Matchings in a marriage market under limited foresight

by
Aron van Woerkom
Student number: 2069559

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Econometrics and Mathematical Economics
Tilburg School of Economics

Supervisors: prof. dr. Jean-Jacques Herings, dr. Ruud Hendrickx

Date final version: February 23, 2023

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Acknowledgement

I am deeply thankful to my supervisors prof. dr. Jean-Jacques Herings and dr. Ruud Hendrickx, for their ongoing support and guidance throughout the writing process. Their encouragement and patience have been essential in finishing my master's thesis and have played a crucial role in obtaining the results of this research project. Without their expertise and support, I would not have been able to finish the final project of my master's program.

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Abstract

Existing research on one-to-one-matching problems has mainly focused on either assuming myopia or full farsightedness by players in the game. This paper proposes to introduce *limited foresight* as an assumption on the players by examining each player's incentives to change the matching by establishing or dissolving links in the matching. Each player is assumed to have the ability to initiate these changes, and depending on the level of foresight, each player can evaluate the effect of further changes that may follow his change. Based on these new assumptions, I define a new stable set that always exists in a one-to-one matching problem and is unique. Next to assuming limited foresight, I propose the notion of *stochasticity* that players are assumed to be aware of. This stochasticity makes them less optimistic about the consequences of their actions. I show the relation of the stable sets under both conditions and show that both stable sets may not be subsets of each other. Last, I reconsider the assumptions under stochasticity and show the key takeaways that arise under the new assumptions.

Key words: matchings; marriage market; limited foresight; stochasticity; stable matching; stochastic marriage market.

1 Introduction

Matching problems are typically presented in the form of a marriage market where the players of the game can be divided into two separate groups. In the context of a marriage market, these are referred to as men and women. Each player has preferences over the players of the opposite sex. A matching is a complete allocation in which each player is allocated a player of the opposite sex or is single. If a player a is allocated a player b, then the player b is allocated player a. Furthermore, each player can only be allocated one player or is single. The players in the marriage market play the game by forming or breaking links between players of the opposite sex. These links comprise the allocation as described. Each player has the incentive to form a link with a partner as mostly preferred as possible and the formation and deletion of links are determined by the preferences of players in the game.

The matching literature started with the publication of the paper by Gale and Shapley (1962) which was the first paper to consider matching games. Next to considering marriage market games, they also considered school admission problems in which school applicants have preferences over schools and schools have preferences over applicants. Gale and Shapley (1962) distinguished marriage market games from school admission problems by allowing for schools to be allocated more than one student. In school admission problems, the number of schools can then be of larger size than the number of applicants. The marriage market game can, however, be interpreted as a restricted version of the school admission problem in which the number of schools is set equal to the number of applicants such that each school is allocated at most one student and each student is allocated at most one school. The marriage market game is usually referred to as a one-to-one matching problem because each player can be allocated one player of the opposite sex or is single.

A key concept introduced by Gale and Shapley (1962) is the concept of stability in the marriage market. In a stable matching, no player prefers being single over being matched to his allocated partner. Also, there is no pair of players that both prefer each other over their allocated partner in the matching. In other words, there is no player that can independently improve by breaking with his allocated player, and there is no pair of players that can jointly decide to break their links and form a new link between them such that both end up with a more preferred partner. Gale and Shapley (1962) also showed that, in any marriage market game, a stable matching exists. They showed this by providing an iterative procedure for finding a stable set of marriages that can be applied in every marriage market game.

A key assumption in the paper of Gale and Shapley (1962), however, is that players of the marriage market game act in a 'myopic' way. This 'myopia' or 'near-sightedness' breaks down to players considering only the direct consequence of the formation or deletion of a link. Consequently, players do not foresee that, through such an action, other players in the game may start to act which could be resulting in new matchings. The matching resulting from an action performed in the matching created by the initial player may not necessarily be in line with the interest of the player initially deciding to act. Furthermore, such an action could result in a path of matchings following from subsequent actions by players in the game. Depending on the assumptions made on the myopia of the players of the game, this path of events may or may not be foreseen by the player performing the

action.
The literature presented so far on matchings can be subdivided into two streams. Players are either assumed to be myopic or they are assumed to be farsighted. In the latter case, players can fully foresee the consequences of the formation or the deletion of a link. An example of a paper that assumes myopia includes Ehlers (2007) which shows several results in one-to-one matching problems in relation to stable sets. Others deriving results related to stability and stable sets under the assumption of myopic players include Wako (2010) and Demuynck, Herings, Saulle, and Seel (2019). Next to papers assuming myopia, Mauleon, Vannetelbosch, and Vergote (2011) show several results related to stable sets under the assumption of farsighted players. Another paper showing results with regard to stability and stable sets is by Mauleon, Molis, Vannetelbosch, and Vergote (2014).
The problem on the subdivision in the literature based on the assumption of myopia or farsightedness does not only apply to matching problems. Namely, in network problems, the focus on either myopia or farsightedness has also been present. Matching games can be seen as a restricted version of network games. Namely, in network games, no sex is defined and players can usually form links with multiple players. Furthermore, players have preferences over the networks and not necessarily over other players. Herings and Khan (2022) address the issue of the sole focus on either myopia or farsightedness in network games. In their research, they define the concept of 'limited foresight' that is assumed to be present by players in the network game. This limited foresight means that players can foresee the consequences of their actions up to a certain level. Herings and Khan (2022) state that players have foresight K and thus, players can foresee the subsequent $K - 1$ actions, possibly by other players, that are involved in the action. The key assumption here is that the degree of foresight gradually declines until 1 is reached which corresponds to the last action that can be foreseen by a player.

In one-to-one matchings problems, or in a wider context of matchings, assuming full foresight or myopia by players might not actually always be close to the setting of real-life problems. Namely, it does not automatically go beyond human reasoning to observe that one single action by a player can be foreseen by the acting player. The player could consequently observe the result of his action and analyse whether the new resulting situation might serve his benefit. Furthermore, it might be interesting to observe to what extent the assumption of limited foresight affects the outcome of matching problems. Therefore, in this paper, I assume limited foresight by players in the one-to-one matching game that is described as a marriage market. To this purpose, I translate the approach of Herings and Khan (2022) in network games to the context of matching problems.
The translation of the approach by Herings and Khan (2022) in my paper results in some key conclusions that can be drawn about the outcome of the marriage market game. I define the stable set of matchings that can be seen as the collection of all possible outcomes of the marriage market game and I show that this stable set always exists, for any level of foresight assumed to be present over the players. Obviously, this stable set depends on the level of foresight assumed. A key property of the stable set is that it is impossible for this stable set to be left by consecutive actions of the players. Furthermore, it should always be possible to get to a matching in the stable set from any matching outside the set by a finite sequence of the players' actions. Also, I show that the set of stable matchings, as defined initially by Gale and Shapley (1962), equals the stable set under level-1 foresight.

Next, I show that the stable set under level-1 foresight must be a subset of the stable set under level-2 foresight. The theorem in this section shows under which condition the level of foresight assumed to be present by the players does not influence the stable set of matchings.
In the part following the previously mentioned results, I change the assumption of the belief by the players on the outcomes of their actions by the introduction of stochasticity. Namely, in the set-up in networks by Herings and Khan (2022) players are assumed to be very optimistic about the game's outcome following a possible action. In fact, players always assume that, out of all matchings resulting from an action, a matching occurs that makes them end up with a more preferred partner. However, following the initial formation or deletion of a link, several paths might occur and not all the matchings at the end of these paths might benefit the initial deviator. Therefore, I propose a new notion of limited foresight in which players account for the existence of different paths. I refer to this assumption as 'stochasticity.' Under stochasticity, players act according to the fact that an action may make them end up with a less preferred partner than initially and do no longer assume that they always end up with the player they foresee to end up with. In other words, players attach probabilities to these different paths and act accordingly when forming and breaking links. Examples, definitions, and illustrations are given under stochasticity. Under this new assumption, I show that other deviations occur resulting in other stable sets. Under level-1 foresight, the outcome of the game is the same as in the initial set-up and I show the relation of the stable sets under both assumptions under level-2 foresight. Nevertheless, I show that deviations by players can be different for other levels of foresight and hence no relation between the outcomes of both assumptions can be drawn in a general setting. Also, I show under which condition the level of foresight assumed to be present over the players does not impact the stable set.
In the last part of this paper, I reconsider assumptions initially made when defining stochasticity to make the actions of the players even more 'credible.' I show that deviations under all set-ups defined can be different, leading to different outcomes of the game. In the specific case of the so-called 'utility-maximising' players, I draw conclusions on the outcome under level-2 foresight.

In this paper, I start off by formally defining the marriage market set-up under limited foresight in Section 2 with necessary definitions and explanations. Next, in Section 3, I give the results on this topic. In Section 4, I reconsider the assumptions from Section 2 by introducing stochasticity and the subsequent results are given in Section 4.4. In Section 5, I reconsider assumptions made in Section 4 and I give examples of consequences and also some new results. Last, I conclude my main findings in Section 6.

2 Definitions

2.1 Marriage market

The marriage market framework that is described here is similar to the approach taken by Herings, Mauleon, and Vannetelbosch (2020).
A marriage market consists of a finite set of players N that is divided into a set of men M and women W. In this setting, $N = M \bigcup W$ and $M \bigcap W = \emptyset$. For each player $i \in N$ it holds that he or she has a complete and transitive *preference ordering* $\succ_i$ over players of the opposite sex and over him or herself, which is equivalent to having no partner. Preferences are strict. $((\succ_m)_{m \in M}, (\succ_w)_{w \in W})$ is the preference profile that is denoted by $\succ$. Hence, a marriage market problem consists of men, women, and a preference order $(M, W, \succ)$.
A *matching* μ is a function : $N \rightarrow N$ that satisfies following conditions:

1. For all $m \in M, \mu(m) \in W \bigcup \{m\}$.

2. For all $w \in W, \mu(w) \in M \bigcup \{w\}$.

3. For all $i \in N, \mu(\mu(i)) = i$.

A matching implies that each player in $i \in N$ is matched to exactly one player of the opposite sex or is matched to himself. If a player i is *single*, he or she is matched to him or herself, which is described as $\mu(i) = i$. Furthermore, it holds for all $i \in N$, that if i is matched to j, then j is matched to i. This is made sure by condition **3**.
I denote the set of all possible matchings in a defined marriage market problem as $\mathbb{M}$. A matching μ is *individually rational* if each player prefers his partner over not being matched or is not matched at all. Hence, it must hold that $\forall i \in N, \mu(i) \succeq_i i$. In case of a matching $\mu \in \mathbb{M}$ that is not individually rational, because for player i $\mu(i) \prec_i i$, then player i would simply break up with his partner, resulting in a matching $\nu \in \mathbb{M}$ where i is single and $\nu(i) = i$.
A matching μ is *stable* if it is individually rational and it cannot be blocked by any pair of players that are not matched in μ, but both prefer each other over their partners in matching μ. Hence, it must hold for m and w to be blocking μ: $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$. The pair (m, w) is defined as the *blocking pair* in such a situation. Please note that for all $i \in \{m, w\}$, it may hold: $\mu(i) = i$, so a blocking pair may consist of players that are not matched.
Similarly, I say that a matching $\mu \in \mathbb{M}$ cannot be blocked by a group of players $S \subseteq N$, referred to as the *blocking coalition*, if there does not exist a matching μ' in which it holds that for all $i \in S \subseteq N$ $\mu'(i) \succ_i \mu(i)$ and for all $i \in S$: $\mu'(i) \in S$. Hence, it must hold for all players in any power set S of N that they cannot leave their partners and match with another player in S such that each of these players is better off. It does not need to hold for all $i \in S$ that $\mu(i) \in S$ because each player in the blocking coalition may leave his partner irrespective of whether that partner is in the blocking coalition. The *core* consists of all matchings for which such a blocking coalition S does not exist. Roth and Sotomayor (1992) show that the core of a marriage market equals the set of stable matchings and Gale and Shapley (1962) show that the core, and thus set of stable matchings, of a marriage market problem, is nonempty.

Let matching $\mu \in \mathbb{M}$ be the matching in which $m \in M$ and $w \in W$ are not matched, and let $\mu' \in \mathbb{M}$ be the matching that is exactly the same as μ but now with (m, w) being matched. Then I write: $\mu' = \mu + (m, w)$. If links existed by either m or w or both in μ, then writing $\mu' = \mu + (m, w)$ implies that these links are deleted in μ'. In a similar manner, let matching $\mu \in \mathbb{M}$ be the matching in which $m \in M$ and $w \in W$ are matched, and let $\mu' \in \mathbb{M}$ be the matching that is exactly the same as μ but now with (m, w) not being matched. Then I write $\mu' = \mu - (m, w)$.
In the following sections of this paper, sequences of matchings will be treated that follow each other by breaking and/ or forming links. Therefore, it is necessary to define which actions can be performed in one step when one goes from one matching to another. A step initiated by player $i \in N$, affecting players j, k and ℓ while $i \neq j \neq k \neq \ell$, in matching μ_0 resulting in matching μ_1 can be one of the following actions:

1. Player i breaks his link with player j and remains unmatched resulting in: $\mu_1(i) = i$ and $\mu_1(j) = j$.

2. Player i breaks his link with player j and matches with k that was not matched in μ_0, resulting in: $\mu_1(i) = k$, $\mu_1(j) = j$ and $\mu_1(k) = i$.

3. Player i breaks his link with player j and matches with k that was matched with player ℓ in μ_0, resulting in: $\mu_1(i) = k$, $\mu_1(j) = j$, $\mu_1(k) = i$ and $\mu_1(\ell) = \ell$.

4. Players i and j are unmatched in μ_0 and they form a link in μ_1, resulting in: $\mu_1(i) = j$ and $\mu_1(j) = i$.

5. Player i that is unmatched in μ_0 matches with player j that was matched with player k in μ_0, resulting in: $\mu_1(i) = j$, $\mu_1(j) = i$ and $\mu_1(k) = k$.

So, a player that was already matched in μ_0 can propose to be matched in μ_1 with another player that was also already matched in μ_0 (action **3**). Any step is allowed to happen under certain conditions that are defined in the following sections. Furthermore, it is good to make clear that simultaneous actions are not allowed to happen in one step. A step is always initiated by only one player that proposes to form a new link or to break a link. For instance, if, in μ_0, player i is matched with j and player k is matched with ℓ, then players i and k cannot simultaneously break their links and remain unmatched (action **1**). However, players i and k are allowed to form a new link between each other and automatically break their links in μ_0 (action **3**). Such an action is to be proposed by one of these players. When matching μ' can be created by one of the five described actions performed in μ, I say that μ' is a neighbour of μ.

2.2 Deviations

In the setting of matchings, each player may have the incentive to change the matching by performing an action as defined in Section 2.1. From here on, I refer to the action of cutting or forming a link as a *deviation*. The reason for a player to deviate is that he foresees ending up with a partner that is more preferred than his partner in the matching without his deviation. A player may also foresee that a deviation by him will trigger more deviations and that his partner at the end of this sequence of triggered deviations is more preferred than the partner he has in the matching he considers deviating from. Before defining the concept of deviating in anticipation of further deviations, I first define the concept of deviating without the foresight of triggered deviations. After this, I define the concept of deviating with foresight about deviations that may follow a deviation.

Definition 2.1. The deviation $\mu_0 \to_S \mu_1$ is a *level*-1 *deviation for player* $i \in S$, if $\mu_1(i) \succ_i \mu_0(i)$.

Definition 2.2. The deviation $\mu_0 \to_S \mu_1$ is a *level*-1 *deviation* if, for every player $i \in S$, it is a *level*-1 *deviation.*

If the deviation from μ_0 to μ_1 involves the addition of a link, then both players m and w need to switch to a more preferred partner in μ_1. Hence, for both $i \in \{m, w\}$ it must hold: $\mu_1(i) \succ_i \mu_0(i)$. In other words, also the opposing player needs to agree on the addition of the link to make the addition a possible level-1 deviation. However, if the deviation from μ_0 to μ_1 involves the deletion of a link, then for at least one player $i \in \{m, w\}$, it must hold: $\mu_1(i) \succ_i \mu_0(i)$. In other words, only one player needs to agree on the deletion of the link to make the deletion a possible level-1 deviation for that player. The collection of matchings that can be reached by a level-1 deviation from a matching μ_0 is denoted by $f_1(\mu_0)$, while $f_1(M) = \bigcup_{\mu_0 \in M} f_1(\mu_0)$ is the collection of matchings that can be reached by a level-1 deviation from matchings in the collection $M \subseteq \mathbb{M}$.
Similarly, a level-K deviation can be defined, for all $K \in \mathbb{N}$. The collection of matchings that can be reached by a level-K deviation from μ_0 is defined as $f_K(\mu_0)$, while $f_K(M) = \bigcup_{\mu_0 \in M} f_K(\mu_0)$ is defined as the set of matchings that can be reached from all matchings in $M \subseteq \mathbb{M}$.

Definition 2.3. Let $K \geq 2$. The deviation $\mu_0 \to_S \mu_1$ is a *level-K deviation for player* $i \in S$ if one of the following two cases holds:

- **(i)** There exists a finite sequence of matchings $\mu_2, ..., \mu_K$ such that for each $k \in \{1, ..., K-1\}$, $\mu_{k+1} \in f_{K-k}(\mu_k)$ and $\mu_K(i) \succ_i \mu_0(i)$.
- **(ii)** There exists a $K' \in \{1, ..., K-1\}$ and a finite sequence of matchings $\mu_2, ..., \mu_{K'}$, such that: (a) $\forall k \in \{1, ..., K'-1\}$, $\mu_{k+1} \in f_{K-k}(\mu_k)$, (b) $f_{K-K'}(\mu_{K'}) = \emptyset$, and (c) $\mu_i(\mu_{K'}) \succ_i \mu_i(\mu_0)$.

Definition 2.4. Let $K \geq 2$. The deviation $\mu_0 \to_S \mu_1$ is a *level-K deviation* if, for every player in S, it is a level-K deviation.

When deviating from μ_0 to μ_1, player $i \in N$, uses in his reasoning process the implied deviations that follow from his deviation to μ_1. I define each of these implied deviations following the deviation to μ_1 as an *induced deviation*. It is good to mention that the matchings following the induced deviations may not necessarily be reached because once μ_1 is reached, all players again have level-K foresight meaning at this point other choices can be made by players that do not lead to the matching μ_2 from μ_1 that is an induced deviation when player $i \in N$ deviates from μ_0 to μ_1. The deviation from μ_1 to μ_2 would be possible if every player had level-$K-1$ foresight at μ_1, meaning, with level-K foresight, the deviation from μ_1 to μ_2 is not necessarily possible. The deviation $\mu_0 \to \mu_1$ is referred to as the *actual deviation*. When player i deviates from μ_0 to μ_1, he can foresee the

impact of the following $K-1$ induced deviations. This leads to two distinct possibilities: In case **(i)** of Definition 2.3, player i forms a series of $K-1$ induced deviations following his deviation from μ_0 to μ_1. For each $k \in \{1, ..., K-1\}$, the k'th induced deviation is a level-$K-k$ deviation. This means that for each $k \in \{1, ..., K-1\}$, player i can foresee the level-$K-k$ deviation $\mu_k \rightarrow \mu_{k+1}$. Hence, the first deviation $\mu_1 \rightarrow \mu_2$ is a level-$K-1$ deviation and the second deviation $\mu_2 \rightarrow \mu_3$ will be a $K-2$ deviation. This will continue until the deviation $\mu_{K-1} \rightarrow \mu_K$, which is a level-1 deviation. The following matching μ_K is the furthest matching that can be foreseen by player i and it marks the end of his reasoning process. I refer to this matching as the *terminal matching*. To make the deviation $\mu_0 \rightarrow \mu_1$ a desirable level-K deviation for player i, the terminal matching μ_K must be more preferred than the matching μ_0 by player i. Hence, it must hold: $\mu_K(i) \succ_i \mu_0(i)$. I refer to this deviation as a *level-K deviation with complete support*.
However, it may be that, in the reasoning process of player i, I end up in an induced matching $\mu_{K'}$ ($1 \leq K' < K$) from which no more level-$K-K'$ deviations exist. If this holds, then I can say: $f_{K-K'}(\mu_{K'}) = \emptyset$, which corresponds to condition (b) in case **(ii)** of Definition 2.3. In this event, player i is not able to form a sequence of $K-1$ induced deviations such that for each $k \in \{1, ..., K-1\}$ $\mu_k \rightarrow \mu_{k+1}$ is a level-$K-k$ deviation. However, for all $k \in \{1, ..., K'\}$, the k'th induced deviation is a level-$K-k$ deviation. In this case, $\mu_{K'}$ is the terminal matching of player i's reasoning process because from $\mu_{K'}$ no level-$K-K'$ deviations exist anymore. To make the deviation $\mu_0 \rightarrow \mu_1$ a desirable level-K deviation for player i in this case, the terminal matching $\mu_{K'}$ must be more preferred than the matching μ_0 by player i. Hence, it must hold: $\mu_{K'}(i) \succ_i \mu_0(i)$. I refer to this deviation as a *level-K deviation with incomplete support*.

Example 2.5. Let me now consider an example in the context of matchings in which it is checked whether a level-K deviation may exist by some player i in some matchings, with $K \geq 2$.
The following marriage problem $(M, W, \succ)$ with men $M = \{m_1, m_2\}$ and women $W = \{w_1, w_2\}$ is considered. The following preferences are present by the players of the game:

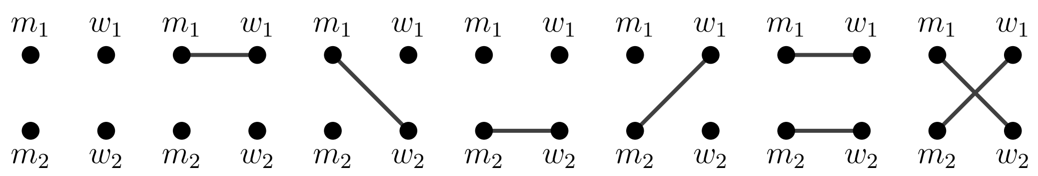
$\succ_{m_1}$: w_1, w_2, m_1

$\succ_{m_2}$: w_2, w_1, m_2

$\succ_{w_1}$: m_2, m_1, w_1

$\succ_{w_2}$: m_1, m_2, w_2

Hence, the following seven matchings are present in this game:



μ_1 μ_2 μ_3 μ_4 μ_5 μ_6 μ_7

Clearly, matchings μ_6 and μ_7 do not have any possible level-1 deviations because it is impossible for any pair of opposite players to both get a more preferred partner by a level-1 deviation. Hence, Definition 2.1 does not hold for any player. All other matchings do have a deviation according to Definition 2.1 for at least 2 players.
Let me now check whether any level-2 deviations exist from the matching without level-1 deviations μ_7, to be written as ν_0 in this example. For players $\{w_1, w_2\}$, this matching is optimal, so in any case, Definition 2.3 cannot be met for these players. Hence, let me consider player m_1 that could deviate by breaking link $\{m_1, w_2\}$, resulting in matching μ_5. In μ_5, called ν_1 here, going to matching μ_7 is a level-1 deviation (could be initiated by both m_1 and w_2) as well as going to μ_4 (could be initiated by m_2 and w_2), so it holds for this second resulting matching: $f_1(\nu_1) \neq \emptyset$. Hence, condition (ii) in Definition 2.3 does not hold. Although μ_4, $\mu_7 \in f_1(\nu_1)$, it does hold that the resulting matching $\nu_2(m_1) \not\succ_{m_1} \mu_7(m_1) = \nu_0(m_1)$. I have now shown that there is no level-2 deviation possible for player m_1 in μ_7 and, by symmetry, this is neither possible for m_2. Furthermore, by symmetry, there are no level-2 deviations present for all players in $\{w_1, w_2\}$ in μ_6.
Moreover, I can establish that, for all players, deviating from $\mu_1 = \nu_0$ to any matching ν_1 in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ is a level-2 deviation. Namely, deviating to μ_6 is a level-1 deviation from matchings μ_2 and μ_4, while deviating to μ_7 is a level-1 deviation from matchings μ_3 and μ_5. Furthermore, it holds for all matchings $\nu_2 \in \{\mu_6, \mu_7\}$ that, for all players $i \in M \bigcup W$, $\nu_2(i) \succ_i \nu_0(i) = \mu_1(i) = i$. Hence, when at μ_1, for any player, forming a link with a player of the opposite sex, leading to a matching in A, is level-2 deviation. ▲

Following Definition 2.4 and Example 2.4, the question arises whether larger degrees of foresight lead to more possible deviations for players in the marriage market. Furthermore, if a level-K deviation is always also a level-$K+1$ deviation, then this result may contribute to conclusions about the matchings the marriage market problem results in. However, in Example 2.6, I show that it cannot always be established that $f_K(\mu_0) \subseteq f_{K+1}(\mu_0)$ for all $K \in \mathbb{N}$ and all $\mu_0 \in \mathbb{M}$. In Example 2.7, I show that it cannot always be established that $f_{K+1}(\mu_0) \subseteq f_K(\mu_0)$ for all $K \in \mathbb{N}$ and all $\mu_0 \in \mathbb{M}$.

Example 2.6. In this example, I give a counterexample to establish that $f_K(\mu) \subseteq f_{K+1}(\mu)$ does not always hold. This example has the exact same set-up as Example 2.5 and I show $f_1(\mu_2) \subset f_2(\mu_2)$.
For matching μ_2, it holds that $f_1(\mu_2) = \{\mu_5, \mu_6\}$. Namely, deviating from μ_2 to μ_5 is a level-1 deviation for both m_2 and w_1, because both m_2 and w_1 are better off in μ_5 than in μ_2. Furthermore, deviating from μ_2 to μ_6 is a level-1 deviation as well for m_2 and w_2, because both m_2 and w_2 are better off in μ_6 than in μ_2. No other level-1 deviations exist from matching μ_2.
Going from μ_2 to μ_5 is also a level-2 deviation for m_2 and w_1, because a level-1 deviation exists from μ_5 to μ_7 for m_1 and w_2, while both m_2 and w_1 are better off in μ_7. Next, going from μ_2 to μ_6 is also a level-2 deviation for m_2 and w_2, because no level-1 deviation exists from μ_6 while both m_2 and w_2 are better off in μ_6. Going from μ_2 to μ_1 is a level-2 deviation for w_1 because going from μ_1 to μ_5 is a level-1 deviation for both m_2 and w_1 while in μ_5, w_1 is better off. Hence, $\{\mu_1, \mu_5, \mu_6\} \subseteq f_2(\mu_2)$. Knowing that $f_1(\mu_2) = \{\mu_5, \mu_6\}$, it can be established that $f_1(\mu_2) \subset f_2(\mu_2)$. ▲

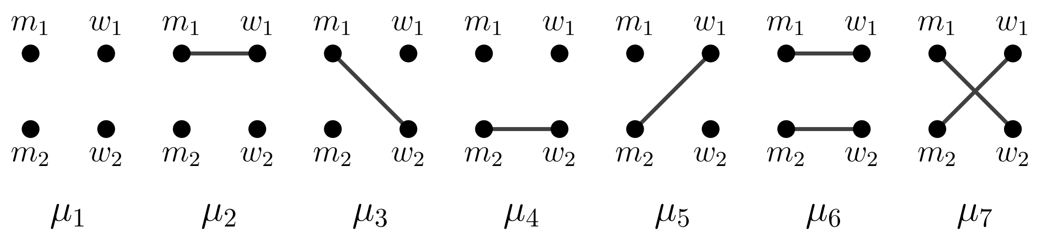
Example 2.7. In this example, I give a counterexample to establish that $f_{K+1}(\mu) \subseteq f_K(\mu)$ does not always hold. I use the same marriage market set-up as in Example 2.5, but with different preferences. I show in this specific example that there exists some matching $\mu_0 \in \mathbb{M}$ for which $f_2(\mu_0) \subset f_1(\mu_0)$. The preferences are as follows:

$\succ_{m_1}$: m_1, w_2, w_1

$\succ_{m_2}$: m_2, w_1, w_2

$\succ_{w_1}$: m_2, m_1, w_1

$\succ_{w_2}$: w_2, m_1, m_2



Let me first consider the level-2 deviations that are possible from μ_6. A deviation from μ_6 to μ_2 is a level-2 deviation for both m_2 and w_2 because the only level-1 deviation existing from μ_2 is to μ_1 and both players are better off in μ_1. Furthermore, deviating to μ_3 is a level-2 deviation for m_1 and w_2 because the only level-1 deviation existing from μ_3 is to μ_1 and both players are better off in μ_1. Also, deviating to μ_4 is a level-2 deviation for m_1 because the only level-1 deviation existing from μ_4 is to μ_1 and m_1 is better off in μ_1. No other level-2 deviations exist from μ_6 (going to μ_5 is not a level-2 deviation for w_1) and hence $f_2(\mu_6) = \{\mu_2, \mu_3, \mu_4\}$.
Next, I check the possible level-1 deviations from μ_6. Deviating to μ_2 is a level-1 deviation for m_2 and w_2, while deviating to μ_4 is a level-1 deviation for m_1. Furthermore, deviating to μ_3 is a level-1 deviation for both m_1 and w_2 because both players are better off in μ_3. Furthermore, deviating to μ_5 is a level-1 deviation for both m_2 and w_1 because both players are better off in μ_5. Therefore, $f_1(\mu_6) = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ and now it has been established that $f_2(\mu_6) \subset f_1(\mu_6)$. ▲

Through these two counterexamples, it has now been shown that, in general, the set of matchings that can be reached by a level-K deviation is not necessarily a subset of the set of matchings that can be reached by a level-$K+1$ deviation and vice versa.

