is not able to find solutions to this set of parameters. This area is treated identically to the same area in Figure 8 as no information is available in this part of the plot.

Moreover, contrary to the investment capacities in the single switch model, the effect of the parameters on which investment capacity at the triggers is bigger is not monotone. Namely, for $\sigma_2 = 0.145$ and $\mu_2 = 0.01$, $Q_1^* > Q_2^*$ and sufficiently large increases or decreases in the drift of regime 2 cause Q_2^* to increase with respect to Q_1^*. This may occur as the effect of the drift on the investment trigger is twofold and thus the effect on the investment capacity at the trigger is also twofold.

However, as the effect of μ_2 for larger values is analyzed, Q_2^* increases compared to Q_1^* if μ_2 increases. Therefore, an increase in the drift in regime 2 of the stochastic process causes the firm to invest with a higher capacity at the investment trigger for this regime. Additionally, a higher value for volatility in regime 2 causes the firm to also invest with a higher capacity for that particular regime. Therefore, increases in the expectation of the revenue of the firm make it invest with a higher capacity to profit from this more favorable state. Furthermore, the effect of an increase in volatility has a similar explanation as in the single switch model, because now for higher values of volatility, the firm invests later and thus with a higher capacity. Hence, the drift rate and the volatility in the two regimes still affect the investment decision.

4.3 Investment Decision for different Regime Switch Rates

Apart from variable parameters of the Brownian motion in the different regimes, the effect of the regime switch rates is also studied as these affect the investment decision. In Table 1 and Table 2, the investment decision is computed for different regime switch rates. Whereas the single switch model is applied to analyze the investment decision using the product life cycle theory, geopolitical unrest and business cycles are in line with the continuous switch model. Therefore, the continuous regime switch model is analyzed in particular for various regime switch rates.

Table 1: The investment triggers in both regimes, where the two numbers in the cells correspond to (X_1^*, X_2^*) for the different change rate in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta = 0.25$, $\eta = 0.05$

	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(0.0464, 0.0419)	(0.0454, 0.0415)	(0.0451, 0.0415)	(0.0449, 0.0415)	(0.0448, 0.0417)
$\lambda_1 = 0.4$	(0.0467, 0.0425)	(0.0457, 0.0420)	(0.0453, 0.0418)	(0.0450, 0.0418)	(0.0448, 0.0419)
$\lambda_1 = 0.6$	(0.0467, 0.0429)	(0.0459, 0.0424)	(0.0454, 0.0421)	(0.0451, 0.0420)	(0.0449, 0.0420)
$\lambda_1 = 0.8$	(0.0467, 0.0431)	(0.0459, 0.0426)	(0.0455, 0.0424)	(0.0451, 0.0422)	(0.0449, 0.0422)
$\lambda_1 = 1.0$	(0.0466, 0.0432)	(0.0459, 0.0428)	(0.0455, 0.0425)	(0.0452, 0.0424)	(0.0450, 0.0423)

Table 2: The investment capacities in both regimes, where the three numbers in the cells correspond to $(Q_1^*, Q_2^*, Q_2^{'})$ for the different change rate in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta = 0.25$, $\eta = 0.05$

	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(6.67, 6.62, 6.94)	(6.29, 6.17, 6.51)	(6.09, 5.94, 6.27)	(5.97, 5.81, 6.12)	(5.89, 5.73, 6.02)
$\lambda_1 = 0.4$	(7.05, 6.96, 7.23)	(6.67, 6.54, 6.82)	(6.43, 6.28, 6.56)	(6.27, 6.11, 6.39)	(6.16, 6.00, 6.26)
$\lambda_1 = 0.6$	(7.25, 7.14, 7.37	(6.90, 6.76, 7.01)	(6.66, 6.51, 6.76)	(6.49, 6.34, 6.58)	(6.36, 6.20, 6.45)
$\lambda_1 = 0.8$	(7.36, 7.25, 7.46)	(7.05, 6.92, 7.14)	(6.82, 6.68, 6.91)	(6.65, 6.50, 6.73)	(6.52, 6.37, 6.59)
$\lambda_1 = 1.0$	(7.44, 7.32, 7.52)	(7.16, 7.03, 7.23)	(6.94, 6.81, 7.02)	(6.78, 6.64, 6.85)	(6.64, 6.50, 6.71)

Analyzing the output of Table 2, an increase in λ_1 causes both investment capacities at the triggers (Q_1^* and Q_2^*) and the maximum investment capacity for the investment range (Q_2') to increase and an increase in λ_2 leads to decreases in all investment capacities. The reason for this is that an increase in λ_1 relative to λ_2 increases the long-term probability of being in the more profitable regime 2. Therefore, to capitalize on this increase in net present value, the firm increases its investment capacity universally. Likewise, an increase in λ_2, relative to λ_1 increases the long-run probability of being in the less fruitful regime 1 and the firm lowers its investment capacity. Hence, the long-term perspective is the most important in determining the investment capacity. A reason for this result is the lumpiness of the investment as the firm can only invest once with a certain capacity.

The effect of the regime switch rate on the investment trigger is not monotone. In Table 1, the investment triggers increase for increases in λ_1 except for the investment trigger in regime 1 if $\lambda_2 = 0.2$. In this anomaly, if $\lambda_1 > 0.2$, increases in λ_1 coincide with decreases in X_1^*. A similar phenomenon is observed for λ_2, where an increase in λ_2 does not universally correspond to a decrease in the investment trigger in regime 2. For instance, if $\lambda_1 = 0.2$, X_2^* does not decrease for higher values of λ_2. However, higher values of λ_2 cause the investment trigger in regime 1 to decrease for all values of λ_1 and higher values of λ_1 jibe with increases in X_2^*. Thus, a sufficiently low regime switch rate in one regime while the other regime switch rate increases has a non-monotone effect on the investment trigger in the regime with an increased regime switch rate.

To study the effects of the regime switch rates on the investment trigger, a plot is made of the investment trigger against the regime switch rates. In Figure 10 and Figure 11 these graphs are plotted. From now on with the current and new regimes, this is meant with respect to the regime switch rate. For instance, the current regime of λ_1 is regime 1 and the new regime of λ_1 is regime 2.

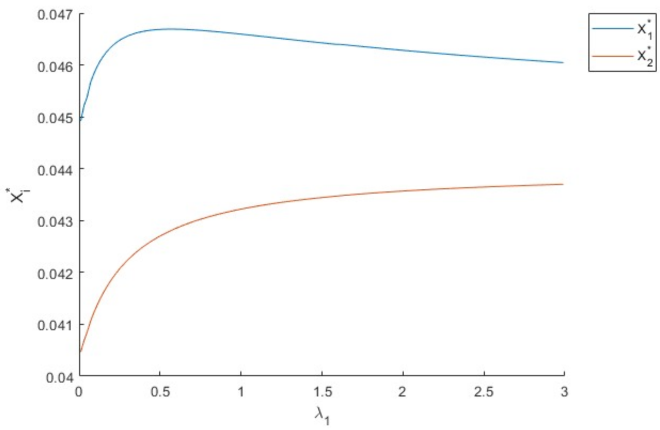


Figure 10: Plot of the investment triggers for variable λ_1 and $\lambda_2 = 0.2$

In Figure 10, the non-monotone relation between λ_1 and X_1^* from Table 1 is observed. Namely, before a certain value for λ_1, the investment trigger increases with increases in λ_1, whereafter the investment trigger decreases with increases in λ_1. A similar but opposite effect of λ_2 on X_2^* is observed in Figure 11. This indicates that the effect of the regime switch rate on the investment decision in the current regime is twofold.

For low values of the switch rates, the probability that the firm stays in the current regime is bigger than for higher switch rates. Hence, the firm bases the investment strategy on the current regime. However, the investment capacity captures the long-term perspective. Therefore, to include the change in perspective of the investment decision, the investment capacity is adjusted

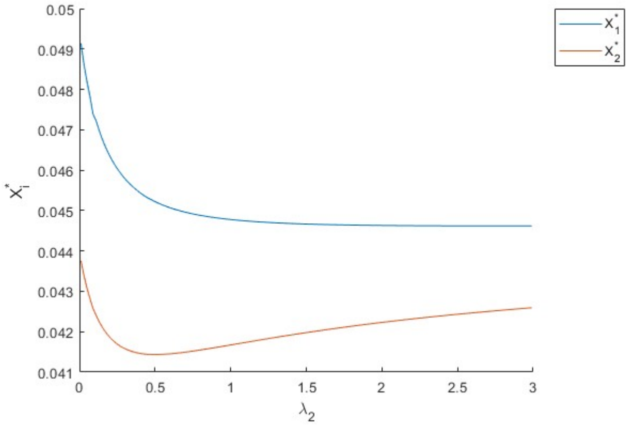


Figure 11: Plot of the investment triggers for variable λ_2 and $\lambda_1 = 0.2$

at the expense of the investment trigger. This result is supported by the effect of low switch rates on the investment capacity (see e.g. Figure 31a and Figure 31d in Appendix D where at lower regime switch rates, changes in switch rates coincide with big changes in investment capacity). At low switch rates, advantageous changes in regime switch rates thus negatively affect the timing of the investment in the current regime because the firm prioritizes the investment capacity over the timing.

Higher regime switch rates, on the contrary, cause a faster convergence to the long-run distribution of the regimes. Hence, the firm does not focus on the current regime as much as for lower regime switch rates. Now, the firm invests with a long-term perspective instead of emphasizing the current regime. Since the investment capacity captures the long-term perspective but converges as the regime switch rates tend to infinity, the investment capacity can not fully capture change in NPV (This confinement of the investment capacity is also observed in Figure 31a and Figure 31d, where higher regime switch rates correspond to small changes in investment capacities.) Therefore, the change in net present value due to the switch parameters is reflected in the other aspect of the investment: timing. Now, increases in the advantageous regime switch rate decrease the current investment trigger. Likewise, the current investment trigger increases for a higher disadvantageous regime switch rate.

In Figure 10, an increase in λ_1 results in an increase in X_2^*. Hence, the investment trigger in regime 2 increases as the regime switch rate to the advantageous regime increases. This relation may result from the effect of λ_1 on the duration of the firm being in regime 2. λ_1 namely does not affect the duration of the firm being in regime 2. λ_1 only affects the expected duration of being in state 1. Since the firm emphasizes the current regime for $\lambda = 0.2$ as shown in figure 10, the change in NPV is reflected in the investment capacity at the expense of the timing. A similar but opposite effect is observed of λ_2 on X_1^* in Figure 11, where the investment trigger is monotone and decreasing with increases in λ_2 due to the capacity effect.

To study whether the capacity effect is always leading in the new regime, the investment triggers in the new regime are plotted for bigger regime switch rates. E.g. if $\lambda_2 = 2$ for a variable λ_1, the capacity effect would decrease as the probability of being in regime 2 decreases.

In Figure 12, at lower values for λ_1, the investment trigger in regime 2 decreases. This suggests that the NPV effect dominates if λ_2 is large compared to λ_1. Therefore, the NPV effect is also present in X_2^* if λ_2 is sufficiently big. However, if the difference is not sufficient, the capacity effect dominates. Therefore, the regime switch parameter in a particular regime determines which effect of the switch rates on the timing of the investment in that regime is preeminent.

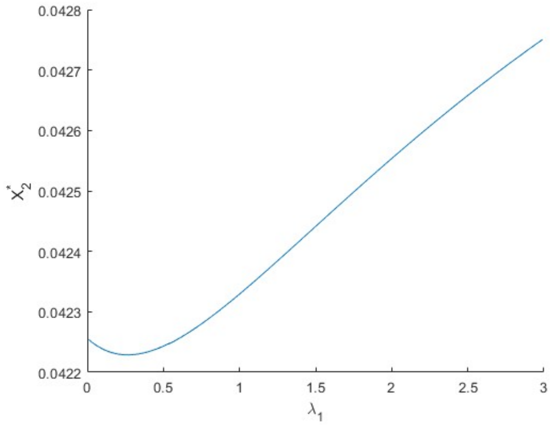


Figure 12: X_2^* for variable λ_1 and $\lambda_2 = 2$

In conclusion, the regime-switch rates affect both the timing of the investment and the investment capacity of the firm. The investment capacity depends on the long-term investment strategy of the firm as the firm cannot change the capacity post-investment. Hence, increases in the advantageous regime switch rates increase the investment capacity and increases in the disadvantageous regime switch rate decrease the investment capacity.

The effect of the regime switch rates on the timing of the investment is twofold. For small regime switch rates, the firm is likely to stay in the current regime. Hence, the firm alters its investment to the current regime. Yet, the investment capacity is based on the long-term perspective of the firm. Thus, the firm adjusts the investment capacity at the expense of timing to conform to the short-term perspective. In this case, advantageous changes in the switch rates increase the investment capacity, and the investment is postponed even though the investment becomes more profitable. For an increase in the disadvantageous regime switch rate when this rate is relatively small, a similar but opposite effect is observed. Therefore, for lower regime switch rates, the so-called capacity effect is preeminent in the timing of the investment.

For large regime switch rates, the investment capacity does not change majorly if the switch rates change. To capture the change in the net present value of the investment because of the change in switch rates, the firm adjusts the timing in the current regime. Now, an increase in the advantageous regime switch rate decreases the investment trigger in the current regime as the net present value of the investment becomes leading in the investment decision. This effect is called the NPV effect.

The effect of the current regime switch rate on the investment trigger in the new regime depends on the switch rate in the new regime as the increase in the current switch rate does not affect the duration of the firm being in the new regime. However, the switch rate in the new regime should be large compared to the switch rate in the current regime to overcome the capacity effect.

Lastly, if both regime switch rates increase simultaneously, the difference in investment triggers decreases because, for higher regime switch rates, more regime switches occur resulting in a smaller difference in the value of waiting and NPV at the moment of investment in the two regimes.

5 Combined model of lead time and Regime Switching

The central objective of this thesis is to find a real options model that includes both geopolitical unrest and time to build in the investment decision. In this section, a combined model that allows for extensions is derived. To derive a regime switch model that includes time to build, the adjustments from the time to build and regime switch models are combined separately. On the one hand, including time to build adjusts the value of the firm at the moment of investment as shown in Section 3. On the other hand, Section 4 illustrates that both the value of investing and the value of waiting change in the regime switch model. Therefore, the value of the firm at the moment of investment in the regime switch model is altered to include lead time. Since the value of waiting depends on the value of investing in the transient region, the value of waiting is also modified.

Section 3 shows that allowing for a variable time to build does not have a large effect on the investment trigger and investment capacity as this effect is inferior to the expected duration of the project. This section will extend the regime switch model to include a constant installation period of the investment. The model for a variable time-to-build is an extension of the constant time-to-build model. The stochastic lead time model is also derived in this Section. However, this model is not analyzed in particular as including a variable time to build corresponds to a slight adjustment in the constant time to build.

5.1 Net Present Value of the Investment

Finding the net present value (NPV) of the investment is different from the approach in the previous sections as the regime can switch during the time it takes to install the investment. Therefore, the firm may be in a new region where it has invested but the investment does not generate revenues yet: the installation period. In this region, $T \geq T + s < T + \theta$. After this period, the firm is in a different region where the plant is built where $T + s \geq T + \theta$.

If $s \geq \theta$, the revenue generated by the firm is identical to the value derived in the regime switch model without lead time as the firm does not have to wait until the investment yields revenue. The period after the installation period is thus equivalent to the stopping region in a regime switch model without lead time. Hence, the effect of the installation period on the value of investing is analyzed in this section because this region is the additional region to previously derived models.

During the installation period, the Bellman equation is applied to find the NPV of the revenue in regime i (R_i). This gives the following set of differential equations

$$\begin{cases} rR_1(X,s) = \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) + R_{1,s}(X,s) \\ \qquad\qquad + \lambda_1(R_2(X,s) - R_1(X,s)) \\ rR_2(X,s) = \mu_2 X R_{2,X}(X,s) + \frac{1}{2}\sigma_2^2 X^2 R_{2,XX}(X,s) + R_{2,s}(X,s) \\ \qquad\qquad + \lambda_2(R_1(X,s) - R_2(X,s)) \end{cases} \quad \text{if } s < \theta$$

$$\begin{cases} rR_1(X,s) = Q(1-\eta Q)X + \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) \\ \qquad\qquad + \lambda_1(R_2(X,s) - R_1(X,s)) \\ rR_2(X,s) = Q(1-\eta Q)X + \mu_2 X R_{2,X}(X,s) + \frac{1}{2}\sigma_2^2 X^2 R_{2,XX}(X,s) \\ \qquad\qquad + \lambda_2(R_1(X,s) - R_2(X,s)) \end{cases} \quad \text{if } s \geq \theta$$

Note that an additional term $R_{i,s}(X,s)$ enters the Bellman equation. This term is the derivative of R_i with respect to s and it follows from Ito's Lemma as the dimension of time has to be taken into account. Before it was not necessary to include it because only the dimension of the stochastic process was present in the equation to determine the revenue of the firm.

5.2 Single Switch Model with Constant Time to Build

Starting with the single switch model with constant lead time results in state 2 being the absorbing state, like in the single switch model without lead time. Therefore, if the firm is in state 2, the no-switch time-to-build model from Section 3 can be applied to determine the investment decision. This makes understanding the derivations in state 1 easier as expressions for R_2, V_2, F_2, X_2^*, and Q_2 are known.

5.2.1 Derivation of the Value of the Firm at the Moment of Investment

First, the NPV of the revenue in state 2 ($R_2(X)$) is determined. Substituting this expression into the system of equations yields $R_1(X) = \Lambda Q(1-\eta Q)X$ for $s \geq \theta$ because the firm is in the same region as in the single switch model without lead time. If $s < \theta$, $R_2(X,s) = \frac{Q(1-\eta Q)X}{r-\mu_2} exp((\mu_2 - r)(\theta - s))$ as the lead time model without regime switching is applied. This expression for R_2 is substituted into the differential equation for $R_1(X,s)$ in the region $s < \theta$:

$$
\begin{aligned}
rR_1(X,s) &= \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) + R_{1,t}(X,s) + \lambda(R_2(X,s) - R_1(X,s)) \\
(r+\lambda)R_1(X,s) &= \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) + R_{1,t}(X,s) \\
&\quad + \lambda \frac{Q(1-\eta Q)X}{r-\mu_2} exp((r-\mu_2)(s-\theta))
\end{aligned}
$$

Since $R_2(X,s)$ is a function of the form $bXexp(c(s-\theta))$, the first guess of R_1 has this form as well. Substituting this into the expression above yields

$$
\begin{aligned}
(r+\lambda)bXexp(c(s-\theta)) &= \mu_1 bXexp(c(s-\theta)) + cbXexp(c(s-\theta)) \\
&\quad + \lambda \frac{Q(1-\eta Q)X}{\mu - r} exp((r-\mu_2)(s-\theta)) \\
(r+\lambda-\mu_1-c)bXexp(c(s-\theta)) &= \lambda \frac{Q(1-\eta Q)X}{\mu_2 - r} exp((r-\mu_2)(s-\theta)) \\
b &= \lambda \frac{Q(1-\eta Q)}{(r-\mu_2)(r+\lambda-\mu_1-c)} exp((r-\mu_2-c)(s-\theta))
\end{aligned}
$$

Furthermore, the boundary condition for $s = \theta$ has to hold. Hence, $b = \Lambda Q(1-\eta Q)$ and

$$
\begin{aligned}
\Lambda Q(1-\eta Q) &= \lambda \frac{Q(1-\eta Q)}{(r-\mu_2)(r+\lambda-\mu_1-c)} exp((r-\mu_2-c)(\theta-\theta)) \\
\Lambda &= \frac{\lambda}{(r-\mu_2)(r+\lambda-\mu_1-c)} \\
\frac{r+\lambda-\mu_2}{(r-\mu_2)(r+\lambda-\mu_1)} &= \frac{\lambda}{(r-\mu_2)(r+\lambda-\mu_1-c)} \\
(r+\lambda-\mu_2)(r+\lambda-\mu_1-c) &= \lambda(r+\lambda-\mu_1) \\
(r+\lambda-\mu_2)(r+\lambda-\mu_1) - c(r+\lambda-\mu_2) &= \lambda(r+\lambda-\mu_1) \\
(r-\mu_2)(r+\lambda-\mu_1) &= c(r+\lambda-\mu_2) \\
c &= \frac{(r-\mu_2)(r+\lambda-\mu_1)}{r+\lambda-\mu_2} = \Lambda^{-1}
\end{aligned}
$$

Therefore, $R_1(X,s) = \Lambda Q(1-\eta Q)Xexp(\Lambda^{-1}(s-\theta))$ and the value of the firm at the moment of investment in regime 1 is

$$V_1(X) = R_1(X,0) - \delta Q = \Lambda Q(1-\eta Q)X exp(-\theta/\Lambda) - \delta Q$$

The economic interpretation for the value of the firm at the moment of investment is that the firm discounts with $r-\mu_2$ in state 2 and by $1/\Lambda$ in state 1 over the lead time. These discount values are the expected lost revenues in these regimes by including lead time. Therefore, using this discount function for lead time does not allow for a regime switch during the building time.

To allow for regime-switching during the building time, a different form of R_1 is assumed: $Q(1-\eta Q)X\Big(b_1 exp(c_1(s-\theta)) + d_1 exp((r-\mu_2)(s-\theta))\Big)$. This form is similar to the adjustment in the value of waiting to allow for regime-switching as the solution of the differential equation consists of two terms instead of one. The differential equation is defined as

$$\begin{aligned}(r+\lambda)R_1(X,s) &= \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) + R_{1,t}(X,s)\\ &\quad + \lambda\frac{Q(1-\eta Q)X}{r-\mu_2} exp((r-\mu_2)(s-\theta))\\ (r+\lambda-\mu_1)R_1(X,s) &= Q(1-\eta Q)X\Big(c_1 b_1 exp(c_1(s-\theta)) + (r-\mu_2)d_1 exp((r-\mu_2)(s-\theta))\Big)\\ &\quad + \lambda\frac{Q(1-\eta Q)X}{r-\mu_2} exp((r-\mu_2)(s-\theta))\end{aligned}$$

The differential equation is split up into two parts for the different variables in the exponential. These parts are:

$$\begin{cases}(r+\lambda-\mu_1)b_1 exp(c_1(s-\theta)) = c_1 b_1 exp(c_1(s-\theta))\\ (r+\lambda-\mu_1)d_1 exp((r-\mu_2)(s-\theta)) = (r-\mu_2)d_1 exp((r-\mu_2)(s-\theta)) + \lambda\frac{1}{r-\mu_2} exp((r-\mu_2)(s-\theta))\end{cases}$$

$$\begin{cases}c_1 = r+\lambda-\mu_1\\ (\lambda-\mu_1+\mu_2)d_1 = \frac{\lambda}{r-\mu_2}\end{cases}$$

$$\begin{cases}c_1 = r+\lambda-\mu_1\\ d_1 = \frac{\lambda}{(r-\mu_2)(\lambda-\mu_1+\mu_2)}\end{cases}$$

From the boundary condition at $s=\theta$, $b_1 + d_1 = \Lambda$ has to hold. Hence,

$$\begin{aligned}b_1 &= \Lambda - d_1\\ &= \frac{r+\lambda-\mu_2}{(r-\mu_2)(r+\lambda-\mu_1)} - \frac{\lambda}{(r-\mu_2)(\lambda-\mu_1+\mu_2)}\\ &= \frac{(r+\lambda-\mu_2)(\lambda-\mu_1+\mu_2) - \lambda(r+\lambda-\mu_1)}{(r-\mu_2)(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}\\ &= \frac{r\lambda - r\mu_1 + r\mu_2 + \lambda^2 - \lambda\mu_1 + \lambda\mu_2 - \lambda\mu_2 + \mu_1\mu_2 - \mu_2^2 - r\lambda - \lambda^2 + \mu_1\lambda}{(r-\mu_2)(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}\\ &= \frac{-r\mu_1 + r\mu_2 + \mu_1\mu_2 - \mu_2^2}{(r-\mu_2)(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}\\ &= \frac{r(\mu_2-\mu_1) + \mu_2(\mu_1-\mu_2)}{(r-\mu_2)(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}\\ &= \frac{(r-\mu_2)(\mu_2-\mu_1)}{(r-\mu_2)(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)} = \frac{\mu_2-\mu_1}{(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}\end{aligned}$$

Allowing for a regime switch during the building time results in a slightly different discount factor than in a model without a regime switch during the building time. The value of the firm at the moment of investment in regime one is, therefore, equal to

$$V_1 = Q(1-\eta Q)X\Big(\frac{\mu_2-\mu_1}{(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}exp(-(r+\lambda-\mu_1)\theta)+\frac{\lambda}{(r-\mu_2)(\lambda-\mu_1+\mu_2)}exp(-(r-\mu_2)\theta)\Big)$$

Since this expression is quite ornate, Γ is defined as

$$\Gamma = \Big(\frac{\mu_2-\mu_1}{(r+\lambda-\mu_1)(\lambda-\mu_1+\mu_2)}exp(-(r+\lambda-\mu_1)\theta)+\frac{\lambda}{(r-\mu_2)(\lambda-\mu_1+\mu_2)}exp(-(r-\mu_2)\theta)\Big)^{-1}$$

To find the optimal investment capacity, the first-order condition is applied. This condition yields

$$\begin{aligned}\frac{\partial V_1(X)}{\partial Q} &= 0\\ \frac{(1-2\eta Q)X}{\Gamma}-\delta &= 0\\ 1-2\eta Q &= \frac{\delta\Gamma}{X}\\ Q_1^*(X) &= \frac{1}{2\eta}\Big(1-\frac{\delta\Gamma}{X}\Big)\end{aligned}$$

To determine whether this investment capacity yields a global maximum, the SOC is applied:

$$-\frac{2\eta X}{\Gamma} < 0$$

This inequality holds because η and X are defined as bigger than zero. To see whether Γ is bigger than zero, the NPV of the revenue is studied. As this value should always be bigger than zero, it follows that Γ is also bigger than zero. Hence, the inequality holds and the FOC yields a global maximum and the formula for the investment capacity thus yields a global maximum of the value of the firm.

Substituting the optimal capacity into the value of the firm at the moment of investment yields

$$\begin{aligned}V_1(X) &= Q^*(X)(1-\eta Q^*(X))X\Gamma^{-1}-\delta Q^*(X)\\ &= \frac{X}{4\eta}\Big(1-\frac{\delta\Gamma}{X}\Big)\Big(1+\frac{\delta\Gamma}{X}\Big)\Gamma^{-1}-\frac{\delta}{2\eta}\Big(1-\frac{\delta\Gamma}{X}\Big)\\ &= \frac{X^2-(\delta\Gamma)^2}{4\eta\Gamma X}-\Big(\frac{2\delta X\Gamma-2(\delta\Gamma)^2)}{4\eta\Gamma X}\Big)\\ &= \frac{X^2-2\delta\Gamma X+\delta^2\Gamma^2}{4\eta\Gamma X}\\ &= \frac{(X-\delta\Gamma)^2}{4\eta\Gamma X}\end{aligned}$$

5.2.2 Derivation of the Value of Waiting

The derivation of the value of waiting is done similarly as in Section 4.1. For an investment-discouraging new regime, the same solution to the system of differential equations of the value of waiting is found as it does not depend on the value of the firm at the moment of investment.

To find the investment trigger in regime 1, the smooth-pasting and value-matching conditions are applied with the new value of the firm at the moment of investment. These are

$$\begin{cases} A_1(X_1^*)^{\beta_1} + A_3(X_1^*)^{\gamma} = \frac{(X_1^*-\delta\Gamma)^2}{4\eta\Gamma X_1^*} \\ \beta_1 A_1(X_1^*)^{\beta_1} + \gamma A_3(X_1^*)^{\gamma} = \frac{X_1^*}{4\eta\Gamma} - \frac{\delta^2\Gamma}{4\eta X_1^*} \end{cases} \tag{22}$$

Below is a plot of an investment-discouraging new regime including lead time. With the defined parameters, $X_1^* = 0.0168$ with $Q_1^* = 7.9290$ and $X_2^* = 0.0179$ with $Q_2^* = 7.7257$. Compared to the graph from Section 4.1, the two thresholds both have increased and are further apart from each other, suggesting that the investment triggers and the investment range increase with lead time.

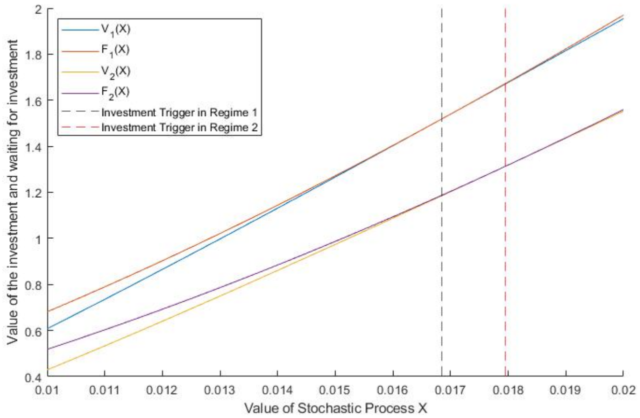


Figure 13: Plot of the value of waiting and the value of investment in both regimes as a function of X with parameter values: $\mu_1 = 0.08$, $\mu_2 = 0.06$, $\sigma_1 = 0.08$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda = 0.1$, $\delta = 0.1$ and $\eta = 0.05$ and $\theta = 0.5$

Contrary to the investment-discouraging new regime, the solution to the set of differential equations for the investment-stimulating new regime depends on the value of investing. The solution to this set of differential equations is different because the value of investing has been adjusted. Yet, the same homogeneous solution is found for the system of differential equations. Hence, it rests to find the particular solution to the differential equation in the investment-stimulating new regime. The differential equation is now

$$(r+\lambda)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda V_2(X)$$
$$(r+\lambda)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda\frac{(X - \delta(r-\mu_2)exp((r-\mu_2)\theta))^2}{4\eta(r-\mu_2)Xexp((r-\mu_2)\theta)}$$

In the transient region, the firm invests at the moment of a regime switch. This switch results in the value of the firm being equal to $V_2(X)$ for different values of X. Since this expression is a polynomial of the form $aX + b + c/X$ if the firm invests with an optimal capacity, the particular solution to the system of differential equations is assumed to have this form as well. Therefore,

$$(r+\lambda)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda\frac{(X-\delta(r-\mu_2)exp((r-\mu_2)\theta))^2}{4\eta(r-\mu_2)Xexp((r-\mu_2)\theta)}$$

$$(r+\lambda)(aX+b+c/X) = \mu_1 X(a-c/X^2) + \frac{1}{2}\sigma_1^2 X^2(2c/X^3) + \lambda\frac{(X-\delta(r-\mu_2)exp((r-\mu_2)\theta))^2}{4\eta(r-\mu_2)Xexp((r-\mu_2)\theta)}$$

$$\begin{cases}(r+\lambda-\mu_1)a = \frac{\lambda}{4\eta(r-\mu_2)exp((r-\mu_2)\theta)}\\ (r+\lambda)b = -\frac{\lambda\delta}{2\eta}\\ (r+\lambda+\mu_1-\sigma_1^2)c = \frac{\lambda\delta^2(r-\mu_2)exp((r-\mu_2)\theta)}{4\eta}\end{cases}$$

$$\begin{cases}a = \frac{\lambda}{4\eta(r-\mu_2)(r+\lambda-\mu_1)exp((r-\mu_2)\theta)}\\ b = -\frac{\lambda\delta}{2(r+\lambda)\eta}\\ c = \frac{\lambda\delta^2(r-\mu_2)exp((r-\mu_2)\theta)}{4\eta(r+\lambda+\mu_1-\sigma_1^2)}\end{cases}$$

The same continuity and smooth-pasting conditions are applied to find the investment triggers and investment capacity implicitly. For the continuity conditions, the adjusted values for a, b, and c are used. However, for the smooth-pasting conditions, the value of the firm at the moment of investment is different. Thus, the smooth-pasting conditions are adjusted and defined as

$$\begin{cases}H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(X_1^*-\delta\Gamma)^2}{4\eta\Gamma X_1^*}\\ \beta_1 H_1(X_1^*)^{\beta_1-1} + \beta_2 H_2(X_1^*)^{\beta_2-1} = \frac{1}{4\eta\Gamma} - \frac{\delta^2\Gamma}{4\eta(X_1^*)^2} - a + c/(X_1^*)^2\end{cases} \tag{23}$$

Below is a plot of an investment-stimulating new regime with lead time.

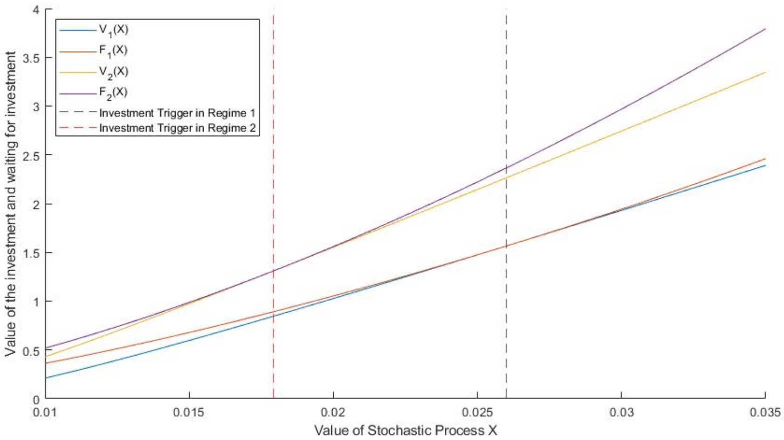


Figure 14: Plot of the value of waiting and the value of the firm at the moment of investment in both regimes as a function of X with parameter values: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.25$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda = 0.1$, $\delta = 0.1$ and $\eta = 0.05$ and $\theta = 0.5$

With the defined parameters, $X_1^* = 0.0260$ with $Q_1^* = 7.9698$, $X_2^* = 0.0179$ with $Q_1^* = 7.7257$ and $Q_2^{'} = 8.4611$

5.3 Continuous Regime Switch Model with Constant Time to Build

The same approach is used to extend the regime-switch model with lead time to allow for continuous switching as in the single-switch model with lead time. The regime-switch model is thus altered to include time to build by adjusting the value of the firm at the moment of investment. The installation period is thus added to the regime-switching model. Since the value of waiting depends on the value of the firm in the transient state, the value of waiting is also revised.

5.3.1 Derivation of the Value of the Firm at the Moment of Investment

The system of differential equations during and after the installation period needs to hold. This system after the investment has been installed is equivalent to the system of differential equations for the NPV of the investment without lead time because in both instances the investment has taken place and yields a certain payoff. Therefore, if $s \geq \theta$, $R_i(X,s) = \Lambda_i Q(1-\eta Q)X$, where Λ_i is similarly defined as in Section 4.2.

Now the NPV of the revenue has to be determined during the installation period. Therefore, the following system of stochastic differential equations has to be solved

$$\begin{cases} rR_1(X,s) = \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) \\ \qquad + R_{1,s}(X,s) + \lambda_1(R_2(X,s) - R_1(X,s)) \\ rR_2(X,s) = \mu_2 X R_{2,X}(X,s) + \frac{1}{2}\sigma_2^2 X^2 R_{2,XX}(X,s) \\ \qquad + R_{2,s}(X,s) + \lambda_2(R_1(X,s) - R_2(X,s)) \end{cases}$$

Like in the single switch model, it is assumed first that $R_i(X,s) = b_i X exp(c_i(s-\theta))$. Substituting this form of the revenue into the differential equation for R_1 yields

$$\begin{aligned} rR_1(X,s) &= \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) \\ &\quad + R_{1,s}(X,s) + \lambda_1(R_2(X,s) - R_1(X,s)) \\ (r+\lambda_1)R_1(X,s) &= \mu_1 X R_{1,X}(X,s) + \frac{1}{2}\sigma_1^2 X^2 R_{1,XX}(X,s) \\ &\quad + R_{1,s}(X,s) + \lambda_1 R_2(X,s) \\ (r+\lambda_1)b_1 X exp(c_1(s-\theta)) &= \mu_1 X b_1 exp(c_1(s-\theta)) + 0 + c_1 b_1 X exp(c_1(s-\theta)) \\ &\quad + \lambda_1 b_2 X exp(c_2(s-\theta)) \\ (r+\lambda_1-\mu_1-c_1)b_1 X exp(c_1(s-\theta)) &= \lambda_1 b_2 X exp(c_2(s-\theta)) \\ b_1 &= \frac{\lambda_1 b_2}{(r+\lambda_1-\mu_1-c_1)} exp((c_2-c_1)(s-\theta)) \end{aligned}$$

Furthermore, R_i is continuous in s. Thus, if $s=\theta$, $R_i = \Lambda_i Q(1-\eta Q)X$, and $b_i = \Lambda_i Q(1-\eta Q)$.

It follows from this boundary condition that

$$\begin{aligned}
\Lambda_1 &= \frac{\lambda_1 \Lambda_2}{(r+\lambda_1-\mu_1-c_1)} exp((c_2-c_1)(\theta-\theta)) \\
(r+\lambda_1-\mu_1-c_1)\Lambda_1 &= \lambda_1\Lambda_2 \\
(r+\lambda_1-\mu_1)\Lambda_1 - c_1\Lambda_1 &= \lambda_1\Lambda_2 \\
c_1\Lambda_1 &= (r+\lambda_1-\mu_1)\Lambda_1 - \lambda_1\Lambda_2 \\
c_1 &= (r+\lambda_1-\mu_1) - \lambda_1\Lambda_2/\Lambda_1 \\
c_1 &= (r+\lambda_1-\mu_1) - \lambda_1 \frac{r+\lambda_1+\lambda_2-\mu_1}{r+\lambda_1+\lambda_2-\mu_2} \\
&= \frac{(r+\lambda_1-\mu_1)(r+\lambda_1+\lambda_2-\mu_2) - \lambda_1(r+\lambda_1+\lambda_2-\mu_1)}{r+\lambda_1+\lambda_2-\mu_2} \\
&= \frac{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) + \lambda_1(r+\lambda_1-\mu_1) - \lambda_1(r+\lambda_1-\mu_1) - \lambda_1\lambda_2}{r+\lambda_1+\lambda_2-\mu_2} \\
&= \frac{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) - \lambda_1\lambda_2}{r+\lambda_1+\lambda_2-\mu_2} = \Lambda_1^{-1}
\end{aligned}$$

Similarly, $c_2 = \Lambda_2^{-1}$ is found and $R_i(X,s) = \Lambda_i Q(1-\eta Q)X exp((s-\theta)/\Lambda_i)$. The value of the firm at the moment of investment is thus

$$V_i(X) = R_i(X,0) - \delta Q = \Lambda_i Q(1-\eta Q)X exp(-\theta/\Lambda_i) - \delta Q$$

Therefore, the value of the firm in regime i decreases by a factor $exp(-\theta/\Lambda_i)$ if time to build is included in the investment decision. Intuitively, the NPV of the revenue is discounted over the lead time by the expected real rate of return of the investment in that particular regime. However, this does not allow for a regime switch during installation as such a regime switch changes the lead time's discount factor.

To include the occurrence of a regime switch during the building time, a different form of the value of the firm in the installation period is assumed. Hence, the form of R_i is now $R_i = qX(b_i exp(c_i(s-\theta)) + d_i exp(e_i(s-\theta))) = qX/\Gamma_i$, where q is the payoff function. This form, like in the single switch model, consists of two terms instead of one to include regime switching. In the model using a linear demand function, the payoff function is $Q(1-\eta Q)$. Hence,

$$\begin{cases} R_1(X,s) = qX\Big(b_1 exp(c_1(s-\theta)) + d_1 exp(c_2(s-\theta))\Big) \\ R_2(X,s) = qX\Big(b_2 exp(c_1(s-\theta)) + d_2 exp(c_2(s-\theta))\Big) \end{cases}$$

This new form still has to satisfy the system of stochastic differential equations. Substituting these expressions of the NPV of the revenue into the differential equations from the beginning of the section gives

$$\begin{cases} (r+\lambda_1)qX\Bigg(b_1exp(c_1(s-\theta))+d_1exp(c_2(s-\theta))\Bigg) = \\ \mu_1Xq\Bigg(b_1exp(c_1(s-\theta))+d_1exp(c_2(s-\theta))\Bigg)+qX\Bigg(c_1b_1exp(c_1(s-\theta))+c_2d_1exp(c_2(s-\theta))\Bigg) \\ +\lambda_1qX\Bigg(b_2exp(c_1(s-\theta))+d_2exp(c_2(s-\theta))\Bigg) \\ (r+\lambda_2)qX\Bigg(b_2exp(c_1(s-\theta))+d_2exp(c_2(s-\theta))\Bigg) = \\ \mu_2Xq\Bigg(b_2exp(c_1(s-\theta))+d_2exp(c_2(s-\theta))\Bigg)+qX\Bigg(c_1b_2exp(c_1(s-\theta))+c_2d_2exp(c_2(s-\theta))\Bigg) \\ +\lambda_2qX\Bigg(b_1exp(c_1(s-\theta))+d_1exp(c_2(s-\theta))\Bigg) \end{cases}$$

$$\begin{cases} (r+\lambda_1-\mu_1)\Bigg(b_1exp(c_1(s-\theta))+d_1exp(c_2(s-\theta))\Bigg) = \\ \Bigg(c_1b_1exp(c_1(s-\theta))+c_2d_1exp(c_2(s-\theta))\Bigg)+\lambda_1\Bigg(b_2exp(c_1(s-\theta))+d_2exp(c_2(s-\theta))\Bigg) \\ (r+\lambda_2-\mu_2)\Bigg(b_2exp(c_1(s-\theta))+d_2exp(c_2(s-\theta))\Bigg) = \\ \Bigg(c_1b_2exp(c_1(s-\theta))+c_2d_2exp(c_2(s-\theta))\Bigg)+\lambda_2\Bigg(b_1exp(c_1(s-\theta))+d_1exp(c_2(s-\theta))\Bigg) \end{cases}$$

This system of equations can be split up into two parts with different terms in the exponential. This step yields

$$\begin{cases} (r+\lambda_1-\mu_1)b_1exp(c_1(s-\theta)) = c_1b_1exp(c_1(s-\theta))+\lambda_1b_2exp(c_1(s-\theta)) \\ (r+\lambda_2-\mu_2)b_2exp(c_1(s-\theta)) = c_1b_2exp(c_1(s-\theta))+\lambda_2b_1exp(c_1(s-\theta)) \\ (r+\lambda_1-\mu_1)d_1exp(c_2(s-\theta)) = c_2d_1exp(c_2(s-\theta))+\lambda_1d_2exp(c_2(s-\theta)) \\ (r+\lambda_2-\mu_2)d_2exp(c_2(s-\theta)) = c_2d_2exp(c_2(s-\theta))+\lambda_2d_1exp(c_2(s-\theta)) \end{cases}$$

$$\begin{cases} (r+\lambda_1-\mu_1-c_1)b_1 = \lambda_1b_2 \\ (r+\lambda_2-\mu_2-c_1)b_2 = \lambda_2b_1 \\ (r+\lambda_1-\mu_1-c_2)d_1 = \lambda_1d_2 \\ (r+\lambda_2-\mu_2-c_2)d_2 = \lambda_2d_1 \end{cases} \tag{24}$$

$$\begin{cases} (r+\lambda_1-\mu_1-c_1)b_1 = \lambda_1\frac{\lambda_2b_1}{r+\lambda_2-\mu_2-c_1} \\ (r+\lambda_1-\mu_1-c_2)d_1 = \lambda_1\frac{\lambda_2d_1}{r+\lambda_2-\mu_2-c_2} \end{cases}$$

$$\begin{cases} (r+\lambda_1-\mu_1-c_1) = \frac{\lambda_1\lambda_2}{r+\lambda_2-\mu_2-c_1} \\ (r+\lambda_1-\mu_1-c_2) = \frac{\lambda_1\lambda_2}{r+\lambda_2-\mu_2-c_2} \end{cases}$$

$$\iff \begin{cases} (r+\lambda_1-\mu_1-c_1)(r+\lambda_2-\mu_2-c_1) = \lambda_1\lambda_2 \\ (r+\lambda_1-\mu_1-c_2)(r+\lambda_2-\mu_2-c_2) = \lambda_1\lambda_2 \end{cases}$$

Therefore, c_1 and c_2 are the roots of the same quadratic equation, namely

$$(r+\lambda_1-\mu_1-c)(r+\lambda_2-\mu_2-c) - \lambda_1\lambda_2 = 0$$

Hence,

$$c^2 - c(r+\lambda_1-\mu_1+r+\lambda_2-\mu_2) + (r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) - \lambda_1\lambda_2 = 0$$

For this quadratic equation, the determinant is

$$\begin{aligned}
D &= b^2 - 4ac \\
&= (r+\lambda_1-\mu_1+r+\lambda_2-\mu_2)^2 - 4\Big((r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) - \lambda_1\lambda_2\Big) \\
&= (r+\lambda_1-\mu_1)^2 + 2(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) + (r+\lambda_2-\mu_2)^2 - 4(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) + 4\lambda_1\lambda_2 \\
&= (r+\lambda_1-\mu_1)^2 - 2(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2) + (r+\lambda_2-\mu_2)^2 + 4\lambda_1\lambda_2 \\
&= (r+\lambda_1-\mu_1-(r+\lambda_2-\mu_2))^2 + 4\lambda_1\lambda_2 \\
&= \Big((\lambda_1-\lambda_2)-(\mu_1-\mu_2)\Big)^2 + 4\lambda_1\lambda_2 \\
&= (\lambda_1+\lambda_2)^2 - 2(\lambda_1-\lambda_2)(\mu_1-\mu_2) + (\mu_1-\mu_2)^2 > 0
\end{aligned}$$

The values for $c_{1,2}$ are thus equal to

$$c_{1,2} = r + \frac{(\lambda_1+\lambda_2)-(\mu_1+\mu_2) \pm \sqrt{(\lambda_1+\lambda_2)^2 - 2(\lambda_1-\lambda_2)(\mu_1-\mu_2) + (\mu_1-\mu_2)^2}}{2}$$

Furthermore,

$$\begin{aligned}
D &= (\lambda_1+\lambda_2)^2 - 2(\lambda_1-\lambda_2)(\mu_1-\mu_2) + (\mu_1-\mu_2)^2 \\
&= (\lambda_1+\lambda_2)^2 - 2(\lambda_1+\lambda_2)(\mu_1-\mu_2) + (\mu_1-\mu_2)^2 + 4\lambda_2(\mu_1-\mu_2) \\
&= \Big((\lambda_1+\lambda_2)-(\mu_1-\mu_2)\Big)^2 + 4\lambda_2(\mu_1-\mu_2)
\end{aligned}$$

Thus, since $\lambda_i > 0$, if $\mu_1 > \mu_2$, $D > \Big((\lambda_1+\lambda_2)-(\mu_1-\mu_2)\Big)^2$ and $c_2 < r-\mu_2$. In this case, c_1 is

$$\begin{aligned}
c_1 &= r + \frac{(\lambda_1+\lambda_2)-(\mu_1+\mu_2)+\sqrt{D}}{2} \\
&> r + \frac{(\lambda_1+\lambda_2)-(\mu_1+\mu_2)+\Big((\lambda_1+\lambda_2)-(\mu_1-\mu_2)\Big)}{2} \\
&= r-\mu_1+\lambda_1+\lambda_2 > r-\mu_1
\end{aligned}$$

Similarly, if $\mu_1 < \mu_2$, $D < \Big((\lambda_1+\lambda_2)-(\mu_1-\mu_2)\Big)^2$ and $c_2 > r-\mu_2$, $c_1 < r-\mu_1+\lambda_1+\lambda_2$. Moreover, R_i has to be continuous. Therefore, the boundary condition $R_i(X,\theta) = XQ(1-$

$\eta Q)\Lambda_i = qX\Lambda_i$ has to hold. This results in the following set of conditions:

$$\begin{cases} b_1 + d_1 = \Lambda_1 \\ b_2 + d_2 = \Lambda_2 \end{cases}$$

From Equation 24, the following conditions also have to be statisfied

$$\begin{cases} b_2 = \frac{\lambda_2 b_1}{r+\lambda_2-\mu_2-c_1} \\ d_2 = \frac{\lambda_2 d_1}{r+\lambda_2-\mu_2-c_2} \end{cases}$$

Solving this system of equations using substitution yields

$$\begin{aligned}
&\frac{\lambda_2 b_1}{r+\lambda_2-\mu_2-c_1} + \frac{\lambda_2 d_1}{r+\lambda_2-\mu_2-c_2} = \Lambda_2 \\
&\frac{\lambda_2 b_1(r+\lambda_2-\mu_2-c_2)}{(r+\lambda_2-\mu_2-c_1)(r+\lambda_2-\mu_2-c_2)} + \frac{\lambda_2 d_1(r+\lambda_2-\mu_2-c_1)}{(r+\lambda_2-\mu_2-c_2)(r+\lambda_2-\mu_2-c_1)} = \Lambda_2 \\
&\frac{\lambda_2 b_1(r+\lambda_2-\mu_2-c_2) + \lambda_2 d_1(r+\lambda_2-\mu_2-c_1)}{(r+\lambda_2-\mu_2-c_1)(r+\lambda_2-\mu_2-c_2)} = \Lambda_2 \\
&\frac{\lambda_2(b_1+d_1)(r+\lambda_2-\mu_2) - \lambda_2(c_2 b_1 + c_1 d_1)}{-\lambda_1\lambda_2} = \Lambda_2 \\
&\frac{\Lambda_1(r+\lambda_2-\mu_2) - (c_2 b_1 + c_1(\Lambda_1 - b_1))}{-\lambda_1} = \Lambda_2 \\
&\Lambda_1(r+\lambda_2-\mu_2-c_1) - (c_2-c_1)b_1 = -\lambda_1\Lambda_2 \\
&(c_1-c_2)b_1 = -\lambda_1\Lambda_2 - \Lambda_1(r+\lambda_2-\mu_2-c_1) \\
&\sqrt{D}b_1 = (-\lambda_1\Lambda_2 - \Lambda_1(r+\lambda_2-\mu_2-c_1)) \\
&b_1 = \frac{(-\lambda_1\Lambda_2 - \Lambda_1(r+\lambda_2-\mu_2-c_1))}{\sqrt{D}}
\end{aligned}$$

with $d_1 = \Lambda_1 - b_1$. Similarly, b_2 is derived and is equal to

$$b_2 = \frac{(-\lambda_2\Lambda_1 - \Lambda_2(r+\lambda_1-\mu_1-c_1))}{\sqrt{D}}$$

with $d_2 = \Lambda_2 - b_2$. Hence, the NPV is defined as

$$R_i = Q(1-\eta Q)X\Big(b_i exp(c_1(s-\theta)) + d_i exp(c_2(s-\theta))\Big)$$

with

$$b_i = \frac{\lambda_i\Lambda_{3-i} - \Lambda_i(r+\lambda_{3-i}-\mu_{3-i}-c_1)}{\sqrt{D}}$$

and $d_i = \Lambda_i - b_i$. Finally, we define Γ_i as

$$\Gamma_i = \Big(b_i exp(-c_1\theta) + d_i exp(-c_2\theta)\Big)^{-1}$$

The value of the firm at the moment of investment is $V_i(X,Q) = \frac{Q(1-\eta Q)X}{\Gamma_i} - \delta Q$.[5].

The expressions for the constants of Γ_i seem quite elaborate. Hence, these constants can

[5]If lead time is a random variable, the value of the firm becomes $V_i = \mathbb{E}_\theta[\frac{Q(1-\eta Q)X}{\Gamma_i} - \delta Q] = \mathbb{E}_\theta[Q(1-\eta Q)X((b_i exp(-c_1\theta) + d_i exp(-c_2\theta))] - \delta Q = Q(1-\eta Q)X\Big(b_i M_\theta(-c_1) + d_i M_\theta(-c_2)\Big) - \delta Q$. Hence, in that case $\Gamma_i = b_i M_\theta(-c_1) + d_i M_\theta(-c_2)$

also be written as a system of linear equations where the constants b_1, b_2, d_1, d_2 constitute the column vector $\mathbf{b} = \begin{bmatrix} b_1 & d_1 & b_2 & d_2 \end{bmatrix}'$, that solves $A\mathbf{b} = \mathbf{s}$, with matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\frac{\lambda_2}{r+\lambda_2-\mu_2-c_1} & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{r+\lambda_2-\mu_2-c_2}{\lambda_2} \end{bmatrix}$$

and $\mathbf{s} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & 0 & 0 \end{bmatrix}'$.