2.3 k-fold iteration

To introduce the topic of k-fold iterations, I first give an example of a k-fold iteration in the setting of level-1 deviations. Later in this section, this will be generalised in a definition and another example is given in the setting of level-2 deviations with appropriate notation.

Example 2.8. The marriage market problem here is exactly the same as in Example 2.5. Considering matching μ_1, deviating to any matching in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ is a level-1 deviation. For each matching in A, the deviation to this matching from μ_1 could be initiated by two players. For instance, deviating to μ_2 could be initiated by m_1 and w_1 A, so: $\mu_1 \rightarrow_{\{m_1,w_1\}} \mu_2$. Deviating to any matching in A from μ_1 is a level-1 deviation because for each element in A, Definition 2.1 holds for the 2 players that are matched in the matching $\mu \in A$. Hence, I can say $\mu \in f_1(\mu_1)$ for each $\mu \in A$. Furthermore, matching μ_6 can be reached by a level-1 deviation from μ_2 and μ_4, so $\mu_6 \in f_1(\{\mu_2, \mu_4\})$, while μ_7 can be reached from a level-1 deviation from μ_3 and μ_5, so $\mu_7 \in f_1(\{\mu_3, \mu_5\})$. Hence, for any $\nu_2 \in \{\mu_6, \mu_7\}$ it holds that ν_2 can be reached by two sequential level-1 deviations that start in μ_1. The first level-1 deviation is from μ_1 to some matching in A, while the second level-1 deviation is from some matching in A to some matching in $\{\mu_6, \mu_7\}$. ▲

The k-fold iteration is the concept that k sequential level-K deviations may follow each other. From a starting matching, different k sequential level-K deviations exist and hence the set of possible matchings that may be reached after k sequential level-K deviations may have cardinality larger than one, as is the case in Example 2.8. Furthermore, in this same example, it has been established that $\{\mu_6, \mu_7\}$ is a subset of all matchings that may be reached after 2 sequential level-1 deviations starting in μ_1.
In general, when an actual level-K deviation in matching μ_0 is performed by some player leading to matching μ_1, new level-K deviations may follow this deviation. In Section 2.2, I defined the set of matchings that can be reached by a level-K deviation from μ_0 as $f_K(\mu_0)$. Once $\mu_1 \in f_K(\mu_0)$ is reached, all players again have level-K foresight and hence the actual deviation from μ_1 is again a level-K deviation. Continuing in this manner, I can construct a sequence of matchings that follow each other by sequential level-K deviations. This sequence may not be unique for some $K \geq 1$ and hence the set of matchings that can be reached from k sequential level-K deviations can have more than one element but it can also be empty if no level-K deviations exist from μ_0 or if after $1 < j < k$ level-K deviations no more level-K deviations exist.
The k-fold iteration of f_K from matching μ_0 is defined as the set of matchings that can be reached by k sequential level-K deviations starting in μ_0 and is denoted by $f_K^k(\mu_0)$. For instance, μ_1 will be a possible resulting matching from a level-K deviation starting in μ_0 ($\mu_1 \in f_K(\mu_0)$) and μ_2 will be a possible resulting matching from a level-K deviation starting in μ_1 ($\mu_2 \in f_K^2(\mu_0)$ and $\mu_2 \in f_K(\mu_1)$). This can be further generalised in a new definition. Similarly, $f_K^k(M)$ is the set of all matchings that can be reached by k sequential level-K deviations starting in any matching in the set $M \subseteq \mathbb{M}$.

Definition 2.9. For $\mu_k \in \mathbb{M}$ and $M \subseteq \mathbb{M}$, $\mu_k \in f_K^k(M)$, when there exists a $\mu_{k-1} \in \mathbb{M}$ such that $\mu_{k-1} \in f_K^{k-1}(M)$ and $\mu_k \in f_K(\mu_{k-1})$, with $K \geq 1$ and $k \geq 2$.

In Definition 2.9, we may have that M consists of only one element, say $M = \{\mu_0\}$ and $\mu_0 \in \mathbb{M}$. The condition $\mu_{k-1} \in f_K^{k-1}(M)$ makes sure that matching μ_{k-1} can be reached by $k-1$ sequential level-K deviations, starting in some matching $\mu_0 \in M$. The condition $\mu_k \in f_K(\mu_{k-1})$ makes sure that μ_k can be reached by a single level-K deviation starting in μ_{k-1}. Please note that $f_K(\mu_{k-1}) \subseteq f_K^k(\mu_0)$, for some $\mu_0 \in M$ because $f_K^k(\mu_0)$ may contain matchings that are reached through a path of sequential level-K deviations of which μ_{k-1} is not necessarily part, while μ_{k-1} is part of the path that leads to μ_k.
Next, I specify the set of matchings that can be reached by any larger than zero number of sequential level-K deviations from deviations starting in set $M \subseteq \mathbb{M}$.

Definition 2.10. The collection of all matchings that can be reached by the composition of a finite number of sequential level-K deviations from any matching in the set of matchings $M \subseteq \mathbb{M}$ is denoted by: $f_K^{\mathbb{N}}(M) = \bigcup_{k \in \mathbb{N}} f_K^k(M)$.

In Definition 2.10, it may be that M consists of only one element, say $M = \{\mu_0\}$ and $\mu_0 \in \mathbb{M}$.

Example 2.11. Let me now consider an example regarding k-fold iterations using level-2 deviations, using the same setting as in Example 2.5 with identical preferences:

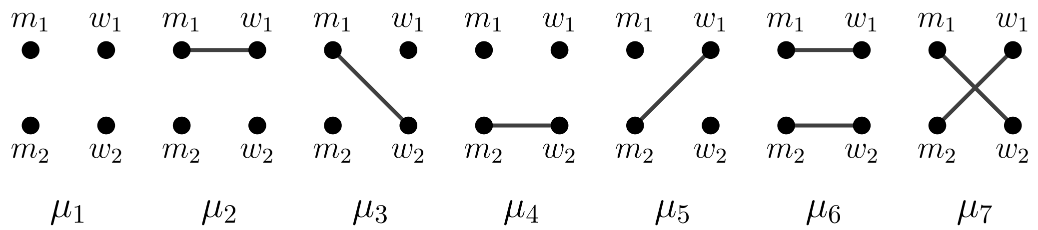
$\succ_{m_1}$: w_1, w_2, m_1

$\succ_{m_2}$: w_2, w_1, m_2

$\succ_{w_1}$: m_2, m_1, w_1

$\succ_{w_2}$: m_1, m_2, w_2

All possible matchings in this setting are shown again:



Deviating from μ_1 to any matching in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ is a level-2 deviation. This holds because, when at μ_1, each player may initiate a deviation because each player foresees that a level-2 deviation to some matching in A can lead to a level-1 deviation that leads to a matching in $\{\mu_6, \mu_7\}$. In any matching in $\{\mu_6, \mu_7\}$, it holds that each player is better off than in μ_1, so for any player, a level-2 deviation to a matching in A is a fruitful deviation. Hence, I can write, for each $\mu \in A$: $\mu \in f_2(\mu_1)$. However, deviating from any matching in A to a matching in $\{\mu_6, \mu_7\}$ is also a level-2 deviation. This has already been shown in Example 2.5. In fact it holds: $\mu_6 \in f_2(\{\mu_2, \mu_4\})$ and $\mu_7 \in f_2(\{\mu_3, \mu_5\})$. Hence, for any $\nu_2 \in \{\mu_6, \mu_7\}$ there exists a $\nu_1 \in \mathbb{M}$ such that $\nu_1 \in f_2^1(\mu_1)$ and $\nu_2 \in f_2(\nu_1)$, satisfying the condition in Definition 2.9. Therefore, both μ_6 and μ_7 can be reached by two sequential level-2 deviations starting in μ_1. Using the notation from Definition 2.9, I can write: for each $\nu_2 \in \{\mu_6, \mu_7\}$: $\nu_2 \in f_2^2(\mu_1)$. ▲

2.4 Stability

Definition 2.12. Let $K \in \mathbb{N}$. The collection $M_K \subseteq \mathbb{M}$ is a *level-K stable set* if it satisfies the following three conditions:

1. **Deterrence of external deviations:** $f_K(M_K) \subseteq M_K$.
2. **Iterated external stability:** For all $\mu \notin M_K$, $f_K^{\mathbb{N}}(\mu) \bigcap M_K \neq \emptyset$.
3. **Minimality:** There is no proper subset $M \subseteq M_K$ satisfying conditions 1 and 2.

In a level-K stable set $M_K \subseteq \mathbb{M}$, a level-K deviation from any matching $\mu \in M_K$ does not lead to a matching outside this set, as required by deterrence of external deviations, which also implies that a finite number of sequential level-K deviations from a matching in M_K never leads to a matching outside M_K. Furthermore, iterated external stability makes sure that sequential level-K deviations from any matching $\nu \in \mathbb{M} \setminus M_K$ ultimately lead to a matching $\mu \in M_K$. The minimality condition is required to make sure that the level-K stable set is not unnecessarily large, in fact, $\mathbb{M}$ satisfies conditions 1 and 2, but also that the level-K stable set is unique which will be shown later on.
Deterrence of external deviations implies that, if this is satisfied by a set of matchings $M_K \subseteq \mathbb{M}$, then there does not exist a subset of matchings in $\mathbb{M} \setminus M_K$ for which iterated external stability holds. This holds because, for any $\mu \in M_K$, it is no longer possible to reach a matching outside M_K by means of sequential level-K deviations. Similarly, iterated external stability implies that, if this is satisfied by a set of matchings $M_K \subseteq \mathbb{M}$, then there does not exist a subset of matchings in $\mathbb{M} \setminus M_K$ for which deterrence of external deviations holds. This holds because it is possible to reach M_K by sequential level-K deviations from any matching in $\mathbb{M} \setminus M_K$. From these properties, it follows that any set satisfying deterrence of external deviations and iterated external stability must have a non-empty intersection with the level-K stable set itself. Namely, in the level-K stable set also both deterrence of external deviations and iterated external stability hold and there does not exist any subset in $\mathbb{M}$ outside the level-K stable set for which both conditions can hold. From the third condition of minimality, it then follows that this set must be unique. Later in this paper, I formally prove that the level-K always exists and is unique. In the next definition, I define the concept of level-K cycles.

Definition 2.13. Let $K \in \mathbb{N}$. The non-empty set $M \subseteq \mathbb{M}$ is a *level-K cycle* if it is a minimal set satisfying deterrence of external deviations.

If a level-K cycle M is a singleton, then the only element in set M is a level-K stable matching.

Example 2.14. Let me now consider an example regarding stability in a matching, using the same setting as in Example 2.5 with identical preferences:

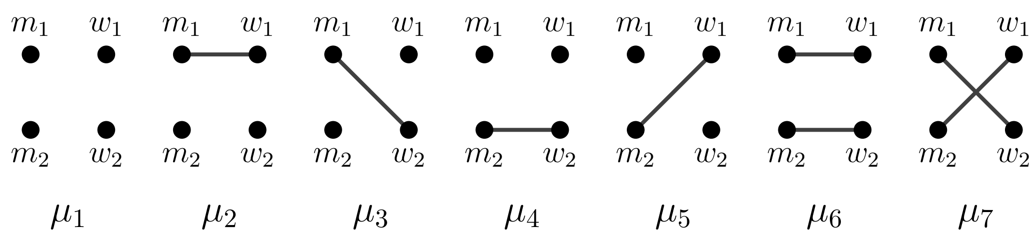
$\succ_{m_1}$: w_1, w_2, m_1

$\succ_{m_2}$: w_2, w_1, m_2

$\succ_{w_1}$: m_2, m_1, w_1

$\succ_{w_2}$: m_1, m_2, w_2

All possible matchings in this setting are shown again:



The purpose of this example is to show that the set $M_1 = M_2 = \{\mu_6, \mu_7\}$ is both a level-1 and level-2 stable set but not a level-K stable set for $K > 2$.
Firstly, I will state that the set of matchings $M_1 = \{\mu_6, \mu_7\}$ is a level-1 stable set. Clearly, for both matchings μ_6, μ_7 there exists no level-1 deviation by any player to a matching outside M_1. Hence, for both elements the deterrence of external deviations is satisfied. Furthermore, for the matchings in the set $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ a level-1 deviation exists that results in a matching in M_1 and for μ_1 a level-1 deviation exists that leads to a matching in A (so $\mu_6, \mu_7 \in f_1^2(\mu_1)$, in fact $\mu_6, \mu_7 \in f_1^{\mathbb{N}}(A \bigcup \{\mu_1\})$. Hence, for both elements in M_1, iterated external stability is satisfied. Also, there exists no proper subset of M_1 satisfying both conditions. For example, let's assume $M_1 = \{\mu_6\}$, then it holds for $\mu_7 \notin M_1$ that $f_K^{\mathbb{N}}(\mu_7) \bigcap M_1 = \emptyset$, because no level-1 deviations are possible from μ_7. By symmetry, the same holds for assuming $M_1 = \{\mu_7\}$. Therefore, also minimality is satisfied and it can be concluded that M_1 is a level-1 stable set.
Secondly, it can be established that the same set of matchings, now called, $M_2 = \{\mu_6, \mu_7\}$ is also a level-2 stable set. In Example 2.5, I showed that there exist no level-2 deviations from both elements within this set. Hence, the deterrence of external deviations is satisfied. Furthermore, for matchings in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ deviating to a matching in M_2 is a level-2 deviation and because it is also a level-1 deviation, deviating from μ_1 to a matching in A is a level-2 deviation. Hence, it can be concluded that iterated external stability is satisfied for M_2. By the same logic that M_1 satisfies minimality when showing that it is a level-1 stable set, it can be reasoned that minimality is satisfied for M_2. Therefore, it can be concluded that M_2 is a level-2 stable set.
Let me now consider the case with $K = 3$. Going from μ_7 to μ_5 is a level-3 deviation by m_1 because from μ_5 a level-2 deviation exists to μ_4 for m_2 and w_2 while from μ_4 a level-1 deviation exists to μ_6 for m_1 and w_1. In μ_6, m_1 is better off than in μ_7 and hence this is a fruitful deviation for m_1. By symmetry, going from μ_7 to μ_3 is a level-3 deviation and, by symmetry, going from μ_6 to μ_2 and μ_4 are level-3 deviations. Furthermore, three possible level-3 deviations exist from μ_5. Firstly, going from μ_5 to μ_7 is a level-3 deviation for m_1 and w_2 because in μ_7 both players are better off and no level-2 deviations exist from μ_7. Secondly, going from μ_5 to μ_4 is a level-3 deviation for m_2 and w_2, because going from μ_4 to μ_6 is a level-2 deviation for m_1 and w_1 and no level-1 deviations exist in μ_6, while both players are better off in μ_6. Thirdly, going from μ_5 to μ_1 is a level-3 deviation for m_2 because going from μ_1 to any matching in $\{\mu_2, \mu_4\}$ is a level-2 deviation and from both these matchings there exists a level-1 deviation to μ_6, while m_2 is better off in any matching in μ_6 than in μ_5. From μ_1 going to any matching in A for any player is a level-3 deviation because from any matching in A there exists a level-2 deviation to some matching in $\{\mu_6, \mu_7\}$ from where no level-2 deviations exist and where any player is better off than in μ_1. Considering the possible level-3 deviations from μ_1, μ_5 and μ_7 and the symmetry that follows from μ_5 and μ_7, the following possible level-3 deviations exist from each matching in $\mathbb{M}$:

$f_3(\mu_1) = \{\mu_2, \mu_3, \mu_4, \mu_5\}$
$f_3(\mu_2) = \{\mu_1, \mu_5, \mu_6\}$
$f_3(\mu_3) = \{\mu_1, \mu_2, \mu_7\}$
$f_3(\mu_4) = \{\mu_1, \mu_3, \mu_6\}$
$f_3(\mu_5) = \{\mu_1, \mu_4, \mu_7\}$

$f_3(\mu_6) = \{\mu_2, \mu_4\}$
$f_3(\mu_7) = \{\mu_3, \mu_5\}$

This means that all matchings in $\mathbb{M}$ need to be in the level-3 stable set to satisfy the deterrence of external deviations. Hence, the level-3 stable set here is equal to $\mathbb{M}$.
Let me now consider the general case for $K \geq 4$. Going from μ_5 to μ_1 is a level-4 deviation for m_2 because going from μ_1 to some matching in $\{\mu_2, \mu_4\}$ is a level-3 deviation and from each matching in $\{\mu_2, \mu_4\}$ there exists a level-2 deviation to μ_6 from which no more level-1 deviations exist, while m_2 is better off in μ_6. Furthermore, going from μ_1 to μ_5 is a level-4 deviation by w_1 and m_2 because a level-3 deviation exists from μ_5 to μ_1 from which a level-2 deviation exists to some matching in $\{\mu_2, \mu_4\}$ from which a level-1 deviation exists to μ_6. Hence, $\mu_1 \to \mu_5$ and $\mu_5 \to \mu_1$ are both level-4 deviations, because from the matching after the deviation, a sequence of induced matchings exists (starting with a level-3 deviation) leading to μ_6, where the couple or single player initiating the deviation is better off than in the starting matching. However, if $\mu_1 \to \mu_5$ and $\mu_5 \to \mu_1$ are both level-4 deviations then they are also both level-5 deviations because from the matching after the deviation (μ_1 to μ_5 or vice versa) a sequence of induced matchings exists (starting with a level-4 deviation) leading to μ_6.
This reasoning can be iterated as follows for $K \geq 3$: going from μ_5 to μ_1 is a level-K deviation because from the matching after the deviation a sequence of induced matchings exists (starting with a level-$K-1$ deviation) leading to μ_6. By symmetry, this also holds for deviations from other matchings in A to μ_1 (that may have terminal matching μ_6 or μ_7). This same reasoning can also be applied to a level-K deviation from μ_5 to μ_4. Namely, deviating from μ_5 to μ_4 is a level-K deviation, with $K \geq 3$ because from μ_4 a sequence of induced matchings exists (starting with a level-$K-1$ deviation) leading to μ_6. This same reasoning can also be applied from the deviation $\mu_5 \to \mu_7$. However, if $K = 3$, then there is no induced sequence of deviations (but we are already in $\{\mu_6, \mu_7\}$), while if $K \geq 4$, more deviations exist from μ_7. Therefore, by the symmetry involved in this setting, the following holds for $K \geq 3$:

$f_K(\mu_1) = \{\mu_2, \mu_3, \mu_4, \mu_5\}$
$f_K(\mu_2) = \{\mu_1, \mu_5, \mu_6\}$
$f_K(\mu_3) = \{\mu_1, \mu_2, \mu_7\}$
$f_K(\mu_4) = \{\mu_1, \mu_3, \mu_6\}$
$f_K(\mu_5) = \{\mu_1, \mu_4, \mu_7\}$
$f_K(\mu_6) = \{\mu_2, \mu_4\}$
$f_K(\mu_7) = \{\mu_3, \mu_5\}$

This means that all matchings in $\mathbb{M}$ need to be in the level-K stable set to satisfy the deterrence of external deviations. Hence, the level-K stable set here is equal to $\mathbb{M}$, for $K \geq 3$. ▲

2.5 α-reducibility

In this section, I introduce the concept of α-reducibility. This concept is used in Section 3 to show a result that holds under marriage markets satisfying α-reducibility. This concept

was first described by Alcalde (1994) in the context of roommate problems.
The concept of α-reducibility implies that the marriage market problem $(M, W, \succ)$ has the property that the set of players $N = M \bigcup W$ can be partitioned in ℓ coalitions S_j, $j = 1, ..., \ell$, such that $N = \bigcup_{j=1}^{\ell} S_j$ and $S_j \bigcap S_k = \emptyset$, for each $j \neq k$. These coalitions S_j consist of either one player or two players of the opposite sex. Now, α-reducibility requires that the player(s) in S_1 are their top preferred partner. The player(s) in S_2 must be their top preferred partner when the set of players S_1 is discarded from the marriage market. Hence, the player(s) in S_2 are their top preferred partner when the players in S_2 are only allowed to have preferences over players of the opposite sex and themselves in $N \setminus S_1$. This pattern continues for further coalitions. Hence, the player(s) in S_j are their top preferred partner when the players in S_j are allowed to have preferences only over players of the opposite sex and themselves in $N \setminus S_1 \bigcup ... \bigcup S_{j-1}$, for $j = 1, ..., \ell$. As has been described, the ordering of the coalitions $S_1...S_\ell$ is important and will also be used when deriving a property in α-reducible marriage markets in Section 3.
Alcalde (1994) showed in his paper that, when α-reducibility is satisfied, the core is unique and consists of a single stable matching μ. Because there exists only one stable matching, the partition $S_1, ..., S_\ell$ must also be unique. Namely, in each coalition, the players in the coalition are matched to each other or the single player in the coalition remains single if the coalition is of size one. Now only one stable matching exists, there must also be only one way of partitioning N. Hence, only one partition exists in α-reducible marriage market problems. In Example 2.15, I show the coalitions S_j and the stable matching of the marriage market that satisfies α-reducibility.

Example 2.15. The purpose of this example is to illustrate the concept of α-reducibility. Below, I first show the preferences of the marriage market with 6 players. I show how stable matching μ evolves in the marriage market from the matching μ' in which for all $i \in N$ $\mu'(i) = i$. I make use of the coalitions S_j.

$\succ_{m_1}$: w_2, w_1, w_3, m_1

$\succ_{m_2}$: w_2, m_2, w_3, w_1

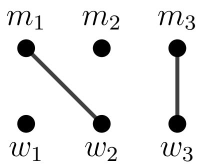
$\succ_{m_3}$: w_3, w_2, w_1, m_3

$\succ_{w_1}$: m_2, m_1, m_3, w_1

$\succ_{w_2}$: m_1, m_3, m_2, w_2

$\succ_{w_3}$: m_1, m_2, m_3, w_3

This marriage market does satisfy α-reducibility and hence, N can be partitioned in several coalitions S_j. First, it can be observed that m_1 and w_2 are their first choice. Hence, they form the coalition $S_1 = \{m_1, w_2\}$ and are matched to each other in stable matching μ. The top choice of player m_2 is w_2. However, when considering only $N \setminus S_2$, m_2's top choice is remaining single. Therefore, m_2 remains single and $S_2 = \{m_2\}$. Player m_3's top choice in $N \setminus S_1 \bigcup S_2$ is w_3, while w_3's top choice in $N \setminus (S_1 \bigcup S_2)$ is m_3. Therefore, both players match and $S_3 = \{m_3, w_3\}$. Now the only remaining player is w_1 that remains single and so $S_4 = \{w_1\}$. The resulting matching μ is stable and is shown below:



μ

▲

3 Results

In Section 2, I have introduced the context of matchings, the definition of deviating and the conditions under which deviations can happen, and the k-fold iteration. Last, combining these three concepts, the level-K stable set has been defined. So far, the level-K stable set has been described extensively, but nothing is known yet about its existence and other properties. Therefore, in this section, several theorems are established regarding the properties of the level-K stable set.
First of all, in Theorem 3.1, I prove that there always exists such a level-K stable set and that it is unique and, in Theorem 3.2, I prove that the level-K stable set is equal to the union of level-K cycles. Both theorems have been derived from similar proofs in the context of networks from the paper by Herings and Khan (2022). In the setting defined so far, the exact same steps as in the original proofs can be performed, while in the proofs given here, usually more explanation on each step is given and on the reason why certain properties hold. In Theorem 3.3, the proof is provided that a level-1 stable set only consists of singleton level-1 cycles and in Theorem 3.4 it is proved that any matching is part of the level-1 stable set if and only if the matching is stable. In Theorem 3.5, it is proved that the level-1 stable set is a subset of the level-2 stable set. Last, in Theorem 3.6, I prove that, in α-reducible marriage markets, for any $K > 0$, the level-K stable set equals the stable matching.

Theorem 3.1. For each $K \in \mathbb{N}$, there exists a unique level-K stable set.