In Figure 15, Γ_i for different forms are plotted for a varying time to build. The single solution to the differential equations corresponds to the form $b_i exp(-c_i\theta)$ and the dual solution corresponds to $b_i exp(-c_1\theta) + d_i exp(-c_2\theta)$.

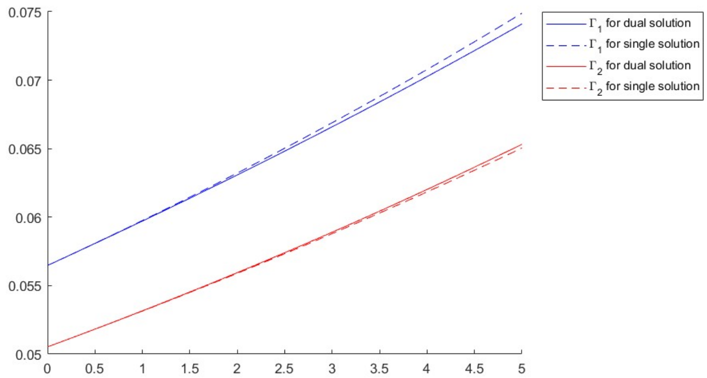


Figure 15: Γ_i plotted for different values of lead time with parameters $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, and $r = 0.1$.

This figure shows that Γ_1 is overestimated if the single form is used compared to the dual form. This result illustrates that the value of the firm at the moment of investment in regime 1 is underestimated if a single form is used. Similarly, Γ_2 is underestimated if the single form is used compared to the dual form and the value of the firm at the moment of investment in regime 2 is overestimated in the single form.

This bias in the single form arises from the possibility of a regime switch occurring during the installation period. However, for small values of time to build, the difference is relatively small as the probability of a switch occurring during the installation period is small. Therefore, the assumption that a regime switch occurs during the installation period is not likely to be violated.

Finding the optimal investment capacity is similar to finding the optimal investment capacity in regime 1 for the single switch model, but now with Γ_i instead of Γ. Therefore, the optimal investment capacity for a given level of the stochastic process is denoted as

$$Q_i^*(X) = \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_i}{X}\Big)$$

Substituting this optimal investment capacity in the formula for the value of the firm at the moment of investment gives the same derivation as for the value of the firm in regime 1 in the single switch model but with Γ_i instead of Γ. Therefore,

$$V_i(X) = \frac{(X - \delta\Gamma_i)^2}{4\eta\Gamma_i X}$$

5.3.2 Derivation of the Value of Waiting

As the value of the firm at the moment of investment is derived, the value of waiting can be determined. In the regime switch model without lead time, the values of waiting do not depend on the value of investing in the general waiting region. Thus, applying the steps from the continuous regime switch model without lead time yields the same expression for the values of waiting in regime 1 and regime 2 for $X \in [0, X_2^*)$. Since the value of waiting in regime 2 is only defined on this interval, an expression for $F_2(X)$ is found.

However, the value of waiting in regime 1 changes in the transient region ($X_t \in [X_2^*, X_1^*)$) as the regime may switch from regime 1 to regime 2. This results in the firm investing immediately with capacity $Q_2(X)$ corresponding to the current value of the stochastic process. Since the value of the firm at the moment of investment changes when time to build is included, the value of waiting in the transient region has a different value than in the model without lead time. Therefore, the value of waiting is adjusted to include the change in the value of investing due to the installation time. First, the differential equation for the value of waiting in regime 1 in the transient region is defined:

$$(r + \lambda_1)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 V_2(X)$$

The homogeneous solution to this differential equation is equal to the homogeneous solution to the differential equation in the continuous regime switch model without lead time (from Section 4.2). Hence, the only adjustment is in the particular solution of this differential equation.

Since the firm is in the transient region with a flexible investment capacity, $V_2(X)$ has the form $aX + b + c/X$. Therefore, the particular solution to this differential equation has a similar form. Substituting the form of the particular solution into the differential equation yields

$$(r + \lambda_1)(aX + b + c/X) = \mu_1 X(a - c/X^2) + \frac{1}{2}\sigma_1^2 X^2 * 2 * c/X^3 + \lambda_1\Big(\frac{X}{4\eta\Gamma_2} - \frac{\delta}{2\eta} + \frac{\delta^2\Gamma_2}{4\eta X}\Big)$$
$$(r + \lambda_1 - \mu_1)aX + (r + \lambda_1)b + (r + \lambda_1 + \mu_1 - \sigma_1^2)c/X = \lambda_1\Big(\frac{X}{4\eta\Gamma_2} - \frac{\delta}{2\eta} + \frac{\delta^2\Gamma_2}{4\eta X}\Big)$$

$$\begin{cases} (r + \lambda_1 - \mu_1)a = \frac{\lambda_1}{4\eta\Gamma_2} \\ (r + \lambda_1)b = -\frac{\lambda_1\delta}{2\eta} \\ (r + \lambda_1 + \mu_1 - \sigma_1^2)c = \frac{\lambda_1\delta^2\Gamma_2}{4\eta} \end{cases}$$
$$\begin{cases} a = \frac{\lambda_1}{4\eta\Gamma_2(r+\lambda_1-\mu_1)} = \frac{\lambda_1}{g_1(1)}\frac{1}{4\eta\Gamma_2} \\ b = -\frac{\lambda_1\delta}{2\eta(r+\lambda_1)} = -\frac{\lambda_1}{g_1(0)}\frac{\delta}{2\eta} \\ c = \frac{\lambda_1\delta^2\Gamma_2}{4\eta(r+\lambda_1+\mu_1-\sigma_1^2)} = \frac{\lambda_1}{g_1(-1)}\frac{\delta^2\Gamma_2}{4\eta} \end{cases}$$

where $g_1(.)$ is similarly defined as in Section 4.2. Furthermore, the continuity conditions need to hold in X_2^*. These remain the same as in the continuous switch model without lead time but with adjusted values for a, b, and c.

5.3.3 Finding the Investment Trigger and Optimal Investment Capacity

To find solutions for the investment triggers X_1^* and X_2^* and their corresponding investment capacities Q_1^* and Q_2^*, the smooth-pasting conditions are applied. In regime 2 these are

$$\begin{cases} F_2(X_2^*) = V_2(X_2^*) \\ \frac{\partial F_2(X)}{\partial X}\Big|_{X=X_2^*} = \frac{\partial V_2(X)}{\partial X}\Big|_{X=X_2^*} \end{cases}$$

$$\iff$$

$$\begin{cases} \frac{\lambda_2}{g_2(\gamma_1)} A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1 (X_2^*)^{\gamma_2} = \frac{(X-\delta\Gamma_2)^2}{4\eta\Gamma_2 X} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1 (X_2^*)^{\gamma_1 - 1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1 (X_2^*)^{\gamma_2 - 1} = \frac{1}{4\eta\Gamma_2} - \frac{\delta^2\Gamma_2}{4\eta(X_2^*)^2} \end{cases}$$

$$\iff$$

$$\begin{cases} \frac{\lambda_2}{g_2(\gamma_1)} A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1 (X_2^*)^{\gamma_2} = \frac{(X_2^*-\delta\Gamma_2)^2}{4\eta\Gamma_2 X_2^*} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1 (X_2^*)^{\gamma_2} = \frac{X_2^*}{4\eta\Gamma_2} - \frac{\delta^2\Gamma_2}{4\eta X_2^*} \end{cases}$$

and in regime 1 the smooth-pasting conditions are

$$\begin{cases} F_1(X_1^*) = V_1(X_1^*) \\ \frac{\partial F_1(X)}{\partial X}\Big|_{X=X_1^*} = \frac{\partial V_1(X)}{\partial X}\Big|_{X=X_1^*} \end{cases}$$

$$\iff$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + aX_1^* + b + c/X_1^* = \frac{(X_1^*-\delta\Gamma_1)^2}{4\eta\Gamma_1 X_1^*} \\ \beta_1 H_1(X_1^*)^{\beta_1 - 1} + \beta_2 H_2(X_1^*)^{\beta_2 - 1} + a - c/(X_1^*)^{-2} = \frac{1}{4\eta\Gamma_1} - \frac{\delta^2\Gamma_1}{4\eta(X_1^*)^2} \end{cases}$$

$$\iff$$

$$\begin{cases} H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} = \frac{(X_1^*-\delta\Gamma_1)^2}{4\eta\Gamma_1 X_1^*} - aX_1^* - b - c/X_1^* \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} = \frac{X_1^*}{4\eta\Gamma_1} - \frac{\delta^2\Gamma_1}{4\eta X_1^*} - aX_1^* + c/X_1^* \end{cases}$$

Hence, the full system of equations consists of the continuity condition of F_1 in X_2^*, the smooth pasting, value matching conditions at the investment triggers is

$$\begin{cases} \textit{Continuity conditions of the value of waiting in regime one at } X_2^* \\ A_1(X_2^*)^{\gamma_1} + B_1(X_2^*)^{\gamma_2} = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + aX_2^* + b + c/X_2^* \\ \gamma_1 A_1(X_2^*)^{\gamma_1} + \gamma_2 B_1(X_2^*)^{\gamma_2} = \beta_1 H_1(X_2^*)^{\beta_1} + \beta_2 H_2(X_2^*)^{\beta_2} + aX_2^* - c/X_2^* \\ \textit{Smooth pasting and value matching conditions in } X_2^* \\ \frac{\lambda_2}{g_2(\gamma_1)} A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1 (X_2^*)^{\gamma_2} = \frac{(X_2^*-\delta\Gamma_2)^2}{4\eta\Gamma_2 X_2^*} \\ \frac{\lambda_2}{g_2(\gamma_1)} \gamma_1 A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} \gamma_2 B_1 (X_2^*)^{\gamma_2} = \frac{X_2^*}{4\eta\Gamma_2} - \frac{\delta^2\Gamma_2}{4\eta X_2^*} \\ \textit{Smooth pasting and value matching conditions in } X_1^* \\ H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} = \frac{(X_1^*-\delta\Gamma_1)^2}{4\eta\Gamma_1 X_1^*} - aX_1^* - b - c/X_1^* \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} = \frac{X_1^*}{4\eta\Gamma_1} - \frac{\delta^2\Gamma_1}{4\eta X_1^*} - aX_1^* + c/X_1^* \end{cases} \tag{25}$$

With these six equations, the system of equations is perfectly identified and can be solved numerically to find the thresholds in the two regimes. With these thresholds, investment capacities in the two regimes are determined using the optimal capacity functions. However, these are not the only capacities the firm may invest with as a regime switch from regime 1 to regime 2 if $X \in [X_2^*, X_1^*)$ causes the firm also to invest with investment capacity $Q_2(X)$. Below is a plot of the solution to this system of equations.

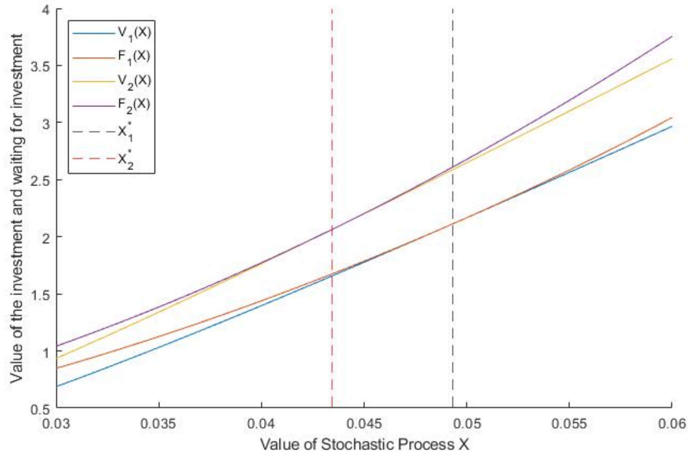


Figure 16: Plot of the value of waiting and the value of investment in the both regimes with in a continuous regime switch model as a function of X with parameter values: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\theta = 0.5$, $\delta = 0.25$ and $\eta = 0.05$

For these parameters, $X_1^* = 0.0493$ with $Q_1^* = 7.0538$, $X_2^* = 0.0435$ with $Q_2^* = 7.0191$ and $Q_2' = 7.3717$. Compared to the regime switch model without time to build, both X_1^* and X_2^* have increased. This is in line with the results from Section 3, where the investment is postponed because of the time to build.

Furthermore, investment capacity in regime 1 and the maximum investment capacity in the investment range (Q_1^* and Q_2') also increase when lead time is included. Interestingly, Q_2^* decreases if time to build is also taken into consideration compared to the regime switch model without time to build. Hence, contrary to the no-switch model, lead time affects both the timing and capacity of the investment.

The investment capacities in the regime switch model including lead time are also briefly analyzed. Analytically, only the relation between the upper bound of the investment capacity range and the investment capacity at the triggers can be studied. Namely,

$$\begin{aligned} Q_2' = Q_2(X_1^*) &= \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_2}{X_1^*}\Big) \\ &> \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_2}{X_2^*}\Big) = Q_2^* \text{ as } X_1^* > X_2^* \end{aligned}$$

Next to this, $Q_2' > Q_1^*$ or $Q_2' < Q_1^*$. The upper bound of the investment capacity range is

$$\begin{aligned} Q_2' &= \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_2}{X_1^*}\Big) \\ Q_2' &< \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_1}{X_1^*}\Big) = Q_1^* \iff \Gamma_1 < \Gamma_2 \\ Q_2' &> \frac{1}{2\eta}\Big(1 - \frac{\delta\Gamma_1}{X_1^*}\Big) = Q_1^* \iff \Gamma_1 > \Gamma_2 \end{aligned}$$

Therefore, whether the investment capacity at the trigger in regime 1 is bigger or smaller than the maximum investment capacity depends on the semi-parameters Γ_1 and Γ_2. Since c_1 and c_2 are equivalent in the formulas for Γ_i, it depends on b_1, b_2, d_1, and d_2.

The case that $\mu_1 < \mu_2$ is analyzed first. Now, $\Lambda_1 < \Lambda_2$. Hence, $b_1 + d_1 < b_2 + d_2$. However, this does not directly indicate that $\Gamma_1 > \Gamma_2$ as the factor $exp(-c_i\theta)$ is also important. From the formula for c_1 and c_2, $c_1 > c_2$. Therefore, $exp(-c_1\theta) < exp(-c_2\theta)$. From Equation 24,

$$\begin{aligned}
\lambda_2 d_1 &= (r + \lambda_2 - \mu_2 - c_2)d_2 \\
\frac{d_1}{d_2} &= \frac{r + \lambda_2 - \mu_2 - c_2}{\lambda_2} \\
&= \frac{\lambda_2 - \mu_2 - \frac{1}{2}\Big((\lambda_1 + \lambda_2) - (\mu_1 + \mu_2) - \sqrt{D}\Big)}{\lambda_2} \\
&= \frac{\frac{1}{2}(\lambda_2 - \lambda_1) - \frac{1}{2}(\mu_2 - \mu_1) + \frac{1}{2}\sqrt{D}}{\lambda_2} \\
&= \frac{1}{2}\frac{(\lambda_2 - \lambda_1) - (\mu_2 - \mu_1) + \sqrt{D}}{\lambda_2} \\
&< \frac{1}{2}\frac{(\lambda_2 - \lambda_1) - (\mu_2 - \mu_1) + (\lambda_1 + \lambda_2) - (\mu_1 - \mu_2)}{\lambda_2} \\
&= \frac{1}{2}\frac{\lambda_2 + \lambda_2}{\lambda_2} = 1
\end{aligned}$$

I.e., $d_1 < d_2$. Since $exp(-c_1\theta) < exp(-c_2\theta)$, $\Gamma_1 > \Gamma_2$ and $Q_2^{'} > Q_1^*$. The opposite also holds as $\mu_1 > \mu_2$ results in $\Gamma_1 < \Gamma_2$ and $Q_2^{'} < Q_1^*$.

However, whether Q_1^* is smaller or bigger than Q_2^* depends on the investment triggers and Γ_i. It follows that $Q_2^* > Q_1^*$ if $\frac{\Gamma_1}{X_1^*} > \frac{\Gamma_2}{X_2^*}$ and that $Q_1^* > Q_2^*$ if $\frac{\Gamma_2}{X_2^*} > \frac{\Gamma_1}{X_1^*}$. Since only an implicit expression for the investment triggers is found, a definite conclusion about the investment capacity at the triggers using the parameters cannot be stated.

In Section 4, the investment capacities are computed for different parameters to see in which regime the investment capacity is bigger. A similar relation is expected to hold if time to build is also taken into consideration as including time to build only scales the value of the firm at the moment of investment. The effect of the parameters of the stochastic process on investment triggers is also expected not to differ greatly from the regime-switching model without lead time.

6 Analysis of Relationships in the Continuous Regime Switch Model including Time to Build

In the previous section, a real options model is derived for a monopoly that enters a new market and optimizes its one-time investment capacity allowing for regime switching and an installation period. In this model, the revenue of the firm can switch between two states with different parameters of the stochastic process which denotes the correlation between price and capacity. In addition, the lead time is also included in this model. In this part, whether the extended model is still in line with earlier models and the effects of the added parameters on the investment decision are studied.

6.1 Special Cases

The factor that changes the value of the firm at the moment of investment in the new model compared to the baseline model from Section 2 is Γ_i. Hence, this section studies the effect of certain parameter values on Γ_i. This factor is defined in Section 5.3 as

$$\Gamma_i = (b_i exp(-c_1\theta) + d_i exp(-c_2\theta))^{-1}$$

with $b_i + d_i = \Lambda_i$ and Λ_i is defined as

$$\Lambda_i = \frac{r + \lambda_1 + \lambda_2 - \mu_1 - \mu_2 + \mu_i}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2}$$

Here i denotes the regime the firm is in. In the described model, i can be 1 or 2. Furthermore, θ denotes the installation time of the investment and λ_1 is the rate at which the regime switches from regime 1 to regime 2 and λ_2 is the rate at which the regime switches from regime 2 to regime 1. Moreover, in the continuous regime switch models, regime 2 stimulates investment, and regime 1 discourages investment, i.e. $X_1^* > X_2^*$.

6.1.1 Special Cases for Regime Switching Models

λ_1 and λ_2 are adjusted to have certain values that coincide with models from the previous sections. Namely, if $\lambda_1 = 0$, $\Lambda_1 = \frac{1}{r-\mu_1}$ and $\Lambda_2 = \frac{r+\lambda_2-\mu_1}{(r+\lambda_2-\mu_2)(r-\mu_2)}$. Hence, the value of the firm at the moment of investment is similar to this value in the single regime switch model. Consequently, the homogeneous solution for the value of waiting for these parameters in regime 1 is

$$F_1(X) = A_1 X^{\gamma_1} + 0$$

where γ_1 is the positive root of $g_1(\gamma)g_2(\gamma) = 0$. Note that this is for $X \in [0, X_1^*)$ as this differential equation only has a homogeneous solution because, in regime 1, a switch back to regime 2 cannot occur.

The solution for value of waiting in regime 2 for $X \in [0, X_2^*)$ is

$$F_2(X) = A_2 X^{\gamma_2} + \frac{\lambda_2}{g_2(\gamma_1)} A_1 X^{\gamma_1}$$

where γ_2 is the positive root of $g_1(\gamma)g_2(\gamma) = 0$. Hence, the same values for γ_1 and γ_2 are found. However, the value of waiting depends on $\frac{1}{g_2(\gamma_1)}$, thus $g_2(\gamma_1)$ cannot be equal to 0. Therefore, γ_1 is the positive root of $g_1(\gamma) = r - \mu_1\gamma - \frac{1}{2}\sigma_1^2\gamma(\gamma - 1)$ and γ_2 is the positive root of $g_2(\gamma) = r + \lambda_2 - \mu_2\gamma - \frac{1}{2}\sigma_2^2\gamma(\gamma - 1)$.

These parameters coincide with the β and α in the single regime switch model. The values of waiting in the two regimes are thus equal to the values of waiting in the single switch model for a discouraging new regime. Therefore, the model is now equivalent to the single switch model for a discouraging new regime, where state 1 is the new regime and state 2 is the initial regime.

Likewise, if $\lambda_2 = 0$, $\Lambda_1 = \frac{r+\lambda_1-\mu_2}{(r+\lambda_1-\mu_1)(r-\mu_2)}$ and $\Lambda_2 = \frac{1}{r-\mu_2}$. Thus, the value of the firm at the moment of investment is identical to the value of the firm at the moment of investment in the single regime switch model. Furthermore, for $X \in [X_2^*, X_1^*)$, the expression for the value of waiting is the same as in the single regime switch model. The homogeneous solution for the value of waiting in regime 2 for $X \in [0, X_2^*)$ is

$$F_2(X) = A_2 X^{\gamma_2}$$

where γ_2 is the positive root of $g_2(\gamma) = r - \mu_2\gamma - \frac{1}{2}\sigma_2^2\gamma(\gamma-1)$ using a similar argumentation as for the case where $\lambda_1 = 0$.

The solution to the differential equation for the value of waiting in regime 1 is

$$F_1 = A_1 X^{\gamma_1} + \frac{\lambda_1}{g_1(\gamma_2)} A_2 X^{\gamma_2}$$

where γ_1 is the positive root of $g_1(\gamma) = r + \lambda_1 - \mu_1\gamma - \frac{1}{2}\sigma_1^2\gamma(\gamma-1)$ for $X \in [0, X_2^*)$. Since the value of waiting in the transient region does not change, the value of waiting in the two regimes is similar to the value of waiting in the single switch model for a stimulating new regime. Therefore, in this instance, the model is a single switch model with a stimulating new regime.

Subsequently, if $\lambda_1 = 0$ and $\lambda_2 = 0$, $\Lambda_1 = \frac{1}{r-\mu_1}$ and $\Lambda_2 = \frac{1}{r-\mu_2}$. Hence, the value of the firm at the moment of investment is equal to two separate cases of a real options model without regime switching. Now the value of waiting in regime 1 is equal to

$$F_1(X) = A_1 X^{\gamma_1}$$

where γ_1 is the positive root of $g_1(\gamma) = r - \mu_1\gamma - \frac{1}{2}\sigma_1^2\gamma(\gamma-1)$.

Similarly,

$$F_2(X) = A_2 X^{\gamma_2}$$

where γ_2 is the positive root of $g_2(\gamma) = r - \mu_2\gamma - \frac{1}{2}\sigma_2^2\gamma(\gamma-1)$. Hence, the value of waiting in the two regimes is equal to two separate no-switch models. Therefore, this case is identical to two separate models of Huisman and Kort (2015). It now depends on the starting state to determine the investment decision.

However, it may also be that one of the regime switch rates tends to infinity. Economically, in case λ_1 tends to infinity compared to λ_2, the probability of being in state 2 becomes very close to 1 and the investment strategy becomes similar to a no-switch model with the parameters from regime 2. Similarly, if λ_2 tends to infinity with respect to λ_1, the probability of being in regime 1 becomes very close to 1 and the investment decision becomes analogous to the one determined in their model with the parameters from regime 1.

Furthermore, this interpretation of the special cases of the regime switch parameters is also argued from a mathematical perspective. First, the instance $\lambda_1 \to \infty$ is analyzed. Now, Λ_i becomes

$$\begin{aligned}
\lim_{\lambda_1\to\infty} \Lambda_i &= \lim_{\lambda_1\to\infty} \frac{r+\lambda_1+\lambda_2-\mu_1-\mu_2+\mu_i}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2} \\
&= \lim_{\lambda_1\to\infty} \frac{r+\lambda_1+\lambda_2-\mu_1-\mu_2+\mu_i}{(r-\mu_1)(r-\mu_2)+\lambda_1(r-\mu_2)+\lambda_2(r-\mu_1)} \\
&= \frac{\lambda_1}{\lambda_1(r-\mu_2)} = \frac{1}{r-\mu_2}
\end{aligned}$$

The value of the firm at the moment of investment in both regimes becomes analogous to the value of the firm in regime 2 for a no-switch model. The differential equation for the value of waiting in regime 1 if $X_t < X_2^*$ becomes

$$\begin{aligned}
rF_1(X) &= \lim_{\lambda_1\to\infty} \Big(\mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1(F_2(X) - F_1(X))\Big) \\
\lim_{\lambda_1\to\infty}(r+\lambda_1)F_1(X) &= \lim_{\lambda_1\to\infty} \Big(\mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 F_2(X)\Big) \\
\lambda_1 F_1(X) &= \lambda_1 F_2(X) \\
F_1(X) &= F_2(X)
\end{aligned}$$

Now, the differential equation of the value of waiting in regime 2 becomes

$$\begin{aligned}
\lim_{\lambda_1\to\infty} rF_2(X) &= \lim_{\lambda_1\to\infty} \mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X) + \lambda_2 * (F_1 - F_2) \\
rF_2(X) &= \mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X) + \lambda_2 * \lim_{\lambda_1\to\infty}(F_1 - F_2) \\
rF_2(X) &= \mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X)
\end{aligned}$$

This is equivalent to the differential equation for the value of waiting without regime switching. The value of waiting is now equal to the value of waiting without regime switching with the parameters in regime 2.

Lastly, the value of waiting in the transient region is analyzed. This is

$$\begin{aligned}
rF_1(X) &= \lim_{\lambda_1\to\infty} \Big(\mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1(V_2(X) - F_1(X))\Big) \\
\lim_{\lambda_1\to\infty}(r+\lambda_1)F_1(X) &= \lim_{\lambda_1\to\infty} \Big(\mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 V_2(X)\Big) \\
\lambda_1 F_1(X) &= \lambda_1 V_2(X) \\
F_1(X) &= V_2(X) = V_1(X)
\end{aligned}$$

Hence, the firm cannot be in the transient region if the switch rate in regime 1 tends to infinity. Since the value of the firm at the moment of investment and the value of waiting in both regimes are identical, the model is equivalent to a no-switch model. Furthermore, the values of waiting and investing are tantamount to the values in regime 2. Therefore, the firm is always in regime 2.

The case that $\lambda_2 \to \infty$ has a similar derivation for $X_t < X_2^*$ as the case $\lambda_1 \to \infty$. Hence, $F_2(X) = F_1(X)$ and $V_2(X) = V_1(X)$. To find X_2^*, the smooth pasting and value matching conditions are applied. This investment trigger is equal to the investment trigger in regime 1 as the value of waiting and the value of the firm at the moment of investment are the same and the firm cannot be in the transient region. Therefore, if $\lambda_2 \to \infty$, the model becomes equivalent to a no-switch model where the firm is in regime 1, and thus the firm ends up in a no-switch model if one of the regime switch rates tends to infinity.

Additionally, if both λ_1 and λ_2 tend to infinity with a similar convergence rate (i.e. $\lambda_1/\lambda_2 \to c$

where $c \notin \{0, \infty\})$, $\lambda_2 = c\lambda_1$ as $\lambda_1, \lambda_2 \to \infty$. Λ_i becomes

$$\begin{aligned}
\lim_{\lambda_1,\lambda_2\to\infty} \Lambda_i &= \lim_{\lambda_1,\lambda_2\to\infty} \frac{r+\lambda_1+\lambda_2-\mu_1-\mu_2+\mu_i}{(r+\lambda_1-\mu_1)(r+\lambda_2-\mu_2)-\lambda_1\lambda_2} \\
&= \lim_{\lambda_1\to\infty} \frac{r+\lambda_1+c\lambda_1-\mu_1-\mu_2+\mu_i}{(r+\lambda_1-\mu_1)(r+c\lambda_1-\mu_2)-\lambda_1\lambda_2} \\
&= \lim_{\lambda_1\to\infty} \frac{r+(1+c)\lambda_1-\mu_1-\mu_2+\mu_i}{(r-\mu_1)(r-\mu_2)+\lambda_1(r-\mu_2)+c\lambda_1(r-\mu_1)} \\
&= \frac{(1+c)\lambda_1}{\lambda_1(r-\mu_2)+c\lambda_1(r-\mu_1)} \\
&= \frac{1+c}{r-\mu_2+c(r-\mu_1)} \\
&= \frac{1}{r-\frac{c\mu_1+\mu_2}{1+c}} = \frac{1}{r-\tilde{\mu}}
\end{aligned}$$

with $\tilde{\mu} = \frac{c\mu_1+\mu_2}{1+c}$. Hence, the value of the firm at the moment of investment is similar in both regimes if both regime switch rates tend to infinity. Furthermore, this expression is equal to the value of the firm at the moment of investment for a no-switch model with μ being the weighted average of the drift of the stochastic process in the two regimes.

The approach for deriving the value of waiting in both regimes is similar to a single regime switch rate tending to infinity. The differential equation in regime 1 as both switch rates tend to infinity is

$$\begin{aligned}
\lim_{\lambda_1,\lambda_2\to\infty}(r+\lambda_1)F_1(X) &= \lim_{\lambda_1,\lambda_2\to\infty} \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 F_2(X) \\
\Bigg(F_1(X) &= \lim_{\lambda_1,\lambda_2\to\infty} \frac{\mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 F_2(X)}{r+\lambda_1}\Bigg) \\
\lim_{\lambda_1\to\infty}(r+\lambda_1)F_1(X) &= \lim_{\lambda_1,\lambda_2\to\infty} \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) \\
&\quad + \lambda_1\Big(\frac{\mu_2 X F_2'(X) + \frac{1}{2}\sigma_2^2 X^2 F_2''(X) + \lambda_2 F_1(X)}{r+\lambda_2}\Big) \\
\lim_{\lambda_1,\lambda_2\to\infty}(r+\lambda_1-\frac{\lambda_1\lambda_2}{r+\lambda_2})F_1(X) &= \lim_{\lambda_1,\lambda_2\to\infty}\Big(\mu_1 F_1'(X) + \frac{\lambda_1}{r+\lambda_2}\mu_2 F_2'(X)\Big)X \\
&\quad + \frac{1}{2}X^2\Big(\sigma_1^2 F_1''(X) + \frac{\lambda_1}{r+\lambda_2}\sigma_2^2 F_2''(X)\Big) \\
\lim_{\lambda_1\to\infty}(r+\frac{\lambda_1(r+c\lambda_1)-c\lambda_1^2}{r+c\lambda_1})F_1(X) &= \lim_{\lambda_1\to\infty}\Big(\mu_1 F_1'(X) + \frac{\lambda_1}{r+c\lambda_1}\mu_2 F_2'(X)\Big)X \\
&\quad + \frac{1}{2}X^2\Big(\sigma_1^2 F_1''(X) + \frac{\lambda_1}{r+c\lambda_1}\sigma_2^2 F_2''(X)\Big) \\
\lim_{\lambda_1\to\infty}\Big(1+\frac{\lambda_1}{r+c\lambda_1}\Big)rF_1(X) &= \Big(\mu_1 F_1'(X) + \frac{1}{c}\mu_2 F_2'(X)\Big)X + \frac{1}{2}X^2\Big(\sigma_1^2 F_1''(X) + \frac{1}{c}\sigma_2^2 F_2''(X)\Big) \\
\Big(1+\frac{1}{c}\Big)rF_1(X) &= \Big(\mu_1 F_1'(X) + \frac{1}{c}\mu_2 F_2'(X)\Big)X + \frac{1}{2}X^2\Big(\sigma_1^2 F_1''(X) + \frac{1}{c}\sigma_2^2 F_2''(X)\Big) \\
(1+c)rF_1(X) &= \Big(c\mu_1 F_1'(X) + \mu_2 F_2'(X)\Big)X + \frac{1}{2}X^2\Big(c\sigma_1^2 F_1''(X) + \sigma_2^2 F_2''(X)\Big)
\end{aligned} \tag{26}$$

and the value of waiting in regime 2

$$\begin{aligned}
\lim_{\lambda_2\to\infty}(r+\lambda_2)F_2(X) &= \lim_{\lambda_2\to\infty}\mu_2 X F_2'(X)+\frac{1}{2}\sigma_2^2X^2F_2''(X)+\lambda_2F_1(X)\\
\lim_{\lambda_2\to\infty}(r+\lambda_2)F_2(X) &= \lim_{\lambda_2\to\infty}\lambda_2F_1(X)\\
\lim_{\lambda_2\to\infty}\lambda_2F_2(X) &= \lim_{\lambda_2\to\infty}\lambda_2F_1(X)\\
F_2(X) &= F_1(X)
\end{aligned} \tag{27}$$

Similar to the case that a single regime switch rate tends to infinity, the value of waiting in the transient region in regime 1 tends to the value of investing in regime 2. Furthermore, the value of investing in the two regimes tends toward each other. The firm can, therefore, almost never be in the transient region and the transient region disappears.

Hence, the two values of waiting in the two regimes converge toward each other if the regime switch rates tend to infinity: $F_1, F_2(X) \to F(X)$ as $\lambda_1, \lambda_2 \to \infty$. The values of waiting in the two regimes thus converge to $F(X)$ with its differential equation equal to

$$\begin{aligned}
rF(X) &= \frac{c\mu_1+\mu_2}{1+c}XF'(X)+\frac{1}{2}X^2F''(X)\frac{c\sigma_1^2+\sigma_2^2}{1+c}\\
&= \tilde{\mu}XF'(X)+\frac{1}{2}\tilde{\sigma}^2X^2F''(X)
\end{aligned} \tag{28}$$

with $\tilde{\mu}$ similarly defined as in the derivation of the value of the firm at the moment of investment and $\tilde{\sigma}^2 = \frac{c\sigma_1^2+\sigma_2^2}{1+c}$.

Therefore, the value of waiting becomes dependent on the mean drift rate and the mean volatility of the Brownian motions in the two regimes. Since the value of the firm and the value of waiting in the two regimes are equal, the model is equivalent to a no-switch model with parameters that are an average of the parameters in the two regimes. Consequently, for high rates of regime switches, the investment decisions in the different regimes become more similar as they both converge to the average no-switch model.

In conclusion, if one of the $\lambda_i = 0$, the model becomes a single switch model. Intuitively, this is what one would expect as a rate of 0 indicates that a transition away from this state is not possible. This state is thus the absorbing state. This corresponds to the previous single-switch and no-switch models. As both $\lambda_i = 0$, the model is similar to the model of Huisman and Kort (2015). Furthermore, if $\lambda_i \to \infty$ the firm ends up no-switch model, which is in line with the economic reasoning that the probability of being in a certain state becomes negligible. If both regime switch rates tend to infinity with a similar convergence rate, the limit of the stochastic process goes to a no-switch model with weighted average parameter values.

6.1.2 Special Cases for Time to Build Models

The model of Huisman and Kort (2015) is also extended by including time to build. One variable is added for a deterministic lead time: θ. The only interesting case is when $\theta = 0$ because if $\theta \to \infty$ (i.e. the installation period tends to infinity), the investment trigger in the no-switch model tends to infinity and the firm would never invest. Interestingly, the investment capacity remains constant in this case.

Furthermore, if $\theta = 0$, it follows that $\Gamma_i = \frac{1}{b_i+d_i} = \frac{1}{\Lambda_i}$ in the regime switch model. If the value of Γ_i is substituted into the system of equations to find the investment trigger and optimal investment capacities, exactly the same system of equations is obtained as in Section 4.2. Therefore, a regime switch model including lead time where the lead time is negligible is the same as a regime switch model without time to build. The time-to-build model, therefore, includes the real options model without time to build.

A stochastic installation time results in additional variables in the real options to denote the distribution of lead time. These parameters affect the MGF of θ. Hence, they affect the timing in the no-switch model but not the investment capacity. Moreover, the installation time is nonnegative. This should be reflected in the distribution of θ and thus the MGF of θ. Γ_i is adjusted accordingly to meet this change in the distribution of θ. A special case of a stochastic installation time is when the variance of the installation time is close to zero. Then, the model becomes equivalent to the deterministic lead time model.

6.1.3 Special Cases of the Regime Switch Model including Time to Build

As shown above, the regime switch model and the time-to-build model separately have special cases if $\lambda_i \to \infty$, $\lambda_i = 0$, or $\theta = 0$. In this section, the value of the firm changes for different regime switch rates and lead times. In the previous segment, the special cases of lead time are studied. However, only $\theta = 0$ is interesting and has already been analyzed in the regime switch model without lead time. Therefore, in this segment, special cases of the regime switch parameters are studied in the combined model.

Since the value of waiting depends on the value of investing in the transient state, the main focus of this section is the analysis of the value of investing. Γ_i captures the full effect of time to build and the regime switch parameters on the value of the firm at the moment of investment. Hence, this semi-parameter is analyzed for special cases of the regime switch parameters. Starting with $\lambda_1 = 0$, the parameters of Γ_i become

$$\begin{aligned}
c_1 &= r - \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\lambda_2 + \frac{1}{2}\sqrt{(-\lambda_2 - \mu_1 + \mu_2)^2 + 0} \\
&= r - \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\lambda_2 + \frac{1}{2}(-\lambda_2 - \mu_1 + \mu_2) \\
&= r + \mu_1
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= r - \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\lambda_2 - \frac{1}{2}\sqrt{(-\lambda_2 - \mu_1 + \mu_2)^2 + 0} \\
&= r - \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\lambda_2 - \frac{1}{2}(-\lambda_2 - \mu_1 + \mu_2) \\
&= r + \lambda_2 - \mu_2
\end{aligned}$$

Hence, the c_1 and c_2 coincide with those in the single-switch model including lead time. From here, similar derivations follow to obtain b_i and d_i as A becomes

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\frac{\lambda_2}{\lambda_2 - (\mu_2 - \mu_1)} & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and $s = \begin{bmatrix} \frac{1}{r-\mu_1} & \frac{r+\lambda_2-\mu_1}{(r+\lambda_2-\mu_2)(r-\mu_1)} & 0 & 0 \end{bmatrix}'$.

This results in $d_1 = 0$ and $b_1 = \frac{1}{r-\mu_1}$. b_2 is equal to

$$\begin{aligned}
b_2 &= \frac{\lambda_2}{\lambda_2 - (\mu_2 - \mu_1)} b_1 \\
&= -\frac{\lambda_2}{(\lambda_2 - \mu_2 + \mu_1)(r - \mu_1)}
\end{aligned}$$

d_2 is given by the second row of the matrix and the second element of s. This is

$$d_2 = \frac{r + \lambda_2 - \mu_1}{(r + \lambda_2 - \mu_2)(r - \mu_2)} - b_2$$

It is trivial that the value of d_2 corresponds to the value in the single switch model, but now with the opposite subscripts. Hence, the model is equivalent to a single regime switch model with a discouraging new regime as also the value of waiting converges to a single regime switch model.

If $\lambda_2 = 0$, the same Γ_i are found, but now with the opposite subscripts compared to the case that $\lambda_1 = 0$ (thus the same subscripts as in the single regime switch model). Since regime 2 is investment stimulating, the model is equivalent to a single regime switch model with an investment-stimulating new regime.

When $\lambda_1 = 0$ and $\lambda_2 = 0$, $c_1 = r - \mu_1$ and $c_2 = r - \mu_2$, and $b_1 = \frac{1}{r-\mu_1}$, $d_1 = 0$, $b_2 = 0$ and $d_2 = \frac{1}{r-\mu_1}$. These correspond to the value of the firm in no-switch models including lead time like in the case of the regime switch model without lead time.

Therefore, if one of the $\lambda_i \to 0$, Γ_i becomes equivalent to the factor in a single regime switch model. The value of the firm at the moment of investment is thus similar to a firm's value at the moment of investment in a single regime switch model. The value of waiting is also identical to a single switch model if one of the switch rates is zero. If both regime switch rates are negligible, the firm is in a no-switch model as Γ_i is equal to $(r - \mu_i)exp(-(r - \mu_i)\theta)$.

However, it may also occur that one of the regime switch rates tends to infinity. Intuitively, as $\lambda_1 \to \infty$, $c_2 \to r - \mu_2$ as the regime switches back immediately if a regime switch occurs from regime 2 to regime 1. Therefore, c_2 should converge. To show this, $f(\lambda_1)$ and $h(\lambda_1)$ are defined as

$$\begin{aligned} f(\lambda_1, \lambda_2) &= (\lambda_1 + \lambda_2) - (\mu_1 - \mu_2) \\ h(\lambda_1, \lambda_2) &= \sqrt{(\lambda_1 + \lambda_2)^2 - 2(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} \end{aligned}$$

It is trivial to see that $c_1 = r - \mu_2 + \frac{1}{2}(f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2))$ and $c_2 = r - \mu_2 + \frac{1}{2}(f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2))$. Now, it rests to show that $\lim_{\lambda_1 \to \infty} f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2) = 0$. First,

$$\begin{aligned} \lim_{\lambda_1 \to \infty} f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2) &= \lim_{\lambda_1 \to \infty} (\lambda_1 + \lambda_2) - (\mu_1 - \mu_2) + \sqrt{(\lambda_1 + \lambda_2)^2 - 2(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} \\ &\to \infty \end{aligned}$$

Hence, $c_1 \to \infty$ as $\lambda_1 \to \infty$. Since b_1 and b_2 have a polynomial relation with c_1 and the term $exp(-c_1\theta)$ is an exponential with respect to λ_1, the term $\lim_{\lambda_1 \to \infty} b_i exp(-c_1\theta) = 0$. Therefore, the remaining term in Γ_i as λ_1 tends to infinity is $d_i exp(-c_2\theta)$. To find c_2 as $\lambda_1 \to \infty$,

$f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2)$ is analyzed as $\lambda_1 \to \infty$:

$$\begin{aligned}
\lim_{\lambda_1 \to \infty} f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2) &= \lim_{\lambda_1 \to \infty} (f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2)) \frac{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \\
&= \lim_{\lambda_1 \to \infty} \frac{(f(\lambda_1, \lambda_2))^2 - (h(\lambda_1, \lambda_2))^2}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \\
&= \lim_{\lambda_1 \to \infty} \left(\frac{\Big((\lambda_1 + \lambda_2) - (\mu_1 - \mu_2)\Big)^2}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \right. \\
&\left. - \frac{(\lambda_1 + \lambda_2)^2 - 2(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \right) \\
&= \lim_{\lambda_1 \to \infty} \frac{-2(\lambda_1 + \lambda_2)(\mu_1 - \mu_2) + 2(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \\
&= \lim_{\lambda_1 \to \infty} \frac{-4\lambda_2(\mu_1 - \mu_2)}{f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)} \\
&= 0
\end{aligned}$$

Therefore, c_2 converges to $r - \mu_2$ as $\lambda_1 \to \infty$. Furthermore, as $\lambda_1 \to \infty$, $\Lambda_i \to \frac{1}{r-\mu_2}$ resulting in $d_1 = d_2 = \frac{1}{r-\mu_2}$. Hence, $\Gamma_i = \frac{exp(-(r-\mu_2)\theta)}{r-\mu_2}$. Thus the value of the firm in the two regimes becomes identical. As shown in the regime switch model without installation time, the values of waiting in the two regimes also tend toward each other as $\lambda_1 \to \infty$. Thus, the model is similar to a no-switch model including lead time. If $\lambda_2 \to \infty$, the same holds, but now $c_2 \to r - \mu_1$ and $d_i \to \frac{1}{r-\mu_1}$.

Lastly, it may also be that both regime switch rates tend to infinity with a similar convergence rate. Like in the regime switch model without time to build, $\lambda_2 \to c\lambda_1$ as $\lambda_2, \lambda_1 \to \infty$ with $c \geq 0$. Now, $f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2)$ tends to

$$\begin{aligned}
\lim_{\lambda_1, \lambda_2 \to \infty} f(\lambda_1, \lambda_2) + h(\lambda_1, \lambda_2) &= \lim_{\lambda_1, \lambda_2 \to \infty} (\lambda_1 + \lambda_2) - (\mu_1 - \mu_2) \\
&+ \sqrt{(\lambda_1 + \lambda_2)^2 - 2(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} \\
&= \lim_{\lambda_1 \to \infty} (\lambda_1 + c\lambda_1) - (\mu_1 - \mu_2) \\
&+ \sqrt{(\lambda_1 + c\lambda_1)^2 - 2(\lambda_1 - c\lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} \\
&\to \infty
\end{aligned}$$

Hence, c_1 tends to infinity like in the single convergence case and $\Gamma_i = d_i exp(-c_2\theta)$ as $\lambda_1, \lambda_2 \to \infty$. c_2 converges to a constant depending on $f(\lambda_1, \lambda_2) - h(\lambda_1, \lambda_2)$. Writing this expression out as $\lambda_1, \lambda_2 \to \infty$ yields

$$
\begin{aligned}
\lim_{\lambda_1,\lambda_2\to\infty} f(\lambda_1,\lambda_2) - h(\lambda_1,\lambda_2) &= \lim_{\lambda_1\to\infty} f(\lambda_1, c\lambda_1) - h(\lambda_1, c\lambda_1) \\
&= \lim_{\lambda_1\to\infty} \Big(f(\lambda_1, c\lambda_1) - h(\lambda_1, c\lambda_1)\Big) \frac{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)} \\
&= \lim_{\lambda_1\to\infty} \frac{(f(\lambda_1, c\lambda_1))^2 - (h(\lambda_1, c\lambda_1))^2}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)} \\
&= \lim_{\lambda_1\to\infty} \frac{\Big((1+c)\lambda_1 - (\mu_1 - \mu_2)\Big)^2}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)} \\
&- \frac{\Big(((1+c)\lambda_1)^2 - 2(1-c)\lambda_1(\mu_1-\mu_2) + (\mu_1-\mu_2)^2\Big)}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)} \\
&= \lim_{\lambda_1\to\infty} \frac{-2(1+c)\lambda_1(\mu_1-\mu_2) + 2(1-c)\lambda_1(\mu_1-\mu_2)}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)} \\
&= \lim_{\lambda_1\to\infty} \frac{-4c\lambda_1(\mu_1-\mu_2)}{f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)}
\end{aligned}
\tag{29}
$$

To find the convergence rate of $f(\lambda_1, c\lambda_1) + h(\lambda_1, c\lambda_1)$, the ratio of h and f as $\lambda_1 \to \infty$ is analyzed. This is

$$
\begin{aligned}
\lim_{\lambda_1\to\infty} \frac{h(\lambda_1, c\lambda_1)}{f(\lambda_1, c\lambda_1)} &= \lim_{\lambda_1\to\infty} \frac{\sqrt{((1+c)\lambda_1)^2 - 2(1-c)\lambda_1(\mu_1-\mu_2) + (\mu_1-\mu_2)^2}}{(1+c)\lambda_1 - (\mu_1-\mu_2)} \\
&= \lim_{\lambda_1\to\infty} \sqrt{\frac{((1+c)\lambda_1)^2 - 2(1-c)\lambda_1(\mu_1-\mu_2) + (\mu_1-\mu_2)^2}{(1+c)^2\lambda_1 - 2(1+c)\lambda_1(\mu_1-\mu_2) + (\mu_1-\mu_2)^2}} \\
&= \lim_{\lambda_1\to\infty} \sqrt{1 - \frac{4c\lambda_1(\mu_1-\mu_2)}{(1+c)^2\lambda_1 - 2(1+c)\lambda_1(\mu_1-\mu_2) + (\mu_1-\mu_2)^2}} \to \sqrt{1-0} = 1
\end{aligned}
$$

Therefore, $\lim_{\lambda_1\to\infty} h(\lambda_1, c\lambda_1) = \lim_{\lambda_1\to\infty} f(\lambda_1, c\lambda_1)$. Substituting this into the expression above yields

$$
\begin{aligned}
\lim_{\lambda_1,\lambda_2\to\infty} f(\lambda_1,\lambda_2) - h(\lambda_1,\lambda_2) &= \lim_{\lambda_1\to\infty} \frac{-4c\lambda_1(\mu_1-\mu_2)}{2f(\lambda_1, c\lambda_1)} \\
&= \lim_{\lambda_1\to\infty} \frac{-4c\lambda_1(\mu_1-\mu_2)}{2\Big((1+c)\lambda_1 - (\mu_1-\mu_2)\Big)} \\
&\to \frac{-4c\lambda_1(\mu_1-\mu_2)}{2(1+c)\lambda_1} = \frac{-2c(\mu_1-\mu_2)}{1+c}
\end{aligned}
$$

Thus, c_2 converges as $\lambda_{1,2} \to \infty$ and is equal to

$$
\begin{aligned}
\lim_{\lambda_1,\lambda_2\to\infty} c_2 &= \lim_{\lambda_1,\lambda_2\to\infty} \Big(r - \mu_2 + \frac{1}{2}(f(\lambda_1,\lambda_2) - h(\lambda_1,\lambda_2))\Big) \\
&= r - \mu_2 + \lim_{\lambda_1,\lambda_2\to\infty} \frac{1}{2}\Big(f(\lambda_1,\lambda_2) - h(\lambda_1,\lambda_2)\Big) \\
&= r - \mu_2 + \frac{1}{2} * \frac{-2c(\mu_1-\mu_2)}{1+c} \\
&= r - \frac{(1+c)\mu_2 + c(\mu_1-\mu_2)}{1+c} \\
&= r - \frac{\mu_2 + c\mu_1}{1+c} = r - \tilde{\mu}
\end{aligned}
$$

With $\tilde{\mu}$ similarly defined as in the case for $\lambda_1 \to \infty$ in the regime switch model without lead time. Like in the regime switch model without lead time, $\Lambda_i \to \frac{1}{r-\tilde{\mu}}$. Since, $b_i exp(-c_1\theta) \to 0$ as $\lambda_1, \lambda_2 \to \infty$, $d_i = \Lambda_i \to \frac{1}{r-\tilde{\mu}}$. Thus, $\Gamma_i = \frac{exp(-(r-\tilde{\mu})\theta)}{r-\tilde{\mu}}$. Therefore, the drift of the stochastic process is the weighted average drift in the two regimes as both regime switch rates tend to infinity. This results in a similar value of the firm at the moment of investment in both regimes. Since the value of waiting is analogous in the two regimes as the regime switch rates tend to infinity with weighted averaged parameters for the drift rate and volatility of the stochastic process, the firm is in a no-switch model including lead time with a mean drift, $\tilde{\mu} = \frac{c\mu_1+\mu_2}{1+c}$, and mean volatility, $\tilde{\sigma}^2 = \frac{c\sigma_1^2+\sigma_2^2}{1+c}$.

Furthermore, if a model with negligible lead time is combined with special cases for the regime switch model, the same models arise, but without time to build. Therefore, having a negligible lead time coincides with the models from Section 4. For instance, if both $\theta = 0$, $\lambda_1 = 0$ and $\lambda_2 = 0$, the model by Huisman and Kort (2015) can be applied where the parameters of the starting state determine the investment trigger and investment capacity. Intuitively, this makes sense as the probability of any regime switch is now zero and the investment immediately generates revenue. These special cases are assumptions in their model.

Therefore, the previously derived models are included in the combined model. This combined model is equivalent to the model of Huisman and Kort (2015) if lead time and the switch rates are negligible. Their monopoly model is thus part of the combined model. No anomalies arise in the combined model compared to the models in Section 2, Section 3, and Section 4.

6.2 Analysis of the Regime Switch Model including Time to Build

Since an explicit expression is not obtained for the relationship between the additional parameters and the investment decision in the combined model, computations are used to analyze the effect of lead time and the regime switch rate on the investment trigger and capacity. First, the effect of lead time on the investment decision is studied. Secondly, the effect of the rate parameters on the investment decision is analyzed in the combined model to find whether auxiliary effects of the rate parameters and time to build are present in the investment decision.

6.2.1 Effect of Time to Build on the Investment Decision

From Figure 16, the hypothesis is obtained that an increase in lead time increases the investment triggers and minorly affects the investment capacities in both regimes. Using a variable lead time, the investment triggers and investment capacities are computed in Table 3.