Proof. This proof consists of two parts. Firstly, it is proved that the level-K stable set exists. Secondly, it is shown that it is unique by assuming two level-K stable sets. In this second part is shown that the intersection of these two sets also satisfies the three properties for a level-K stable set and that therefore both sets must be equal.
Take any $K \in \mathbb{N}$.
The set $M^0 = \mathbb{M}$ meets the deterrence of external deviations and iterated external stability conditions.
Assume there is no level-K stable set. Since $\mathbb{M}$ does not satisfy minimality, there exists a set $M^1 \subset M^0$ that satisfies deterrence of external deviations and iterated external stability. We can continue this reasoning for any $k \in \mathbb{N}$ by saying that, for each subset $M^k \subseteq \mathbb{M}$ that satisfies deterrence of external deviations and iterated external stability, there exists a subset $M^{k+1} \subset M^k$ that also satisfies deterrence of external deviations and iterated external stability. Ultimately, this would lead to a matching with negative cardinality because the cardinality of $\mathbb{M}$ is finite. Hence, this leads to a contradiction because in this context, sets with negative cardinality do not exist. Consequently, it can be concluded that level-K stable sets exist.
Let me now assume that there exist two level-K stable sets $M^1, M^2 \subseteq \mathbb{M}$. Deterrence of external deviations implies that, if this is satisfied by a set of matchings $M_k \subseteq \mathbb{M}$, then there does not exist a subset of matchings in $\mathbb{M} \setminus M_k$ for which iterated external stability holds. This holds because, for any $\mu \in M_k$, it is no longer possible to reach a matching outside M_k by means of sequential level-K deviations. Similarly, iterated external stability implies that, if this is satisfied by a set of matchings $M_k \subseteq \mathbb{M}$, then there does not exist a subset of matchings in $\mathbb{M} \setminus M_k$ for which deterrence of external deviations holds. This holds because it is possible to reach M_k by sequential level-K deviations from any

matching in $\mathbb{M} \setminus M_k$.
Both M^1 and M^2 satisfy the deterrence of external deviations, so by definition, I can say: $f_K(M^1) \subseteq M^1$ and $f_K(M^2) \subseteq M^2$. This means that the set of matchings that can be reached by a level-K deviation from the intersection of M^1 and M^2 is a subset of the intersection of the matchings that can be reached by a level-K deviation from M^1 and the matchings that can be reached by a level-K deviation from M^2. Furthermore, the latter is also a subset of the intersection of M^1 and M^2. Therefore, the following relation holds: $f_K(M^1 \bigcap M^2) \subseteq f_K(M^1) \bigcap f_K(M^2) \subseteq M^1 \bigcap M^2$. Therefore, $M^1 \bigcap M^2$ satisfies deterrence of external deviations condition.
It also needs to be shown that $M^1 \bigcap M^2$ satisfies iterated external stability. So for any matching μ outside of $M^1 \bigcap M^2$ it needs to be shown that there is a finite number of sequential level-K deviations that will lead to a matching in $M^1 \bigcap M^2$. So, take any $\mu \notin M^1 \bigcap M^2$. Three situations for μ can hold that need to be considered:
Situation 1: $\mu \in M^1 \setminus M^2$. M^2 satisfies iterated external stability, so M^2 must be reachable from any matching outside M^2 by sequential level-K deviations. Hence, it must hold: $f_K^{\mathbb{N}}(\mu) \bigcap M^2 \neq \emptyset$. M^1 satisfies deterrence of external deviations, so: $f_K^{\mathbb{N}}(M^1) \subseteq M^1$. Since $\mu \in M^1$: $f_K^{\mathbb{N}}(\mu) \bigcap M^1 \neq \emptyset$. Hence, because from μ, it is possible to reach both M^1 and M^2, iterated external stability is satisfied for $M^1 \bigcap M^2$, and $f_K^{\mathbb{N}}(\mu) \bigcap (M^1 \bigcap M^2) \neq \emptyset$.
Situation 2: $\mu \in M^2 \setminus M^1$. I can draw the same conclusion as in situation 1, because of the symmetry.
Situation 3: $\mu \notin M^1 \bigcup M^2$. By the iterated external stability of M^1, it holds: $f_K^{\mathbb{N}}(\mu) \bigcap M^1 \neq \emptyset$. Let $\mu' \in f_K^{\mathbb{N}}(\mu) \bigcap M^1$. If $\mu' \in M^2$, it holds: $f_K^{\mathbb{N}}(\mu) \bigcap (M^1 \bigcap M^2) \neq \emptyset$. If $\mu' \in M^1 \setminus M^2$, then $f_K^{\mathbb{N}}(\mu') \bigcap (M^1 \bigcap M^2) \neq \emptyset$, which follows from situation one and also implies: $f_K^{\mathbb{N}}(\mu) \bigcap (M^1 \bigcap M^2) \neq \emptyset$.
Now it has been shown that $M^1 \bigcap M^2$ satisfies both deterrence of external deviations and iterated external stability. Since $M^1 \bigcap M^2 \subset M^1$, $M^1 \bigcap M^1 \subset M^2$ and $M^1 \neq M^2$, I end up with a contradiction of the minimality of both M^1 and M^2. Consequently, a unique level-K stable set exists. □

Now it has been established that the level-K stable set always exists and is unique, I can use this result in the next proof. Namely, in that proof, I show that the level-K stable set is equal to the union of the level-K cycles of the marriage market problem.

Theorem 3.2. Let $K \in \mathbb{N}$. The level-K stable set is equal to the union of the level-K cycles of the marriage market problem.

Proof. This proof consists of two parts. Firstly, it is shown that the union of level-K cycles is a subset of the level-K stable set. Secondly, it is shown that the union of level-K cycles is equal to the level-K stable set.
Let me denote the level-K stable set by M_K and the union of the level-K cycles by C_K. Assume there is a level-K cycle C that is not a subset of M_K. Let $\mu \in C \setminus M_K$. Because of the iterated external stability of M_K, for any matching in $C \setminus M_K$, it is possible to go to a matching in M_K, so there exists a $\mu' \in M_K$ such that $\mu' \in f_K^{\mathbb{N}}(\mu)$. Since C is a cycle and thus satisfies deterrence of external deviations, it holds $\mu' \in C$ because any finite sequence of level-K deviations from C always leads to a matching in C. For the same reason, (because I know $\mu' \in C$ and C is a cycle), it must hold: $f_K^{\mathbb{N}}(\mu') \subseteq C$. It must also be possible to reach μ by a finite number of sequential level-K deviations starting in μ', so it holds $\mu \in f_K^{\mathbb{N}}(\mu')$. Namely, if this does not hold, then $f_K^{\mathbb{N}}(\mu')$ would be a subset of C

satisfying deterrence of external deviations and hence contradicting the minimality of C. I also know, by definition, that M_K satisfies deterrence of external deviations, so it holds: $f_K^{\mathbb{N}}(\mu') \subseteq M_K$. This, along with $\mu \in f_K^{\mathbb{N}}(\mu')$, implies $\mu \in M_K$. Namely, $f_K^{\mathbb{N}}(\mu') \subseteq M_K$ implies that by a finite number of level-K deviations starting in M_K, it will never be possible to get outside of M_K and $\mu \in f_K^{\mathbb{N}}(\mu')$ implies that $\mu \in C \setminus M_K$ must also be an element of the set of all matchings that can be reached by a finite sequence of matchings starting in M_K. $\mu \in M_K$, however, contradicts $\mu \in C \setminus M_K$. Consequently, every level-K cycle C is a subset of M_K, and hence $C_K \subseteq M_K$ for the union of all level-K cycles C_K.
Because we know that C_K is a union of cycles, it clearly satisfies deterrence of external deviations, but it still needs to be shown that it also satisfies iterated external stability. Essentially, I need to show that $C_K = M_K$, while so far I have shown $C_K \subseteq M_K$.
Let's assume C_K does not satisfy iterated external stability. Then a matching $\mu \notin C_K$ exists from which it is not possible to get to a matching in C_K by sequential level-K deviations and $f_K^{\mathbb{N}}(\mu) \bigcap C_K = \emptyset$. It also holds that $f_K^{\mathbb{N}}(\mu) \neq \emptyset$, because otherwise $\{\mu\}$ would be a level-K cycle and then it should be in the union of all level-K cycles C_K. The set of matchings $f_K^{\mathbb{N}}(\mu)$ satisfies deterrence of external deviations because, by definition, $f_K(f_K^{\mathbb{N}}(\mu)) \subseteq f_K^{\mathbb{N}}(\mu)$. Since $f_K^{\mathbb{N}}(\mu)$ is a finite set, there exists a non-empty collection of matchings $M \subseteq f_K^{\mathbb{N}}(\mu)$ that is a minimal set satisfying deterrence of external deviations, because I know that there does exist at least one matching in M from where it is not possible to reach any matching in C_K by a finite sequence of level-K deviations. Hence, I can conclude that M contains at least one level-K cycle, so it must hold that M is in the union of level-K cycles: $M \subseteq C_K$. This is, however, contradicting $f_K^{\mathbb{N}}(\mu) \bigcap C_K = \emptyset$. Hence, it must hold: $f_K^{\mathbb{N}}(\mu) \bigcap C_K \neq \emptyset$ and thus C_K is satisfying iterated external stability.
I know now that C_K satisfies iterated external stability and deterrence of external deviations and that is a subset of M_K. Knowing that M_K is a level-K stable set, it must hold that it satisfies the minimality condition. Hence, I conclude $C_K = M_K$. □

Now it has been established that the level-K stable set consists of the union of level-K cycles, I can use this result in the next proof. Namely, in that theorem, I show that the level-1 stable set is equal to the union of singleton level-1 cycles of the marriage market problem.

Theorem 3.3. The level-1 stable set is equal to the union of singleton level-1 cycles.

Proof. From Theorem 3.2, I know that the level-1 stable set is equal to the union of level-1 cycles. Hence, to show that the level-1 stable set is equal to the union of singleton level-1 cycles, I only need to show that there exist no level-1 cycles with cardinality larger than 1. If this has been shown, then it has been made clear that the level-1 stable equals the union of singleton level-1 cycles because from Theorem 3.1 it is known that the level-1 stable set exists. In this proof, I first introduce a result by Roth and Vande Vate (1990) regarding paths to stable matchings and I show that stable matchings are also in the level-1 stable set. Last, using this result, I prove that assuming level-1 cycles with cardinality larger than 1 leads to a contradiction.
Roth and Vande Vate (1990) show that, for any matching $\mu \in \mathbb{M}$, there exists a finite sequence of matchings $\mu_1, ..., \mu_k$, such that $\mu = \mu_1$ and such that μ_k is a stable matching. Furthermore, for each $i = 1, ..., k-1$, there exists a blocking pair $(m_i, w_i) \in M \times W$ for μ_i such that μ_{i+1} is obtained from μ_i by satisfying the blocking pair (m_i, w_i). In other words, from each matching, it is possible to reach a stable matching by a sequence of

matchings and in each step in the sequence a blocking pair is matched. In the paper of Roth and Vande Vate (1990), a blocking pair may also consist of one player and, in that case, a player is a blocking pair with 'him or herself.'
Let $S \subseteq \mathbb{M}$ be the set of matchings containing all stable matchings. The result of Roth and Vande Vate (1990) implies that the deterrence of external deviations cannot be satisfied for any subset of matchings in $\mathbb{M} \setminus S$ because from any matching in $\mathbb{M}$ there leads a path to a stable matching. For any stable $\mu \in S$ it holds that there is no blocking (m, w) pair, so for each $i \in N$ deviating by matching a player of the opposite sex is impossible. Furthermore, μ is individually rational, so any player $i \in N$ cannot deviate by becoming single. For these two reasons, no level-1 deviations exist in matching μ. If I now assume that μ is not in the level-1 stable set, then the iterated external stability is not satisfied for the set, because from μ no level-1 deviations are possible. Therefore, μ must be part of the level-1 stable set.
Now it has been established that any matching in the set of stable matchings S is in the level-1 stable set. Furthermore, each matching in $\mu \in S$ does not allow for level-1 deviations to any matching in $\mathbb{M} \setminus \{\mu\}$ and is thus satisfying deterrence of external deviations. Hence, the set of stable matchings contains only singleton level-1 cycles.
Let's now assume there exists a level-1 cycle C with cardinality larger than 1. This cycle satisfies deterrence of external deviations and can hence not be a subset of $\mathbb{M} \setminus S$. Now there are two situations to consider:
Situation 1: there is at least one matching $\mu \in C$ (and at most $|C| - 1$) matchings) that is in $\mathbb{M} \setminus S$. All matchings in S do not allow for level-1 deviations. Hence, this would lead to a contradiction because it is not possible to go from μ to some matching in S and back.
Situation 2: $C \subseteq S$. This would lead to a contradiction because $|C| \geq 2$ and this means level-1 deviations should be possible to and from matchings in C. However, no level-1 deviations exist from any matching in S because it contains only singleton level-1 cycles. This means that assuming the existence of level-1 cycles with cardinality larger than 1 leads to a contradiction. From Theorem 3.2 it is known that the level-1 stable set equals the union of level-1 cycles. Therefore, the level-1 stable set equals the union of singleton level-1 cycles. □

The fact that a matching $\mu \in \mathbb{M}$ is in the level-1 stable set that consists of singleton level-1 cycles gives me the result of the equivalence of the stable set and the level-1 stable set. I show this equivalence in the next proof.

Theorem 3.4. For each $\mu \in \mathbb{M}$, it holds that μ is part of the level-1 stable set $M_1 \subseteq \mathbb{M}$ if and only if μ is stable.

Proof. This proof consists of two parts. The first part proves that all matchings of a level-1 stable set are stable. The second part proves that all stable matchings belong to the level-1 stable set.
According to the framework defined so far, a level-1 deviation from μ_1 to μ_2 is possible in two situations. The first situation: i breaks a link and is better off single than with his partner in μ_1, then it must hold: $i \succ_i \mu(i)$. The second situation: i breaks with j (or breaks with himself because he is single) and forms a link with k with whom he is better off, then it must hold: $\mu_2(i) \succ_i \mu_1(i)$ and $\mu_2(k) \succ_k \mu_1(k)$. In the first situation, i is the blocking player and in this situation, μ_1 is not individually rational. In the second

situation, there exists a blocking man-woman pair because both players can improve by matching with each other. If there is no level-1 deviation possible, both situations 1 and 2 are absent. Therefore, the absence of a possible level-1 deviation in matching $\mu_1 \in \mathbb{M}$ implies stability. I have proved in Theorem 3.3 that any level-1 stable set contains only singleton level-1 cycles. From these cycles, no level-1 deviations exist. Therefore, all matchings in a level-1 stable set are stable.
Let's now assume that matching $\mu \in \mathbb{M}$ is stable. Because μ is stable, there is no blocking (m, w) pair, so for each $i \in N$ deviating by matching a player of the opposite sex is impossible. Furthermore, μ is individually rational, so any player $i \in N$ cannot deviate by becoming single. For these two reasons, no level-1 deviations exist in matching μ. If I now assume that μ is not in the level-1 stable set, then the iterated external stability is not satisfied for the set, because from μ no level-1 deviations are possible. Therefore, μ is part of the level-1 stable set. □

Now it has been shown that the level-1 stable set consists of singleton level-1 cycles and that this set equals the stable set, I relate these results to the level-2 stable set. I show in the next proof that the level-1 stable set is a subset of the level-2 stable set.

Theorem 3.5. The level-1 stable set is a subset of the level-2 stable set.

Proof. In this proof, I call the level-1 stable set Λ_1 and the level-2 stable set Λ_2. I prove $\Lambda_1 \subseteq \Lambda_2$ by contradiction. Let's assume: $\Lambda_1 \nsubseteq \Lambda_2$, so: $\exists \mu \in \Lambda_1$ and $\mu \notin \Lambda_2$. In μ, no level-1 deviations exist, which is known by Theorem 3.3. However, by the iterated external stability of Λ_2, there exists a sequence of level-2 deviations from μ to some matching $\nu \in \Lambda_2$. Therefore, in μ a level-2 deviation must exist. This level-2 deviation, from μ to, let's say, μ' with induced matching to μ'', by player $i \in N$ must be one of the following situations:
Situation 1: player i gets single in μ' and intends to be matched with j in μ''. In this case, $\{i, j\}$ is a blocking pair in μ.
Situation 2: player i matches player j in μ' and intends to be matched with k in μ''. In this case, $\{i, k\}$ is a blocking pair in μ.
Situation 3: player i gets matched with j in μ' and both do not deviate in μ'. In this case, $\{i, j\}$ is a blocking pair in μ. If j now intends to be matched with $k \neq i$ in μ'', then $\{j, k\}$ is a blocking pair in μ.
Situation 4: player i gets single in μ' and intends to remain single (does not deviate) in μ''. In this case, $\{i\}$ is a blocking player in μ.
Situation 5: player i matches player j in μ' and intends to get single in μ''. In this case, $\{i\}$ is a blocking player in μ.
Hence, this means there must exist a blocking pair or player in μ and level-2 deviations only exist if there exists a blocking pair or blocking player. Therefore, it cannot hold that μ is in Λ_1 while not in Λ_2, which contradicts assuming $\mu \in \Lambda_1$ and $\mu \notin \Lambda_2$. Therefore, it must hold that $\Lambda_1 \subseteq \Lambda_2$. □

The last proof that is shown in this section is about the relation between α-reducibility, as described in Section 2.5 and foresight. I prove that the level of foresight has no impact in α-reducible marriage markets because the level-K stable set always contains only the stable matching in α-reducible problems for any $K > 0$ as is shown next. In Theorem 3.6, I use the same notation for the coalitions $S_1, ..., S_\ell$ as has been used in the description

of α-reducible problems in Section 2.5. Furthermore, I make use of the property that in α-reducible problems, there exists a single stable matching and a unique partition for this stable matching, as was shown by Alcalde (1994).
In the proof next, I first show that the stable matching μ must be in the level-K stable set for any $K > 0$ to let the level-K stable set satisfy iterated external stability. In the second part, I show that the set containing the stable matching only is the level-K stable set.

Theorem 3.6. Let $(M, W, \succ)$ be a marriage market problem satisfying α-reducibility. Then, for any $K > 0$, the level-K stable set equals the stable matching.

Proof. Consider players in S_1. In the stable matching μ in the α-reducible marriage market $(M, W, \succ)$, all players in S_1 have their top choice. Therefore, for all $i \in S_1$ there does not exist a $\mu' \in \mathbb{M} \setminus \{\mu\}$ for which it holds that $\mu'(i) \succ_i \mu(i)$. Therefore, there exists no matching in $\mu' \in \mathbb{M} \setminus \{\mu\}$ that they would be willing to deviate to from μ. Irrespective of the level of foresight K, all players in S_1 cannot improve. Hence, it holds for all $i \in S_1$ and for all $K > 0$ that for all $\mu' \in f_K^{\mathbb{N}}(\mu)$ that $\mu'(i) = \mu(i)$.
Now consider players in S_2. All players in S_2 could only improve by deviating such that there exists an induced path that matches them with someone in S_1. However, there exists no single matching in $\mathbb{M} \setminus \{\mu\}$ to which a player in S_1 would deviate from μ, irrespective of K. It holds for all players $i \in S_1$ that for all $\mu' \in f_K^{\mathbb{N}}(\mu)$ that $\mu'(i) = \mu(i)$. Therefore, all players in S_2 cannot improve in μ, irrespective of K. Hence, it holds for all $i \in S_1 \bigcup S_2$ and for all $K > 0$ that for all $\mu' \in f_K^{\mathbb{N}}(\mu)$ that $\mu'(i) = \mu(i)$.
Now consider players in S_k, for each $k \in \{3, ..., \ell\}$. All players in S_k could only improve by deviating such that there exists an induced path that matches them with someone in $S_1 \bigcup ... \bigcup S_{k-1}$. However, there exists no single matching in $\mathbb{M} \setminus \{\mu\}$ to which a pair of players in $S_1 \bigcup ... \bigcup S_{k-1}$ would deviate by forming a link from μ or a single player by only dissolving a link, irrespective of K. It holds for all players $i \in S_1 \bigcup ... \bigcup S_{k-1}$ that for all $\mu' \in f_K^{\mathbb{N}}(\mu)$ that $\mu'(i) = \mu(i)$. Therefore, all players in $S_1 \bigcup ... \bigcup S_{k-1}$ cannot improve in μ, irrespective of K. Hence, it holds for all $i \in S_1 \bigcup ... \bigcup S_k$ and for all $K > 0$ that for all $\mu' \in f_K^{\mathbb{N}}(\mu)$ that $\mu'(i) = \mu(i)$.
Since $N = S_1 \bigcup ... \bigcup S_\ell$ there exist no level-K deviations for any player from stable matching μ for $K > 0$. Hence, μ must be in the level-K stable set to let the level-K stable set satisfy iterated external stability. Also, because no level-K deviations exist from μ for any $K > 0$, $f_K(\{\mu\}) \subseteq \{\mu\}$ and so the set $\{\mu\}$ must satisfy deterrence of external deviations.

To show that the level-K stable set contains only the stable matching μ in the α-reducible marriage market $(M, W, \succ)$, I show that there exists a path from every matching $\mu' \neq \mu$ to μ, such that $\{\mu\}$ also satisfies iterated external stability.
By Theorem 3.2, I know that μ is a singleton cycle in the level-K stable set because no level-K deviations exist in μ. Furthermore, it is known that μ' is unstable as μ is the only stable matching in the marriage market. Therefore, there exists at least one blocking pair in μ'. In fact, there exists at least one S_k in $S_1, ..., S_\ell$ in which the players in S_k are not matched in μ' if $|S_k| = 2$, or in which the player is not single in μ' if $|S_k| = 1$.
Let S_k be the first in $S_1, ..., S_\ell$ for which this holds. For all players in S_k there always exists at least one player in $S_1 \bigcup ... \bigcup S_k$ that they prefer more than their partner in μ'.

However, since all players in $S_1 \bigcup ... \bigcup S_{k-1}$ have no level-K deviations in each matching in which the implied link of each S_n is formed for $1 \leq n \leq k-1$, all players in S_k cannot deviate such that they end up with someone in $S_1 \bigcup ... \bigcup S_{k-1}$. Nonetheless, all players in S_k could improve by deviating such that they end up with themselves (if $|S_k| = 1$) or with the other player in S_k (if $|S_k| = 2$). Hence, forming the implied link in S_k is a level-K deviation for all players in S_k from μ'. Once the players in S_k have matched the opposite player in S_k (or himself if $|S_k| = 1$), no more level-K deviations exist by these players, for any $K > 0$. Namely, players in $S_1 \bigcup ... \bigcup S_{k-1}$ do not match someone in S_k and the player(s) in S_k do not prefer being matched with a player in $S_{k+1} \bigcup ... \bigcup S_\ell$. Therefore, forming the implied link from S_k is a level-K deviation for each $i \in S_k$, for any $K > 0$. If now some $i \in S_k$ matches some $j \notin S_k$, then from $\mu' + (i, j)$, there still must be a level-K deviation that matches all players in S_k with each other. Hence, a path exists such that all players in S_k are matched from $\mu' + (i, j)$, with $i \in S_k$ and $j \notin S_k$.
Once the implied link in S_k has been formed, the same process can be repeated for the next S in $S_{k+1}, ..., S_\ell$ for which the implied link in S is not formed. This process can be continued for any $K > 0$ until the stable matching μ is reached through consecutive level-K deviations. Once μ has been reached, no more level-K deviations exist. Now there exists a path from each $\mu' \neq \mu$ in $\mathbb{M}$ to μ by a sequence of consecutive level-K deviations, while from μ no deviations exist. Consequently, the set $\{\mu\}$ satisfies iterated external stability. In the first part of this proof, I showed the deterrence of external deviations of the set $\{\mu\}$. Hence, knowing that the level-K stable set must exist, by minimality, μ is the only matching in the level-K stable set. □

4 Stochastic behaviour of the marriage market

4.1 Motivation

In the setting so far defined, there exists a problem with the credibility of some deviations by some players. I would like to illustrate this by giving the next example:

Example 4.1. The marriage market set-up is the same as in previous examples and is shown here again.

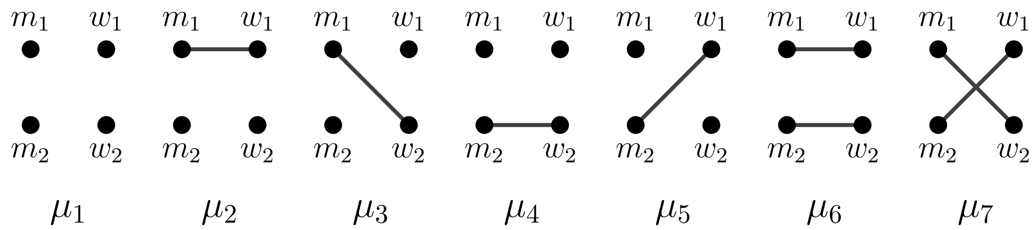
$\succ_{m_1}$: w_1, w_2, m_1

$\succ_{m_2}$: w_2, w_1, m_2

$\succ_{w_1}$: m_2, m_1, w_1

$\succ_{w_2}$: m_1, m_2, w_2

All possible matchings in this setting are shown again:



Let me consider matching μ_3 out of the set of all matchings. In this matching, player m_1 could perform a level-2 deviation to matching μ_1 because from μ_1 there exists a level-1 deviation to matching μ_2 and it holds: $\mu_2(m_1) \succ_{m_1} \mu_3(m_1)$. According to the current set-up, this would be a valid level-2 deviation for player m_1. However, this seems not to be resembling reality a lot. Namely, it is not assured that the system transits from μ_1 to μ_2. In fact, there exist four level-1 deviations in μ_1 of which only the one to μ_2 is an improvement for m_1. Considering the symmetry that is present in this defined marriage market problem, it would be intuitive to assume that each matching in $f_1(\mu_1) = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ has a 0.25 probability to be reached. This means that m_1 has a chance of only 25 % to improve and a chance of 50 % to end up with a less preferred partner by his deviation. Considering this stochastic behaviour of the game that would be plausible here, the deviation from μ_2 to μ_1 would not be very credible for m_1 at first sight.
However, it has not been established by how much m_1 prefers w_1 more than w_2. If w_1 is preferred considerably more than w_2, while w_2 is only a bit more preferred than being single, the deviation to μ_1 is realistic. Nevertheless, if w_1 is preferred only slightly more than w_2, while w_2 is preferred considerably more than being single, the deviation to μ_1 is unrealistic. ▲

In the approach that I follow to tackle the problem defined in Example 4.1, I propose to define a Markov chain that describes the probabilities of the system evolving from one matching to the other in one step. The probabilities in this chain depend on the level of foresight K and on the preferences of each of the players. Also, the probabilities depend on the utilities that each player gives to each partner. Defining utilities in the marriage market problem with stochastic deviations is necessary to make sure that a planned deviation from a player always implies a positive expected change in utility. First, next to some minor first notation, I give two more examples under level-1 and level-2 foresight to illustrate the new set-up and the definitions that follow in Section 4.2.

Example 4.2. Let me now consider the same set-up as in Example 4.1. I describe how the system might evolve under level-1 foresight. Under level-1 foresight, no utilities over players need to be defined because each player only deviates if the consequence of his deviation makes the player end up with a more preferred partner. From matching μ_1, each player has a level-1 deviation to two matchings in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$, while each matching in A can be reached by a level-1 deviation of two players. When saying that each player has an equal probability to be given the option to deviate then for each $\mu \in A$: $P_1(\mu|\mu_1) = 0.25$. Deviating from μ_1 to any matching in $B = \{\mu_6, \mu_7\}$ is impossible because these matchings are not a neighbour of μ_1, and therefore: $P_1(\mu_6|\mu_1) = P_1(\mu_7|\mu_1) = 0$.
From μ_3, two level-1 deviations exist, to μ_2 by m_1 and w_1 and to μ_7 by m_2 and w_1. Now, it is crucial to define which deviation would be most credible. Clearly, both deviations involve player w_1. Previously, both deviations were defined to be credible by player w_1. Nevertheless, matching with m_2 gives the highest utility to w_1. Although this deviation would be more credible, I still define $P_1(\mu_2|\mu_3) = P_1(\mu_7|\mu_3) = 0.5$. Later, this concept could be further ennobled to let players only deviate to a matching giving them the highest utility. Now, by symmetry, $P_1(\mu_5|\mu_2) = P_1(\mu_6|\mu_2) = P_1(\mu_3|\mu_4) = P_1(\mu_6|\mu_4) = P_1(\mu_4|\mu_5) = P_1(\mu_7|\mu_5) = 0.5$.
Considering all described deviations in the marriage market problem, the following matrix P_1 can be constructed. In this matrix, the element (i,j) represents the probability that the system evolves from matching μ_i to μ_j by a level-1 deviation. Matrix P_1 is as follows:

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0 & 0 & 0 & 0 & 0.50 \\ 0 & 0 & 0.50 & 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

▲

Example 4.3. In this example, I consider the same marriage market problem as in the previous example. However, now I consider level-2 deviations in the context of the system evolving in a stochastic manner. In this example, I define utilities that each player has over the opposite set of players and being single to be decreasing in equal steps from the most preferred partner to the least preferred. This means that if player i has preference ordering $\succ_i$: j, k, ℓ, with $i \in \{j, k, \ell\}$, then j is equally preferred over k as k is over ℓ. When utilities were to be defined in this example, then if i attaches utility a to j and $a - b$ to k, it follows that $a - 2b$ utility is attached to ℓ, while $b > 0$.
Let me consider matching μ_1 and player m_1. This player can match with w_1, resulting in μ_2 or with w_2, resulting in μ_3. Player m_1 has level-2 foresight and foresees that deviating

to μ_2 results in matching μ_5 or μ_6 through a level-1 deviation with equal probability. Hence, with probability 0.5, m_1 ends up with a more preferred partner than in μ_1, while the probability of having the same utility is 0.5 and thus, there is an expected increase in utility with this deviation. Therefore, deviating to μ_2 is a level-2 deviation for player m_1. Deviating to μ_3 ultimately leads to a level-1 deviation to μ_2 or μ_7, resulting in an increase in utility with probability 1 for m_1, and so this is also a level-2 deviation for him. For w_1, deviating to μ_2 is a level-2 deviation, because from μ_2, under level-1 foresight, the system evolves to μ_5 or to μ_6 with equal probability. Player w_1 is better off than in μ_1 in both matchings and hence, deviating to μ_2 from μ_1 is a level-2 deviation from for w_1. For w_2, deviating to μ_3 is a level-2 deviation, because from μ_3, under level-1 foresight, the system evolves to μ_2 or to μ_7 with equal probability. Player w_2 is better off than in μ_7 than μ_1 and has the same partner in μ_2 as in μ_1. Hence, this deviation results in a positive expected increase in utility and so deviating to μ_3 from μ_1 is a level-2 deviation for w_2. Deviating to μ_2 would be the most preferred option by m_1 compared to deviating to μ_3. However, for now, I assume that both deviations have equal probabilities.
Previous reasoning implies that, if m_1 were to be picked as the player to be starting the deviation, deviations of the system from μ_1 to μ_2 or μ_3 would be credible, and are assumed to be equally credible here. Similar reasoning could be done for the other players and therefore, by the symmetry here, each matching in $A = \{\mu_2, \mu_3, \mu_4, \mu_5\}$ has an equal probability to be reached by a deviation from μ_1. The key underlying assumption here is that each player has an equal chance to initiate the deviation. Therefore, $P_2(\mu|\mu_1) = 0.25$, for all $\mu \in A$.
Now I consider matching μ_2. Player m_1 cannot deviate because he has he most preferred option. Player w_1 can deviate. In the setting defined in previous sections, deviating to μ_1 would be a fruitful level-2 deviation for w_1 since from μ_1 a deviation exists to μ_5. However, in this newly defined setting, w_1 knows that from μ_1, the system evolves to some matching in A and each matching in A has a 25 % chance to be reached from μ_1 by a level-1 deviation. Only in μ_5 w_1 is better off and therefore, under the current assumption of the utilities, deviating to μ_1 does not have an expected increase in utility for player w_1. Now w_1 can also deviate to μ_5 which would need approval by m_2. From μ_5 the system goes to μ_4 or μ_7, with both a 50 % probability. For m_2, this would be a positive expected increase in utility. For w_1 however, the probability of a more preferred partner is 50 % and is equaling the probability of a less preferred partner. Therefore, the expected increase in utility is zero under the current assumption of utilities and this is not a fruitful level-2 deviation for w_1. In case the utilities had been defined differently, however, this could have been a fruitful deviation for w_1. Now only players m_2 and w_2 can deviate in μ_2. The only thing they can do is form a new link between them, resulting in μ_6. From μ_6 no more level-1 deviations exist and both players are better off in μ_6. Therefore, the only level-2 deviation from μ_2 is to μ_6 and $P_2(\mu_6|\mu_2) = 1$. By symmetry, I can now say $P_2(\mu_7|\mu_3) = P_2(\mu_6|\mu_4) = P_2(\mu_7|\mu_5) = 1$.
From μ_6 and μ_7 no level-2 deviations exist and therefore, I can construct the following matrix P_2.

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In this example, it has now also been shown that a level-2 deviation in the set-up as described in Section 2 does not necessarily need to imply a level-2 deviation in this section. For instance, in Section 2 $\mu_2 \rightarrow_{\{m_2,w_1\}} \mu_5$ would be a level-2 deviation. However, in this example, this is not a level-2 deviation. ▲

In the next Section 4.2, I introduce new definitions that are necessary to tackle the issues that have appeared in this section. Before starting that section, I briefly summarise the motivation for the newly introduced topics in that section.
Firstly, it has been shown that in the set-up of Section 2, players are very optimistic about their chances of improving on partners. Namely, the only requirement for a level-K deviation by some player i was that a path of induced deviations existed that led to a better partner, without any indication of the likelihood that this path would be realistic. Therefore, in the next section, I introduce probabilities of switching to matchings that players can account for when deciding on fruitful level-K deviations. This will result in a marriage market set-up in which the transition from and to matchings is stochastic and thus will become a Markov chain with each matching equivalent to a state in the chain.
Secondly, as a consequence of the loss of the opportunistic belief by the players that a deviation always leads to an improved partner, a deviation by a player may depend on the extent to which partners are preferred over others. For instance, a small probability of improving on a partner following a deviation might actually lead to a deviation when this partner is way more preferred than the current partner of the possible deviator. To indicate the extent to which partners are preferred over others, however, it is necessary to introduce utilities that are attached to possible partners.
Now, by describing this stochastic set-up in the next sections, I tackle the issues described in this section leading to more realistic deviations and thus eventually to more realistic outcomes of the marriage market problem.

4.2 Definitions

4.2.1 Level-1 and level-2 foresight

Now the new framework has been extensively illustrated through several examples, I give new definitions that describe the set-up of the marriage market. I do so by first intuitively describing the new set-up under level-1 and level-2 foresight.
In the previous examples, a key underlying assumption was that one player is given the opportunity to initiate a deviation. In this new framework, I assume that a player is randomly chosen over all players to initiate a deviation. Clearly, if a player is picked that, when in matching μ, has no fruitful deviation in μ, then a random player is chosen until there is some player that can deviate. This set-up leads to each player having a chance of being given the opportunity to deviate as 1 divided by all players that could deviate in μ.

Consequently, all players are aware of this 'random draw' when considering their options to deviate.
Another key underlying assumption in the previous examples was that each possible deviation by a player given the opportunity to deviate has an equal chance of being chosen. Hence, even though deviating from μ_x to μ_y by player i results in a matching with a higher expected utility for player i according to his reasoning process than deviating from μ_x to μ_z, it still holds that these deviations are given equal chance to. Later, I will restrict this and only allow players to deviate to their most preferred matching. However, at this stage, this is not done in order to stick close to the approach of Section 2.
Another assumption that was made use of in the examples defined was the assumption of utilities over the opposite players and over being single for each $i \in N$. As has been illustrated, these utilities can be essential to the end result of the matching. Therefore, I define the utility that player i gets in μ when $\mu(i) = j$ as $U^i(j)$. Hence, for each player $i \in N$ there exists a complete row vector of utilities over players of the opposite sex and himself. $((U^m)_{m\in M}, (U^w)_{w\in W})$ is a complete matrix of utilities that is denoted by U with each row corresponding to a player. This means that in this section, the marriage market problem consists of men, women, and a matrix with row vectors consisting of utilities: (M, W, U). Also, I define $V_{i\mu_y,1} = U^i(\mu_y(i))$, which is the expected utility player i gets from a level-1 deviation to μ_y. Later, I generalise this expected utility for general levels of foresight. In a level-1 setting, this just the utility i gets from the player he is matched to in μ_y.
Before formally defining the deviations in the new setting, I first describe the calculation of probabilities that make the system move from one matching to the other. I describe these probabilities by first illustrating the probabilities for $K = 1$. Clearly, when $K = 1$ and starting in $\mu_x \in \mathbb{M}$, in the marriage market several players might have the possibility to deviate and each of these players may have several possible deviations that do increase his utility when deviating from μ_x to μ_y. As mentioned, out of all possible deviators a player is randomly picked with equal possibility over all deviators. Thereafter, out of all his or her possible deviations, a deviation is randomly picked, with all possible deviations having an equal probability to be picked. If a deviation involves the addition of a link, then the addition should be a deviation by the opposite player as well. I put these probabilities in a matrix P_1 in which the element of the x'th row and y'th column (x, y) is the probability $P_1(\mu_y|\mu_x)$, and where 1 is the level of foresight. Hence, say in μ_x, $L_{1,\mu_x} \subseteq N$ is the set of players that can perform a level-1 deviation and let $I_{i\mu_x\mu_y,1}$ be the indicator function that is 1 if i has a level-1 deviation to μ_y from μ_x that is also a level-K deviation by the opposite player if it involves the addition of a link and 0 otherwise. Furthermore, let $D_{i\mu_x,1}$ be the total number of level-1 deviations by i from μ_x for which it holds that this is also a deviation by the opposite player if it involves the addition of a link. Now, given $|D_{i\mu_x,1}| > 0$, the probability that the system evolves to μ_y by a level-1 deviation of player i equals $\frac{1}{|L_{1,\mu_x}|} * \frac{I_{i\mu_x\mu_y,1}}{D_{i\mu_x,1}}$. If I now sum over i, I get the probability that the system moves from μ_x to μ_y, so: $P_1(\mu_y|\mu_x) = \frac{1}{|L_{1,\mu_x}|} \sum_{i\in L_{1,\mu_x}} \frac{I_{i\mu_x\mu_y,1}}{D_{i\mu_x,1}}$. If $|L_{1,\mu_x}| = 0$, no level-1 deviations exist and for all $\mu_y \neq \mu_x$, $P_1(\mu_y|\mu_x) = 0$ and $P_1(\mu_x|\mu_x) = 1$.
Now, I consider the probability that the system moves from $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ through a level-2 deviation. With level-2 foresight, players can foresee the first deviation following their level-2 deviation. In this setting of stochastic deviations, it means that they are

aware of the probabilities of the system moving from μ_y to each other matching and use these probabilities to determine their own deviations. Next, I show how the probabilities and deviations are calculated for $K = 2$ and $K = 3$.
I consider a setting in which three matchings μ_1, μ_2 and μ_3 exist. Matrix $P_1 \in \mathbb{R}^3 \times \mathbb{R}^3$ can be taken as given with matchings μ_1, μ_2 and μ_3. The probabilities in that matrix are calculated as aforementioned. I consider a possible level-2 deviation by player i that has utilities $U^i(\mu_1(i))$, $U^i(\mu_2(i))$, $U^i(\mu_3(i))$. Matrix P_1 is defined as follows:

$$\begin{bmatrix} P_1(\mu_1|\mu_1) & P_1(\mu_2|\mu_1) & P_1(\mu_3|\mu_1) \\ P_1(\mu_1|\mu_2) & P_1(\mu_2|\mu_2) & P_1(\mu_3|\mu_2) \\ P_1(\mu_1|\mu_3) & P_1(\mu_2|\mu_3) & P_1(\mu_3|\mu_3) \end{bmatrix}$$

In this matrix, $P_1(\mu_y|\mu_x)$ is the probability that the system moves from matching μ_x to matching μ_y by a level-1 deviation. Now consider a possible level-2 deviation from $\mu_x \in \mathbb{M}$ by player i in the marriage market problem. The utility that player i gets by a deviation to μ_y, say $V_{i\mu_y,2}$, equals $P_1(\mu_1|\mu_y) * U^i(\mu_1(i)) + P_1(\mu_2|\mu_y) * U^i(\mu_2(i)) + P_1(\mu_3|\mu_y) * U^i(\mu_3(i))$. Knowing $V_{i\mu_y,1} = U^i(\mu_y(i))$, I can also write: $V_{i\mu_y,2} = P_1(\mu_1|\mu_y) * V_{i\mu_1,1} + P_1(\mu_2|\mu_y) * V_{i\mu_2,1} + P_1(\mu_3|\mu_y) * V_{i\mu_3,1}$. Now the deviation to $\mu_y \in \mathbb{M} \setminus \{\mu_x\}$ is a fruitful level-2 deviation by player i if $V_{i\mu_y,2} = P_1(\mu_1|\mu_y) * U^i(\mu_1(i)) + P_1(\mu_2|\mu_y) * U^i(\mu_2(i)) + P_1(\mu_3|\mu_y) * U^i(\mu_3(i)) > U^i(\mu_x(i))$.
In this manner, for every player, it can be concluded whether a level-2 deviation exists from μ_x to μ_y. Now, the elements in matrix P_2 can be calculated in a similar manner to the elements of P_1. The element $P_2(\mu_y|\mu_x)$ gives the probability that the system moves from μ_x to μ_y by a level-2 deviation. Hence, say in μ_x, $|L_{2,\mu_x}| \leq N$ players can perform a level-2 deviation that is also a level-2 deviation by the opposite player if applicable and let $I_{i\mu_x\mu_y,2}$ be the indicator function that is 1 if i has a level-2 deviation to μ_y from μ_x that is also a level-2 deviation by the opposite player if applicable and 0 otherwise, which is equivalent with $V_{i\mu_y,2} > U^i(\mu_x(i))$. If the deviation now involves the addition of a link with say player $j \neq i$, then, for j, it must also be that $V_{j\mu_y,2} > U^j(\mu_x(j))$. Furthermore, let $D_{i\mu_x,2}$ be the total number of level-2 deviations by i from μ_x for which it holds that this is also a deviation by the opposite player if it involves the addition of a link. Now, the probability that the system evolves to μ_y by a level-2 deviation of player i is equal to, given $D_{i\mu_x,2} > 0$, $\frac{1}{|L_{2,\mu_x}|} * \frac{I_{i\mu_x\mu_y,2}}{D_{i\mu_x,2}}$. If I now sum over i, I get the probability that the system moves from μ_x to μ_y, so given $|L_{2,\mu_x}| > 0$, the probability that the system moves from μ_x to μ_y under level-2 foresight is $P_2(\mu_y|\mu_x) = \frac{1}{|L_{2,\mu_x}|} \sum_{i \in L_{2,\mu_x}} \frac{I_{i\mu_x\mu_y,2}}{D_{i\mu_x,2}}$. In this manner, all elements of P_2 can be calculated. If now $|L_{2,\mu_x}| = 0$, then no one can improve by becoming single and there exist no additions of links that can improve the utility of the involved players. Hence, in that case $P_2(\mu_x|\mu_x) = 1$ and so the system stays in μ_x under level-2 foresight.
Now consider a possible level-3 deviation from μ_x. The deviation from μ_x to μ_y is a fruitful level-3 deviation by player i if $V_{i\mu_y,3} = P_2(\mu_1|\mu_y) * V_{i\mu_1,2} + P_2(\mu_2|\mu_y) * V_{i\mu_2,2} + P_2(\mu_3|\mu_y) * V_{i\mu_3,2} > U^i(\mu_x(i))$. In this sum, $V_{i\mu_\ell,2}$ is the expected utility that a player i gets when the system evolves to matching μ_ℓ by a level-2 deviation, with the key assumption of level-2 foresight in μ_y. Multiplying this utility by $P_2(\mu_\ell|\mu_y)$, the probability of a level-2 deviation from μ_y to μ_ℓ, gives a term that should be summed over to get the expected utility of the level-3 deviation to μ_y. Namely, in this sum over all $\mu_\ell \in \mathbb{M}$, each term is the utility that

i gets from each respective matching μ_ℓ times the probability that μ_ℓ is reached from μ_y by a level-2 deviation. Therefore, this sum gives the expected utility for i of the level-3 deviation from μ_x to μ_y.

4.2.2 Arbitrary level of foresight

In the previous subsection, some new notation has been introduced for specific settings of $K = 1$ and $K = 2$. Now, in order to present a complete overview of the new notation, I summarise the new notation for general $K > 0$. In the overview presented, I consider a deviation in the stochastic setting under level-K foresight from some matching $\mu_x \in \mathbb{M}$ to some other matching $\mu_y \neq \mu_x$ in $\mathbb{M}$:

- **(i)** $V_{i\mu_y,K}$ is the expected utility for player i after a level-K deviation to μ_y.
- **(ii)** $I_{i\mu_x\mu_y,K}$ is the indicator function that is 1 if i has a level-K deviation from μ_x to μ_y that is also a deviation by the opposite player if it involves the addition of a link and 0 otherwise.
- **(iii)** $L_{K,\mu_x} \subseteq N$ is the set of players with a level-K deviation in μ_x, while for all deviations that involve the addition it holds that this is also a deviation by the opposite player.
- **(iv)** $D_{i\mu_x,K}$ is the number of level-K deviations by i in μ_x for which it holds that this is also a deviation by the opposite player if it involves the addition of a link.
- **(v)** $P_K(\mu_y|\mu_x)$ is the probability that the system evolves from μ_x to μ_y through a deviation under the assumption of stochasticity and level-K foresight.

The definition for L_{K,μ_x} in point **(iii)** can be mathematically written as $L_{K,\mu_x} = \{i \in N | V_{i\mu_x-(i,\mu_x(i)),K} > U^i(\mu_x(i))\} \bigcup \{i \in N : \exists j \in N | V_{i\mu_x+(i,j),K} > U^i(\mu_x(i)) \wedge V_{j\mu_x+(i,j),K} > U^j(\mu_x(j))\}$. The formula for the calculation of $P_K(\mu_y|\mu_x)$ in **(v)** is, given $|L_{K,\mu_x}| > 0$, $P_K(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x}|}\sum_{i\in L_{K,\mu_x}} \frac{I_{i\mu_x\mu_y,K}}{D_{i\mu_x,K}}$. If $|L_{K,\mu_x}| = 0$, $P_K(\mu_y|\mu_x) = 0$, for all $\mu_y \neq \mu_x$ and $P_K(\mu_x|\mu_x) = 1$. In the next paragraph, I present the exact description of the calculation of $P_K(\mu_y|\mu_x)$. Thereafter, I show how the parameters in the formula for $P_K(\mu_y|\mu_x)$ are calculated in a recursive manner.

In a level-K setting in some $\mu_x \in \mathbb{M}$, out of all deviators, a deviator is randomly picked. Hence, the probability of a random deviator to be picked is $\frac{1}{|L_{K,\mu_x}|}$. Out of all his deviations, which are also deviations by the opposite player if it involves the addition of a link, a deviation is randomly picked. Hence, the probability that the system moves from μ_x to μ_y by a deviation of player i is, given $|D_{i\mu_x,K}| > 0$: $\frac{1}{|L_{K,\mu_x}|}\frac{I_{i\mu_x\mu_y,K}}{D_{i\mu_x,K}}$. If I now sum over i, I obtain the probability that the system evolves from μ_x to μ_y under level-K foresight. Hence, given $|L_{K,\mu_x}| > 0$, $P_K(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x}|}\sum_{i\in L_{K,\mu_x}} \frac{I_{i\mu_x\mu_y,K}}{D_{i\mu_x,K}}$. Now if $|L_{K,\mu_x}| = 0$, no player can gain by dissolving a link and no two players can agree on the addition of a link. Consequently, under level-K foresight, $P_K(\mu_x|\mu_x) = 1$. Next to considering probabilities of the system evolving from matching to matching, I also consider $P_K(M_y|M_x)$ as the probability that the system evolves through a deviation under the assumption of stochasticity and level-K foresight from a set of matchings M_x to a set of matchings M_y. Now the formal notation needed in the description of the stochastic set-up has been given, I intuitively describe the formal stochastic set-up in the next paragraphs. Also, the exact

calculations of the introduced parameters are given. In Section 4.2.4, I compactly denote all formal equations and definitions.
The probability of the evolution of the system that just has been given depends on the utilities that each player gives to each matching and consequent deviations. For $K = 1$, every player knows which deviations improve his utility. With that information, for each $\mu_x \in \mathbb{M}$, $\mu_y \in \mathbb{M}$ and $i \in N$, parameters $D_{i\mu_x,1}$, $I_{i\mu_x\mu_y,1}$ and L_{1,μ_x} can be calculated and the probabilities $P_1(\mu_y|\mu_x) = \frac{1}{|L_{1,\mu_x}|}\sum_{i\in L_{1,\mu_x}} \frac{I_{i\mu_x\mu_y,1}}{D_{i\mu_x,1}}$. For $K > 1$, deviations depend on induced deviations as well. In other words, the expected utility that a player gives to a deviation iteratively depends on the expected utility in induced deviations. Hence, because the expected utilities for player i following a level-K deviation iteratively depend on the expected utilities for i following possible level-$K-\ell$ deviations, for $1 \leq \ell \leq K-1$, I first write down the utility for i following a level-2 deviation.
I say that matching μ_y is the matching that follows the level-2 deviation and so only level-1 deviations are foreseen in μ_y in line with the foresight. Now, the utility player i gets from that deviation is an explicit function of the real utilities that he gets in each matching and the probabilities of the system evolving to each of these matchings through a level-1 deviation from μ_y. So, the expected utility that player $i \in N$ gets when the system evolves to $\mu_y \in \mathbb{M}$ with level-2 foresight is $V_{i\mu_y,2} = \sum_{\mu_\ell\in\mathbb{M}} P_1(\mu_\ell|\mu_y) * V_{i\mu_\ell,1} = \sum_{\mu_\ell\in\mathbb{M}} P_1(\mu_\ell|\mu_y) * U^i(\mu_\ell(i))$. Under level-2 foresight, the utility of a deviation to each possible matching can now be derived. Based on the utility following a deviation, each player knows which deviations improve his utility and so his deviations can be derived. Consequently, in μ_x, for each $\mu_y \in \mathbb{M}$, $I_{i\mu_x\mu_y,2}$, $D_{i\mu_x,2}$ and L_{2,μ_x} can be derived. Knowing $P_2(\mu_y|\mu_x) = \frac{1}{|L_{2,\mu_x}|}\sum_{i\in L_{2,\mu_x}} \frac{I_{i\mu_x\mu_y,2}}{D_{i\mu_x,2}}$, this probability can be calculated for each $\mu_y \in \mathbb{M}$.
Now, say we are in matching μ_x again, under level-3 foresight. For all $i \in N$ and for all $\mu_y \in \mathbb{M}$, it has been shown how to calculate $P_2(\mu_y|\mu_x)$ and $V_{i\mu_y,2}$. This is what is needed to calculate the expected utility of a level-3 deviation to some matching μ_y: $V_{i\mu_y,3} = \sum_{\mu_\ell\in\mathbb{M}} P_2(\mu_\ell|\mu_y) * V_{i\mu_\ell,2}$. By filling in each $V_{i\mu_\ell,2}$, the exact utility of such a deviation can be obtained. Knowing all these utilities, $I_{i\mu_x\mu_y,3}$, $D_{i\mu_x,3}$ and L_{3,μ_x} can be derived to calculate $P_3(\mu_y|\mu_x)$ for each $\mu_y \in \mathbb{M}$.
This pattern can be continued for any $K > 0$. Therefore, for any $K > 0$, the expected utility that player $i \in N$ gets when the system evolves to $\mu_y \in \mathbb{M}$ with level-K foresight and $K > 1$ is $V_{i\mu_y,K} = \sum_{\mu_\ell\in\mathbb{M}} P_{K-1}(\mu_\ell|\mu_y) * V_{i\mu_\ell,K-1}$.

Each player in the marriage market knows the expected utility of a possible deviation, while each player is willing to deviate from μ_x to μ_y when the expected utility in μ_y is higher than the utility the player gets in μ_x. Therefore, when assuming level-1 foresight, a player i deviates to μ_y from μ_x when $U^i(\mu_y(i)) > U^i(\mu_x(i))$. Hence, the deviation $\mu_x \to_S \mu_y$ is a *stochastic level-1 deviation for player* $i \in S$, if $U^i(\mu_y(i)) > U^i(\mu_x(i))$. Because a level-1 deviation does not consider any further deviations after the deviation, stochastic level-1 deviations are equivalent to level-1 deviations defined in Section 2. Knowing for each $\mu_x \in \mathbb{M}$ which deviations exist for each $i \in N$ under level-1 foresight, the probabilities on the evolution of the system can be derived.
Now for $K = 2$, deviations depend on induced deviations under level-1 foresight. Hence, knowing how to calculate $V_{i\mu_y,2}$, the deviation $\mu_x \to \mu_y$ is a *stochastic level-2 deviation for player* $i \in S$, if $V_{i\mu_y,2} > U^i(\mu_x(i))$. Consequently, the probabilities of the evolution

of the system can be derived under level-2 foresight. This reasoning can be iterated for larger levels of foresight by knowing the probabilities of the evolution of the system under lower levels of foresight and the expected utilities under lower levels of foresight. Hence, for general $K > 1$, the deviation $\mu_x \to_S \mu_y$ is a *stochastic level-K deviation for player* $i \in S$ if $V_{i\mu_y,K} > U^i(\mu_x(i))$.

In this paragraph, I briefly summarise the calculation of the matrix P_K, for some $K > 0$. When a player considers deviating under level-K foresight, he foresees all following $K-1$ deviations. In the stochastic setting, he calculates the probabilities of possible deviations corresponding to the level-K foresight. These probabilities are iteratively calculated as has been explained. This iterative calculation starts with the calculation of the matrix P_1 that belongs to the last induced deviation. P_1 depends on the utilities that each player gets of being matched with each possible partner in the marriage market. Out of all players with a level-1 deviation in some matching, a player is randomly picked and out of all his deviations, a deviation is randomly picked. If the deviation involves the addition of a link, then both players must agree on the addition. In this manner, for each matching, the probability of going to each other matching can be determined and P_1 can be calculated. Knowing P_1, the utility of a deviation to each other matching can be calculated for each player under level-2 foresight. Now, for each player, the deviations can be derived under level-2 foresight in each matching in $\mathbb{M}$. By the rule that, out of all deviators, a player is randomly picked and, out of all his deviations, a deviation is randomly picked, the subsequent matrix P_2 can be derived. Here again, if a deviation involves the addition of a link, both players should agree on that addition. Consequently, the utilities of a level-3 deviation can be derived, and with them the matrix P_3. This pattern can be continued until the matrix P_K can be calculated, for some $K > 0$.
When defining stochastic level-K deviations, it is no longer necessary to take the level-K deviation with incomplete support into account as is done in Definition 2.4. Namely, when deviating, player $i \in N$ takes in his reasoning process into account that the induced level-K' deviation, with $2 \leq K' < K$, may result in a matching from which no level-$K'-1$ deviations exist. If that happens, then the utility in that matching is part of i's calculation and is considered together with the probability of getting to and from that matching. Throughout the rest of the paper, when there exists the need to distinguish between the two types of deviations, I refer to a level-K deviation as defined in Section 2 as an optimistic level-K deviation. I use the word optimistic because, in that section, each player has the 'optimistic' belief that the path, following the induced matchings, he foresees is reached.

4.2.3 Stability and cycles

Having intuitively explained the deviations and probabilities in the stochastic context, I consider the k-fold iterations in the stochastic context which is necessary to define stable sets and cycles in the stochastic context. The concept in the stochastic setting is very similar to the setting given in Section 2.3. However, now the matrix P_K has been defined that contains the probabilities of going from matching μ_x to μ_y by a stochastic level-K deviation, for all $\mu_x, \mu_y \in \mathbb{M}$. If now two sequential stochastic level-K deviations happen, starting in μ_x, then it holds that the probability that the system is in μ_y equals element (x, y) in matrix P_K^2. In general, the probability that the system is in μ_y after k sequential stochastic level-K deviations when starting in μ_x is the element (x, y) in matrix P_K^k, which is following from the properties of a Markov chain. In Section 2, μ_y is in the k-fold iteration of μ_x, when μ_y could have been reached by k sequential level-K deviations. In this setting, this is equivalent with $P_K^k(\mu_y|\mu_x) > 0$.