Table 3: The effect of different values for lead time on the investment triggers and investment capacity for parameters values: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\delta = 0.25$ and $\eta = 0.05$

θ	X_1^*	X_2^*	Q_1^*	Q_2^*	$Q_2^{'}$
0	0.04736	0.04250	7.0189	7.0277	7.3327
0.1	0.04773	0.04269	7.0257	7.0260	7.3403
0.2	0.04811	0.04288	7.0323	7.0243	7.3478
0.3	0.04849	0.04308	7.0388	7.0227	7.3550
0.4	0.04886	0.04327	7.0450	7.0212	7.3620
0.5	0.04924	0.04347	7.0511	7.0198	7.3688
0.6	0.04961	0.04367	7.0571	7.0184	7.3754
0.7	0.04999	0.04388	7.0629	7.0170	7.3818
0.8	0.05036	0.04408	7.0685	7.0158	7.3881
0.9	0.05074	0.04429	7.0740	7.0145	7.3941
1	0.05111	0.04450	7.0793	7.0134	7.4000

In Table 3, an increase in lead time increases both investment triggers. In this case, the investment trigger in regime 1 increases more than in regime 2. Economically, this may be explained by the drift of the revenue as this value is bigger in regime 2 than in regime 1. This results in a larger discounting of the revenue corrected for the expected drift rate in regime 1 than in regime 2. The value of the firm at the moment of investment is thus lower in regime 1 than in regime 2. To compensate for this loss in revenue, the firm postpones the investment more in regime 1 than in regime 2 for an increase in lead time.

Moreover, the effect of lead time is increasing and convex. These conclusions are also found in the lead time model without regime switching, where the investment trigger is increased by a factor that is convex in θ.

A change in θ also affects the investment capacities according to the results in Table 3. Q_1^* and $Q_2^{'}$ increase as θ increases. Yet, Q_2^* decreases with an increase in θ. This indicates that the investment range widens for increases in installation time of the investment as Q_2^* decreases and $Q_2^{'}$ increases. Contrary to the findings in the time-to-build model without regime switching, the investment capacities thus change for different values of lead time. The mathematical reason for this dependence of the investment capacity on lead time is that the model is dynamic without a constant value for the investment capacity. The investment capacity is ever-changing and dependent on the current level of the stochastic process X_t because the firm does not only have investment triggers but also a transient region.

Economically, an increase in lead time increases the probability that a regime switch occurs during the installation period. Therefore, the firm adjusts its investment capacities to hedge the risk of an increase in switch probability. This effect results in a decrease in the investment capacity in the more profitable regime 2 and an increase in investment capacity for the more unprofitable regime 1 for increases in lead time. This so-called ‘hedging’ effect is in line with economic intuition as this increase in regime switch probability during the installation period causes the firm to mitigate between the expected revenue of the firm in the current regime and the long-term expected revenue.

The upper bound of the investment capacity range increases, because the firm postpones investment in regime 1 and it is now in a more advantageous position than at the investment trigger in regime 2. To profit from the more profitable position, the firm invests with an increased capacity. This ‘improved position’ effect is bigger than the ‘hedging’ effect for ending up in the less profitable regime when the investment generates revenue. This could emanate from the larger volatility in regime 1 increasing the investment trigger in regime 1 more than in regime 2.

6.2.2 Effect of the Rates of Regime Switching on the Investment Decision

Apart from analyzing the effect of lead time, the effect of regime-switching on the investment decision is studied too. The investment decision is computed for different switch rates in Table 4 and Table 5 on the next page. Positive regime switch rates are used as a model that includes geopolitical unrest can switch continuously between states of turmoil and prosperity. A continuous regime switch model includes thus geopolitical unrest. A switch rate of zero for one of the regimes corresponds to a single switch model, which is not the focus of this thesis.[6]

[6]The single switch model including lead time may be interesting in other applications such as the patenting of products. For instance, requesting a patent takes time, and after a certain period, the patent expires. In this application, the request time could correspond to the lead time and the occurrence of a single regime switch corresponds to the expiring of the patent

Table 4: The investment triggers in both regimes, where the two numbers in the cells correspond to (X_1^*, X_2^*) for the different change rate in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta = 0.25$, $\eta = 0.05$, $\theta = 0.5$

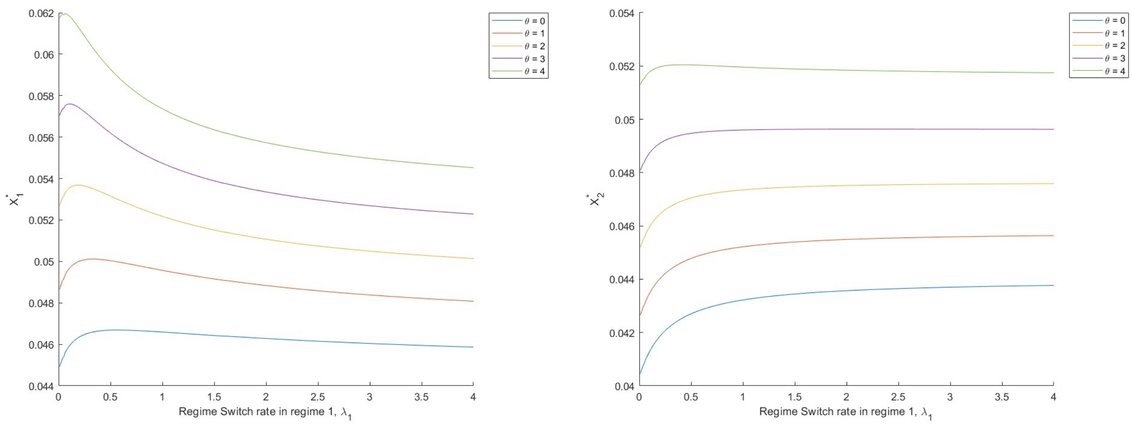
	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(0.04818, 0.04291)	(0.04722, 0.04261)	(0.04683, 0.04268)	(0.04664, 0.04284)	(0.04653, 0.04302)
$\lambda_1 = 0.4$	(0.04840, 0.04353)	(0.04745, 0.04312)	(0.04697, 0.04303)	(0.04670, 0.04307)	(0.04653, 0.04316)
$\lambda_1 = 0.6$	(0.04837, 0.04386)	(0.04752, 0.04344)	(0.04703, 0.04329)	(0.04673, 0.04326)	(0.04654, 0.04329)
$\lambda_1 = 0.8$	(0.04828, 0.04406)	(0.04752, 0.04367)	(0.04705, 0.04349)	(0.04675, 0.04342)	(0.04654, 0.04341)
$\lambda_1 = 1.0$	(0.04816, 0.04420)	(0.04750, 0.04384)	(0.04705, 0.04365)	(0.04675, 0.04355)	(0.04654, 0.04352)

Table 5: The investment capacities in both regimes, where the three numbers in the cells correspond to $(Q_1^*, Q_2^*, Q_2^{'})$ for the different change rate in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta = 0.25$, $\eta = 0.05$, $\theta = 0.5$

	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(6.6923, 6.6029, 6.9750)	(6.3043, 6.1519, 6.5277)	(6.1032, 5.9241, 6.2852)	(5.9816, 5.7919, 6.1345)	(5.9009, 5.7080, 6.0324)
$\lambda_1 = 0.4$	(7.0806, 6.9486, 7.2562)	(6.6897, 6.5266, 6.8438)	(6.4479, 6.2703, 6.5827)	(6.2855, 6.1018, 6.4044)	(6.1696, 5.9843, 6.2756)
$\lambda_1 = 0.6$	(7.2734, 7.1319, 7.3998)	(6.9193, 6.7578, 7.0358)	(6.6773, 6.5057, 6.7834)	(6.5030, 6.3270, 6.5996)	(6.3722, 6.1951, 6.4605)
$\lambda_1 = 0.8$	(7.3875, 7.2454, 7.4861)	(7.0713, 6.9145, 7.1646)	(6.8406, 6.6760, 6.9277)	(6.6661, 6.4979, 6.7471)	(6.5302, 6.3610, 6.6055)
$\lambda_1 = 1.0$	(7.4624, 7.3226, 7.5431)	(7.1791, 7.0277, 7.2567)	(6.9625, 6.8048, 7.0362)	(6.7928, 6.6319, 6.8623)	(6.6568, 6.4948, 6.7223)

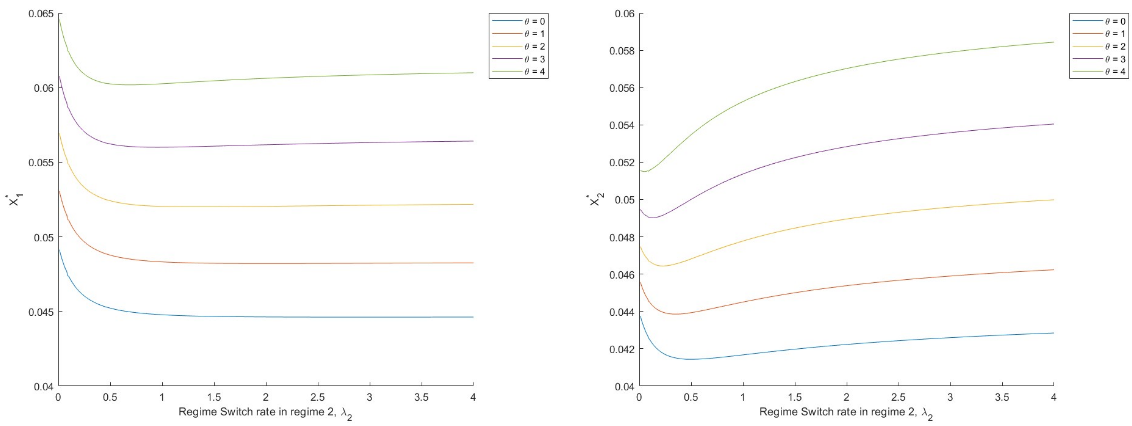
In Table 4 and Table 5, the same effects of the regime switch rates are observed as in the table for the continuous regime switch model without lead time. The capacity effect and NPV effect are still present in the investment decision. Furthermore, the investment capacities still depend on the long-run investment distribution of the investment. The hedging effect is also prevailing as the investment capacity at the trigger in regime 1 increases and at the trigger in regime 2 decreases compared to the output in Table 2. However, the hedging effect is not as prevalent in the investment capacity as the NPV effect of the regime switch rates. This inferiority of the hedging effect results from the assumption that the firm cannot alter the capacity post-investment. Hence, the long-term perspective is more dominating in the investment strategy. Compared to Table 1, the differences in investment triggers are bigger as the corrected discounting of the investment is bigger in regime 1 than in regime 2.

In Figure 17 and Figure 18, the investment triggers are plotted for different values of lead time against the regime switch rates to analyze the effect of lead time on the effect of the regime switch rates. The investment triggers are plotted in separate figures to maintain a clear overview of the effect of lead time.



(a) X_1^* (b) X_2^*

Figure 17: Plots of the investment triggers for $\lambda_2 = 0.2$ and the same parameters as defined in the plot and in Table 4



(a) X_1^* (b) X_2^*

Figure 18: Plots of the investment triggers for $\lambda_1 = 0.2$ and the same parameters as defined in the plot and in Table 4

In Figure 17 and Figure 18, larger lead times increase the investment triggers in regime 1 universally. This is also observed in the previous tables where the investment decision is computed for different values of lead time. This suggests that increases in lead time correspond to increases in investment triggers like in the no-regime switch model including lead time.

Additionally, in Figure 17a, increases in lead time change the effect of the regime switch rate in regime 1. For lower lead times the effect is not monotone like in Section 4. However, for high lead times, the increase in investment trigger by increases in the regime switch rates is more limited. This suggests that the capacity effect is less preeminent for high lead times. Moreover, for high regime switch rates and high lead times, the investment trigger decreases significantly. Therefore, the timing of the investment becomes more leading as the installation time increases.

This effect is also observed in Figure 18b. Here, an increase in lead time decreases the capacity effect and causes the NPV effect to become more preeminent. Yet, the effect is opposite to the one in 17a as an increase in λ_2 worsens the overall position of the firm, whereas a high λ_1 improves the NPV of the investment.

In Figure 17b, increases in lead time also alter the effect of the regime switch rate. For lower lead times, the investment trigger in regime 2 is strictly increasing with respect to the regime switch rate in regime 1. However, for high lead times, the effect of the regime switch rate is twofold. For high lead times and a lower regime switch rate in regime 1, the investment trigger in regime 2 increases for increases in λ_1. For high lead times and high switch rates, an increase in regime switch rates in regime 1 results in a decrease in the investment trigger of regime 2. This indicates that the NPV effect in the timing of the investment becomes more preeminent with higher lead times.

When Figure 18a is analyzed, a similar effect is observed at high lead times. Yet, the effect of the regime switch rate in regime 2 on the investment trigger in regime 1 is inverse compared to the effect of λ_2 on X_1^* for the same reason that the effect of λ_1 on X_1^* is reversed compared to the effect of λ_2 on X_2^*.

Therefore, including lead time in the regime switch model causes the NPV effect to become more preeminent in the timing of the investment. This is also supported if the investment capacities are analyzed as these do not differ majorly at high lead times for changes in lead time (see Appendix D). This result is similar to the effect of lead time on the investment capacity in the no-regime-switch model in Section 3, where the investment capacity does not depend on the installation time.

This auxiliary effect of lead time on the effect of the rate parameters results from the increase in switch probability during the installation period. The current regime is now less leading in the investment decision as a switch is more likely to occur before the investment has been installed. Adjusting the investment capacity based on the current regime at the expense of timing is therefore less important for the firm. To still correct for the change in NPV because of a change in regime switch rates, the timing is adjusted. The NPV effect is more prevalent than the capacity effect in the timing of the investment at high lead times. The investment capacities are also affected by lead time as the firm emphasizes the NPV of the investment and mitigates the risk of ending up in the other regime at the moment the investment is installed. Though, the investment capacity is affected more by the change in NPV than the hedging effect as the firm can only invest once.

6.3 General Cases for Different Regime Switch Rates

Next to the special cases and the analysis of the model, more general cases of the real options regime switch model with time to build are studied. Another application of the regime switch model is the inclusion of business cycles in the investment decision. Three cases of business cycles are distinguished: favorable, neutral, and idle business cycles.

In the favorable business cycle, the regime switch rate to the advantageous regime is bigger than the rate to the disadvantageous regime. In the idle business cycles, the regime switch rate to the advantageous regime is smaller than the rate to the disadvantageous regime. Lastly, in the neutral business cycle, the regime switch rates to the advantageous and the disadvantageous regime are equal. In this section, the effect of lead time on the investment decision is analyzed for these business cycles.

This implies that in the favorable business cycle, the firm expects to be longer in the advantageous regime than in the disadvantageous regime. For the idle business cycle, the opposite holds, where the expected duration in the disadvantageous regime is bigger than the expected duration in the advantageous regime. In the neutral business cycle, the firm expects to be as long in the advantageous regime as in the disadvantageous regime.

In Figure 19 and Figure 20 below, the investment triggers and investment capacities at the triggers are plotted for the different business cycle cases and time to build. Here, the favorable business cycle has regime switch rates $\lambda_1 = 1$ and $\lambda_2 = 0.2$, the neutral business cycle has regime switch rates $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{1}{3}$ and the idle business cycles has $\lambda_1 = 0.2$ and $\lambda_2 = 1.0$. This parameterization corresponds to an expected switch once every 1 and 5 years in the favorable and idle business cycles and once every 3 years in the neutral business cycle. In all cases, a full business cycle takes an average of six years.

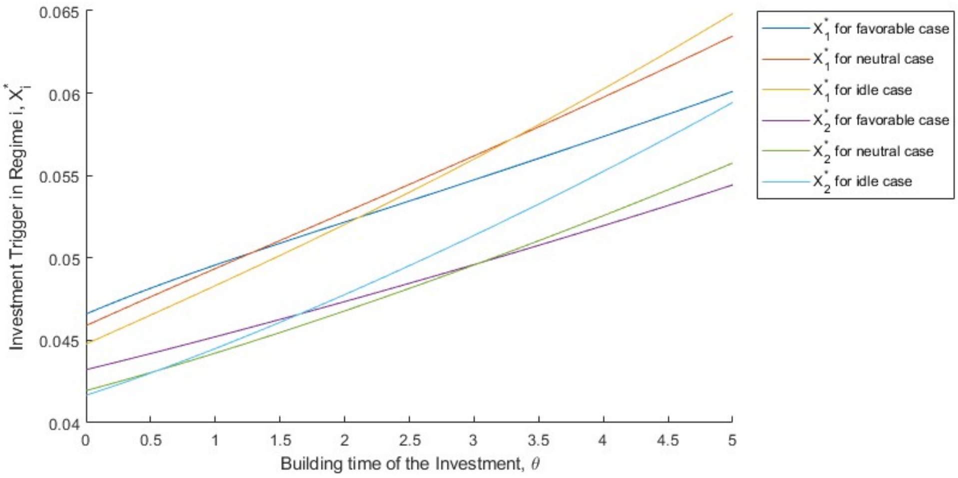


Figure 19: The Investment Triggers for the different cases of Business Cycles and variable lead time with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\eta = 0.05$, $\delta = 0.25$

In Figure 19, the investment triggers in the advantageous and disadvantageous regimes of the different business cycles increase universally for high lead times as the firm discounts the investment more if the installation time increases. To compensate for this discounting, the firm postpones the investment. Moreover, the investment triggers increase less for more profitable business cycles for all lead times. This suggests that lead time has a bigger effect on the investment triggers in idler business cycles, because the NPV factor in regime i, Γ_i, increases more with increases in lead time for idler business cycles.

However, this does not indicate that the investment triggers in the idler business cycles are always bigger than in more favorable business cycles. For instance, at lower lead times, the firm invests at lower investment triggers in the idler business cycles. In idler business cycles, the firm has an increased probability of the firm being in the less profitable regime at the moment of payout. Therefore, it adjusts the investment capacity to the investment strategy in the current regime. Hence, the possibility of altering the investment capacity causes the firm to be more willing to invest in idler business cycles for smaller lead times. This effect is in line with the

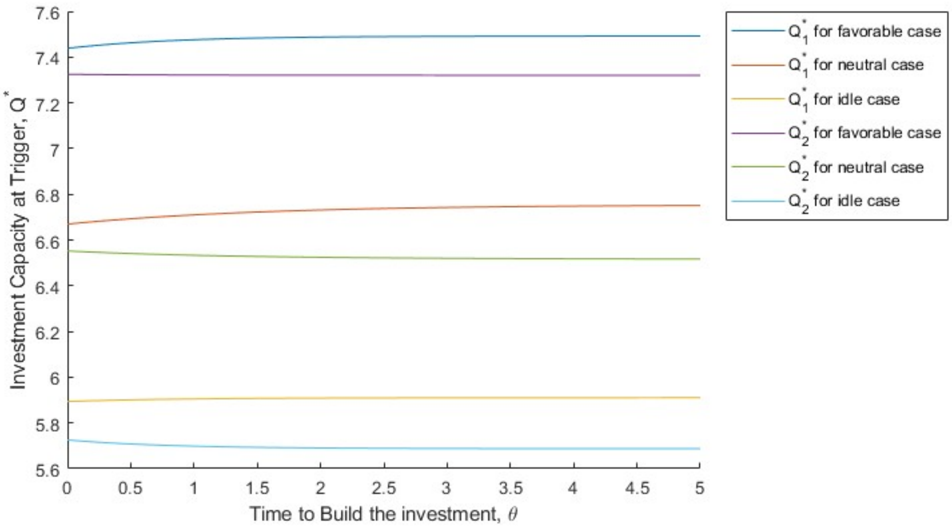


Figure 20: The Investment Capacities at the triggers for the different cases of Business Cycles and variable lead time with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\eta = 0.05$, $\delta = 0.25$

capacity effect, which causes the firm to invest earlier in less favorable regimes at low lead times.

This effect of lead time on the regime in which payout starts is also present when the effect of high lead times on the investment capacity is analyzed. Increases in lead time cause the regime at the moment of payoff to be more in line with the long-run regime distribution as the probability of switches occurring during the installation period increases. This increase in switch probability causes the firm to emphasize the NPV effect in the timing of the investment while the investment capacity is only minorly adjusted by the hedging effect. Therefore, the NPV effect is more prevalent than the capacity effect in the timing of the investment for high lead times.

As one can see in Figure 19 and Figure 20 at large lead times, increases in time to build result in increases of the investment triggers, whereas the investment capacities stay mostly constant. The reason the investment capacities stay almost constant is that the effect of an increase in switch probability is limited at high lead times. Hence, the hedging effect is not as prevalent as at high lead times.

Finally, the reason that Q_1^* and Q_2^* do not converge is that the firm invests at different moments at X_1^* and X_2^* for high lead times. In regime 1, the firm waits with investing until it is in a better position than in regime 2 and this difference in timing increases with lead time. Hence, the investment triggers do not converge thus the investment capacities do not converge either.

7 Robustness Check using Iso-elastic Demand

One of the assumptions made is that the price function of the firm is linear in demand. However, in other studies, an iso-elastic demand function is often used. In this section, the implications of an iso-elastic demand function on the model are analyzed to study whether the same effects are at play in the investment decision. An iso-elastic price function is defined as

$$P(t) = X_t Q_t^{-\zeta}$$

where X_t is the stochastic process from the previous sections and can follow a normal GBM, but can also allow for regime switching. In the paragraphs below, the continuous regime-switching model including time build is derived. This model is the main focus of this section as for certain parameter values, it is equivalent to a no-regime switch model with lead time, a regime-switch model without lead time, or similar to the monopoly model of Huisman and Kort (2015) as shown in Section 6.[7] Hence, these particular models are included in the derived model.

7.1 Finding the Value of the Firm at the Moment of Investment

A different demand function alters the value of the firm at the moment of investment. This adjustment affects the value of waiting in the transient region. Therefore, the value of the firm at the moment of investment is adjusted to adhere to the iso-elastic demand function. This adjustment is used to alter the value of waiting in the transient region.

The value of the firm if it invests still follows

$$V(X) = \max_{T \geq 0, Q(t) \geq 0} \mathbb{E}\Big[\int_{t=T}^{\infty} Q(t)P(t)exp(-rt)dt - I(Q)exp(-rT)|X(0) = X\Big]$$

Though, from here it differs in price function. The revenue of the firm for a one-time interval is

$$\begin{aligned} R_t(X) &= P_t Q_t \Delta t \\ &= X_t Q_t^{-\zeta} Q_t \Delta t \\ &= X_t Q_t^{1-\zeta} \Delta t \end{aligned}$$

instead of $Q(t)(1 - \eta Q(t))X_t\Delta t$. Furthermore, Balter et al. (2022) have shown that the cost function should be of the form $\delta_0 + \delta_1 Q$ if an iso-elastic demand function is used because in case that $\delta_0 = 0$, the investment trigger becomes 0 in the model without time to build and regime switching. This restriction also holds in the model with lead time (see Appendix B for the derivation of this finding). Hence, the cost function is also adjusted. Substituting this into the value of a monopoly at the moment of investment given it invests once, yields

$$V_i(X) = \frac{Q^{1-\zeta} X}{\Gamma_i} - \delta_0 - \delta_1 Q$$

[7] Appendix B contains the derivation of the time-to-build model using a normal GBM as a stochastic process. In this model, the same effects of lead time are found in the model using the linear demand function.

Applying FOC to find the optimal investment capacity gives

$$\begin{aligned}
\frac{\partial V_i}{\partial Q} &= 0 \\
(1-\zeta)Q^{-\zeta}X\Gamma_i^{-1} - \delta_1 &= 0 \\
Q^{-\zeta} &= \frac{\delta_1\Gamma_i}{(1-\zeta)X} \\
Q_i^*(X) &= \left(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\right)^{\frac{1}{\zeta}}
\end{aligned}$$

To determine whether this capacity is optimal, the SOC is applied:

$$\frac{\partial^2 V_i}{\partial Q^2} = \frac{-\zeta(1-\zeta)Q^{-(1+\zeta)}X}{\Gamma_i} < 0$$

This holds because the investment capacity, Q, the stochastic process, X, and the NPV factor, Γ_i, are all bigger than zero and the price elasticity parameter, ζ is between zero and one. Hence, the FOC yields a universally optimal investment capacity.

Substituting the optimal investment capacity into the value of the firm at moment of investment gives

$$\begin{aligned}
V_i(X) &= \frac{(Q_i^*(X))^{1-\zeta}X}{\Gamma_i} - (\delta_0 + \delta_1 Q_i^*(X)) \\
&= \frac{X}{\Gamma_i}\left(\Big(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\Big)^{\frac{1}{\zeta}}\right)^{1-\zeta} - \delta_0 - \delta_1\Big(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\Big)^{\frac{1}{\zeta}} \\
&= \frac{X}{\Gamma_i}\left(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\right)^{\frac{1}{\zeta}-1} - \delta_1\Big(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\Big)^{\frac{1}{\zeta}} - \delta_0 \\
&= \frac{(1-\zeta)^{\frac{1}{\zeta}-1}X^{\frac{1}{\zeta}}}{\delta_1^{\frac{1}{\zeta}-1}\Gamma_i^{\frac{1}{\zeta}}} - \frac{\Big((1-\zeta)X\Big)^{\frac{1}{\zeta}}}{\delta_1^{\frac{1}{\zeta}-1}\Gamma_i^{\frac{1}{\zeta}}} - \delta_0 \\
&= \Big((1-\zeta)^{-1} - 1\Big)\frac{(1-\zeta)^{\frac{1}{\zeta}}X^{\frac{1}{\zeta}}}{\delta_1^{\frac{1}{\zeta}-1}\Gamma_i^{\frac{1}{\zeta}}} - \delta_0 \\
&= \Big(\frac{1}{1-\zeta} - \frac{1-\zeta}{1-\zeta}\Big)\delta_1\left(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\right)^{\frac{1}{\zeta}} - \delta_0 \\
&= \frac{\zeta\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X}{\delta_1\Gamma_i}\right)^{\frac{1}{\zeta}} - \delta_0
\end{aligned}$$

Compared to the results of Huisman and Kort (2015), the value of investing is quite similar. The only difference is that Γ_i pops up in the derivation above, whereas Huisman and Kort (2015) had the term $r-\mu$. In Section 6.1.2, it is proven that if the installation time is negligible and both switch rates equal zero, $\Gamma_i = r-\mu$. In the model of Huisman and Kort (2015), the factor Γ_i is thus equal to $r-\mu$ for the current regime. Therefore, the value of investing that takes into account installation time and regime switching is an eloquent extension of the value of the firm derived in previous studies that did not include lead time and regime switching.