Now that level-K deviations and k-fold iterations have been illustrated in the context of the stochastic evolution of the marriage market, I can define stability. In Section 2.4 the level-K stable set must satisfy deterrence of external deviations and iterated external stability. Also, there must not be a subset of the level-K stable set that does also satisfy these criteria. Furthermore, by Theorem 3.2 it is known that this set consists of the union of cycles satisfying deterrence of external deviations.

Deterrence of external deviations for some set of matchings means in the new context that the probability that the system evolves to a matching outside the set is zero. Hence, say that the set of matchings $M_K \subseteq \mathbb{M}$ satisfies stochastic deterrence of external deviations. It must hold for each matching in M_K that the probability that the system evolves to another matching in M_K is 1. Hence for all $\mu_x \in M_K$, $\sum_{\mu_y \in M_K} P_K(\mu_y|\mu_x) = 1$ and $\sum_{\mu_y \notin M_K} P_K(\mu_y|\mu_x) = 0$.

In Section 2, a set of matchings $M_K \subseteq \mathbb{M}$ satisfies iterated external stability if from any matching in $\mathbb{M}$, it is possible to reach M_K through sequential level-K deviations. In the new setting, a set M_K satisfies stochastic iterated external stability if, for any matching in $\mathbb{M}$, there exists a $k \in \mathbb{N}$ such that there exists a strictly positive probability that the system evolves to some matching in M_K through k sequential stochastic level-K deviations. Namely, in that case, there is a path to M_K and this also implies that, if k goes to infinity, in a marriage market with a finite number of players, the probability that the system has been in some matching in M_K converges to 1. As before, the collection $M_K \subseteq \mathbb{M}$ is a *level-K stochastically stable set* if it satisfies stochastic deterrence of external deviations and stochastic iterated external stability and there is no strict subset of M_K that satisfies both criteria.

The rationale behind the level-K stochastically stable set in this context is that, ultimately, starting in any matching in $\mathbb{M}$ the system must evolve to some matching in M_K through sequential stochastic level-K deviations. For this reason, in a marriage market with a finite number of players, the probability that M_K has been reached converges to 1 as $k \to \infty$. Furthermore, it must be made sure that it is impossible to reach any matching from M_K in $\mathbb{M} \setminus M_K$ by the deterrence of external deviations. Therefore, for all $\mu_x \in M_K$: $\sum_{\mu_y \in M_K} P_K(\mu_y|\mu_x) = 1$ and $\sum_{\mu_y \notin M_K} P_K(\mu_y|\mu_x) = 0$. Similar to distinguishing optimistic level-K and stochastic level-K deviations, I use the term optimistic deterrence of external deviations and optimistic iterated external stability to refer to these specific concepts as defined in Section 2. Also, I call the level-K stable set of that section the level-K optimistically stable set.

Last, I also define the concept of a stochastic level-K cycle that is used in deriving results in the framework of stochastic deviations. Hence, let $K \in \mathbb{N}$. The non-empty set $M \subseteq \mathbb{M}$ is a *level-K cycle* if it is a minimal set satisfying stochastic deterrence of external deviations. If a stochastic level-K cycle M is a singleton, then the only element in set M is a level-K stochastically stable matching. Again, when distinguishing between the stochastic level-K cycle and the level-K cycle of Section 2, I refer to the latter one as the optimistic level-K cycle.

For each matching μ in a cycle, it holds that the probability of getting to μ again when starting in μ approaches 1 as k goes to infinity, in a marriage market with a finite number of players. Since I only consider marriage markets with a finite number of players, it must hold that each matching in a cycle is a positive recurrent state in the Markov chain that is considered here.

4.2.4 Overview

In previous sections, I intuitively defined all necessary notation and concepts in the context of stochasticity. In this last part of the subsection, I summarise the new set-up and compactly write all the definitions and equations in a formal way. Notation that is used in this section has been summarised at the beginning of Section 4.2.2. I briefly summarise this notation again in the next paragraph.

$V_{i\mu_y,K}$ is the expected utility for player i after a level-K deviation to μ_y. $I_{i\mu_x\mu_y,K}$ is the indicator function that is 1 if i has a level-K deviation from μ_x to μ_y that is also a deviation by the opposite player if it involves the addition of a link and 0 otherwise. $L_{K,\mu_x} \subseteq N$ is the set of players with a level-K deviation in μ_x, with for all deviations that involve the addition of a link it holds that this is also a deviation by the opposite player. $D_{i\mu_x,K}$ is the number of level-K deviations by i in μ_x for which it holds that this is also a deviation by the opposite player if it involves the addition of a link. Last, $P_K(\mu_y|\mu_x)$ is the probability that the system evolves through a deviation under the assumption of stochasticity and level-K foresight from μ_x to μ_y.

As described, under stochasticity, deviations under higher levels of foresight depend on deviations belonging to the system evolution under lower levels of foresight. Therefore, I start by formally defining deviations under level-1 foresight.

Definition 4.4. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-*1 *deviation for player* $i \in S$, if $U^i(\mu_y(i)) > U^i(\mu_x(i))$.

Definition 4.5. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-*1 *deviation* if, for every player $i \in S$, it is a *stochastic level-*1 *deviation.*

Now, knowing what stochastic level-1 deviations exist by each player in matching $\mu_x \in \mathbb{M}$, the parameters $I_{i\mu_x\mu_y,1}$, L_{1,μ_x} and $D_{i\mu_x,1}$ can be calculated for every $i \in N$. $I_{i\mu_x\mu_y,1}$ can be calculated for each $\mu_y \in \mathbb{M}$ for which $\mu_y \neq \mu_x$. These parameters are necessary to calculate the probability $P_1(\mu_y|\mu_x)$ that the system moves from μ_x to every $\mu_y \in N$ under level-1 foresight. I present this probability in Equation 4.6.

Equation 4.6. The probability that the system goes from matching $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ under level-1 foresight, when $|L_{1,\mu_x}| > 0$, is $P_1(\mu_y|\mu_x) = \frac{1}{|L_{1,\mu_x}|} \sum_{i \in L_{1,\mu_x}} \frac{I_{i\mu_x\mu_y,1}}{D_{i\mu_x,1}}$.

If now $|L_{1,\mu_x}| = 0$, $P_1(\mu_x|\mu_x) = 1$ and $P_1(\mu_y|\mu_x) = 0$, for all $\mu_y \neq \mu_x$.

Having formally written down the probability under level-1 foresight, I define the concept of expected utility. Under higher levels of foresight, players base deviations on expected utility that depends on probabilities belonging to induced deviations under lower levels of foresight. Because of the recursive structure, I first write down the expected utility under level-2 foresight in Equation 4.7. All necessary parameters in the equation have been defined previously in this section.

Equation 4.7. The expected utility that player $i \in N$ gets when the system evolves to $\mu_y \in \mathbb{M}$ under level-2 foresight is $V_{i\mu_y,2} = \sum_{\mu_\ell \in \mathbb{M}} P_1(\mu_\ell|\mu_y) * V_{i\mu_\ell,1} = \sum_{\mu_\ell \in \mathbb{M}} P_1(\mu_\ell|\mu_y) * U^i(\mu_\ell(i))$.

Now, knowing the expected utility under level-2 foresight, stochastic level-2 deviations can be derived.

Definition 4.8. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-2 deviation for player* $i \in S$, if $V_{i\mu_y,2} > U^i(\mu_x(i))$.

Definition 4.9. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-2 deviation* if, for every player $i \in S$, it is a *stochastic level-2 deviation.*

Knowing what stochastic level-2 deviations exist by each player in matching $\mu_x \in \mathbb{M}$, the parameters $I_{i\mu_x\mu_y,2}$, L_{2,μ_x} and $D_{i\mu_x,2}$ can be calculated for every $i \in N$. $I_{i\mu_x\mu_y,2}$ can be calculated for each $\mu_y \in \mathbb{M}$ for which $\mu_y \neq \mu_x$. These parameters are necessary to calculate the probability $P_2(\mu_y|\mu_x)$ that the system moves from μ_x to every $\mu_y \in N$ under level-2 foresight. I present this probability in Equation 4.10.

Equation 4.10. The probability that the system goes from matching $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ under level-2 foresight, when $|L_{2,\mu_x}| > 0$, is $P_2(\mu_y|\mu_x) = \frac{1}{|L_{2,\mu_x}|} \sum_{i \in L_{2,\mu_x}} \frac{I_{i\mu_x\mu_y,2}}{D_{i\mu_x,2}}$.

If now $|L_{2,\mu_x}| = 0$, $P_2(\mu_x|\mu_x) = 1$ and $P_2(\mu_y|\mu_x) = 0$, for all $\mu_y \neq \mu_x$.
In previous equations and definitions, I have described how the evolution of the marriage market can be calculated under levels 1 and 2 foresight. If the evolution of the marriage market needs to be described for some higher levels of foresight, then this depends on the probabilities under lower levels of foresight that ultimately depend on the utilities each player has over the opposite partners. The calculations under level-1 and level-2 foresight have been given in the first part of this section. The calculations for higher levels of foresight are done in the same manner as for level-2 foresight while assuming that necessary parameters under lower levels of foresight are known. Therefore, when giving the equations and definitions for general levels of K in this last part, I assume that the necessary parameters belonging to lower levels of foresight are known.

Definition 4.11. Let $K \geq 2$. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-K deviation for player $i \in S$* if $V_{i\mu_y,K} > U^i(\mu_x(i))$.

Definition 4.12. Let $K \geq 2$. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-K deviation* if, for every player in S, it is a level-K deviation.

Knowing which deviations exist under level-$K-1$ foresight, players in the marriage market can determine their utility of a deviation under level-K foresight. I give the formula for expected utility in Equation 4.13.

Equation 4.13. The expected utility that player $i \in N$ gets when the system evolves to $\mu_y \in \mathbb{M}$ under level-K foresight is $V_{i\mu_y,K} = \sum_{\mu_\ell \in \mathbb{M}} P_{K-1}(\mu_\ell|\mu_y) * V_{i\mu_\ell,K-1}$.

Now, knowing what stochastic level-K deviations exist by each player in matching $\mu_x \in \mathbb{M}$, the parameters $I_{i\mu_x\mu_y,K}$, L_{K,μ_x} and $D_{i\mu_x,K}$ can be calculated for every $i \in N$. $I_{i\mu_x\mu_y,K}$ can be calculated for each $\mu_y \in \mathbb{M}$ for which $\mu_y \neq \mu_x$. These parameters are necessary to calculate the probability $P_K(\mu_y|\mu_x)$ that the system moves from μ_x to every $\mu_y \in \mathbb{M}$ under level-K foresight. I present this probability in Equation 4.14.

Equation 4.14. The probability that the system goes from matching $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ under level-K foresight, when $|L_{K,\mu_x}| > 0$, is $P_K(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x}|} \sum_{i \in L_{K,\mu_x}} \frac{I_{i\mu_x\mu_y,K}}{D_{i\mu_x,K}}$.

If now $|L_{K,\mu_x}| = 0$, $P_K(\mu_x|\mu_x) = 1$ and $P_K(\mu_y|\mu_x) = 0$, for all $\mu_y \neq \mu_x$.
In Section 4.2.3, I intuitively explained the concept of k-fold iterations under the assumption of stochasticity, after having described deviations for any level of foresight in that section. Consequently, I described stable sets that need to meet stochastic iterated external stability, stochastic deterrence of external deviations, and the minimality condition. The formal definition of the level-K stochastically stable set is given in the next Definition 4.15.

Definition 4.15. Let $K \in \mathbb{N}$. The collection $M_K \subseteq \mathbb{M}$ is a *level-K stochastically stable set* if it satisfies the following three conditions:

1. **Stochastic deterrence of external deviations:** for all $\mu_x \in M_K$: $\sum_{\mu_y \in M_K} P_K(\mu_y|\mu_x) = 1$.
2. **Stochastic iterated external stability:** for all $\mu_x \notin M_K$, there exists a $k \in \mathbb{N}$ and $\mu_y \in M_K$, such that $P_K^k(\mu_y|\mu_x) > 0$.
3. **Minimality:** there is no proper subset $M \subseteq M_K$ satisfying conditions 1 and 2.

Last, I give the formal definition of a stochastic level-K cycle in Definition 4.16.

Definition 4.16. Let $K \in \mathbb{N}$. The non-empty set $M \subseteq \mathbb{M}$ is a *stochastic level-K cycle* if it is a minimal set satisfying stochastic deterrence of external deviations.

4.3 Example on stochastic set-up

Now the entire stochastic set-up has been defined, I give an example of the advantage of employing the stochastic set-up. I do so in the next Example 4.17. The set-up is compared to the optimistic set-up of Example 2.14 where I show that for $K = 3$ the level-K stable set equals $\mathbb{M}$ such that nothing can be concluded about the outcome of the marriage market problem. In the example next, it is shown that assuming stochasticity reduces the size of the level-K stable set significantly when comparing it to the optimistic set-up when $K = 3$.

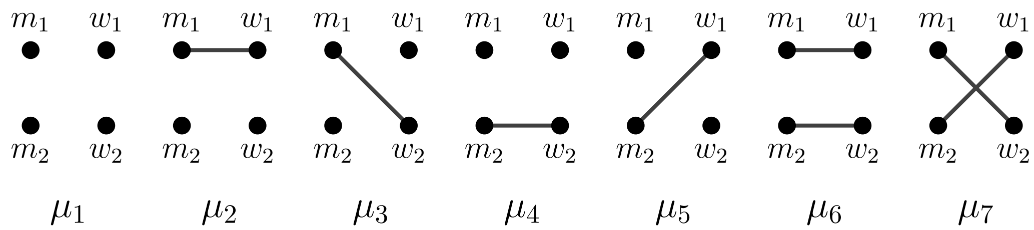
Example 4.17. In this example, I consider the same marriage market that has been used previously. I assume that the utilities are decreasing in equal steps over the preferences. In this example, I use these utilities for making calculations and I show these utilities and existing matchings:

$$m_1 : U^{m_1}(w_1) = 2;\ U^{m_1}(w_2) = 1;\ U^{m_1}(m_1) = 0$$

$$m_2 : U^{m_2}(w_2) = 2;\ U^{m_2}(w_1) = 1;\ U^{m_2}(m_2) = 0$$

$$w_1 : U^{w_1}(m_2) = 2;\ U^{w_1}(m_1) = 1;\ U^{w_1}(w_1) = 0$$

$$w_2 : U^{w_2}(m_1) = 2;\ U^{w_2}(m_2) = 1;\ U^{w_2}(w_2) = 0$$



For each player $i \in N$ and for each matching $\mu \in \mathbb{M}$, the expected utility of a level-1 deviation to μ is $V_{i,\mu 1}$, which is just the utility that i gets from the partner he is matched with in μ. Hence, I can write these in a matrix V_1 in which the first row represents m_1, the second row m_2, the third row w_1, and the fourth row w_2. Then, for $j = 1, 2, ..., 7$ the j'th column represents matching μ_j in $\mathbb{M}$. Hence, by observing which partner each player is matched with, in each matching, this matrix can be derived as follows:

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 & 1 & 2 \end{bmatrix}$$

The matrix P_1 that describes the probabilities of the evolution of the system has been derived in Example 4.2 and is shown below:

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0 & 0 & 0 & 0 & 0.50 \\ 0 & 0 & 0.50 & 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In Example 2.14, I did not derive $V_{i,\mu 2}$ for each i and each μ. Because these utilities are used in this exercise for the calculation of P_3 according to the method presented, I derive these utilities in the next paragraph.

From matchings μ_6 and μ_7 no level-1 deviations exist. Hence, a level-2 deviation to one of these matchings results in a utility that is equal to the utility obtained in each respective matching for each player. Now, from μ_1, under level-1 foresight, the system goes to some matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$, each with equal probability. For each player, it holds that it has a matching in this set with utility 2, a matching with utility 1, and 2 matchings with utility 0. Hence, for each i, the expected utility of a deviation to μ_1 is 0.75.

Now in μ_2, the system evolves to μ_5 with a probability of 0.5 and to μ_6 with a probability of 0.5. Hence, for m_1, a level-2 deviation to μ_2 results in an expected utility of $0.5*0+0.5*2 = 1$. For m_2, this results in an expected utility of $0.5*1+0.5*2 = 1.5$. For w_1, this results in an expected utility of $0.5*2+0.5*1 = 1.5$. For w_2, this results in an expected utility of $0.5*0+0.5*1 = 0.5$. Hence, the second column of the matrix representing the utilities under level-2 foresight is $[1, 1.5, 1.5, 0.5]'$. Now, by symmetry, I can construct the columns for μ_3, μ_4, μ_5, resulting in the following matrix V_2:

$$\begin{bmatrix} 0.75 & 1 & 1.5 & 1.5 & 0.5 & 2 & 1 \\ 0.75 & 1.5 & 0.5 & 1 & 1.5 & 2 & 1 \\ 0.75 & 1.5 & 1.5 & 0.5 & 1 & 1 & 2 \\ 0.75 & 0.5 & 1 & 1.5 & 1.5 & 1 & 2 \end{bmatrix}$$

The matrix P_2 that describes the probabilities of the evolution of the system has been derived in Example 4.3. Because this matrix was derived under the assumption of utilities decreasing in equal steps over the preferences, P_2 must be the same as in Example 4.3. I show P_2 again below:

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In Example 2.14, I showed that the level-1 and level-2 optimistically stable sets are: $\{\mu_6, \mu_7\}$, while the level-K stable set for $K = 3$ is $\mathbb{M}$. In the stochastic setting, it holds for both matrices P_1 and P_2 that the probabilities of being in states $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ goes to zero if n goes to infinity when considering P_1^n and P_2^n, irrespective of the starting matching. In matchings μ_6 and μ_7, the probability that the system leaves these matchings is zero. Therefore, both must be in the level-1 and level-2 stochastically stable set and so both sets equal $\{\mu_6, \mu_7\}$.

Now, I consider $K = 3$ in the stochastic setting, and I derive the utilities following a deviation for each player in each matching and P_3.

From matching μ_1, under level-2 foresight, the system moves to a matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$, each with equal probability. Looking at the utilities in the matrix following a level-2 deviation, a level-3 deviation to μ_1 results in an expected utility of $0.25*(1+1.5+1.5+0.5) = 1.125$ for each player.

From μ_2, under level-2 foresight, the system evolves to μ_6 with probability 1. Hence, for each player, a level-3 deviation to μ_2 is just the utility he gets from the partner he is matched to in μ_6. For μ_4 it also holds that the system evolves to μ_6 with probability 1; hence, the same utilities can be attached to a level-3 deviation to μ_4 as for a level-2 deviation to μ_2.

From μ_3, under level-2 foresight, the system evolves to μ_7 with probability 1. Hence, for each player, a level-3 deviation to μ_3 is just the utility he gets from the partner he is matched to in μ_7. For μ_5 it also holds that the system evolves to μ_7 with probability 1; hence, the same utilities can be attached to a level-3 deviation to μ_5 as for a level-2 deviation to μ_3.

From μ_6, under level-2 foresight, the system stays in μ_6 with probability 1. Hence, for each player, a level-3 deviation to μ_6 is just the utility he gets from the partner he is matched to in μ_6.

From μ_7, under level-2 foresight, the system stays in μ_7 with probability 1. Hence, for each player, a level-3 deviation to μ_7 is just the utility he gets from the partner he is matched to in μ_7.

Taking all these utilities together, I can construct the following matrix V_3 of expected utilities under level-3 foresight:

$$\begin{bmatrix} 1.125 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1.125 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1.125 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1.125 & 1 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}$$

Now, I check the probabilities of the system's evolution under level-3 foresight in other matchings in the marriage market. In μ_1, each player has two deviations to a matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$. By the symmetry involved here, each matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$ has probability equal to 0.25 to be reached from μ_1 under level-3 foresight.
In μ_2, under level-3 foresight, player m_1 has his most preferred partner, while w_1 could deviate to μ_1 by breaking (m_1,w_1). As can be seen, deviating to μ_1 results in a higher expected utility than obtained in μ_2 for w_1. Hence, this is a stochastic level-3 deviation for w_1. She could also deviate by matching m_2, resulting in μ_5. A stochastic level-3 deviation to μ_5 results in a higher expected utility for both and hence $\mu_2 \rightarrow_{\{m_2,w_1\}} \mu_5$ is a stochastic level-3 deviation. In μ_2, player m_2 could also propose to w_2, resulting in μ_6. For both, this results in a higher expected utility as seen in the matrix; hence, this is also a stochastic level-3 deviation. Taking all these deviations together, player m_1 could make the system evolve to μ_5 and μ_6, player w_1 could make the system evolve to μ_1 and μ_5, and w_2 could make the system evolve to μ_6 only. Hence, $P_3(\mu_1|\mu_2) = 0.17$, $P_3(\mu_5|\mu_2) = 0.33$ and $P_3(\mu_6|\mu_2) = 0.5$. By symmetry, I can now also derive the probabilities for stochastic level-3 deviations from μ_3, μ_4 and μ_5.
Now in μ_6, player w_1 could deviate to μ_4 by breaking (m_1, w_1). The expected utility of a level-3 deviation to μ_4 equals 1 which is equal to the utility obtained by w_1 in μ_6. Consequently, this is not a stochastic level-3 deviation for w_1. By symmetry, w_2 does neither consider deviating from μ_6, while m_1 and m_2 do not consider deviating from μ_7. Hence, $P_3(\mu_6|\mu_6) = 1$ and $P_3(\mu_7|\mu_7) = 1$.
Taking all these probabilities together, I can construct the following probability matrix P_3.

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0.17 & 0 & 0 & 0 & 0.33 & 0.50 & 0 \\ 0.17 & 0.33 & 0 & 0 & 0 & 0 & 0.50 \\ 0.17 & 0 & 0.33 & 0 & 0 & 0.50 & 0 \\ 0.17 & 0 & 0 & 0.33 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As can be seen from P_3, the level-3 stochastically stable set equals $\{\mu_6, \mu_7\}$. ▲

Through Example 4.17, it has been shown that the level-3 stable set reduces significantly in size compared to the stable set from Example 2.14. As a consequence, it can be concluded what the probable matchings are as a result of the marriage market. Furthermore, probabilities can be attached to each of these matchings. The reason that the level-3 stable set is smaller in Example 2.14 is that players are less optimistic about their chances of improvement. In the optimistic set-up, players always deviate when an improving path is existent, while in the stochastic set-up, players only do so when there is an expected improvement. In Example 4.17, this has become clear by the fact that stochastic level-3 deviations do not exist in matchings in the level-3 stochastically stable set $\{\mu_6, \mu_7\}$

because players observe that this may not necessarily lead to a utility increase. However, in the optimistic set-up, deviations exist in $\{\mu_6, \mu_7\}$ because there exists a path to a better matching for the deviators.
Having obtained P_3 in Example 4.17, it can be seen that $P_3 \neq P_2 \neq P_1$. However, if a probability matrix belonging to a certain level of foresight equals another probability matrix belonging to a different level of foresight, the question could be raised whether there exists a pattern in the matrices.
In general settings, the matrix P_K depends on existing stochastic level-K deviations in all matchings by all players in the system. These do depend on the utilities that each player gets from a specific deviation under level-K foresight and on the utility obtained from the matching that is deviated from. In Equation 4.13, the formula is given for the expected utility of a stochastic deviation under level-K foresight.
Under level-4 foresight, the expected utility of a stochastic deviation to μ_y for player i is $V_{i\mu_y,4} = \sum_{\mu_\ell \in \mathbb{M}} P_3(\mu_\ell|\mu_y)V_{i\mu_y,3}$. The expected utility of a stochastic level-2 deviation to the same matching for i is $V_{i\mu_y,2} = \sum_{\mu_\ell \in \mathbb{M}} P_1(\mu_\ell|\mu_y)V_{i\mu_y,1}$. If now $P_1 = P_3$, $P_1(\mu_\ell|\mu_y) = P_3(\mu_\ell|\mu_y)$, for all possible pairs of μ_ℓ, μ_y in $\mathbb{M}$. However, that does not automatically imply that $V_{i\mu_y,4} = V_{i\mu_y,2}$, for all $\mu_y \in \mathbb{M}$ because then $V_{i\mu_y,1}$ must equal $V_{i\mu_y,3}$ for all $\mu_y \in \mathbb{M}$. In other words, there may exist instances of marriage market problems where $P_1 = P_3$, but where $V_{i\mu_y,3}$ is not equal to $V_{i\mu_y,1}$, for all $i \in N$ and for all $\mu_y \in \mathbb{M}$. Stochastic level-K deviations depend on this expected utility and if all these utilities are exactly the same for different levels of foresight, only then it can be concluded that the probability matrix is the same for different levels of foresight.

4.4 Results stochastic marriage market

Now all definitions have been given in the context of a stochastic evolution of the marriage market, I am ready to present the results in the context of stochastic level-K deviations. Throughout this section, I make use of the fact that level-1 deviations are equivalent to stochastic level-1 deviations. Namely, if a level-1 deviation by player $i \in N$ exists in some matching $\mu \in \mathbb{M}$ to $\mu' \in \mathbb{M}$, then it must hold that player i gets a more preferred partner in μ' and so $\mu'(i) \succ_i \mu(i)$. This is equivalent with $U^i(\mu'(i)) > U^i(\mu(i))$. Therefore, if going from μ to μ' is a level-1 deviation, then this must also be a stochastic level-1 deviation and vice versa, while $\mu' \in f_1(\mu)$ and $P_1(\mu'|\mu) > 0$. Consequently, the level-1 stable set must equal the level-1 stochastically stable set. Both properties are formalised in the following two lemmas.

Lemma 4.18. Going from $\mu \rightarrow_S \mu'$ is a level-1 deviation if and only if $\mu \rightarrow_S \mu'$ is a stochastic level-1 deviation.

Lemma 4.19. The level-1 stochastically stable set is equal to the level-1 stable set.

In Example 4.3, it has been shown that an optimistic level-2 deviation does not need to be a stochastic level-2 deviation and therefore Lemma 4.18 does generally not hold for $K > 1$. However, in Example 4.3, it is the case that all stochastic level-2 deviations are also optimistic level-2 deviations. This raises the question of whether this property holds in general. In the next proofs, I show that each stochastic level-2 deviation must also be an optimistic level-2 deviation. I do so by giving a proof in Theorem 4.20. However, this property does not hold for general K, which is shown through a counterexample in Example 4.22. In this counterexample, however, I assume that players can have preferences over matchings and not necessarily over players, which I need to give the counterexample.

Theorem 4.20. If $\mu_2 \rightarrow_S \mu_1$ is a stochastic level-2 deviation, then it must also be an optimistic level-2 deviation.