7.2 Finding the Value of Waiting

The previous section showed that the value of investing has the form $aX^{\frac{1}{\zeta}} + b$. This section includes this adjustment in the value of waiting in the transient region. Section 4 illustrates that the homogeneous solution to the differential equation of the value of waiting does not depend on the value of investing. Hence, the particular solution to the differential equation of the value of waiting in the transient region is different from the model using a linear demand function. The assumed form of this particular solution is the same as the form of the value of the firm at the moment of investment. The particular solution has the form $aX^{\frac{1}{\zeta}} + b$ and has to satisfy the following differential equation

$$(r + \lambda_1)F_1(X) = \mu_1 X F_1'(X) + \frac{1}{2}\sigma_1^2 X^2 F_1''(X) + \lambda_1 V_2(X)$$

Substituting the form of the particular solution into this differential equation yields

$$(r + \lambda_1)(aX^{\frac{1}{\zeta}} + b) = \frac{\mu_1}{\zeta} X a X^{\frac{1}{\zeta}-1} + \frac{1}{2}\frac{\sigma_1^2}{\zeta}(\frac{1}{\zeta} - 1)X^2 a X^{\frac{1}{\zeta}-2} + \lambda_1\left(\frac{\zeta\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} - \delta_0\right)$$

$$\left(r + \lambda_1 - \frac{\mu_1}{\zeta} - \frac{1}{2}\sigma_1^2\frac{1}{\zeta}\left(\frac{1}{\zeta} - 1\right)\right) aX^{\frac{1}{\zeta}} + (r+\lambda_1)b = \lambda_1\frac{\zeta\delta_1}{1-\zeta}\left(\frac{1-\zeta}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} X^{\frac{1}{\zeta}} - \lambda_1\delta_0$$

$$\iff \begin{cases} a = \frac{\zeta\delta_1\lambda_1}{(1-\zeta)\left(r+\lambda_1-\frac{\mu_1}{\zeta}-\frac{1}{2}\sigma_1^2\frac{1}{\zeta}(\frac{1}{\zeta}-1)\right)}\left(\frac{1-\zeta}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} \\ b = -\frac{\lambda_1}{r+\lambda_1}\delta_0 \end{cases}$$

$$\iff \begin{cases} a = \frac{\lambda_1}{g_1(\frac{1}{\zeta})}\frac{\zeta\delta_1}{(1-\zeta)}\left(\frac{1-\zeta}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} \\ b = -\frac{\lambda_1}{g_1(0)}\delta_0 \end{cases}$$

with g_i similarly defined as in Section 4.2. Hence, the value of waiting in regime 1 is now

$$\begin{cases} F_1(X) & = A_1X^{\gamma_1} + B_1X^{\gamma_2} \text{ if } X_t \in (0, X_2^*) \\ F_1(X) & = H_1X^{\beta_1} + H_2X^{\beta_2} + aX^{\frac{1}{\zeta}} + b \text{ if } X_t \in [X_2^*, X_1^*) \end{cases}$$

and the expression for the value of waiting in regime 2 remains the same as in Section 4. Thus, the expression for the value of waiting in the iso-elastic demand model differs in the particular solution to the differential equation. Additionally, the value of waiting in regime 1 should satisfy the continuity conditions in the investment trigger of regime 2. Hence, the first set of equations in X_2^* is

$$\begin{cases} \lim\limits_{X \to X_2^*} A_1 X^{\gamma_1} + B_1 X^{\gamma_2} = \lim\limits_{X \to X_2^*} H_1 X^{\beta_1} + H_2 X^{\beta_2} + a X^{\frac{1}{\zeta}} + b \\ \lim\limits_{X \to X_2^*} \gamma_1 A_1 X^{\gamma_1} + \gamma_2 B_1 X^{\gamma_2} = \lim\limits_{X \to X_2^*} \beta_1 H_1 X^{\beta_1} + \beta_2 H_2 X^{\beta_2} + \frac{1}{\zeta} a X^{\frac{1}{\zeta}} \end{cases}$$

$$\iff \tag{30}$$

$$\begin{cases} A_1 (X_2^*)^{\gamma_1} + B_1 (X_2^*)^{\gamma_2} = H_1 (X_2^*)^{\beta_1} + H_2 (X_2^*)^{\beta_2} + a (X_2^*)^{\frac{1}{\zeta}} + b \\ \gamma_1 A_1 (X_2^*)^{\gamma_1} + \gamma_2 B_1 (X_2^*)^{\gamma_2} = \beta_1 H_1 (X_2^*)^{\beta_1} + \beta_2 H_2 (X_2^*)^{\beta_2} + \frac{1}{\zeta} a (X_2^*)^{\frac{1}{\zeta}} \end{cases}$$

7.3 Finding the Investment Triggers

The smooth-pasting and value-matching conditions are applied to have an identified system of equations for the number of unknowns. In X_2^*, these conditions are

$$\begin{cases} \lim\limits_{X \to X_2^*} F_2(X) = \lim\limits_{X \to X_2^*} V_2(X) \\ \frac{\partial F_2(X)}{\partial X}\Big|_{X \to X_2^*} = \frac{\partial V_2(X)}{\partial X}\Big|_{X \to X_2^*} \end{cases}$$

$$\iff \tag{31}$$

$$\begin{cases} \frac{\lambda_2}{g_2(\gamma_1)} A_1 (X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1 (X_2^*)^{\gamma_2} = \frac{\zeta \delta_1}{1-\zeta} \left(\frac{(1-\zeta) X_2^*}{\delta_1 \Gamma_2} \right)^{\frac{1}{\zeta}} - \delta_0 \\ \gamma_1 \frac{\lambda_2}{g_2(\gamma_1)} A_1 (X_2^*)^{\gamma_1} + \gamma_2 \frac{\lambda_2}{g_2(\gamma_2)} B_1 (X_2^*)^{\gamma_2} = \frac{\delta_1}{1-\zeta} \left(\frac{(1-\zeta) X_2^*}{\delta_1 \Gamma_2} \right)^{\frac{1}{\zeta}} \end{cases}$$

and in X_1^*, these are

$$\begin{cases} \lim\limits_{X \to X_1^*} F_1(X) = \lim\limits_{X \to X_1^*} V_1(X) \\ \frac{\partial F_1(X)}{\partial X}\Big|_{X \to X_1^*} = \frac{\partial V_1(X)}{\partial X}\Big|_{X \to X_1^*} \end{cases}$$

$$\begin{cases} H_1 (X_1^*)^{\beta_1} + H_2 (X_1^*)^{\beta_2} + a (X_1^*)^{\frac{1}{\zeta}} + b = \frac{\zeta \delta_1}{1-\zeta} \left(\frac{(1-\zeta) X_1^*}{\delta_1 \Gamma_1} \right)^{\frac{1}{\zeta}} - \delta_0 \\ \beta_1 H_1 (X_1^*)^{\beta_1} + \beta_2 H_2 (X_1^*)^{\beta_2} + \frac{1}{\zeta} a (X_1^*)^{\frac{1}{\zeta}} = \frac{\delta_1}{1-\zeta} \left(\frac{(1-\zeta) X_1^*}{\delta_1 \Gamma_1} \right)^{\frac{1}{\zeta}} \end{cases} \tag{32}$$

Together with the continuity conditions for $F_1(X)$, the system consists of six equations and six unknown variables similar to the model using the linear demand function. Thus, the estimates for the investment triggers can be determined using numerical approximation. The full system of equations, therefore, is defined as

$$\begin{cases} A_1(X_2^*)^{\gamma_1} + B_1(X_2^*)^{\gamma_2} = H_1(X_2^*)^{\beta_1} + H_2(X_2^*)^{\beta_2} + a(X_2^*)^{\frac{1}{\zeta}} + b \\ \gamma_1 A_1(X_2^*)^{\gamma_1} + \gamma_2 B_1(X_2^*)^{\gamma_2} = \beta_1 H_1(X_2^*)^{\beta_1} + \beta_2 H_2(X_2^*)^{\beta_2} + \frac{1}{\zeta} a(X_2^*)^{\frac{1}{\zeta}} \\ \\ \frac{\lambda_2}{g_2(\gamma_1)} A_1(X_2^*)^{\gamma_1} + \frac{\lambda_2}{g_2(\gamma_2)} B_1(X_2^*)^{\gamma_2} = \frac{\zeta\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X_2^*}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} - \delta_0 \\ \gamma_1 \frac{\lambda_2}{g_2(\gamma_1)} A_1(X_2^*)^{\gamma_1} + \gamma_2 \frac{\lambda_2}{g_2(\gamma_2)} B_1(X_2^*)^{\gamma_2} = \frac{\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X_2^*}{\delta_1\Gamma_2}\right)^{\frac{1}{\zeta}} \\ \\ H_1(X_1^*)^{\beta_1} + H_2(X_1^*)^{\beta_2} + a(X_1^*)^{\frac{1}{\zeta}} + b = \frac{\zeta\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X_1^*}{\delta_1\Gamma_1}\right)^{\frac{1}{\zeta}} - \delta_0 \\ \beta_1 H_1(X_1^*)^{\beta_1} + \beta_2 H_2(X_1^*)^{\beta_2} + \frac{1}{\zeta} a(X_1^*)^{\frac{1}{\zeta}} = \frac{\delta_1}{1-\zeta}\left(\frac{(1-\zeta)X_1^*}{\delta_1\Gamma_1}\right)^{\frac{1}{\zeta}} \end{cases} \tag{33}$$

Using the investment triggers from the numerical approximation, the investment capacities are derived with the optimal investment capacity function. Hence, the investment decision can be determined using this system of equations.

To numerically find the optimal investment decision, the investment decision without regime switching is used as an initial starting point to decrease computation time. These estimates are used because they roughly coincide with the actual investment decision and are computationally feasible as an explicit expression is found (see Appendix B for the derivation and expression of the no-switch model including installation time). This expression depends on the elasticity parameter ζ and its additional restriction stated by Balter et al. (2022): $\beta\zeta > 1$. If this restriction does not hold, the investment capacity is negative which is not representative of a real-life investment decision. Therefore, this restriction needs to hold for the lowest β calculated for the guesses. [8]

This restriction also has an economic intuition. If $\zeta < \frac{1}{\beta}$ in one of the regimes, the Sharpe ratio in that regime is negative and the firm would not be able to generate money for the project. This condition is studied by J. Thijssen (personal communication, January 27, 2023).[9] He combines the perspective of the financier with the real options model. To include this perspective, different conditions hold compared to the real options model with the perspective of the firm, though these additional conditions can also be related to the real options model using the firm's perspective. Yet, since the studied model is based on the perspective of the firm, the intuition behind these additional assumptions is not of interest here.

7.4 Computation of the Investment Decision

Using the restriction and the guesses, the investment decision for rate parameters, λ_1 and λ_2, and installation time, θ, is computed in Table 6, Table 7 and Table 8.

[8] Note that this β is different from the β_1 and β_2 from the value of waiting in regime 1.

[9] For the exact details, a request can be sent to n.a.bun@tilburguniversity.edu

Table 6: The effect of different values for lead time on the investment triggers and investment capacity for parameters values:$\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $r = 0.1$, $\delta_1 = 0.25$, $\delta_0 = 1$, $\zeta = 0.85$

θ	X_1^*	X_2^*	Q_1^*	Q_2^*	$Q_2^{'}$
0	0.1653	0.1483	1.940	1.946	2.211
0.1	0.1666	0.1490	1.945	1.945	2.218
0.2	0.1679	0.1497	1.950	1.944	2.225
0.3	0.1692	0.1504	1.955	1.943	2.232
0.4	0.1705	0.1510	1.960	1.942	2.239
0.5	0.1718	0.1517	1.964	1.941	2.246
0.6	0.1731	0.1524	1.969	1.939	2.253
0.7	0.1744	0.1532	1.973	1.938	2.259
0.8	0.1757	0.1539	1.978	1.938	2.265
0.9	0.1770	0.1546	1.982	1.937	2.271
1.0	0.1783	0.1553	1.986	1.936	2.277

The same effects are observed in Figure 6 as in the linear demand case. Namely, an increase in θ result in an increase in X_1^*, X_2^*, Q_1^*, and $Q_2^{'}$ and in a decrease in Q_2^*. Therefore, the investment is postponed if lead times are included in the investment decision and the investment capacities are adjusted to mitigate the risk of regime switching during the building time. The hedging effect and improved position effect are thus present in determining the investment capacity for the iso-elastic demand function.

Lead time is not the only variable that affects the investment decision in the combined model. The switch parameters play an important factor in the investment decision as well. In Table 7 and Table 8, the investment trigger and capacity are computed for different regime switch rates. To isolate the effect of lead time, θ is set to zero.

Table 7: The investment triggers in both regimes, where the two numbers in the cells correspond to (X_1^*, X_2^*) for the different switch rates in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta_1 = 0.25$, $\delta_0 = 1$, $\zeta = 0.85$, $\theta = 0$

	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(0.16285, 0.14730)	(0.16167, 0.14819)	(0.16161, 0.14958)	(0.16177, 0.15082)	(0.16198, 0.15186)
$\lambda_1 = 0.4$	(0.16278, 0.14847)	(0.16065, 0.14812)	(0.16016, 0.14886)	(0.16015, 0.14978)	(0.16031, 0.15066)
$\lambda_1 = 0.6$	(0.16294, 0.14949)	(0.16030, 0.14845)	(0.15944, 0.14868)	(0.15921, 0.14928)	(0.15924, 0.14996)
$\lambda_1 = 0.8$	(0.16308, 0.15031)	(0.16022, 0.14889)	(0.15908, 0.14875)	(0.15865, 0.14908)	(0.15854, 0.14957)
$\lambda_1 = 1.0$	(0.16318, 0.15096)	(0.16024, 0.14934)	(0.15892, 0.14894)	(0.15832, 0.14904)	(0.15808, 0.14937)

Table 8: The investment capacities in both regimes, where the three numbers in the cells correspond to $(Q_1^*, Q_2^*, Q_2^{'})$ for the different switch rates in the regimes with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\delta_1 = 0.25$, $\delta_0 = 1$, $\zeta = 0.85$, $\theta = 0.5$

	$\lambda_2 = 0.2$	$\lambda_2 = 0.4$	$\lambda_2 = 0.6$	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$
$\lambda_1 = 0.2$	(1.7136, 1.6869, 1.8984)	(1.5320, 1.4852, 1.6452)	(1.4556, 1.4037, 1.5375)	(1.4140, 1.3611, 1.4782)	(1.3880, 1.3356, 1.4408)
$\lambda_1 = 0.4$	(1.9652, 1.8939, 2.1105)	(1.7146, 1.6461, 1.8111)	(1.5951, 1.5300, 1.6675)	(1.5258, 1.4639, 1.5839)	(1.4809, 1.4217, 1.5294)
$\lambda_1 = 0.6$	(2.1291, 2.0321, 2.2488)	(1.8536, 1.7703, 1.9377)	(1.7096, 1.6346, 1.7746)	(1.6217, 1.5526, 1.6749)	(1.5629, 1.4983, 1.6079)
$\lambda_1 = 0.8$	(2.2436, 2.1309, 2.3455)	(1.9627, 1.8689, 2.0374)	(1.8050, 1.7226, 1.8642)	(1.7048, 1.6300, 1.7538)	(1.6357, 1.5666, 1.6777)
$\lambda_1 = 1.0$	(2.3279, 2.2050, 2.4164)	(2.0505, 1.9492, 2.1177)	(1.8858, 1.7975, 1.9400)	(1.7773, 1.6979, 1.8229)	(1.7008, 1.6279, 1.7401)

In Table 8, the investment capacity depends on the long-term perspective of the firm because an increase in λ_1, thus an increase in the long-run probability of being in regime 2, increases the investment capacity. An increase in λ_2 decreases the investment capacity. These conclusions are in line with the combined model using a linear demand function.

Table 7 shows that the effect of the regime switch rates on the timing of the investment is not straightforward like for the linear demand function. To study the effect of λ_1 on the timing of the investment in regime 1, the investment trigger is plotted for certain values of λ_2 and variable λ_1 in Figure 21.

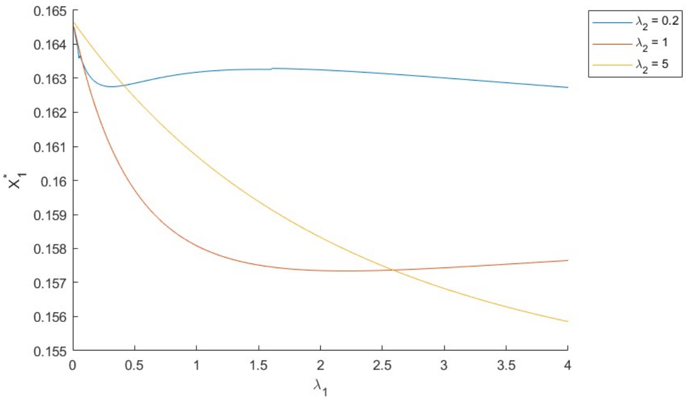


Figure 21: X_1^* for variable λ_1 and certain values of λ_2 using the same parameters as in Table 7

The figure above suggests that a higher λ_2 coincides with horizontal and vertical dilation of the curve with positive scale factors. Therefore, the effect of λ_1 on the timing of the investment seems to follow the same trajectory. The graph is analyzed for one value of λ_2 and different values of λ_1 as different values of λ_2 only dilate the graph. In Figure 22, the investment trigger in regime 1 is plotted only with $\lambda_1 = 0.2$.

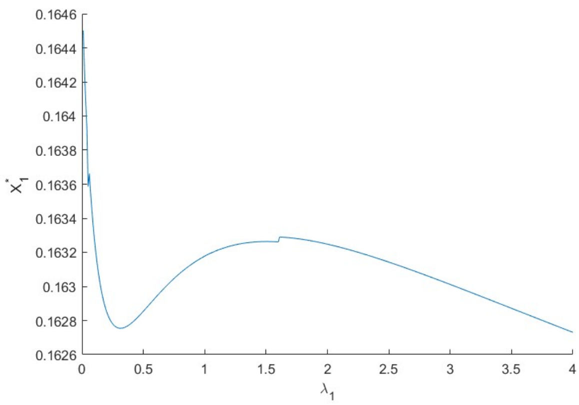


Figure 22: X_1^* for variable λ_1 with $\lambda_2 = 0.2$

In Figure 22, the investment trigger in regime 1 is decreasing for low values of λ_1. If λ_1 is roughly between 0.4 and 1.2, the investment trigger is increasing, whereafter X_1^* decreases again. This latter part is similar to the graph of the model using a linear demand function. If λ_1 is bigger than 0.3, the effect of the regime switch rate on the investment trigger is twofold, like in the model using a linear demand function. At first, capacity is leading in the investment decision as it experiences relatively big changes while the investment trigger increases. After a certain point, the investment trigger decreases while investment capacity stays mostly constant as the long-term perspective is preeminent in the investment decision.

The effect of λ_1 on the first section of the graph may be due to the probability of switching to a more favorable regime. When λ_1 is very close to zero, the current regime (regime 1) is leading as the probability of staying in this regime is relatively high and the investment trigger is close to the investment trigger in a no-switch model (with these parameters, this is 0.1647). For these low switch rates, small changes in the switch rate to the advantageous regime increase the NPV of the investment. To profit from this increase in NPV, the firm invests earlier and with an increased capacity. Hence, contrary to the model with the linear demand function, the firm does not have to choose between the two.

This additional effect in the first section is not found in the model that uses a linear demand function. In the linear demand model, at low switch rates, the capacity effect dominates, and at high switch rates, the NPV effect dominates in determining the investment trigger. This difference probably results from the difference in demand function because, for the linear demand function, a lower investment capacity corresponds to a higher elasticity, and higher investment capacities coincide with a lower elasticity (Balter et al., 2022). Hence, an increase in investment capacity decreases the marginal effect of capacity on the marginal price of the product. Therefore, increasing the investment capacity in the linear demand model is more likely to be justified by compensating in another dimension of the investment decision, here timing, than in the case of an iso-elastic demand function as the iso-elastic demand function has constant elasticity.

The effect of a small λ_1 on the investment trigger in regime 2 also changes the NPV of the investment. Yet, this effect is very limited as the firm is already in the advantageous regime at the moment of investment. This is visualized in Figure 24. At first, the investment trigger in regime 2 decreases slightly with increases in λ_1 because of the NPV effect of λ_2. However, for higher values of the regime switch rate, the investment trigger in regime 2 is increasing like in the model with a linear demand function.

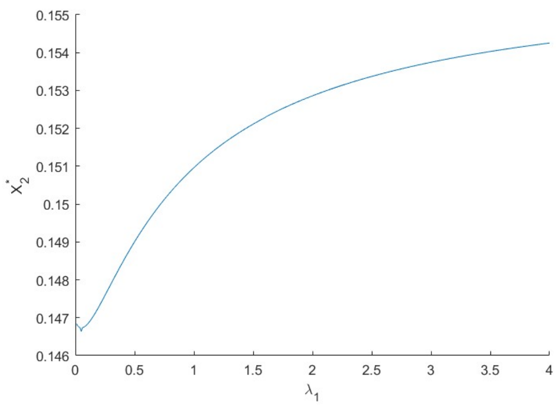


Figure 23: X_2^* for different λ_1 and $\lambda_2 = 0.2$ using an iso-elastic demand function with parameters defined in Tables 7 and 8

Moreover, the effect of the regime switch rate in regime 2 on the investment triggers is also analyzed. In Figure 24, the investment triggers are plotted against λ_2.

The effect of λ_2 on both investment triggers is twofold as at first the investment triggers decrease and after a certain point, they increase. This suggests that the effects of the switch rates on the investment decision are similar to their effects in the linear demand model. Hence, both the capacity and NPV effect of the switch parameters are present in the timing of the investment.

The full effect of the regime switch rates is similar as different regime switch rates cause either the capacity or timing to be leading in the investment. Therefore, the firm also has to choose to focus on the investment capacity or the timing if an iso-elastic demand function is

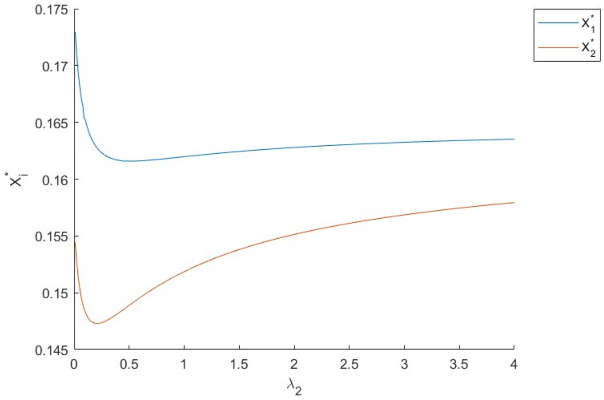


Figure 24: X_2^* for different λ_1 and $\lambda_2 = 0.2$ using an iso-elastic demand function with parameters defined in Tables 7 and 8

used. Albeit that the individual effects of the regime switch rates are different than in the linear demand case.

Furthermore, the effect of different lead times on the investment decision is studied too. In Figure 25 and Figure 32, the investment triggers and investment capacities are plotted like in Section 6 for the linear demand model.

Similar to the model using a linear demand function, the NPV effect on the timing of the investment becomes more preeminent if lead time increases. However, this is different than in the linear demand model. First, the capacity effect of λ_1 only becomes inferior for higher lead times in regime 1. This suggests that here the NPV effect only becomes leading in regime 1 for changes in λ_1. For the investment trigger in regime 2, an increase in lead time reduces the capacity effect of λ_1. Yet, the capacity effect is not universally inferior for $\theta \leq 5$.

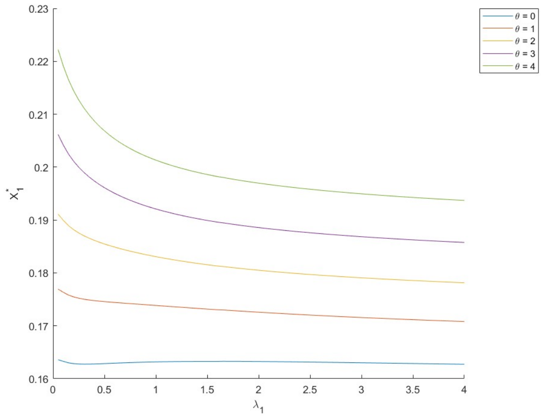
To analyze whether the NPV effect becomes completely leading, the investment trigger in regime 2 is plotted for very high lead times against the regime switch rate in regime 1 in Figure 26.

Here, the effect of λ_1 on the investment trigger is strictly decreasing. Hence, the NPV effect dominates in the regime switch rate in regime 1 for both investment triggers.

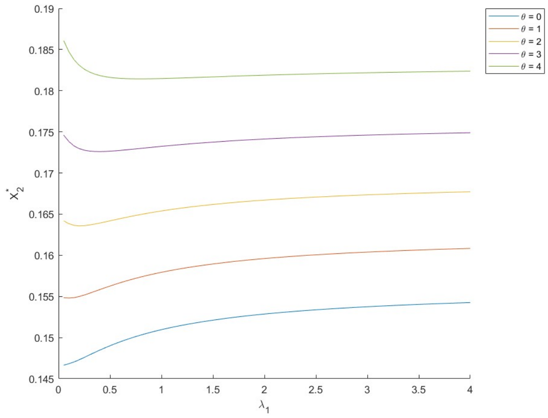
On the contrary, in Figure 25c and Figure 25d, the effect of the regime switch rate in regime 2 remains twofold for varying lead times. This result indicates that the capacity effect of the regime switch rate in regime 2 does not diminish for increases in lead time. Nonetheless, increases in lead time do increase the NPV effect as for large lead times, an increase in λ_2 increases the investment trigger more than at low lead times. Therefore, an increase in lead time causes the firm to reflect the change in NPV of the investment in both aspects of the investment decision. Though, this auxiliary effect of the installation time using the iso-elastic demand function is different from the linear demand model as the capacity effect does not necessarily become less prevalent. This results from the unboundedness of the investment capacity in the iso-elastic demand model compared to the boundedness in the linear demand model.

This conclusion is also supported by Figure 32 in Appendix D. Here, the investment capacity is more 'L'-shaped for changes in λ_2 than λ_1. Hence, the change in investment capacity changes more with changes in λ_2 than λ_1. Furthermore, the investment capacities remain quite similar for different lead times like in the linear demand case.

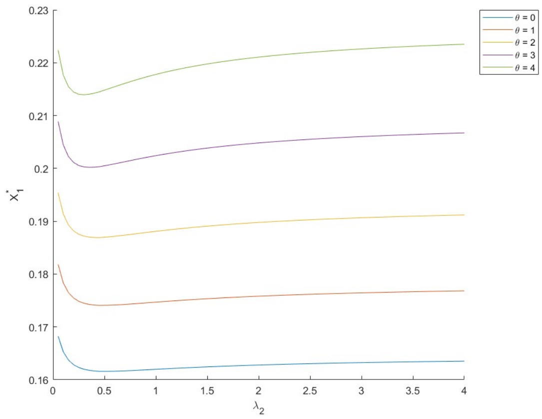
In conclusion, including lead time in the regime switch model using an iso-elastic demand function has similar effects on the investment decision as the model using a linear demand function. The hedging effect and improved position effect are observed in determining the investment capacity. In the timing of the investment, the NPV effect becomes more dominating



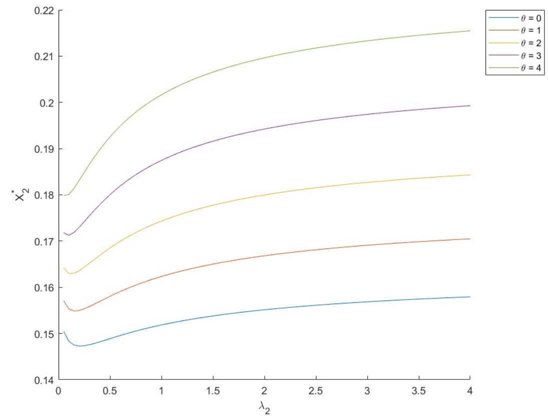
(a) X_1^* for $\lambda_2 = 0.2$



(b) X_2^* for $\lambda_2 = 0.2$



(c) X_1^* for $\lambda_1 = 0.2$



(d) X_2^* for $\lambda_1 = 0.2$

Figure 25: Plots of the investment triggers for and the same parameters as defined in the plot and in Table 7

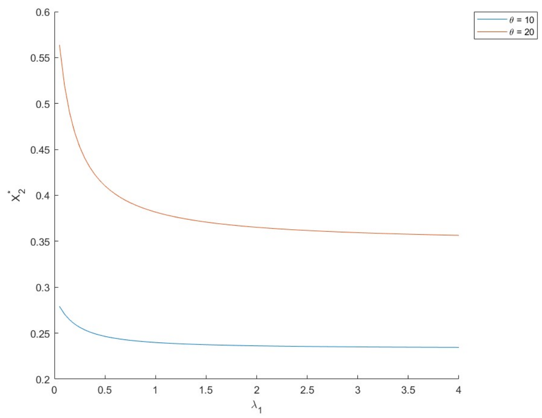


Figure 26: X_2^* for $\lambda_2 = 0.2$ and very high lead times

as the installation time increases. However, the capacity effect does not abate completely for higher lead times because the optimal investment capacity is unbounded, whereas the investment capacity is bounded if a linear demand function is used.

Similar to the linear demand model, the investment decision for different business cycles are analyzed against different values of lead time as well. The same parameter values for the regime switches are used for the business cycles. I.e. The favorable business cycle has $\lambda_1 = 1$ and $\lambda_2 = 0.2$, the neutral business cycle has rate parameters $\lambda_1 = \lambda_2 = \frac{1}{3}$ and the idle business cycle has switch rates $\lambda_1 = 0.2$ and $\lambda_2 = 1$. The investment triggers for the different business cycles are plotted in Figure 27 and the investment capacities are plotted in 28 against time to build.

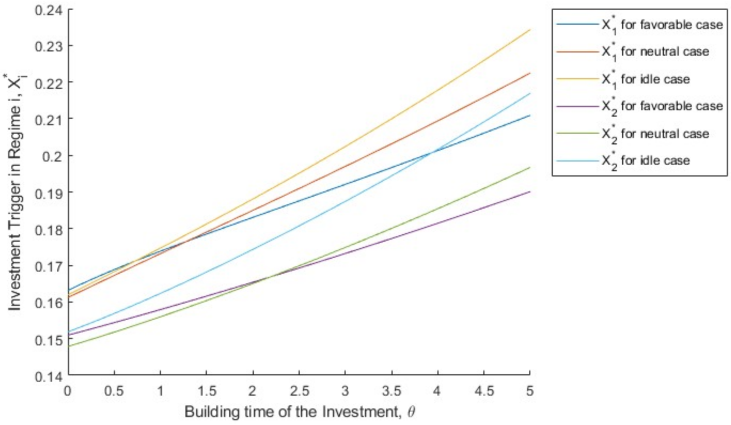


Figure 27: The investment triggers for the different Business Cycles and variable lead time with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\zeta = 0.85$, $\delta_1 = 0.25$, and $\delta_0 = 1$

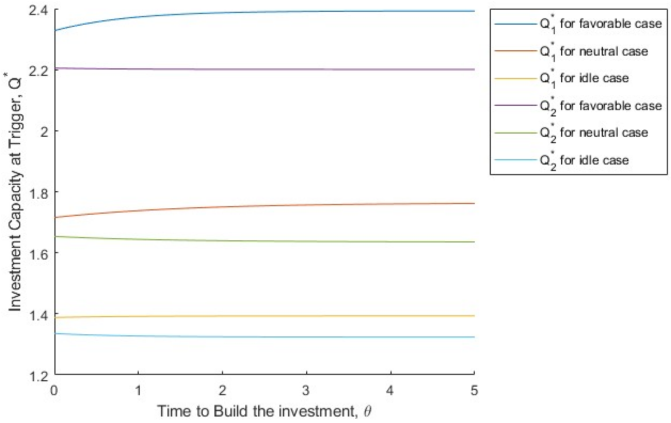


Figure 28: The investment capacities for the different Business Cycles and variable lead time with parameters: $\mu_1 = 0.02$, $\mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.1$, $r = 0.1$, $\zeta = 0.85$, $\delta_1 = 0.25$, and $\delta_0 = 1$

Similar to the regime-switching model using a linear demand function, the slope of the investment triggers in the idler business cycles is bigger than in more favorable business cycles for changes in lead time. Furthermore, Figure 28 illustrates that the investment capacities also converge for larger lead times. This suggests that increases in lead time cause the firm to emphasize the timing of the investment instead of the investment capacity. This effect of lead time on the investment decision is also observed in the linear demand model.

Yet, the investment capacities converge for idler business cycles. This is not observed in the linear demand model and is due to the constant elasticity of the demand function. It results from the investment capacity being unbounded in the iso-elastic demand model, whereas in the linear demand model, the investment capacity is bounded. Therefore, the firm can increase its investment capacity in more favorable business cycles in the iso-elastic demand case. Hence, increases in profitability result in a bigger difference in investment capacity.

8 Emperical Results using Data from ASML

To analyze the applications of the model, public data from ASML from its annual reports are used. In these annual reports, annual data until 2022 is available at the moment of writing. Hence, the regime switch model is applied on an annual basis. The relevant data from these reports are the net system sales (the revenue of ASML in that year), the cost of system sales (the operation costs to make the products in that year), the number of systems shipped (the quantity sold in that year), and the capital expenditures per year (the investment made to increase the investment capacity). Figure 29 illustrates the average profit per product and quantity of the goods sold by ASML from 2005 until 2022. The average profit per product is obtained by dividing the profit of the system sales per year by the number of goods sold. The average profit per product is denoted by the net price.

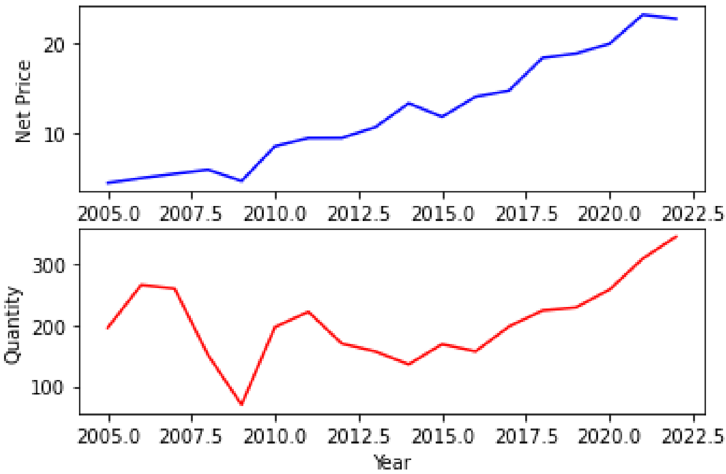


Figure 29: Net Price and Capacity of ASML from 2005 until 2022

Assuming the demand follows a linear relation with capacity and ASML is a monopoly, the parameter η is obtained. This demand function yields $\eta = -0.000865 < 0$. This parameter suggests that price and demand are positively correlated. Hence, one of the key micro-economic assumptions is violated.

Therefore, the iso-elastic demand function is assumed to evade this violation. Now, the elasticity parameter is not negative: $\zeta = 0.223$. Furthermore, the restriction $\zeta \in (0, 1)$ is satisfied. Hence, the trajectory of the stochastic process, X_t, is derived using an iso-elastic demand function in Figure 30. The orange line is the expectation of the stochastic process based on 2005. The red and black vertical lines denote regime switches although determining the current regime of ASML is hard as it is an undefined measure in the annual reports. For now, a negative regime switch is defined as a switch from a period of growth to a period of decline (red line) and a positive regime switch is a switch from a period of decline to a period of growth (black line).

In this figure, a clear overall upward trend in the market position of ASML is observed with ephemeral and limited decreases in the stochastic process. Furthermore, it seems that the process experiences exponential growth as in the beginning the changes in X_t are relatively small whereas these changes are relatively big in more recent years. Thus, the assumption that the stochastic process is a geometric Brownian motion is reasonable. Using this assumption for an ordinary GBM without regime switching, $\mu = 0.0975$ and $\sigma = 0.1283$ are found.

To find an estimate of the discount rate r, the weighted average cost of capital (WACC) for ASML of a recent year is used. In the annual report of 2021, ASML had a WACC of 10.5% on an annual basis. Therefore, the continuous discount rate is $r = ln(1 + WACC_{2021}) = 0.0998$.

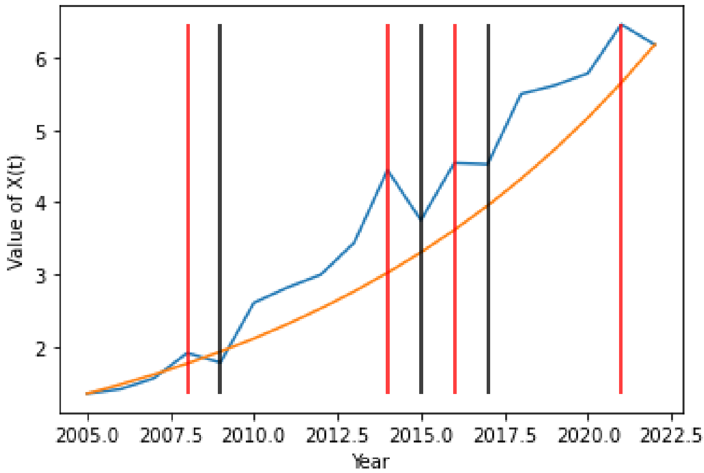


Figure 30: Trajectory of the Stochastic Process X_t for ASML from 2005 until 2022 assuming an iso-elastic demand function

These parameter values indicate that $\beta = 1.022$. Now, it is clear to see that $\beta\zeta < 1$. Therefore, at least in one of the regimes, the restriction $\beta\zeta > 1$ is violated. Hence, the model using an iso-elastic demand function cannot be applied either. Therefore, the models derived in this thesis are not likely to apply to the data from ASML.

This results from several factors. Firstly, ASML has experienced unprecedented revenue growth in the period between 2005 and 2022. Applying real options models may be sensitive to this bias in the data. Secondly, ASML increases its capacity over time. It does not use a one-time investment strategy but an expansive investment strategy. Hence, another assumption in the studied model is violated in the business model of ASML.

Moreover, the data used to apply the model may be a factor. Namely, publicly available data is used. An indirect assumption is that all products produced by ASML are homogeneous. This assumption is not realistic as the firm produces EUV and DUV systems which are different products. Furthermore, the sanctions for EUV systems are different from the sanctions for EUV systems. Thus, these products are distinctly prone to geopolitical tensions. Therefore, EUV and DUV systems are probably heterogeneous products.

Lastly, technological advances are not included in the studied model either. Since ASML has experienced major technological leaps in the past years, the omission of technological advances hinders the applicability of the regime switch model. Hence, adjustments have to be made to apply this model to analyze real-world real options problems.

9 Conclusion

Previous real options models may not accurately represent the current investment environment due to recent developments on the international stage. This limitation may result from the exclusion of relevant variables in previous real options models. This thesis broadens the current understanding of an irreversible investment strategy for a monopoly that invests once with an optimal capacity by adding two new concepts: time to build and regime switching. The model makes two assumptions for these concepts. First, the installation time of the investment is independent of the investment capacity. Second, the regime switch rates stay constant over time. The results are obtained by deriving and analyzing the expression of the investment decision in the models.

A deterministic lead time postpones the investment while the investment capacity remains constant in a no-switch model. This result holds for investment models that use a linear demand function or an iso-elastic demand function. The increase in investment trigger is equal to the discount factor corrected for the drift rate of the investment over the lead time. This discounting results in a similar expected revenue of the investment when the investment is installed as in the model without lead time. Allowing for a stochastic time to build does not affect the investment capacity although increases in the uncertainty of the duration of the installation period incentives the firm to invest earlier. This outcome follows from the convexity of the payoff function. Namely, a longer installation period increases the potential risk less than the potential gain of the investment. Yet, the effect of the uncertainty of the building time is inferior to the effect of the expected building time.

Regime switching, on the other hand, causes the firm to change both the timing and the capacity of the investment. In a regime-switching model, the firm has two investment triggers as the value of waiting and the value of investing are different in the two regimes. Furthermore, the firm can be in an additional transient investment region where it invests in the investment stimulating regime but not in the investment discouraging regime. The firm thus has multiple optimal investment capacities corresponding to the investment triggers and a transient investment region. The investment capacity depends on the long-term perspective as advantageous changes in regime switch rates increase the capacities at the investment triggers as well as the capacities corresponding to the transient region and vice versa. This conclusion results from the assumption that the firm can only invest once.

Furthermore, the magnitude of the effect of the regime switch rates on the investment capacity decrease for larger switch rates because for large regime switch rates, the investment decision converges to a no-switch model and their effect on the capacity is monotone. This convergence arises because, for a large switch rate, a regime switch occurs almost immediately. Hence, if the switch rate in one regime is large, the duration of the firm being in that regime is short. The firm is thus virtually always in one regime. If both regime switch rates tend to infinity with a similar convergence rate, regime switches occur nearly instantly. Hence, the firm is practically in an averaged regime and the regime-switching model becomes equivalent to a no-switch model with averaged parameters.

The regime switch rates' effect on the investment trigger is not monotone. At low regime switch rates, the current regime is important in the investment strategy. However, the long-term investment strategy is captured in the investment capacity. To adjust to this change in perspective, the firm alters the investment capacity at the expense of the timing. Hence, if this so-called capacity effect is leading, an increase in the advantageous regime switch rate causes an increase in the investment trigger to compensate for a large increase in capacity and vice versa. For large regime switch rates, the long-term perspective dominates the investment strategy. Both the investment capacity and trigger thus depend on the long-term view of the firm. Therefore, at large switch rates, a change in the net present value of the investment is preeminent in the timing of the investment.

In the combined model where both building time and regime switching are considered, an increase in lead time causes the firm to postpone the investment and the investment capacities change slightly for different values of lead time. A hedging effect is present in determining the investment capacity to mitigate the risk of a regime switch occurring during the installation period. Yet, the effect of the installation time is inferior to the effect of the regime switch rates as the investment capacity depends mostly on the long-term strategy of the firm. Moreover, lead time affects the effect of the switch parameters on the timing of the investment. Increases in time to build increase the probability of a regime switch occurring during the installation period. Hence, the current regime is less leading in the investment decision and the capacity effect becomes less relevant. Therefore, for long installation periods, the net present value effect becomes preeminent in the timing of the investment, even at low regime switch rates.

These conclusions hold for linear demand functions. For iso-elastic demand functions, most conclusions also hold except that the capacity effect does not become inferior for all regime switch rates. Specifically, the NPV effect becomes more preeminent, but the capacity effect remains leading at low regime switch rates as the optimal investment capacity is unbounded in the iso-elastic demand model, whereas in the linear demand model, the investment capacity is bounded.

10 Discussion and Future Research

To derive a real options model for a monopoly including regime switching and lead time for a one-time, irreversible investment with a flexible capacity, several strict assumptions have been made. The findings in this model give insight into the investment decision in light of these assumptions. If one of these assumptions is not met, the model does not yield a realistic investment decision. This drawback can be seen clearly in Section 8, where the data does not meet the assumptions of the model. This inapplicability of the combined model suggests that the investment strategy of ASML cannot be represented adequately by the model. Hence, relaxing or reviewing the assumptions could be beneficiary to understanding the investments of real-life firms.

Some of the assumptions in the combined model are similar to the assumptions Huisman and Kort (2015) have made in their monopoly model, such as the firm having a one-time investment without expansion possibilities and having constant investment costs depending on the investment capacity. However, these assumptions are not the only limitations in this model as their model is extended by including installation time and regime switching. Therefore, additional assumptions are made to extend their model.

To derive the real options regime-switching model with lead time, the assumption is made that lead time is independent of the investment capacity and similar in the two regimes. However, in practice, the investment capacity might be correlated with the installation time of the investment. This correlation may alter the timing and capacity of the investment. Furthermore, it may also be that the building time in one regime is different from the building time in another regime as supply chains for installing the production facility might be elongated in one regime compared to the other. Moreover, the firm could prepare the investment to decrease lead time while it waits with investing. This semi-investment state is not considered either throughout this thesis.

Additionally, the regime switch rates do not change over time and are equal for every type of geopolitical unrest. However, there are many kinds of geopolitical unrest. For instance, the impact of the COVID pandemic could vary from the impact of market regulations by authorities. In the studied model, there are only two types of regimes, which could be unrepresentative of the effect of different kinds of geopolitical unrest. Besides, regime switch rates could change over time because adjustments in policies and the pace of these adjustments could alter over time.

Furthermore, a regime switch model is studied for a monopoly. In many real-life markets, multiple firms are active and may preempt each other. Another limitation of this thesis is that it assumes that the active firm maximizes its profits. Yet, it may also be that the firm is not purely interested in profit optimization as it applies a stakeholder model or aims to optimize the total welfare of the market. To adjust the objective of the firm, a utility function could be maximized instead of the profits, like in Driffill et al. (2013). Yet, they analyzed the pricing of real options instead of the simulation of the investment decision.

The analysis itself has its limitations as well. The use of numerical estimation limits the set of parameters that can be analyzed. Therefore, the conclusion can only be stated for the studied range of parameters. Furthermore, a system of nonlinear equations is solved numerically. This method may result in an undefined investment decision if this system cannot be solved, like in Section 4. One equation that implies the investment decision could yield a more accurate solution. Hence, if the system of equations is reduced into one equation the model could yield a more viable outcome. Moreover, a real options model is analyzed and the investment perspective of the firm is studied. However, to fully understand the dynamics of the investment, the perspective of the financier of the investment should also be analyzed (see for instance Thijssen (2010) for an arbitrage pricing approach).

It would thus be interesting to relax some of the made assumptions in future research or to deepen the understanding of the regime-switching model with lead time to depict the investment strategy of firms more accurately.

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Appendix

A

The consumer surplus is defined by $\int\limits_{P(Q)}^{X} D(P)dP$. As

$$
\begin{aligned}
P(Q) &= X(1-\eta Q)\\
\frac{P}{X} &= 1-\eta Q\\
\eta Q &= 1-\frac{P}{X}\\
D(P) = Q &= \frac{1}{\eta}\Big(1-\frac{P}{X}\Big)
\end{aligned}
$$

Hence, the instantaneous consumer surplus is

$$
\begin{aligned}
ICS(X,Q) &= \int\limits_{P(Q)}^{X} D(P)dP\\
&= \int\limits_{X(1-\eta Q)}^{X} \frac{1}{\eta}\Big(1-\frac{P}{X}\Big)dP\\
&= \Big[\frac{1}{\eta}\Big(P-\frac{1}{2}\frac{P^2}{X}\Big)\Big]_{P=X(1-\eta Q)}^{P=X}\\
&= \frac{1}{\eta}\Big(X-\frac{1}{2}\frac{X^2}{X}-(X(1-\eta Q)-\frac{1}{2}\frac{(X(1-\eta Q))^2}{X})\Big)\\
&= \frac{1}{\eta}\Big(\frac{1}{2}X-(X-X\eta Q-\frac{1}{2}X(1-\eta Q)^2)\Big)\\
&= \frac{1}{\eta}\left(\frac{1}{2}X-\Big(X-X\eta Q-\frac{1}{2}X(1-2\eta Q+\eta^2Q^2)\Big)\right)\\
&= \frac{1}{\eta}\Big(\frac{1}{2}X\eta^2Q^2\Big)\\
&= \frac{1}{2}XQ^2\eta
\end{aligned}
$$

And the total expected consumer surplus is equal to

$$
\begin{aligned}
ECS(X,Q) &= \mathbb{E}\Big[\int\limits_{t=0}^{\infty} ICS(X(t),Q)exp(-rt)dt|X(0)=X\Big] \\
&= \mathbb{E}\Big[\int\limits_{t=0}^{\infty} \frac{1}{2}X(t)Q^2\eta * exp(-rt)dt|X(0)=X\Big] \\
&= \frac{1}{2}Q^2\eta\mathbb{E}\Big[\int\limits_{t=0}^{\infty} X(t)exp(-rt)dt|X(0)=X\Big] \\
&= \frac{1}{2}Q^2\eta\int\limits_{t=0}^{\infty} \mathbb{E}[X(t)|X(0)=X]exp(-rt)dt \\
&= \frac{1}{2}Q^2\eta\int\limits_{t=0}^{\infty} Xexp((\mu-r)t)dt \\
&= \frac{1}{2}XQ^2\eta\Big[\frac{1}{\mu-r}exp((\mu-r)t)\Big]_{t=0}^{t=\infty} \\
&= \frac{XQ^2\eta}{2(r-\mu)}
\end{aligned}
$$

The expected producer surplus is equal to the value of the firm. This is thus equal to $V(X)$

$$
EPS(X,Q) = V(X) = \frac{XQ(1-\eta Q)}{r-\mu} - \delta Q \tag{34}
$$

And the total expected surplus is

$$
\begin{aligned}
TES(X,Q) &= ECS(X,Q) + EPS(X,Q) \\
&= \frac{XQ^2\eta}{2(r-\mu)} + \frac{XQ(1-\eta Q)}{r-\mu} - \delta Q \\
&= \frac{XQ^2\eta + 2XQ(1-\eta Q)}{2(r-\mu)} - \delta Q \\
&= \frac{XQ^2\eta + 2XQ - 2\eta Q^2X}{2(r-\mu)} - \delta Q \\
&= \frac{2XQ - \eta Q^2X}{2(r-\mu)} - \delta Q \\
&= \frac{XQ(2-\eta Q)}{2(r-\mu)} - \delta Q
\end{aligned}
$$

In a monopoly, the total expected surplus is

$$\begin{aligned}
TES(X^*, Q^*) &= \frac{X^*Q^*(2-\eta Q^*)}{2(r-\mu)} - \delta Q^* \\
&= \frac{\frac{\beta+1}{\beta-1}\delta(r-\mu)\frac{1}{(\beta+1)\eta}(2-\eta\frac{1}{(\beta+1)\eta})}{2(r-\mu)} - \delta\frac{1}{(\beta+1)\eta} \\
&= \frac{\frac{1}{\beta-1}\delta\frac{1}{\eta}(2-\frac{1}{\beta+1})}{2} - \delta\frac{1}{(\beta+1)\eta} \\
&= \frac{\delta(2-\frac{1}{\beta+1})}{2(\beta-1)\eta} - \delta\frac{1}{(\beta+1)\eta} \\
&= \frac{\delta(\frac{2(\beta+1)-1}{\beta+1})}{2(\beta-1)\eta} - \delta\frac{1}{(\beta+1)\eta} \\
&= \frac{\delta(2\beta+1)}{2(\beta-1)(\beta+1)\eta} - \delta\frac{1}{(\beta+1)\eta} \\
&= \frac{\delta(2\beta+1)}{2(\beta-1)(\beta+1)\eta} - \frac{2\delta(\beta-1)}{2(\beta+1)(\beta-1)\eta} \\
&= \frac{2\beta\delta+\delta-2\beta\delta+2\delta}{2(\beta+1)(\beta-1)\eta} \\
&= \frac{3\delta}{2(\beta+1)(\beta-1)\eta}
\end{aligned}$$

The social planner maximizes the total expected surplus, instead of the value of the firm in a monopoly setting. Hence, to determine Q_W^*, the FOC is applied to the total expected surplus and to determine X_W^*, the smooth pasting and value matching conditions are applied to the total expected surplus. Applying the FOC to the total expected surplus gives

$$\begin{aligned}
\frac{\partial}{\partial Q}TES(X,Q) &= 0 \\
\frac{X(1-\eta Q)}{r-\mu} - \delta &= 0 \\
1-\eta Q &= \frac{\delta(r-\mu)}{X} \\
Q_W^*(X) &= \frac{1}{\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big)
\end{aligned}$$

Using the smooth pasting conditions used before, but now with $V(X) = \frac{XQ(2-\eta Q)}{2(r-\mu)} - \delta Q$, the investment trigger equals

$$
\begin{aligned}
\frac{\beta}{X}\Big(\frac{XQ(2-\eta Q)}{2(r-\mu)}-\delta Q\Big) &= \frac{Q(2-\eta Q)}{2(r-\mu)} \\
\frac{\beta Q(2-\eta Q)}{2(r-\mu)}-\frac{Q(2-\eta Q)}{2(r-\mu)} &= \frac{\delta Q\beta}{X} \\
\frac{(\beta-1)Q(2-\eta Q)}{2(r-\mu)} &= \frac{\delta Q\beta}{X} \\
X &= \frac{2\delta\beta(r-\mu)}{(\beta-1)(2-\eta Q)} \\
X &= \frac{2\delta\beta(r-\mu)}{(\beta-1)(2-\eta\frac{1}{\eta}\Big(1-\frac{\delta(r-\mu)}{X}\Big))} \\
X &= \frac{2\delta\beta(r-\mu)}{(\beta-1)(1+\frac{\delta(r-\mu)}{X})} \\
X(\beta-1)(1+\frac{\delta(r-\mu)}{X}) &= 2\delta\beta(r-\mu) \\
X(\beta-1)+(\beta-1)\delta(r-\mu) &= 2\delta\beta(r-\mu) \\
X(\beta-1) &= (\beta+1)\delta(r-\mu) \\
X_W^* &= \frac{\beta+1}{\beta-1}\delta(r-\mu) = X^*
\end{aligned}
\tag{35}
$$

To determine the investment capacity for the social planner, $Q_W^* = Q_W^*(X_W^*)$ is used:

$$
\begin{aligned}
Q_W^* = Q_W^*(X_W^*) &= \frac{1}{\eta}\Big(1-\frac{\delta(r-\mu)}{X_W^*}\Big) \\
&= \frac{1}{\eta}\Big(1-\frac{\delta(r-\mu)}{\frac{\beta+1}{\beta-1}\delta(r-\mu)}\Big) \\
&= \frac{1}{\eta}\Big(1-\frac{\beta-1}{\beta+1}\Big) \\
&= \frac{1}{\eta}\Big(\frac{\beta+1}{\beta+1}-\frac{\beta-1}{\beta+1}\Big) \\
&= \frac{1}{\eta}\Big(\frac{2}{\beta+1}\Big) \\
&= \frac{2}{(\beta+1)\eta} = 2Q^*
\end{aligned}
\tag{36}
$$

Therefore, a social planner invests at the same time as the monopolist, namely as $X_t \geq X^*$, but with twice the capacity.