Proof. Let's assume that there exists a stochastic level-2 deviation $\mu_2 \rightarrow_S \mu_1$ that is not an optimistic level-2 deviation, and $|S| \in \{1, 2\}$. Now, all players in S deviate to μ_1, and thus, for all players in S, it must hold that their expected utility in μ_1 following an optimistic level-2 deviation to μ_1 is higher than their utility in μ_2. Hence, for all $i \in S$ $V_{i\mu_1,2} > U^i(\mu_2(i))$, which is equivalent with $\sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) > U^i(\mu_2(i))$. Now there are two situations to consider following the stochastic level-2 deviation to μ_1:
Situation 1: no more stochastic level-1 deviations exist from μ_1 and hence no more optimistic level-1 deviations exist. In that case $P_1(\mu_1|\mu_1) = 1$. Now to let this be a valid stochastic level-2 deviation, it must hold for all $i \in S$ that $V_{i\mu_1,2} = U^i(\mu_1(i)) * P_1(\mu_1|\mu_1) = U^i(\mu_1(i)) > U^i(\mu_2(i))$, equivalent with $\mu_1(i) \succ_i \mu_2(i)$. However, this leads to a contradiction because I assumed that $\mu_2 \rightarrow_S \mu_1$ is not an optimistic level-2 deviation.
Situation 2: there exists at least one stochastic level-1 deviation in μ_1. Now, following the definition of a stochastic level-2 deviation, for all $i \in S$, $V_{i\mu_1,2} = \sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) > U^i(\mu_2(i))$. Because $\mu_2 \rightarrow_S \mu_1$ is not an optimistic level-2 deviation, it must hold that there does not exist for each $i \in S$ a matching μ_0 such that $\mu_0 \in f_1(\mu_1)$ and $\mu_0(i) \succ_i \mu_2(i)$. Now I know that stochastic level-1 deviations are also always optimistic level-1 deviations, meaning that if $P_1(\mu_0|\mu_1) > 0$, it must be that $\mu_0 \in f_1(\mu_1)$. Now for each matching $\mu_0 \in \mathbb{M}$ for which $P_1(\mu_0|\mu_1) > 0$ it must be that for at least one player $i \in S$: $U^i(\mu_0(i)) \leq U^i(\mu_2(i))$. Namely, if this does not hold, then $\mu_2 \rightarrow_S \mu_1$ is an optimistic level-2 deviation because then, for all $i \in S$, there would exist a $\mu_0 \in f_1(\mu_1)$ with $\mu_1 \in f_2(\mu_2)$ and $\mu_0(i) \succ_i \mu_2(i)$. However, this leads to a contradiction because now $\sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) \leq U^i(\mu_2(i))$ for at least one i because for all μ_0 for which it holds that $P(\mu_0|\mu_1) > 0$, it holds $U^i(\mu_0(i)) \leq U^i(\mu_2(i))$. Therefore, this cannot be a stochastic level-2 deviation for i if this is not an optimistic level-2 deviation and thus leads to a contradiction with the assumption that $\mu_2 \rightarrow_S \mu_1$ is not an optimistic level-2 deviation.
Now it can be concluded that each stochastic level-2 deviation must also be an optimistic level-2 deviation because assuming the existence of a stochastic level-2 deviation that is not an optimistic level-2 deviation leads in each case to a contradiction. □

Through previous theorem, it has been shown that a stochastic level-2 deviation from matching μ to some μ' can only be a stochastic level-2 deviation if this is also an optimistic level-2 deviation. This raises the question of what consequences this has for the relation of the level-2 optimistically stable set and the level-2 stochastically stable set.
By Theorem 3.2 I know that the level-2 optimistically stable set equals the union of optimistic level-2 cycles and that the level-2 stochastically stable set equals the union of stochastic level-2 cycles. I show only cycles can exist that satisfy certain requirements with respect to these stable sets. I name Λ_2 the level-2 optimistically stable set and X_2 the level-2 stochastically stable set.
Let $\mathcal{C}_{\Lambda,2}$ be the set containing all optimistic level-2 cycles as elements such that $\mathcal{C}_{\Lambda,2} =$

$\{C|C$ is a cycle in $\Lambda_2\}$. Also, let $\mathcal{C}_{X,2}$ be the set containing all stochastic level-2 cycles as elements such that $\mathcal{C}_{X,2} = \{C|C$ is a cycle in $X_2\}$.
I refer to $C_{\Lambda,\ell,2}$ as an optimistic level-2 cycle and to $C_{X,\ell,2}$ as a stochastic level-2 cycle. In Figure 1, I refer to 5 different types of stochastic and optimistic level-2 cycles. Cycles 1 and 2 refer to an optimistic level-2 cycle, 3 and 4 refer to a stochastic level-2 cycle and 5 refers to a cycle that is both an optimistic level-2 cycle and a stochastic level-2 cycle. These numbers match the indices given to the cycles in Figure 1. The proof shows that the red cycles in the figure cannot exist and that the union of the level-2 optimistically stable set and the level-2 stochastically stable must therefore be equal to the cycles given the black colour in the figure. For completeness, cycles for which no evidence exists that they cannot occur are also shown in the proof.

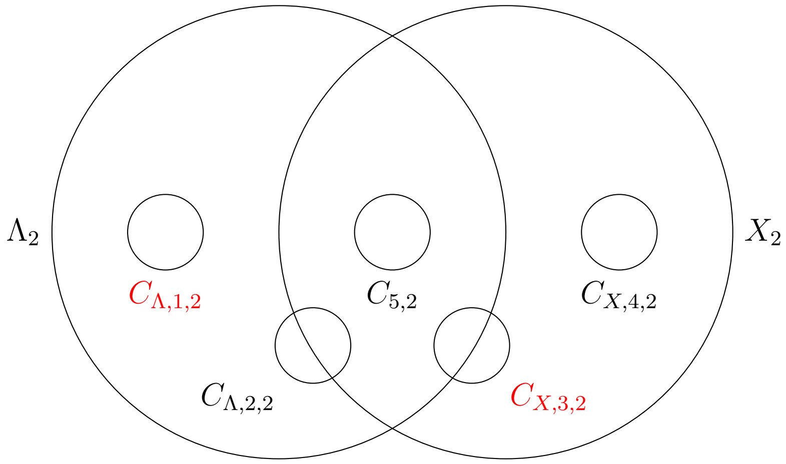


Figure 1: Cycles in the level-2 optimistically stable set Λ_K and the level-2 stochastically stable set X_K. The indices are equivalent to what has been described. Red cycles cannot exist.

Theorem 4.21. $\Lambda_2 \bigcup X_2 = (\mathcal{C}_{\Lambda,2}) \bigcup (\mathcal{C}_{X,2})$:

(i) For all $C \in \mathcal{C}_{\Lambda,2}$: $C \bigcap X_2 \neq \emptyset$.

(ii) For all $C \in \mathcal{C}_{X,2}$: $C \bigcap \Lambda_2 = \emptyset$ or $C \bigcap \Lambda_2 = C$.

Proof. I consider the five situations that a cycle $C_2 \subseteq \mathbb{M}$ that is an optimistic level-2 cycle and/ or a stochastic level-2 cycle can be in with respect to Λ_2 and X_2. Situations 1 and 2 refer to an optimistic level-2 cycle, 3 and 4 refer to a stochastic level-2 cycle and 5 refers to a cycle that is both an optimistic level-2 cycle and a stochastic level-2 cycle.
Situation 1: $C_{\Lambda,\ell,2} \bigcap X_2 = \emptyset$. This is impossible because from $C_{\Lambda,\ell,2}$ the other set X_2 cannot be reached by a finite number of optimistic level-2 deviations and therefore neither by a finite number of stochastic level-2 deviations, which is known from Theorem 4.20 Hence, in this situation, X_2 cannot satisfy stochastic iterated external stability.
Situation 2: $C_{\Lambda,\ell,2} \bigcap X_2 \neq \emptyset$. Now, while staying within this cycle, it is possible to go by sequential stochastic level-2 deviations from an element in $\Lambda_2 \setminus X_2$ to elements in X_2.

However, once in X_2, it is impossible to go back to $\Lambda_2 \setminus X_2$ by stochastic level-2 deviations because of the stochastic iterated external stability of X_2. However, it is still possible to go back to $\Lambda_2 \setminus X_2$ by optimistic level-2 deviations. Considering that optimistic level-2 deviations are a relaxation of the stochastic level-2 deviation, this does not contradict any property of the results so far.
Situation 3: $C_{X,\ell,2} \bigcap \Lambda_2 \neq \emptyset$. This is impossible because, while staying in $C_{X,\ell,2}$, $X_2 \setminus \Lambda_2$ can be reached by sequential stochastic level-2 deviations from $C_{X,\ell,2} \bigcap \Lambda_2$. Stochastic level-$K$ deviations are also optimistic level-2 deviations and therefore Λ_2 does not satisfy deterrence of external deviations which makes this situation contradictory.
Situation 4: $C_{X,\ell,2} \bigcap \Lambda_2 = \emptyset$. In this situation, $\Lambda_2 \setminus X_2$ may be reached from $C_{X,\ell,2}$ by optimistic level-2 deviations but not by stochastic level-2 deviations, to satisfy the iterated external stability of Λ_2. Considering that stochastic level-2 deviations are a restriction of the optimistic level-2 deviations, this does not contradict any property of the results so far.
Situation 5: $C_{X,\ell,2} = C_{\Lambda,\ell,2}$. In this situation, the cycle is both an optimistic level-2 and a stochastic level-2 cycle. This means that the cycle satisfies both stochastic and optimistic deterrence of external deviations, which is not contradictory to any result so far.
Hence, considering all these five situations, optimistic level-2 cycles cannot have an empty intersection with X_2 and the intersection of stochastic level-2 cycles with Λ_2 must be empty or equal to the stochastic level-2 cycle. Therefore, the union of the level-2 optimistically stable set and the level-2 stochastically stable set is equal to the union of stochastic level-2 cycles and optimistic level-2 cycles with a nonempty intersection with the level-2 stochastically stable set. In Figure 1, I graphically depict which cycles can exist and which cannot, visualising the result. □

In Figure 1, X_2 must satisfy stochastic iterated external stability, also from $X_2 \setminus C_{\Lambda,2,2}$. $C_{\Lambda,2,2}$ and $C_{5,2}$ are cycles meaning that $C_{5,2}$ cannot be reached by optimistic level-2 deviations and thus not by stochastic level-2 deviations from $C_{\Lambda,2,2}$. Now by the deterrence of external deviations of $C_{\Lambda,2,2}$, cycle $C_{X_K,4,2}$ cannot be reached from $C_{\Lambda,2,2}$ by stochastic or optimistic level-2 deviations. Now to satisfy the stochastic iterated external stability for X_2 from $C_{\Lambda,2,2} \setminus X_2$, the only possibility is sequential stochastic level-2 deviations from $C_{\Lambda,2,2} \setminus X_2$ to $C_{\Lambda,2,2} \bigcap X_2$.
The result in Theorem 4.21 is based on the property that, if some deviation from μ to μ' is not an optimistic level-2 deviation, then it can also not be a stochastic level-2 deviation by Theorem 4.20. Hence, this result is a consequence of the restriction on the optimistic level-2 deviation and must hold for all settings in which the optimistic deviations are restricted. In the stochastic set-up, I have shown that this restriction holds for $K = 2$. Nonetheless, it is not sure whether this result holds for general K. The reason why this may not hold for general K is that, in each matching, at least as many optimistic level-2 deviations exist as stochastic level-2 deviations. Consequently, one could think of an example with a deviation from μ_3 to μ_2 under level-3 foresight in which all optimistic induced deviations lead to a deterioration for at least one deviator, while no stochastic level-2 deviations exist in μ_2 and all deviators have a more preferred partner in μ_2 compared to μ_3.
In Example 4.22, I show that the result of Theorem 4.20 does not hold under the additional assumption that there may exist players that have preferences over matchings instead of over partners. This additional assumption has not been made anywhere else in this paper. Hence, the game considered in Example 4.22 is different from the games considered so

far. However, as previously, the rules for forming and deleting links and the rule that each player is allocated at most one partner of the opposite sex or is single still hold in the next example.

Example 4.22. In this counterexample, I show that there may exist stochastic level-3 deviations that are not optimistic level-3 deviations. I show this under the assumption that players in the marriage market may have preferences over matchings instead of over partners. In other words, the utility of at least one player $i \in N$ in all matchings $\mu \in \mathbb{M}$ may be affected by not only the player he is matched with but also by links formed by other players in $N \setminus \{i, \mu(i)\}$.
In this specific example with $N = \{m_1, m_2, m_3, w_1, w_2, w_3\}$, I assume that only the utilities of m_2 and w_1 are affected by links formed by other players. For players in $N \setminus \{m_2, w_1\}$, I consider the following utilities that are independent of links formed by other players:

$$m_1 \ : U^{m_1}(w_1) = 3;\ U^{m_1}(w_3) = 2;\ U^{m_1}(w_2) = -10;\ U^{m_1}(m_1) = -11$$

$$m_3 \ : U^{m_3}(w_2) = 3;\ U^{m_3}(m_3) = 2;\ U^{m_3}(w_1) = -10;\ U^{m_3}(w_3) = -11$$

$$w_2 \ : U^{w_2}(m_2) = 3;\ U^{w_2}(m_1) = 2;\ U^{w_2}(m_3) = -10;\ U^{w_2}(w_2) = -11$$

$$w_3 \ : U^{w_3}(w_3) = 3;\ U^{w_3}(m_1) = 2;\ U^{w_3}(m_2) = -10;\ U^{w_3}(m_3) = -11$$

For player m_2, I say that the utility he gets from a matching depends on whether m_1 is also matched. When m_1 is matched, I write the utility of player m_2 being matched to player $i \in N \setminus (M \setminus \{m_2\})$ as $U^{m_2}_+(i)$ and when m_2 is unmatched as $U^{m_2}_-(i)$. So, when m_1 is matched, I consider the following utilities for m_2:

$$m_2 \ : U^{m_2}_+(w_1) = 3;\ U^{m_2}_+(w_2) = 2;\ U^{m_2}_+(w_3) = -10;\ U^{m_2}_+(m_2) = -11$$

When m_1 is unmatched, I reduce each utility by 5. I keep the same order as previously. Hence, the following utilities are considered for m_2:

$$m_2 \ : U^{m_2}_-(w_1) = -2;\ U^{m_2}_-(w_2) = -3;\ U^{m_2}_-(w_3) = -15;\ U^{m_2}_-(m_2) = -16$$

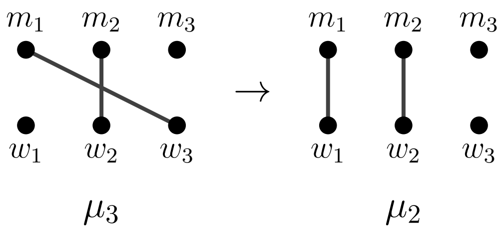
Clearly, now m_2 has the incentive to make sure that m_1 does not end up single.
For player w_1, I say that the utility she gets from a matching depends on whether m_1 and m_2 are also matched. When m_1 are both matched, I write the utility of player w_1 being matched to player $i \in N \setminus (W \setminus \{w_1\})$ as $U^{w_1}_+(i)$ and when both or one player in $\{m_1, m_2\}$ are unmatched as $U^{w_1}_-(i)$. So, when both m_1 and m_2 are matched, I consider the following utilities for w_1:

$$w_1 \ : U^{w_1}_+(m_2) = 3;\ U^{w_1}_+(m_1) = 2;\ U^{w_1}_+(m_3) = -10;\ U^{w_1}_+(w_1) = -11$$

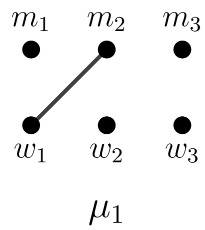
When either m_1 or m_2 is unmatched or when both are unmatched, I reduce each utility by 5. I keep the same order as previously. Hence, the following utilities are considered for w_1:

$$w_1 \ : U^{w_1}_-(m_2) = -2;\ U^{w_1}_-(m_1) = -3;\ U^{w_1}_-(m_3) = -15;\ U^{w_1}_-(w_1) = -16$$

Clearly, now w_1 has the incentive to make sure that m_1 and m_2 do not end up single. Now, I consider the deviation from μ_3 to μ_2 under level-3 foresight by players m_1 and w_1, such that players have level-2 foresight in matching μ_2. I show that this deviation is a stochastic level-3 deviation for both players but not an optimistic level-3 deviation for m_1.



I consider the optimistic setting first. In matching μ_2, under level-2 foresight, players m_1, w_2, and w_3 have their best-preferred partner, such that they cannot deviate and will not accept any proposal. Hence, I consider possible deviations by the remaining players.
Player m_2 could deviate by matching w_1. Under level-1 foresight in $\mu_2 + (m_2, w_1)$, when both m_1 and w_2 are single, the formation of a link between them is a level-1 deviation for both of them. Hence, proposing to w_1 from μ_2 is an optimistic level-2 deviation for m_2 because m_2 can await the formation of this link such that he ends up with his most preferred partner w_1, while he meets the condition that m_1 is also matched in the induced matching. Player m_2 cannot deviate in μ_2 by becoming single since no induced level-1 deviations from $\mu_2 - (m_2, w_2)$ result in a matching where he is matched to w_1, while m_1 is also matched. Only such a matching would mean an improvement for m_2 compared to μ_2.
Player m_3 could now only deviate by proposing to w_1. However, w_1 could only improve with respect to μ_2 by being matched with m_2, while m_1 and m_2 are also matched. No induced matching from $\mu_2 + (m_3, w_1)$ results in a such a matching, meaning forming (m_3, w_1) is not an optimistic level-2 deviation for w_1.
As explained, player w_1 can not deviate in μ_2 by matching m_3. If she deviates by matching m_2, then she prefers that more under the condition that m_1 can be matched in an induced deviation. Under level-1 foresight, when both m_1 and w_2 are single, the formation of a link is a level-1 deviation for both of them. Hence, proposing to m_2 from μ_2 is an optimistic level-2 deviation for w_1. w_1 cannot become single under level-2 foresight from μ_2. Namely, to improve with respect to μ_2, w_1 should be matched to m_2 while m_1 should be matched as well but no induced level-1 deviations exist from $\mu_2 - (m_1, w_1)$ meeting these criteria. Having considered all induced optimistic level-2 deviations from μ_2, it can be concluded that the system evolves to the matching $\mu_1 = \mu_2 + (m_2, w_1)$, as shown below:



In μ_1, under level-1 foresight, player m_2 cannot deviate such that m_1 is also matched. Under the condition that m_1 is unmatched, being matched to w_1 is his best preferred

and he can therefore not deviate. Also, player w_1 cannot deviate such that a matching is created in which both m_1 and m_2 are matched which would be necessary to improve on μ_1. Hence, w_1 can neither deviate from μ_1. Player w_3 has her best-preferred option in μ_1 and will therefore also not deviate.
Player m_1 can deviate by matching w_2, resulting in an improvement for both. Player w_2 can deviate by matching either m_1 or m_3, while m_3 can only deviate by matching w_2.
To conclude the optimistic setting, the induced path of the optimistic level-3 deviation $\mu_3 \to \mu_2$ ends with either $\mu_1 + (m_1, w_2)$ or $\mu_1 + (m_3, w_2)$. Both of these matchings result in a less preferred partner for m_1 and therefore, $\mu_3 \to \mu_2$ is not an optimistic level-3 deviation for m_1.

Having concluded that $\mu_3 \to \mu_2$ is not an optimistic level-3 deviation for m_1, I check whether it is a stochastic level-3 deviation for both m_1 and w_1.
As in the optimistic setting, in μ_2, m_1, w_2, and w_3 all have their best-preferred partner, meaning no proposals or acceptances are to be expected by these players.
In μ_2, player m_2 could propose to w_1. If accepted by w_1, we end up in $\mu_1 = \mu_2 + (m_2, w_1)$ again. Next, I check all possible deviations in μ_1.
In μ_1, under level-1 foresight, player m_2 cannot deviate such that m_1 is also matched. Under the condition that m_1 is unmatched, being matched to w_1 is his best preferred and he can therefore not deviate. Also, player w_1 cannot deviate such that a matching is created in which both m_1 and m_2 are matched which would be necessary to improve on μ_1. Hence, w_1 can neither deviate from μ_1. Player w_3 has her best-preferred option in μ_1 and will therefore also not deviate. In μ_1, player m_1 can deviate by matching w_2, resulting in an improvement for both. Player w_2 can deviate by matching either m_1 or m_3, while m_3 can only deviate by matching w_2. Hence, the induced path of the optimistic level-3 deviation $\mu_3 \to \mu_2$ ends with either $\mu_1 + (m_1, w_2)$ or $\mu_1 + (m_3, w_2)$. Both have a 50 % probability of being reached from μ_1. The first has a utility of 3 for m_2 and the second a utility of -2. Consequently, the expected utility of this deviation is 0.5 which is less than m_2's utility in μ_2. Therefore, deviating from μ_2 to μ_1 is not a stochastic level-2 deviation for m_2.
Player m_2 can also not deviate in μ_2 by becoming single since any induced level-1 deviations from $\mu_2 - (m_2, w_2)$ do not result in a matching where he is matched to w_1, while m_1 is also matched. Only such a matching would mean an improvement for m_2 compared to μ_2.
Now player w_1 cannot match m_2 from μ_2 because m_2 would not accept that proposal and can also not deviate by becoming single. Namely, to improve with respect to μ_2, w_1 should be matched to m_2 and m_1 must be matched as well but no induced level-1 deviations exist from $\mu_2 - (m_1, w_1)$ meeting these criteria. Neither can w_1 match m_3 because no level-1 deviation exists from the resulting matching in which w_1 is matched to m_2, while m_1 is also matched. Consequently, m_3 can not propose to w_1 from μ_2.
Now it can be concluded that there exist no stochastic level-2 deviations from μ_2 and that therefore μ_2 is level-2 stochastically stable. Hence, for both m_1 and w_1, the expected utility of a stochastic level-3 deviation from μ_3 to μ_2 is just the expected utility obtained in μ_2. μ_2 is a matching with a higher utility for both players and therefore deviating from μ_3 to μ_2 is a stochastic level-3 deviation for both players.

Now it can be concluded that $\mu_3 \to \mu_2$ is a stochastic level-3 deviation for both m_1 and w_1, while it is not an optimistic level-3 deviation for both players, under the assumption that players can have preferences matchings instead of over partners. ▲

Now it has been shown that stochastic level-K deviations do not necessarily need to be optimistic level-K deviations in a framework where players may have preferences over matchings instead of over players, I illustrate why the marriage market set-up complicates the construction of a counterexample in which a stochastic deviation is not an optimistic deviation in Example 4.23. In this example, players are not allowed anymore to have preferences over matchings instead of over players as in Example 4.22. A counterexample without players having preferences over matchings or proof that stochastic level-K deviations must always be optimistic level-K deviations for any $K > 0$ is suggested to be presented in future research.

Example 4.23. In this example, I discuss problems in the search for a counterexample of a deviation that is a stochastic deviation but not an optimistic deviation under a certain level of foresight. I do so by intuitively describing the complications.
I consider a game with $N = \{m_1, m_2, m_3, w_1, w_2, w_3\}$ and I define the utilities as in Example 4.22, while players m_2 and w_1 no longer have preferences over matchings but over players as has been assumed in the whole paper except for Example 4.22. To that purpose, the utilities that were reduced by -5 are discarded for m_2 and w_1. Later in the example, I change part of these utilities in the example when describing the complications. The following utilities are considered:

$$m_1 : U^{m_1}(w_1) = 3;\ U^{m_1}(w_3) = 2;\ U^{m_1}(w_2) = -10;\ U^{m_1}(m_1) = -11$$

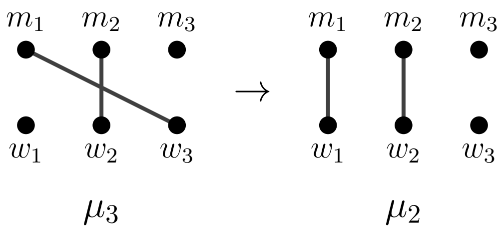
$$m_2 : U^{m_2}(w_1) = 3;\ U^{m_2}(w_2) = 2;\ U^{m_2}(w_3) = -10;\ U^{m_2}(m_2) = -11$$

$$m_3 : U^{m_3}(w_2) = 3;\ U^{m_3}(m_3) = 2;\ U^{m_3}(w_1) = -10;\ U^{m_3}(w_3) = -11$$

$$w_1 : U^{w_1}(m_2) = 3;\ U^{w_1}(m_1) = 2;\ U^{w_1}(m_3) = -10;\ U^{w_1}(w_1) = -11$$

$$w_2 : U^{w_2}(m_2) = 3;\ U^{w_2}(m_1) = 2;\ U^{w_2}(m_3) = -10;\ U^{w_2}(w_2) = -11$$

$$w_3 : U^{w_3}(w_3) = 3;\ U^{w_3}(m_1) = 2;\ U^{w_3}(m_2) = -10;\ U^{w_3}(m_3) = -11$$



As explained, the purpose is to construct an instance that is not an optimistic deviation for at least one player but is still a stochastic deviation for both players. Considering $\mu_3 \to \mu_2$, all induced paths must lead to a deterioration or to the same partner for m_1 compared to μ_2. Additionally, under stochasticity for m_1, the deviation to μ_2 must result in a higher expected utility for m_1 than his utility obtained in μ_3. This could be achieved by, for example, making w_1 not considering any deviations from μ_2 under stochasticity

because she fears becoming single.
Say that in this example, $\mu_3 \to \mu_2$ should not be an optimistic level-3 deviation for m_1. Player w_1 could match m_2 from μ_2, which is a stochastic and optimistic level-2 deviation for both because they prefer each other the most. In μ_2, under level-2 foresight, player w_1 could also unmatch m_1 first with the purpose of matching m_2 from $\mu_2 - (m_1, w_1)$, under level-1 foresight. This can be, however, a path foreseen by player m_1, such that matching w_1 from μ_3 is a level-3 deviation for him. Furthermore, such an induced path can be created under any level-K foresight in μ_3, where K is odd, because w_1 can unmatch m_1 aiming to match m_2 subsequently but then, when single, she can match m_1 again. This is independent of any other existing deviations by players in $N \setminus \{m_1, w_1\}$. This problem can arise in any such a situation, in which w_1 prefers m_1 over being single and m_1 prefers w_1 the most, while w_1 has a deviation to match someone else. Namely, unmatching m_1 (under an even number of foresight) will always be a deviation for w_1 and forming (m_1, w_1) again will always be a deviation for both (under an odd number of foresight). Hence, we should not end up in a matching with the link (m_1, w_1) existent under an even number of foresight.
In Example 4.22, there was an extra incentive for w_1 to have m_1 also matched. Therefore, in μ_2, under level-2 foresight, w_1 would not consider becoming single because from $\mu_2 - (m_1, w_1)$ no level-1 deviations exist that increase w_1's utility with respect to μ_3. From $\mu_2 - (m_1, w_1)$, w_1 could only improve by matching m_2 (with m_1 left single) and by matching m_1, while both deviations result in less utility for w_1 compared to μ_3. Hence, to prevent forming and deleting (m_1, w_1) repeatedly it was necessary in Example 4.22 to also let w_1 have an incentive to let m_1 be matched. Furthermore, also m_2 needed to have an incentive to let m_1 be matched. Otherwise, he could first become single, intending to match w_1 subsequently, and thereby reduce both his and w_1's utility, but then he could form (m_2, w_2) again in $\mu_2 - (m_2, w_2)$, which could be used in m_1's path when matching w_1 from μ_3, such that it ends up in being an optimistic level-3 deviation for him.
Increasing the level of foresight in μ_3 to 4 in this specific example would neither work. Namely, w_1 could first become single from μ_2 resulting in $\mu_2 - (m_1, w_1)$, under level-2 foresight. Subsequently, m_2 could also become become single aiming to match w_1 as an induced deviation, resulting in $\mu_2 - (m_1, w_1) - (m_2, w_2)$ under level-1 foresight. Now m_1 and w_1 could match again, such that there exists an induced path from μ_3 that makes m_1 end up with w_1, meaning $\mu_3 \to \mu_2$ is an optimistic level-4 deviation for him.