The welfare optimizing policy at the moment of investing yields a total welfare of

$$
\begin{aligned}
TES_W = TES(X_W^*, Q_W^*) &= \frac{X_W^* Q_W^* (2 - \eta Q_W^*)}{2(r-\mu)} - \delta Q_W^* \\
&= \frac{\frac{\beta+1}{\beta-1}\delta(r-\mu)\frac{2}{(\beta+1)\eta}(2 - \eta\frac{2}{(\beta+1)\eta})}{2(r-\mu)} - \delta\frac{2}{(\beta+1)\eta} \\
&= \frac{1}{\beta-1}\delta\frac{1}{\eta}(2 - \frac{2}{(\beta+1)}) - \delta\frac{2}{(\beta+1)\eta} \\
&= \frac{\delta(2 - \frac{2}{(\beta+1)})}{(\beta-1)\eta} - \delta\frac{2}{(\beta+1)\eta} \\
&= \frac{2\delta(1 - \frac{1}{(\beta+1)})}{(\beta-1)\eta} - \frac{2\delta}{(\beta+1)\eta} \\
&= \frac{2\delta\beta}{(\beta+1)(\beta-1)\eta} - \frac{2\delta}{(\beta+1)\eta} \\
&= \frac{2\delta\beta - 2\delta(\beta-1)}{(\beta+1)(\beta-1)\eta} \\
&= \frac{2\delta}{(\beta+1)(\beta-1)\eta}
\end{aligned}
$$

If the total expected surplus in a monopoly where the social planner determines the investment is compared to a monopoly where a monopolist determines the investment, the welfare loss is

$$
\frac{\frac{2\delta}{(\beta+1)(\beta-1)\eta} - \frac{3\delta}{2(\beta+1)(\beta-1)\eta}}{\frac{2\delta}{(\beta+1)(\beta-1)\eta}} = \frac{4-3}{4} = 25\%
$$

This is also in line with the results of Huisman and Kort (2015).

B

Derivation of the investment decision for iso-elastic demand function including lead time but without regime switching. Using $P_t = X_t Q_t^{-\zeta}$ and $I = \delta_1 Q + \delta_0$, we find that the value of the firm at the moment of investment is

$$V(X,Q) = \frac{Q^{1-\zeta}X}{r-\mu} exp((\mu - r)\theta) - \delta_1 Q - \delta_0$$

Using the first-order condition, the optimal investment capacity is determined. For a certain value of X, it is equal to

$$Q^*(X) = \left(\frac{(1-\zeta)X}{\delta_1 (r-\mu) exp((r-\mu)\theta)} \right)^{\frac{1}{\zeta}}$$

Substituting this optimal investment capacity into the function for the value of the firm at moment of investment gives

$$V(X) = \frac{\zeta\delta_1}{1-\zeta} \left(\frac{(1-\zeta)X}{\delta_1 (r-\mu) exp((r-\mu)\theta)} \right)^{\frac{1}{\zeta}} - \delta_0$$

Applying the smooth pasting and value matching conditions to the value of the firm at moment of investment and the value of waiting, $F(X) = AX^{\beta}$, where β is the same as in Section 2, we obtain the investment trigger, X^*. The investment trigger is equal to

$$X^* = \left(\frac{\delta_0 \beta (1-\zeta)}{\delta_1 (\zeta\beta - 1)} \right)^{\zeta} \frac{\delta_1 (r-\mu) exp((r-\mu)\theta)}{1-\zeta}$$

We observe that the investment trigger is increased by a factor $exp((r-\mu)\theta)$ if lead time is included in the investment decision. This result is similar to the conclusion in Section 3 for the linear demand function.

To find the investment capacity of the firm, we substitute the investment trigger into the optimal investment capacity function. Doing this yields

$$Q^* = \frac{\delta_0 \beta (1-\zeta)}{\delta_1 (\zeta\beta - 1)}$$

Similar to the linear demand model, the investment capacity remains constant for different values of lead time. It is equal to the investment capacity found by Huisman and Kort (2015) for the investment model with iso-elastic demand without lead time. Hence, for both the linear and iso-elastic demand function, including time only postpones the investment by a factor $exp((r-\mu)\theta)$ while the investment capacity remains constant.

Additionally, if the time to build is a variable, the value of the firm at the moment of investment is $V(X) = \frac{Q^{1-\zeta}X}{r-\mu} M_{\theta}(\mu - r) - \delta_1 Q - \delta_0$ with $M_{\theta}(t)$ as the moment generating function of θ. The investment trigger becomes

$$X^* = \left(\frac{\delta_0 \beta (1-\zeta)}{\delta_1 (\zeta\beta - 1)} \right)^{\zeta} \frac{\delta_1 (r-\mu)}{(1-\zeta) M_{\theta}(\mu - r)}$$

and like for a linear demand function, the investment capacity stays the same as for a constant lead time.

If lead time depends on the investment capacity, the installation time can be written as

$$\theta = g(Q)$$

If it is assumed that $g(Q) = \tau_1 ln(Q) + \tau_2 = ln(Q^{\tau_1}) + \tau_2$ with $\tau_1, \tau_2 \geq 0$ to have a nonnegative installation time, the function for time to build is substituted into the value of the firm at the moment of investment:

$$\begin{aligned} V(X, Q) &= \frac{Q^{1-\zeta}X}{r-\mu} exp((\mu - r)\theta) - \delta_1 Q - \delta_0 \\ &= \frac{Q^{1-\zeta}X}{r-\mu} exp((\mu - r)(ln(Q^{\tau_1}) + \tau_2)) - \delta_1 Q - \delta_0 \\ &= \frac{Q^{1-(\zeta+\tau_1(r-\mu))}X}{r-\mu} exp((\mu - r)\tau_2) - \delta_1 Q - \delta_0 \\ &= \frac{Q^{1-\tilde{\zeta}}X}{r-\mu} exp((\mu - r)\tau_2) - \delta_1 Q - \delta_0 \end{aligned}$$

with $\tilde{\zeta} = \zeta + \tau_1(r - \mu)$. This expression is similar to the value of investing using a constant lead time, but now with differently defined parameters. Hence, the investment trigger is

$$\begin{aligned} X^* &= \left(\frac{\delta_0 \beta(1-\tilde{\zeta})}{\delta_1(\tilde{\zeta}\beta - 1)}\right)^{\tilde{\zeta}} \frac{\delta_1(r-\mu)exp((r-\mu)\tau_2)}{1-\tilde{\zeta}} \\ &= \left(\frac{\delta_0 \beta(1-(\zeta+\tau_1(r-\mu)))}{\delta_1((\zeta+\tau_1(r-\mu))\beta - 1)}\right)^{\zeta+\tau_1(r-\mu)} \frac{\delta_1(r-\mu)exp((r-\mu)\tau_2)}{1-(\zeta+\tau_1(r-\mu))} \end{aligned}$$

and the investment capacity is

$$\begin{aligned} Q^* &= \frac{\delta_0 \beta(1-\tilde{\zeta})}{\delta_1(\tilde{\zeta}\beta - 1)} \\ &= \frac{\delta_0 \beta(1-(\zeta+\tau_1(r-\mu))}{\delta_1((\zeta+\tau_1(r-\mu))\beta - 1)} \end{aligned}$$

Therefore, an analytical solution to model the investment decision is found. It thus depends on the demand function and the function of lead time and capacity whether it is possible to find an analytical solution to the investment decision.

Furthermore, the restrictions of ζ also hold for $\tilde{\zeta}$ as otherwise, the investment decision is not realistic: $\tilde{\zeta} \in (0, 1)$ and $\tilde{\zeta}\beta > 1$. This implies restrictions on the parameter τ_1. First, $\tilde{\zeta} > 0$ always holds as ζ, τ_1 and $r - \mu$ are all bigger than zero. Hence, the remaining restrictions are:

$$\begin{aligned} \tilde{\zeta} &< 1 \\ \zeta + \tau_1(r-\mu) &< 1 \\ \tau_1 &< \frac{1-\zeta}{r-\mu} \end{aligned}$$

and

$$\begin{aligned} \tilde{\zeta}\beta &> 1 \\ \beta\zeta + \beta\tau_1(r-\mu) &> 1 \\ \tau_1 &> \frac{1-\beta\zeta}{(r-\mu)\beta} \end{aligned}$$

Thus, $\frac{1-\beta\zeta}{(r-\mu)\beta} < \tau_1 < \frac{1-\zeta}{r-\mu}$.

Lastly, if τ_2 is a random variable with a known distribution, the investment capacity does not change and the investment trigger becomes

$$X^* = \left(\frac{\delta_0 \beta (1 - (\zeta + \tau_1 (r - \mu))}{\delta_1 ((\zeta + \tau_1 (r - \mu))\beta - 1)} \right)^{\zeta + \tau_1 (r - \mu)} \frac{\delta_1 (r - \mu)}{(1 - (\zeta + \tau_1 (r - \mu)))M_{\tau_2}(\mu - r)}$$

C

A model is derived that uses the approximation technique to obtain an analytical solution to the problem. In this new model, only the stochastic process is adjusted. It is now a jump-diffusion process with N different types of jumps (Martzoukos and Trigeorgis, 2002):

$$dX_t = \mu X_t dt + \sigma X_t dZ_t + X_t \sum_{i=1}^{N} (k_i dq_i) \tag{37}$$

Where $\mathbb{P}[dq_i = 1] = \lambda_i dt$ and $\mathbb{P}[dq_i = 0] = 1 - \lambda_i dt$ and $ln(1 + k_i) \sim N(\gamma_i - 0.5\sigma_{Ji}^2, \sigma_{Ji}^2)$, where $\mathbb{E}[k_i] = exp(\gamma_i) - 1$. Hence, the expression for the value of the firm at the moment of investment is

$$\begin{aligned} V^{AGU}(X) &= \max_{Q \geq 0} \mathbb{E}\Big[\int_{t=0}^{\infty} QX(t)\Big(1 - \eta Q\Big) exp(-rt)dt - \delta Q | X(0) = X \Big] \\ &= \max_{Q \geq 0} \Big(Q(1 - \eta Q) \int_{t=0}^{\infty} \mathbb{E}[X(t)|X(0) = X] exp(-rt)dt - \delta Q \Big) \end{aligned} \tag{38}$$

To solve this problem, $\mathbb{E}[X(t)|X(0) = X]$ has to be derived. This is different than in the model by Huisman and Kort (2015), as X_t is now a jump-diffusion process instead of a diffusion process. Therefore,

$$\begin{aligned} \mathbb{E}[X(t)|X(0) = X] &= \mathbb{E}[\int_0^t dX_s | X_0 = X] \\ &= \mathbb{E}[\int_0^t \Big(\mu X_s ds + \sigma X_s dZ_s + X_s \sum_{i=1}^{N} (k_i dq_i)\Big) | X_0 = X] \\ &= \int_0^t \mathbb{E}[\mu X_s ds | X_0 = X] + \int_0^t \mathbb{E}[\sigma X_s dZ_s | X_0 = X] + \int_0^t \mathbb{E}[X_s \sum_{i=1}^{N} (k_i dq_i) | X_0 = X] \\ &= \int_0^t \mathbb{E}[\mu X_s ds | X_0 = X] + \int_0^t \mathbb{E}[\sigma X_s dZ_s | X_0 = X] + \int_0^t \mathbb{E}[X_s \sum_{i=1}^{N} \mathbb{E}[k_i] \lambda_i ds | X_0 = X] \\ &= \int_0^t \mathbb{E}[(\mu + \sum_{i=1}^{N} \lambda_i \mathbb{E}[k_i]) X_s ds | X_0 = X] + \int_0^t \mathbb{E}[\sigma X_s dZ_s | X_0 = X] \\ &= \int_0^t \mathbb{E}[\tilde{\mu} X_s ds | X_0 = X] + \int_0^t \mathbb{E}[\sigma X_s dZ_s | X_0 = X] \end{aligned} \tag{39}$$

Hence, the expectation of the stochastic process is translated with a new

$$\tilde{\mu} = \mu + \sum_{i=1}^{N} \lambda_i E[k_i]$$

and thus $\mathbb{E}[X(t)|X(0) = X] = X exp(\tilde{\mu} t)$. Consequently,

$$V^{AGU}(X) = \max_{Q \geq 0} \Big(\frac{XQ(1 - \eta Q)}{r - \tilde{\mu}} - \delta Q \Big) \tag{40}$$

and

$$Q^*_{AGU}(X) = \frac{1}{2\eta}\Big(1 - \frac{\delta(r-\tilde{\mu})}{X}\Big) \tag{41}$$

The Bellman equation is now

$$rF(x) = \pi + \frac{1}{dt}\mathbb{E}[dF]$$

As we apply Ito's Lemma, assuming that $F(X) = AX^{\beta}$ and $[X,X]_t$ denoting the quadratic variation of X over t, dF becomes:

$$\begin{aligned}\mathbb{E}[dF] &= \mathbb{E}[F_X dX + \frac{1}{2}F_{XX}d[X,X]_t + F_t dt] \\ &= \mathbb{E}[\beta AX^{\beta-1}(\mu X dt + \sigma X_t dZ_t + X\sum_{i=1}^{N}(k_i dq_i)) + \frac{1}{2}\beta(\beta-1)AX^{\beta-2}d[X,X]_t] \\ &= \beta AX^{\beta-1}\mathbb{E}[\mu X dt + X\sum_{i=1}^{N}(k_i dq_i)] + \frac{1}{2}\beta(\beta-1)AX^{\beta-2}\mathbb{E}[d[X,X]_t]\end{aligned} \tag{42}$$

Here, $\mathbb{E}[d[X,X]_t] = \sigma^2 X^2 dt + X^2\sum_{i=1}^{N} Var(k_i)\lambda_i dt$ because Z_t and dq_i are independent. Hence, the quadratic variance of their sum is the sum of their quadratic variances. Therefore,

$$\begin{aligned}\mathbb{E}[dF] &= \beta F(X)(\mu + \sum_{i=1}^{N}\mathbb{E}[k_i]\lambda_i)dt + \frac{1}{2}\beta(\beta-1)F(X)(\sigma^2 + \sum_{i=1}^{N} Var(k_i)\lambda_i)dt \\ &= \beta F(X)\tilde{\mu}dt + \frac{1}{2}\beta(\beta-1)F(X)\tilde{\sigma}^2 dt\end{aligned} \tag{43}$$

Where $\tilde{\sigma}^2 = \sigma^2 + \sum_{i=1}^{N} Var(k_i)\lambda_i$. The Bellman equation is now

$$rF(X) = \beta F(X)\tilde{\mu} + \frac{1}{2}\beta(\beta-1)F(X)\tilde{\sigma}^2$$

Thus the new β_{AGU} that solves for this system, from now on denoted by $\tilde{\beta}$, is

$$\tilde{\beta} = \frac{1}{2} - \frac{\tilde{\mu}}{\tilde{\sigma}^2} + \sqrt{\Big(\frac{1}{2} - \frac{\tilde{\mu}}{\tilde{\sigma}^2}\Big)^2 + \frac{2r}{\tilde{\sigma}^2}} \tag{44}$$

The threshold and optimal capacity are similarly derived as in Section 2, but with adjusted μ, σ and β;

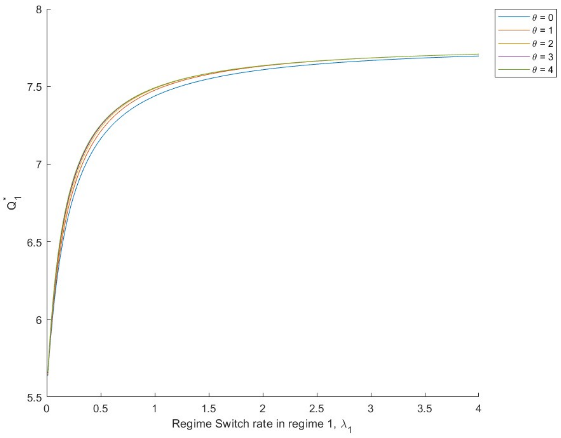
$$X^*_{AGU} = \frac{\tilde{\beta}+1}{\tilde{\beta}-1}\delta(r-\tilde{\mu}) \tag{45}$$

$$Q^*_{AGU} = \frac{1}{(\tilde{\beta}+1)\eta} \tag{46}$$

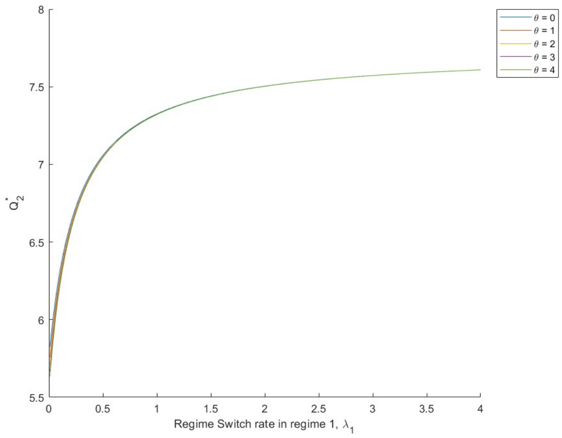
However, it should be noted that this approximation of the jump-diffusion process is not very accurate for all kinds of jumps.

D

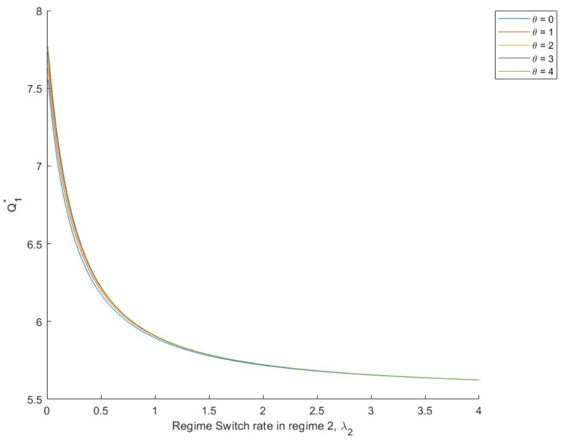
Plots of the investment capacities for different lead times against the regime switch rates for the linear demand model.



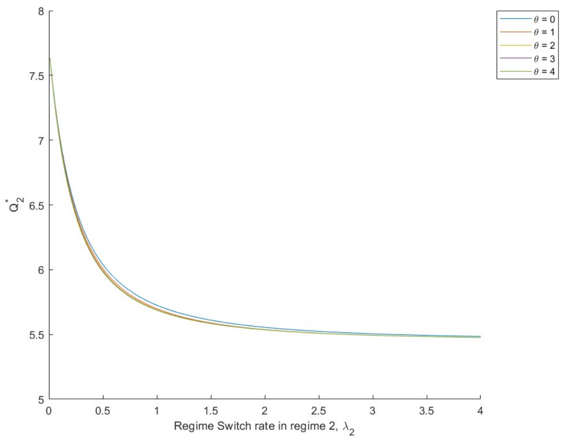
(a) Q_1^* for $\lambda_2 = 0.2$



(b) Q_2^* for $\lambda_2 = 0.2$



(c) Q_1^* for $\lambda_1 = 0.2$



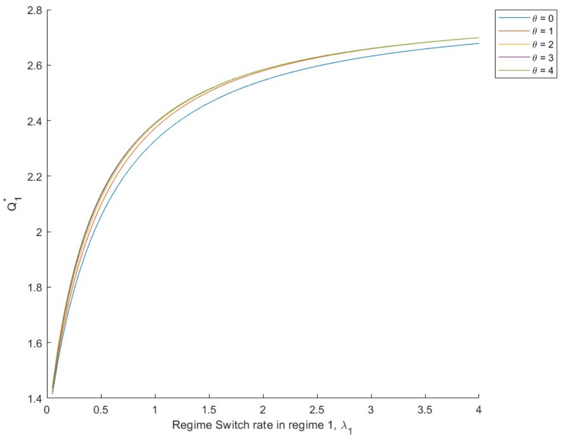
(d) Q_2^* for $\lambda_1 = 0.2$

Figure 31: Plots of the investment capacities for and the same parameters as defined in the plot and in Figure 17 and Figure 18

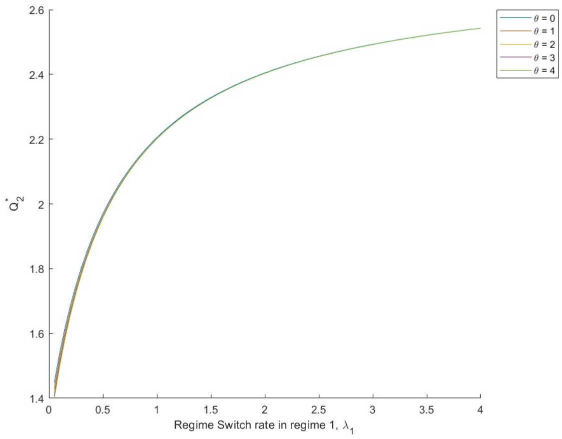
In all figures, the regime switch rates have a much greater effect on the investment capacities than the lead time. Therefore, like in the no regime switch model with lead time, the investment capacities stay relatively constant for different lead times.

Below are lots of the investment capacities for different lead times against the regime switch rates for the iso-elastic demand model.

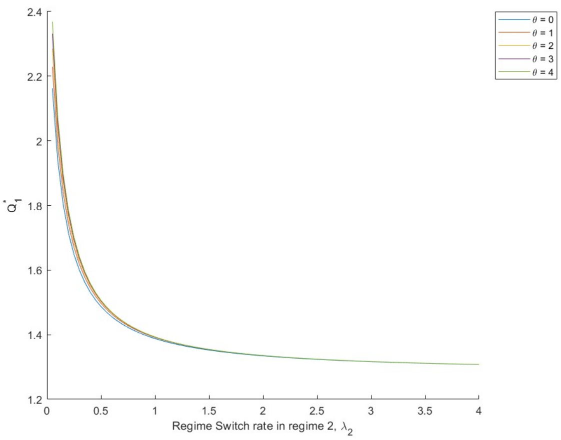
The differences in investment capacities are very small. This suggests that the regime switch rates have a bigger effect on the investment capacity than the lead time. Yet, there are still small differences in investment capacities. This could result from the probability of a regime switch during the lead time, changing the optimal investment capacity.



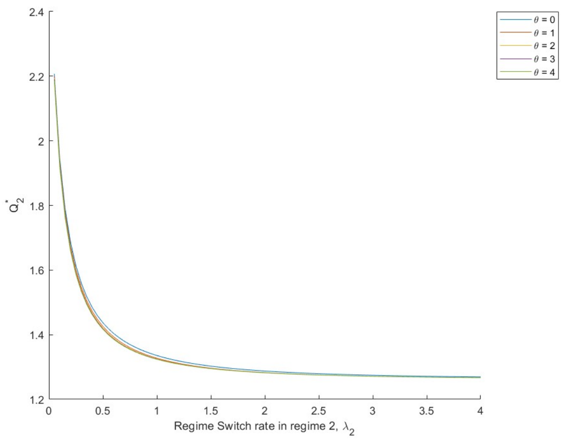
(a) Q_1^* for $\lambda_2 = 0.2$



(b) Q_2^* for $\lambda_2 = 0.2$



(c) Q_1^* for $\lambda_1 = 0.2$



(d) Q_2^* for $\lambda_1 = 0.2$

Figure 32: Plots of the investment capacities for and the same parameters as defined in the plot and in Figure 25