Irrespective of the utilities just presented, ideally, I would come up with a situation in which w_1 does not consider deviating in μ_2 because she risks that she gets single which she does not prefer at all, while all optimistic induced paths from μ_2 lead to a deterioration for m_1. When reformulating utilities, as can be seen, when considering $\mu_3 \to \mu_2$ to be under level-K foresight, with K odd, there always exists an induced path for m_1 in which he still ends up with w_1. This does not need to be the case when K is even. So when creating an instance in which w_1 risks getting single in the last matching in the induced path, she should be left alone again by the player proposed to and then I could set the utility of being single for w_1 very low and the difference in utility of being matched with m_1 and the player she aims to be ending up with should be very low as well. However, this results in the problem that the player she proposes to can already directly perform that deviation before the matching is reached with the implied link with w_1 and her more

preferred partner. In other words, say w_1 can propose to m_3, while she risks that m_3 can propose to w_3, then this deviation can also already be performed directly from μ_2. Now this deviation can be used in m_1's induced path because we end up in a matching with (m_1, w_1) formed under level-K foresight with K even. This means that forming the link is still an optimistic deviation for m_1 because then either (m_1, w_1) can be repeatedly formed or deleted, or w_1 cannot improve anymore such that (m_1, w_1) is not deleted.
The problem of existing deviations by players in $N \setminus \{m_1, w_1\}$ is more general because deviations by external players can also be used in m_1's induced path to make us end up in a matching with (m_1, w_1) formed under an even number of K. In other words, increasing the set of players and subsequently the number deviations goes with the risk that more induced paths exist that can be thought of by m_1 such that he can end up with w_1.

Now, I consider a situation in which w_1 cannot become single in the optimistic set-up under level-2 foresight because there is no path leading to an improvement. This means that possible deviations in μ_2 must be by other players and that m_1 cannot use the repeated deletion and formation of (m_1, w_1) in his induced path. For that purpose, I consider the same utilities for m_1 and w_3, but I change the utilities for the rest of the players compared to the initial utilities. Hence, now I consider the following utilities:

$$m_1 : U^{m_1}(w_1) = 3; U^{m_1}(w_3) = 2; U^{m_1}(w_2) = -10; U^{m_1}(m_1) = -11$$

$$m_2 : U^{m_2}(w_2) = 3; U^{m_2}(w_1) = 2; U^{m_2}(w_3) = -10; U^{m_2}(m_2) = -11$$

$$m_3 : U^{m_3}(m_3) = 3; U^{m_3}(w_1) = 2; U^{m_3}(w_2) = -10; U^{m_3}(w_3) = -11$$

$$w_1 : U^{w_1}(m_2) = 3; U^{w_1}(m_1) = 2; U^{w_1}(m_3) = -10; U^{w_1}(w_1) = -11$$

$$w_2 : U^{w_2}(w_2) = 3; U^{w_2}(m_1) = 2; U^{w_2}(m_2) = -10; U^{w_2}(m_3) = -11$$

$$w_3 : U^{w_3}(w_3) = 3; U^{w_3}(m_1) = 2; U^{w_3}(m_2) = -10; U^{w_3}(m_3) = -11$$

Say that we are under level-2 foresight in matching μ_2, such that $\mu_3 \to \mu_2$ is considered to be a possible level-3 deviation. In μ_2, only w_1 and w_2 do not have their best-preferred partner, meaning no proposals or acceptances are to be expected by the remaining players. Now in matching μ_2, w_1 cannot become single, because, under level-1 foresight in $\mu_2 - (m_1, w_1)$, m_2 would not accept a proposal by w_1. This means that in μ_2, only w_2 can deviate by becoming single. Subsequently, in $\mu_2 - (m_2, w_2)$, the only level-1 deviation is the formation of (m_2, w_1), such that m_1 ends up being single. Hence, the optimistic induced path always leads to w_1 being single. However, also under stochasticity, the only deviation in μ_2 is dissolving (m_2, w_2) with induced deviation the formation of (m_2, w_1). Hence, now the problem occurs that $\mu_3 \to \mu_2$ is neither an optimistic level-3 deviation nor a stochastic level-3 deviation.
One could now think of increasing the level of foresight by 1, such that in μ_2, the level of foresight equals 3. However, now an induced path for m_1 occurs such that he can still end up with w_1 in the optimistic set-up. Namely, in μ_2, under level-3 foresight, w_1 could unmatch m_1 in expectation of w_2 dissolving the link with m_2. However, in $\mu_2 - (m_1, w_1)$, under level-2 foresight, w_1 can deviate by matching m_1 again, such that we are in μ_2 again under level-1 foresight. Now w_1 can no longer deviate by becoming single and m_1 ends up being matched to w_1.

To conclude, when K is odd, the problem of constructing a stochastic deviation for m_1 that is not an optimistic deviation resides in the fact that level-K deviations result in an induced path that can be used by m_1 such that he is matched to w_1 because w_1 can repeatedly match and unmatch m_1. Therefore, this can always be an optimistic deviation for m_1. Now when K is even, still all induced paths must lead to a non-improving partner for m_1 with optimistic players. Ideally, under stochasticity, w_1 would then not deviate because she risks getting single from that deviation. However, then there also exists a deviation by external players that can be used in m_1's induced path that can make deviating to μ_2 an optimistic deviation for him when K is even. Last, in a situation in which w_1 needs to wait on a sole other deviation that needs to occur under level-2 foresight, while she forms a new link in the last deviation, then it is hard to come with an instance in which the deviation would still be a stochastic deviation for m_1. Namely, then under stochasticity, all paths also lead to a worse partner for m_1, such that is neither a stochastic deviation for him. ▲

In the previous example, I have lined out some of the problems that arise in the search for a counterexample. This is an incomplete overview and just serves an illustrative purpose. However, when thinking of the construction of a counterexample in which the formation of (m_1, w_1) is a stochastic but not an optimistic deviation for m_1, one could consider a situation in which w_1 risks getting single when deviating after the formation of (m_1, w_1) under stochasticity. Additionally, it must be made sure that all induced paths do not end up in an improvement for m_1. This could be made sure by allowing for the only deviation under level-2 foresight to be that w_1 matches someone else from m_1, while this should not be possible under higher levels of foresight, if a deviation under these levels is considered. Still, w_1 should then risk to end up single following her level-2 deviation. Such a set-up would then prevent the existence of an induced path of m_1 in which w_1 can repeatedly match and m_1 such that (m_1, w_1) can be an existing link in some end matching.
In Theorem 3.6, I showed that, for any $K > 0$, the level-K stable set consists of the stable matching only in the optimistic setting in α-reducible marriage markets. In the stochastic setting, this same result holds, which is shown in Theorem 4.24. The proof is similar to the proof in Theorem 3.6.

Theorem 4.24. Let (M, W, U) be a marriage market problem satisfying α-reducibility. Then, for any $K > 0$, the level-K stochastically stable set equals the stable matching.

Proof. Consider players in S_1. In the stable matching μ_K in the α-reducible marriage market (M, W, U), all players in S_1 have their top choice. Therefore, for all $i \in S_1$ there does not exist a $\mu_0 \in \mathbb{M} \setminus \{\mu_K\}$ for which $U^i(\mu_0(i)) > U^i(\mu_K(i))$. The expected utility of some possible deviation to $\mu_{K-1} \neq \mu_K$ is $V_{i\mu_{K-1},K}$. Writing out the implied recursion in this formula under level-K foresight, I get the following equation:

$$V_{i\mu_{K-1},K} = \sum_{\mu_{K-2} \in \mathbb{M}} P_{K-1}(\mu_{K-2}|\mu_{K-1}) ... \sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) U^i(\mu_0(i)) \tag{1}$$

Since for all $i \in S_1$ and for all $\mu_0 \in \mathbb{M} \setminus \{\mu_K\}$ $U^i(\mu_0(i)) \leq U^i(\mu_K(i))$, the sum in Equation 1 can never be larger than $U^i(\mu_K(i))$. Hence, there does not exist a matching

$\mu_{K-1} \in \mathbb{M} \setminus \{\mu_K\}$ for which this equation is strictly larger than $U^i(\mu_K(i))$. Therefore, in μ_K and in any matching in which the player(s) in S_1 are matched, for all players in S_1, no stochastic level-K deviations exist, irrespective of the level of foresight K.
Now consider players in S_2. A player i in S_2 could only increase his expected utility by a deviation when there exists a positive probability that the induced path of the deviation ends in a matching where i is matched to someone in S_1. However, for all $K > 0$, all players in S_1 will not perform a stochastic level-K deviation from all matchings in which they are matched. Therefore, there is not a positive probability that the deviation from μ_K to a matching $\mu_{K-1} \in \mathbb{M}$ ends in a matching where i is matched to someone in S_1 when all probabilities of that induced path are considered. Hence, for all $\mu_0 \in \mathbb{M}$ for which $U^i(\mu_0(i)) > U^i(\mu_K(i))$ for at least one $i \in S_2$, the probabilities of getting to μ_0 from μ_K is zero. Therefore, no stochastic level-K deviations exist that are also accepted by the opposite player if it involves a link addition for any player in S_2 in μ_K.
Now consider players in S_k, for each $k \in \{3, ..., \ell\}$. A player i in S_k could only increase his expected utility in μ_K by a deviation when there exists a positive probability that the induced path of the deviation ends in a matching where i is matched to someone in $S_1 \bigcup ... \bigcup S_{k-1}$. However, for all $K > 0$, all players in $S_1 \bigcup ... \bigcup S_{k-1}$ will not perform a stochastic level-K deviation from all matchings in which the implied link of S_n is formed, for each $1 \leq n \leq k-1$. Therefore, there is not a positive probability that the deviation from μ_K to μ_{K-1} ends in a matching where i is matched to someone in $S_1 \bigcup ... \bigcup S_{k-1}$ when all probabilities of that induced path are considered. Hence, for all $\mu_0 \in \mathbb{M}$ for which $U^i(\mu_0(i)) > U^i(\mu_K(i))$ for at least one $i \in S_k$, the probabilities of getting to μ_0 is zero. Therefore, no stochastic level-K deviations exist for any player in S_k that are also accepted by the opposite player if it involves a link addition in μ_K. Since $N = S_1 \bigcup ... \bigcup S_\ell$, there exist no stochastic level-K deviations for any player from stable matching μ_K for $K > 0$. Hence, μ_K must be in the level-K stochastically stable set to let the level-K stochastically stable set satisfy stochastic iterated external stability. Also, from μ_K, no stochastic level-K deviations exist and therefore $\{\mu_K\}$ must satisfy stochastic deterrence of external deviations.

To show that the level-K stochastically stable set only contains the stable matching μ_K in the α-reducible marriage market (M, W, U), I show that there exists a path from every other matching $\mu' \neq \mu_K$ to μ_K such that $\{\mu_K\}$ also satisfies stochastic iterated external stability.
I know that μ_K is a singleton cycle in the level-K stochastically stable set because no stochastic level-K deviations exist in μ_K. Furthermore, it is known that μ' is unstable as μ_K is the only stable matching in the marriage market. Therefore, there exists at least one blocking pair in μ'. In fact, there exists at least one S_k in $S_1, ..., S_\ell$ in which the players in S_k are not matched in μ' if $|S_k| = 2$, or in which the player is not single in μ' if $|S_k| = 1$.
Let S_k be the first in $S_1, ..., S_\ell$ for which this holds. For all players in S_k there always exists at least one player in $S_1 \bigcup ... \bigcup S_k$ that they prefer more than their partner in μ'. However, since all players in $S_1 \bigcup ... \bigcup S_{k-1}$ have no stochastic level-K deviations in each matching in which the implied link of S_n is formed for each $1 \leq n \leq k-1$, all players in S_k cannot deviate such that there exists a positive probability that they end up with someone in $S_1 \bigcup ... \bigcup S_{k-1}$. Nonetheless, all players in S_k could improve by matching someone in

S_k or by deviating such that there exists an induced path to be matched with someone in S_k. The former situation is always a stochastic deviation under any level of foresight by all players in S_k because they can no longer deviate in S_k and must have a more preferred partner than in μ'. Once the players in S_k have matched the opposite player in S_k (or himself if $|S_k| = 1$), no more level-K deviations that could be executed exist by these players, for any $K > 0$. Namely, players in $S_1 \bigcup ... \bigcup S_{k-1}$ do not match someone in S_k and the player(s) in S_k do not prefer being matched with a player in $S_{k+1} \bigcup ... \bigcup S_\ell$. Therefore, forming the implied link from S_k is a stochastic level-K deviation for each $i \in S_k$, for any $K > 0$. Forming the implied link in S_k leads therefore always to a higher expected utility for all players in S_k. Namely, for all $i \in S_k$ the utility of being matched to the other player or to himself if $|S_k| = 1$ is always larger than being matched to all other players in $N \setminus S_1 \bigcup ... \bigcup S_{k-1}$. Hence, both players will not deviate after the formation of the link because the players in $S_1 \bigcup ... \bigcup S_{k-1}$ will not deviate, forming the implied link from S_k is a stochastic level-K deviation for each $i \in S_k$, for any $K > 0$. If now some $i \in S_k$ matches some $j \notin S_k$, then from $\mu' + (i, j)$, there still must be a stochastic level-K deviation from $\mu' + (i, j)$ that matches all players in S_k with each other. Hence, a path exists such that all players in S_k are matched from $\mu' + (i, j)$, with $i \in S_k$ and $j \notin S_k$.
Once the implied link in S_k has been formed, the same process can be repeated for the next S in $S_{k+1}, ..., S_\ell$ for which it holds that the implied link in S is not formed. This process can be continued for any $K > 0$ until the stable matching μ_K is reached through consecutive stochastic level-K deviations. Once μ_K has been reached, no more stochastic level-K deviations exist. Now there exists a path of consecutive stochastic stochastic level-K deviations from any unstable μ' to μ_K, while from μ_K, no stochastic level-K deviations exist. Consequently, the set $\{\mu_K\}$ satisfies stochastic iterated external stability. In the first part of this proof, I showed the stochastic deterrence of external deviations of the set $\{\mu_K\}$. Hence, knowing that the level-K stochastically stable set must exist, by minimality, $\{\mu_K\}$ must be the only matching in the level-K stochastically stable set. □

5 Reconsideration of assumptions in the stochastic marriage market

In Section 4.2, I have made key assumptions about the stochastic behaviour of the marriage market that have impacted the outcomes in Section 4.4. The approach of the stochastic set-up has been motivated by examples in Section 4.1 that show optimistic behaviour by players when using the set-up defined in Section 2. The purpose of the introduction of stochasticity was to avoid these very optimistic deviations by players in the marriage market. When setting up this whole new framework in Section 4.2, assumptions were made that simplify the description of the evolution of the marriage market under the assumption of stochasticity. Also, I intended to stick closely to the approach of Herings and Khan (2022) when introducing stochasticity. Herings and Khan (2022) assumed that players did not consider the utility that can be expected when staying in a matching when deciding on deviations. Also, they did assume that players could deviate to any matching that was expected to result in a more preferred partner. Players were not necessarily assumed to always pick the deviation with an induced path to the most preferred partner. The purpose of this section is to alter the assumptions in the stochastic setting and to show the practical consequences of the reconsideration of the assumptions. Also, when possible, I illustrate to what extent the results drawn from the definitions in Section 4.2 hold.

In Section 4.2, I assumed that players compare the utility following a deviation to the expected utility obtained in the matching that is deviated from, under a certain level of foresight. This assumption is in line with the approach by Herings and Khan (2022). However, if a player does not deviate, another player could deviate which might result in a higher expected utility for that player. Therefore, in Section 5.1, I assume that players plan deviations by comparing the utility of the deviation to the utility they expect to get when staying in that matching. In Section 4.2, I also assumed that players randomly decide on a deviation out of all existing utility-improving deviations. This implies that a player, when having several options on deviations, could perform a deviation that does not imply utility maximisation. Therefore, in Section 5.2, I assume that players only propose to the partner that maximises their utility. In Section 4.2, I also assumed that possible deviators could only be groups of size 1 or 2. That assumption simplifies the description of the marriage market significantly. The relaxation of this assumption by allowing for the formation of larger groups of deviating coalitions has also been considered by Herings and Khan (2022) in the context of networks. Therefore, last, in Section 5.3, I allow for the formation of coalitions of players that can deviate as a coalition and show two examples in both the optimistic and stochastic setting. The purpose of that section is to show that the framework of limited foresight in one-to-one matchings can also be applied in the setting of deviating coalitions.

All alterations that are considered should be read in the context of suggestions for future research. However, when necessary, I make certain assumptions about the precise consequences of the implementation of the new assumptions in order to show examples of the evolution of the marriage market. Also, I introduce some minor notations necessary to describe the marriage market under the new assumptions.

5.1 Opportunity utility

5.1.1 Framework

In this section, I define a new framework that does consider the opportunity utility of staying in a matching. No notation is introduced, except for the subscript 'o' that is added to the parameters to indicate the opportunity utility setting. The key assumption in the stochastic framework was that a stochastic level-K deviation from $\mu_x \in \mathbb{M}$ to neighbouring $\mu_y \in \mathbb{M}$ by a player $i \in N$ could be performed when the consequence of this deviation meant a strict increase in expected utility for player i compared to his utility in μ_x. This is equivalent with $V_{i\mu_y,K} > U^i(\mu_x(i))$ and is formalised in Definition 4.11. Player i decides to deviate by comparing his utility to the utility he gets in μ_x. However, i knows that, if he does not deviate, another player will be the one to deviate if deviations exist by players in $N \setminus \{i\}$. Therefore, it makes sense to let i compare the utility of his possible deviation to the expected utility he gets when not deviating in μ_x. Throughout this section, I usually refer to this expected utility as opportunity utility. I illustrate the new assumption of the comparison to opportunity utility by the presentation of several examples. Furthermore, I theoretically describe this new assumption and its consequences. At the end of this section, I conclude what key takeaways exist when this new assumption was to be implemented in future research.

As before, I say that the expected utility is calculated under the assumption that a random draw is performed over all possible deviators in μ_x. However, now i compares his utility to the expected utility he gets when staying in μ_x. I describe how this utility is calculated under level-1 foresight. Now let $L_{1,\mu_x,o} \subseteq N$, be the set of players that have a level-1 deviation in μ_x that improves their expected utility with respect to the expected utility they get when staying in μ_x and each player in $L_{1,\mu_x,o}$ must have at least one deviation that is also a deviation by the opposite player if it involves adding a link. I add the subscript o to show that we are in the opportunity utility setting. Let $I_{i\mu_x\mu_y,1,o}$ be the indicator function that is one if i has a level-1 deviation to μ_y from μ_x that improves his expected utility compared to the expected utility of staying in μ_x and this must also hold for the opposite player if it involves adding a link. $D_{i\mu_x,1,o}$ is the total number of level-1 deviations for i for which this holds in μ_x. All these players calculate this utility in the same way as i does that is described next.
The probability that the system moves from μ_x to μ_y by a level-1 deviation of player $j \neq i$ is equal to, given that i does not deviate and $|D_{j\mu_x,1,o}| > 0$: $\frac{1}{|L_{1,\mu_x,o} \setminus \{i\}|} * \frac{I_{j\mu_x\mu_y,1,o} - I_{i\mu_x\mu_y,1,o} * I_{j\mu_x\mu_y,1,o}}{D_{j\mu_x,1,o}}$. The term $-I_{i\mu_x\mu_y,1,o} * I_{j\mu_x\mu_y,1,o}$ is added because it needs to be made sure that the system cannot move to μ_y from μ_x if this involves the creation of a link between i and j. Namely, if this term is not added, then deviations that involve the formation of a link with i could still be executed by the opposite player. If I now sum over $j \in N \setminus \{i\}$, I get the probability, $P_{1,i}(\mu_y|\mu_x)$, that the system moves from μ_x to μ_y when $i \in L_{1,\mu_x,o}$ decides to not deviate under level-1 foresight, so: $P_{1,i}(\mu_y|\mu_x) = \frac{1}{|L_{1,\mu_x,o} \setminus \{i\}|} \sum_{j \in L_{1,\mu_x,o} \setminus \{i\}} \frac{I_{j\mu_x\mu_y,1,o} - I_{i\mu_x\mu_y,1,o} * I_{j\mu_x\mu_y,1,o}}{D_{j\mu_x,1,o}}$. If $|L_{1,\mu_x,o} \setminus \{i\}| < 1$, no level-1 deviations exist by players in $N \setminus \{i\}$ and the expected utility that i gets when not deviating is equal to $U^i(\mu_x(i))$. The expected utility that i gets when the system stays in μ_x with i not deviating is: $\sum_{\mu_\ell \in \mathbb{M}} P_{i,1}(\mu_\ell|\mu_x) * V_{i\mu_\ell,1} = \sum_{\mu_\ell \in \mathbb{M}} P_{i,1}(\mu_\ell|\mu_x) * U^i(\mu_\ell(i))$.
The tricky part in this set-up is that players mutually decide whether to deviate based on

the decision of others to deviate. For instance, player $i \in N$ considers his possible benefit of deviating based on the decision of player j and vice versa. In the next example, I make this issue more concrete and I show that it is hard to continue with the assumption that the deviation of some players depends on the deviation of other players and vice versa.

Example 5.1. I consider the example that has been used before in the paper. I show the preferences again:

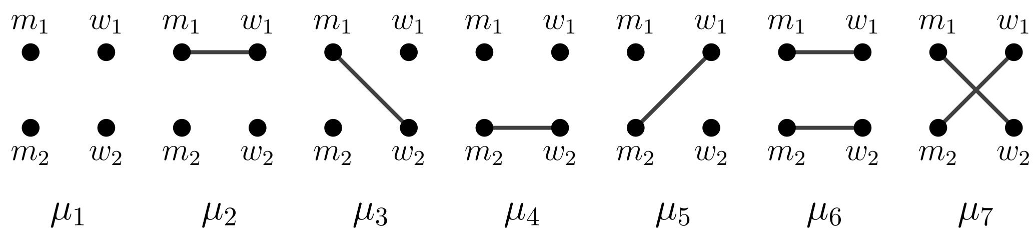
$\succ_{m_1}: w_1, w_2, m_1$

$\succ_{m_2}: w_2, w_1, m_2$

$\succ_{w_1}: m_2, m_1, w_1$

$\succ_{w_2}: m_1, m_2, w_2$

All possible matchings in this setting are also shown again:



Let me consider the expected utility of staying in μ_1 for player m_1 under level-1 foresight. Player m_1 needs to know about the possible deviations of all players in $\{m_2, w_1, w_2\}$ under level-1 foresight to know his expected utility of staying. The possible planned deviations by players in $\{m_2, w_1, w_2\}$ depend on the expected utility of staying in μ_1 that does depend on possible deviations by m_1. Hence, deviations of m_1 depend on that of other players and vice versa. Consequently, it becomes hard to retrieve the expected utility of each player $i \in N$ under the assumption that the deviations by other players, where i's expected utility depends on, also depend on i's expected utility. Therefore, it is problematic to derive probabilities about the evolution of the system under the assumption of the mutual dependence of the utilities. ▲

To solve the problem with the mutual dependence of actions by players, another definition of a stochastic level-K deviation should be given in this context of opportunity utility. In the next paragraph, I redefine this deviation. I do so by assuming that each player decides on a deviation in μ by assuming that this player believes that each other player compares the expected utility of the deviation to the utility in μ and not to opportunity utility. However, each player himself decides on a deviation by comparing the expected utility of that deviation to the expected utility of staying in μ. This assumption takes the mutual dependence away.
For that purpose, I say that $L_{K,\mu_x,o}$ is the set of players that has a stochastic level-K deviation in μ_x that has a larger expected utility than that of staying in μ_x. $L_{K,\mu_x,o}$ is used to describe the probabilities of the system evolution. However, I also still keep on using L_{K,μ_x}, which is the set of players with a stochastic level-K deviation in μ_x, because

players calculate the expected utility of staying under the assumption that other players compare their utility to the utility they get in μ_x, for which L_{K,μ_x} is required. Similarly, $I_{j\mu_x\mu_y,K,o}$ is the indicator function that is one if $j \in N$ has a stochastic level-K deviation from μ_x to μ_y that has a higher expected utility than staying in μ_x and if the deviation involves adding a link, it must be a deviation by the opposite player as well. $D_{j\mu_x,K,o}$ is the number of matchings $\mu_y \in \mathbb{M}$ for which that holds. As for $L_{K,\mu_x,o}$, I also keep on using the definition for the number of stochastic level-K deviations by j in μ_x $D_{j\mu_x,K}$ and the indicator function $I_{j\mu_x\mu_y,K}$ that is one if a stochastic level-K deviation exists from μ_x to μ_y by j. Both are necessary for the calculation of the expected utility of staying in μ_x for an individual i under level-K foresight.
Now, the probability that the system moves from μ_x to μ_y by a stochastic level-K deviation of player $j \neq i$ is equal to, given that i does not deviate and $|L_{K,\mu_x} \setminus \{i\}| > 0$, $\frac{1}{|L_{K,\mu_x}\setminus\{i\}|} * \frac{I_{j\mu_x\mu_y,K} - I_{i\mu_x\mu_y,K} * I_{j\mu_x\mu_y,K}}{D_{j\mu_x,K}}$. The term $-I_{i\mu_x\mu_y,K} * I_{j\mu_x\mu_y,K}$ is added because it needs to be made sure that the system cannot move to μ_y from μ_x if this involves the creation of a link between i and j. Namely, if this term is not added, then deviations that involve the formation of a link with i could still be executed by the opposite player. If I now sum over $j \in N \setminus \{i\}$, I get the probability that the system moves from μ_x to μ_y by a stochastic level-K deviation when $i \in L_{K,\mu_x}$ decides not to deviate, so: $P_{K,i}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x}\setminus\{i\}|} \sum_{j \in L_{K,\mu_x}\setminus\{i\}} \frac{I_{j\mu_x\mu_y,K} - I_{i\mu_x\mu_y,K} * I_{j\mu_x\mu_y,K}}{D_{j\mu_x,K}}$. Note that $P_{K,i}(\mu_y|\mu_x)$ does not represent a probability of the evolution of the system, but is only used in the calculation of the opportunity utility of i. If $|L_{K,\mu_x} \setminus \{i\}| < 1$, no stochastic level-$K$ deviations exist by players in $N \setminus \{i\}$ and the expected utility that i gets when not deviating is equal to $U^i(\mu_x(i))$. Under $|L_{K,\mu_x} \setminus \{i\}| > 0$, the expected utility that i gets when the system stays in μ_x with i not deviating is: $\sum_{\mu_\ell \in \mathbb{M}} P_{i,K}(\mu_\ell|\mu_x) * V_{i\mu_\ell,K}$. Now the deviation to μ_y is a stochastic level-K deviation for i if $V_{i\mu_y,K} > \sum_{\mu_\ell \in \mathbb{M}} P_{i,K}(\mu_\ell|\mu_x) * V_{i\mu_\ell,K}$. Hence, $i \in L_{K,\mu_x,o}$ if $V_{i\mu_y,K} > \sum_{\mu_\ell \in \mathbb{M}} P_{i,K}(\mu_\ell|\mu_x) * V_{i\mu_\ell,K}$. Now the probability that the system moves from μ_x to μ_y under the assumption of the comparison to opportunity utility is equal to $P_{K,o}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x,o}|} \sum_{i \in L_{K,\mu_x,o}} \frac{I_{i\mu_x\mu_y,K,o}}{D_{i\mu_x,K,o}}$.
Next, I give a brief overview of the introduced concepts that are used in Section 5.1.2. The only new notation is $P_{i,K}(\mu_y|\mu_x)$, that is the probability that is believed by player $i \in N$ that the system evolves from μ_x to μ_y when i decides to not deviate. This is used in the calculations of each player when deciding on deviations and does not represent a true probability. Also, I have added subscripts 'o' to the notation to indicate the opportunity utility setting.

Equation 5.2. Let $K \geq 1$. The probability that is believed by player $i \in N$ that the system goes from matching μ_x to μ_y with $i \in N$ deciding to not deviate is, given $|L_{K,\mu_x} \setminus \{i\}| > 0$, $P_{i,K}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x}\setminus\{i\}|} \sum_{j \in L_{K,\mu_x}\setminus\{i\}} \frac{I_{j\mu_x\mu_y,K} - I_{i\mu_x\mu_y,K} * I_{j\mu_x\mu_y,K}}{D_{j\mu_x,K}}$. If $|L_{K,\mu_x} \setminus \{i\}| = 0$, then $P_{i,K}(\mu_y|\mu_x) = 0$, for all $\mu_y \neq \mu_x$ and $P_{i,K}(\mu_x|\mu_x) = 1$.

Equation 5.3. Let $K \geq 1$. The expected utility that player $i \in N$ gets when the system evolves to $\mu_y \in \mathbb{M}$, under opportunity utility, is $\sum_{\mu_\ell \in \mathbb{M}} P_{i,K}(\mu_\ell|\mu_y) * V_{i\mu_\ell,K}$.

Definition 5.4. Let $K \geq 1$. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-K deviation for player $i \in S$ under opporunity utility* if $V_{i\mu_y,K} > \sum_{\mu_\ell \in \mathbb{M}} P_{i,K}(\mu_\ell|\mu_x) * V_{i\mu_\ell,K}$.

Equation 5.5. Let $K \geq 1$. The probability that the system goes from matching $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ under opportunity utility, when $|L_{K,\mu_x,o}| > 0$ is $P_{K,o}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x,o}|} \sum_{i \in L_{K,\mu_x,o}} \frac{I_{i\mu_x\mu_y,K,o}}{D_{i\mu_x,K,o}}$.

5.1.2 Examples and takeaways

The framework that just has been described contributes to a more realistic description of the marriage market. A situation where this could occur is when two players are matched to each other with one of the partners being the most preferred by the other but not vice versa. Previously, the player having his most preferred option did have no incentive to deviate because he compared possible deviations to the utility he received from his most preferred partner. Now, the player matched to his most preferred partner realises that his partner may delete the link and is able to anticipate this by deviating himself. Another situation in which this framework contributes to a more realistic description of the marriage market is when a player can tactically decide to not deviate in expectation of more beneficial deviations by other players. I illustrate both situations in Example 5.6.

Example 5.6. In this example, I show two deviations in a marriage market under the new assumptions. First, I show a level-1 deviation. This deviation follows from players comparing the utility of that deviation by the utility they get when staying in that matching. This deviation is, however, not an optimistic level-1 deviation as defined in previous sections. Second, I show that a player is unwilling to deviate under level-2 foresight when comparing his utility to the expected utility he would get when staying in that matching. It will be shown, however, that optimistic deviations exist in that matching. I consider a marriage market of four players with the following utilities:

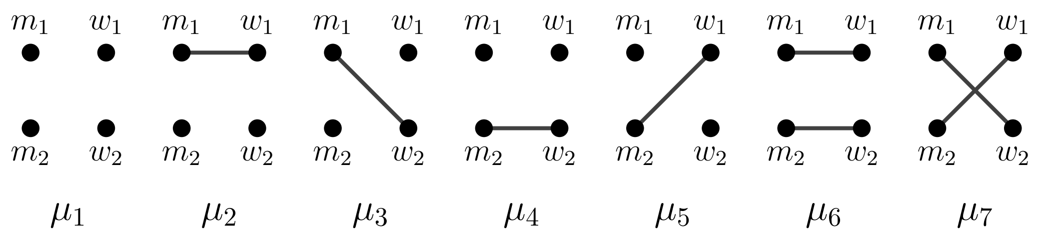
$$m_1 : U^{m_1}(w_1) = 2;\ U^{m_1}(w_2) = 1;\ U^{m_1}(m_1) = 0$$

$$m_2 : U^{m_2}(w_2) = 2;\ U^{m_2}(w_1) = 1;\ U^{m_2}(m_2) = 0$$

$$w_1 : U^{w_1}(m_2) = 2;\ U^{w_1}(m_1) = 1;\ U^{w_1}(w_1) = 0$$

$$w_2 : U^{w_2}(m_2) = 2;\ U^{w_2}(m_1) = 1;\ U^{w_2}(w_2) = 0$$

All possible matchings in this setting are also shown again:



To derive deviations under both level-1 and level-2 foresight, it is useful to construct matrix P_1 of this marriage market (according to the definitions described in Section 4.2). Hence, first, I briefly set up this matrix P_1 in the next paragraph. This matrix is the matrix that players use in the calculation of the probabilities in the system for the purpose of the calculation of expected utility. This matrix does not represent the true probabilities

of the evolution of the system under the new assumptions.
No player prefers to be single, and thus from μ_1 deviations to any matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$ exist that lead to a positive increase in utility under level-1 foresight. In fact, every player has two deviations to a matching in $\{\mu_2, \mu_3, \mu_4, \mu_5\}$ and so each matching in this set has 0.25 probability to be reached from μ_1. Now in μ_2, m_1 will not deviate because he has his most preferred option, m_2 can match with w_1 (leading to μ_5), while m_2 can also deviate and match to w_2 (leading to μ_6). Player w_2 can only match with m_2 in μ_2 which leads to μ_6. Therefore, $P_1(\mu_5|\mu_2) = P_1(\mu_6|\mu_2) = 0.5$. In μ_3, m_1 can match to w_1 to form μ_2 while m_2 can match to w_1 forming μ_7 and with w_2 to form μ_4. Player w_1 can match to m_1 and to m_2 to form μ_2 or μ_7 respectively, while w_2 can match to m_2 to form μ_4. Therefore, $P_1(\mu_2|\mu_3) = P_1(\mu_4|\mu_3) = 0.375$, and $P_1(\mu_7|\mu_3) = 0.25$. In μ_4, m_1 can only match with w_1 to form μ_6 and vice versa because m_2 and w_2 are matched and are their most preferred option. Hence, $P_1(\mu_6|\mu_4) = 1$. In μ_5, m_1 can match with w_2 to form μ_7, while m_2 can form μ_4 by matching w_2. Player w_1 has her most preferred option and w_2 can match with m_1 and m_2 to form μ_4 or μ_7 respectively. Hence, $P_1(\mu_4|\mu_5) = P_1(\mu_7|\mu_5) = 0.5$. In μ_6, m_1 and m_2 have their most preferred option, so $P_1(\mu_6|\mu_6) = 1$. In μ_7, m_1 cannot match, while m_2 can match with w_2 to form μ_4. Player w_1 will not deviate, while w_2 will match to m_2. Hence, $P_1(\mu_4|\mu_7) = 1$. Now taking all these probabilities together, the following matrix P_1 can be constructed:

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0.375 & 0 & 0.375 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Now I check whether a deviation exists by player w_1 under the assumption of opportunity utility in matching μ_7 under level-1 foresight. Player w_1 has her most preferred partner in μ_7. However, she knows that the system evolves to μ_4 with probability 1 if she does not consider a deviation. Hence, the expected utility of staying is 0 under level-1 foresight because $U^{w_1}(\mu_4(w_1)) = 0$. Matching to m_1 would however result in matching μ_2 with corresponding utility equal to 1 because $U^{w_1}(\mu_2(w_1)) = 1$. Now m_1 also knows that the system evolves to μ_4 with probability 1, with utility equal to 0 in μ_4. Therefore, he would be willing to match to w_1 because this deviation results in a higher expected utility than not deviating in μ_7. Hence, the system can also evolve to μ_2 which would not be an optimistic level-1 deviation by some player because w_1 has her most preferred partner in μ_7. According to the framework I have defined, players m_2 and w_2 are unaware of this planned deviation by m_1 and w_1. Therefore, if I were to attach probabilities to the evolution of the system when starting in μ_7, matchings μ_2 and μ_4 both have a probability equal to 0.5 to be reached from μ_7 under the new assumptions.
Now let me consider matching μ_1 under level-2 foresight. I check what the expected utility of m_1 is when not deviating. If m_1 does not deviate, two possible matchings can occur: μ_4 (m_2 matches w_1) or μ_5 (m_2 matches w_2). Both deviations have equal probability because, for all involved players, they result in a positive increase in utility. Now, in μ_4, under level-1 foresight, the system evolves to μ_6 with probability 1, with a utility for m_1 equal

to 2. In μ_5, the system evolves to μ_4 with probability 0.5 and to μ_7 with probability 0.5, leading to an expected utility of this evolution equal to $0.5 * 0 + 0.5 * 1 = 0.5$. Hence, the expected utility for m_1 of not deviating equals $0.5 * 2 + 0.5 * 0.5 = 1.25$.
Now player m_1 could deviate in μ_1 by matching to either w_1 or w_2. Matching w_1 leads to μ_2. From μ_2, the system has a probability of 0.5 to evolve to μ_5 with corresponding utility 0 and a probability of 0.5 to evolve to μ_6 with corresponding utility 2. Hence, m_1's expected utility of matching w_1 is equal to 1. This is lower than the expected utility of not deviating and therefore matching w_1 is not a stochastic level-2 deviation that results in a higher expected utility than not deviating. Matching w_2 leads to matching μ_3. In matching μ_3, under level-1 foresight, the system has a probability of 0.375 to evolve to μ_2 with utility 2, a probability of 0.375 to evolve to μ_4 with utility 0, and a probability of 0.25 to evolve to μ_7 with utility 1. Hence, the expected utility of this deviation is equal to $0.375 * 2 + 0.375 * 0 + 0.25 * 1 = 1$. This is also lower than m_1's expected utility when not deviating. Therefore, m_1's optimal strategy in μ_1 is to not deviate under level-2 foresight. Hence, $m_1 \notin L_{2,\mu_1,o}$. Nevertheless, for m_1, deviating to μ_2 or μ_3 are both optimistic and stochastic level-2 deviations in the old setting without considering opportunity utility. ▲

The first part of the previous example has shown that, under the new assumptions, deviations exist that are no optimistic deviations under a certain level of foresight. In the situation of the example presented, this happens to be even possible under level-1 foresight. The second part showed that an optimistic deviation may exist which is no deviation under the assumptions of this section. The results drawn for stochastic level-2 deviations rely on the fact that a level-2 deviation is also an optimistic level-2 deviation and on the observation that stochastic level-1 deviations are also optimistic level-1 deviations. Therefore, the results drawn in Section 4.4 do not necessarily hold under the new assumptions.
The previous example has also shown that there may exist situations in which a player i decides to not deviate in $\mu_x \in \mathbb{M}$ because anticipating other players deviating results in a higher expected utility than any possible deviation. The key underlying assumption here is that i assumes that other players decide on deviations by comparing their utility to the utility they get in μ_x and not, as i does, by comparing the utility of a possible deviation to the expected utility of not deviating. However, this approach can result in an impasse that I illustrate in Example 5.7.

Example 5.7. Say that in some matching $\mu_1 \in \mathbb{M}$, there are two players, i and j, that can perform a stochastic level-K deviation, such that $L_{K,\mu_x} = \{i, j\}$. Player i can make the system evolve to μ_2, while player j can make the system evolve to μ_3, with $\mu_1 \neq \mu_2 \neq \mu_3$. However, for player i it holds that $V_{i\mu_3,K} > V_{i\mu_2,K} > V_{i\mu_1,K}$ and for player j it holds that $V_{j\mu_2,K} > V_{j\mu_3,K} > V_{j\mu_1,K}$. Both players now anticipate each other's action because they believe that the other player decides on deviating by comparing utility to the utility in μ_1. Now, both are not part of the set of players with a deviation in μ_x under level-K foresight, so: $\{i, j\} \bigcap L_{K,\mu_x,o} = \emptyset$. ▲

In Example 5.7, player i would be better off when he lets j deviate, while for player j it would be better to let i deviate. Nevertheless, for both, staying in μ_1 is not a weakly dominant strategy. When formal definitions were to be written down on this subtopic, it needs to be decided how the system is assumed to evolve in a situation where players mutually anticipate each other's actions. In Example 5.7, two options for the assumption

on the evolution of the system exist in this symmetric setting.
The first option would be that i sees that j will not deviate and therefore decides to deviate himself. However, this reasoning should then also apply to j that sees that i does not deviate and therefore decides to deviate himself as well. In accordance with Section 4.2, a random draw over the possible deviators could be assumed. The outcome would then be that both μ_2 and μ_3 have a 50 % probability to be reached from μ_1 under level-K foresight.
The second option would be that both players keep on anticipating a deviation from the other player. This would result in an impasse with the result that the system stays in μ_1, under the absence of other existing deviations. However, since for both players staying in μ_1 is not a weakly dominant strategy, this assumption is not preferred.

To conclude this section about the implementation of the more realistic assumption of comparing utility to opportunity utility when deciding on deviating, I highlight a few takeaways that need to be taken into account when formal definitions and theory were to be written down on this subtopic.
First, it is problematic to stick to the most realistic situation in which players can also account for their calculations for other players taking into account opportunity utility. Hence, the assumption so far is in line with the notion that players believe that they are the only player that can consider opportunity utility. I have highlighted this in Example 5.1.
Second, it can not be easily concluded which of the results drawn previously for the setting of optimistic and stochastic deviations also hold under the new assumptions. This is a consequence of the fact that level-1 and level-2 deviations under the new assumptions are not always optimistic level-1 and level-2 deviations and vice versa. I have shown this in Example 5.6. As a consequence, different stable sets may appear of which the relation to the stochastic and optimistic stable sets remains unclear.
Third, a decision needs to be made in situations where players await each other's action because they believe that possible deviations by other players result in a higher expected utility. This has been shown in a rather theoretical setting in Example 5.7.

5.2 Utility maximisation

5.2.1 Framework and examples

In Section 4.2, it was assumed that each player calculated his expected utility of a stochastic level-K deviation to matching μ based on the assumption that each other player in induced deviations randomly picked a deviation out of all deviations that were improvements on his expected utility. However, this assumption may not be very credible. Namely, if a player i has a choice between two deviations and can fully oversee the consequences of both deviations, then it makes more sense to assume that i chooses to deviate to the matching that maximises his expected utility. Therefore, in this subsection, I restrict possible deviations described in Section 4.2 by only allowing players to perform a stochastic level-K deviation to the matching maximising expected utility.
To formalise this newly introduced assumption, it is necessary to define deviations in this context. As before, a deviation by player $i \in N$ can consist of two possible actions: a deletion of a link or a creation of a link. For the former, i can do this independently,

while for the latter, the player i proposes to also needs to agree on that formation. If I now allow only this new link to be formed between when the player i proposes to, say j when this is also for j results in a matching that maximises his utility, an impasse in the marriage market can easily occur. I illustrate this in Example 5.8.

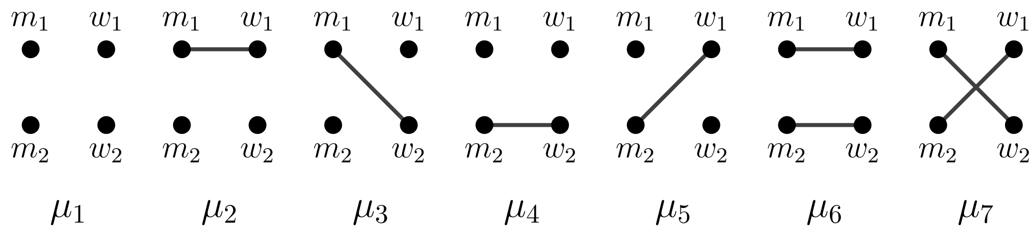
Example 5.8. In this example, I consider the same marriage market set-up as has been used in previous examples. I show the preferences and all matchings in this market again:

$\succ_{m_1}: w_1, w_2, m_1$

$\succ_{m_2}: w_2, w_1, m_2$

$\succ_{w_1}: m_2, m_1, w_1$

$\succ_{w_2}: m_1, m_2, w_2$



Let me consider matching μ_1 in this marriage market. As has been indicated in the description of the assumption of utility-maximising players, players only deviate to the matching that is maximising their utility. In matching μ_1, player m_1 would like to form μ_2 by matching w_1. However, w_1 wants to match m_2 to create μ_5, while m_2 wants to match w_2. This gives rise to a problem because no one gets their most preferred deviation they could create. Staying in μ_1 is for no player a weakly dominant strategy under level-1 foresight. Therefore, a new decision rule should be established to handle this situation. ▲

To avoid the impasse that has been shown in the previous example, I propose to use a different approach for the acceptance of a proposal by the player that is proposed to. In Section 4.2, I described the probabilities of the evolution of the marriage market in the stochastic context. Key underlying assumption there was that a random draw was executed over all players and a subsequent random draw over all their possible deviations when describing the probabilities. In the context of utility-maximising players, I can do the same, but now under the assumption that the player that is picked chooses his most preferred deviation. If he then proposes to a player, this player accepts the proposal when the resulting matching means a utility increase for him. The consequence of this approach is that a creation of a link by the proposal of i to j is a deviation by i but not vice versa because j may propose to another player that results in a higher expected utility for j. However, j still accepts the proposal by i under the condition that this increases his utility.

Now a deviation is a deviation for $i \in N$ when that deviation results in the highest expected utility out of all possible matchings that can be created from μ_x by i and this deviation results in an increase in expected utility. Let, under level-1 foresight,

$A_{i\mu_x,1} \subseteq \mathbb{M}$ be the set of matchings that i can deviate to from μ_x such that for all $\mu \in A_{i\mu_x,1}$ $U^i(\mu(i)) > U^i(\mu_x(i))$. If such matchings do not exist, i has no deviations in μ_x. Now the only matching in $A_{i\mu_x,1}$ that i proposes to deviate to from μ_x is the matching μ_y for which $U^i(\mu_y(i)) > U^i(\mu(i))$ for all $\mu \in A_{i\mu_x,1} \setminus \{\mu_y\}$. If the deviation involves the addition of a link, this addition must be a utility improvement for the opposite player.
Extending the reasoning in the previous paragraph for $K > 1$, let $A_{i\mu_x,K}$ be the set of matchings that i can deviate to from μ_x under level-K foresight such that for all $\mu_y \in A_{i\mu_x,K}$, the expected utility of being in μ_y after a level-K deviation is higher than that of in μ_x. When $K > 1$, however, induced deviations are also taken into account. These induced deviations must also take into account that we are in the utility-maximising setting. Therefore, I add subscript U to $V_{i\mu_y,K}$, which is the expected utility of going to μ_y by a level-K deviation, resulting in $V_{i\mu_y,K,U}$, to indicate that we are in the utility-maximising setting. The calculation of $V_{i\mu_y,K,U}$, for $K > 0$, is exactly the same as described in Section 4.2. Now the only matching in $A_{i\mu_x,K}$ that is proposed by i as a deviation from μ_x is the matching μ_y for which it holds that $V_{i\mu_y,K,U} > V_{i\mu,K,U}$ for all $\mu \in A_{i\mu_x,K} \setminus \{\mu_y\}$. Each player j accepts a proposal by player i in μ_x if the resulting matching μ_y results in a higher expected utility than the utility obtained in μ_x. Using the notation, it should hold for j to accept the proposal that $V_{j\mu_y,K,U} > U^j(\mu_x(j))$.
Now the deviations have been defined in the context of utility maximisation, I can define the probability that the system evolves in a stochastic manner from μ_x to μ_y under level-K foresight under the assumption of utility-maximising players. For that purpose, I do not introduce any new notation but add subscript U to indicate that we are in the utility-maximising setting. I define $I_{i\mu_x\mu_y,K,U}$ that is 1 if player i has a stochastic level-K deviation from μ_x to μ_y that maximises his utility that results in utility improvement by the opposite player if it involves the addition of a link (not necessarily utility-maximising by the opposite player). Furthermore, let $L_{K,\mu_x,U}$ be the set of players for which $I_{i\mu_x\mu_y,K,U} = 1$. Now, the probability that the system moves from μ_x to μ_y by a stochastic level-K deviation of player i under the assumption of utility maximisation is equal to, given that $|L_{K,\mu_x,U}| > 0$, $\frac{1}{|L_{K,\mu_x,U}|} * I_{i\mu_x\mu_y,K,U}$. Now summing over $i \in L_{K,\mu_x,U}$ gives the probability that the system moves from μ_x to μ_y under the new assumptions. Hence, the probability that the system moves from μ_x to μ_y under level-K foresight with utility-maximising players is: $P_{K,U}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x,U}|} \sum_{i \in L_{K,\mu_x,U}} I_{i\mu_x\mu_y,K,U}$. Again, I have added U as a subscript to the sign of the probability to indicate the utility-maximising setting. In Example 5.9, I show the evolution of a marriage market under the assumption of utility-maximising players.

Example 5.9. In this example, I show how matrices $P_{1,U}$ and $P_{2,U}$ are constructed that describe the stochastic behaviour of this specific marriage market. To not elaborate too much in this example, I only show the calculation of the first row of the matrix $P_{2,U}$. The example I consider is the same as Example 5.6. The following utilities are present by the players in the game:

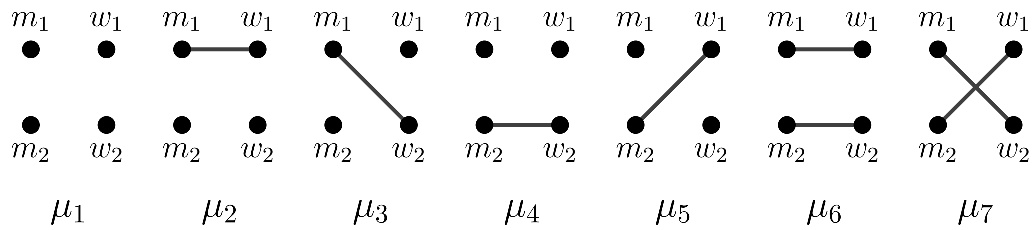
$$m_1 : U^{m_1}(w_1) = 2;\ U^{m_1}(w_2) = 1;\ U^{m_1}(m_1) = 0$$

$$m_2 : U^{m_2}(w_2) = 2;\ U^{m_2}(w_1) = 1;\ U^{m_2}(m_2) = 0$$

$$w_1 : U^{w_1}(m_2) = 2;\ U^{w_1}(m_1) = 1;\ U^{w_1}(w_1) = 0$$

$w_2 \ : U^{w_2}(m_2) = 2;\ U^{w_2}(m_1) = 1;\ U^{w_2}(w_2) = 0$

All possible matchings in this setting are also shown again:



First, I consider the construction of $P_{1,U}$. In matching μ_1, each player proposes to his most preferred partner when he is the player given the chance to deviate. Hence, m_1 proposes to w_1, m_2 to w_2, w_1 to m_2 and w_2 to m_2. All these proposals are accepted because no player prefers to be single. Hence, matchings μ_2 and μ_5 have probability 0.25 to be reached from μ_1 and μ_4 has probability 0.5 to be reached from μ_1. In μ_2, m_1 has his best preferred, m_2 proposes to w_2, w_1 proposes to m_2 and w_2 proposes to m_2. Now, μ_5 has a probability of 1/3 to be reached and μ_6 has a probability of 2/3. In μ_3, m_1 proposes to w_1, m_2 proposes to w_2, w_1 proposes to m_2 and w_2 proposes to m_2. This leads to matching μ_2 having a probability of 0.25, μ_4 a probability of 0.5, and μ_7 a probability of 0.25. In μ_4, m_1 proposes to w_1, m_2 and w_2 do not propose, and w_1 proposes to m_1. This means matching μ_6 is reached with probability 1. In μ_5, m_1 proposes to w_2, m_2 proposes to w_2, w_1 does not propose and w_2 proposes to m_2. This leads to a probability of 2/3 for μ_4 and a probability of 1/3 for μ_7. In μ_6, no player proposes and hence the system stays in μ_6 with probability one. In μ_7, m_1 and w_1 do not propose, and m_2 and w_2 propose to each other, leading to matching μ_4 with probability 1. This leads to following matrix $P_{1,U}$:

$$\begin{bmatrix} 0 & 0.25 & 0 & 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & 0.67 & 0 \\ 0 & 0.25 & 0 & 0.5 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.67 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Now I consider the marriage market under level-2 foresight. In matching μ_1, player m_1 could propose to w_1 and to w_2. Proposing to w_1 leads to μ_2 from where the system evolves, under level-1 foresight, to μ_5 with probability 0.33 and to μ_6 with probability 0.67, resulting in a utility of $0.33 * 0 + 0.67 * 2 = 1.33$. Proposing to w_2 leads to μ_3 with subsequent probabilities of 0.25 to evolve to μ_2, of 0.5 to evolve to μ_4 and 0.25 to μ_7. This leads to an expected utility equal to $0.25 * 2 + 0.5 * 0 + 0.25 * 1 = 0.75$. Hence, m_1 prefers matching to w_1 but would also accept a proposal by w_2 in μ_1.
In μ_1, m_2 could propose to w_1 leading to μ_5. In μ_5, the system goes to μ_4 with probability 0.67 and to μ_7 with probability 0.33, resulting in an expected utility equal to $0.67 * 2 + 0.33 * 1 = 1.67$. Matching to w_2 results in matching μ_4, from where the system goes to μ_6 with probability 1, resulting in an expected utility equal to 2. Hence, m_2 prefers matching to w_2 but would also accept a proposal by w_1 in μ_1.

In μ_1, player w_1 could propose to m_1 leading to μ_2, from where the system evolves, under level-1 foresight, to μ_5 with probability 0.33 and to μ_6 with probability 0.67, resulting in a utility of $0.33 * 2 + 0.67 * 1 = 1.33$. Proposing to m_2 leads to μ_5 with subsequent probabilities of 0.67 to evolve to μ_4 and of 0.33 to evolve to μ_7. This leads to an expected utility equal to $0.67 * 0 + 0.33 * 2 = 0.67$. Hence, w_1 prefers matching to m_1 but would also accept a proposal by m_2 in μ_1.
In μ_1, w_2 could propose to m_1 leading to μ_3. In μ_3, the system goes to μ_2 with probability 0.25, to μ_4 with probability 0.5 and to μ_7 with probability 0.25, resulting in an expected utility equal to $0.25 * 0 + 0.5 * 2 + 0.25 * 1 = 1.25$. Matching to m_2 results in matching μ_4, from where the system goes to μ_6 with probability 1, resulting in an expected utility equal to $1 * 2 = 2$. Hence, w_2 prefers matching to m_2 but would also accept a proposal by m_1 in μ_1.
To conclude, all proposals are accepted because they all lead to an increase in expected utility. This means that m_1 proposes to w_1, m_2 to w_2, w_1 to m_1 and w_2 to m_2. This leads to the following first row corresponding to μ_1 in matrix $P_{2,U}$

$$\begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \end{bmatrix}$$

▲

In the previous example, I have shown how a marriage market may evolve under the new assumption under level-1 and level-2 foresight. Something still unrealistic in this approach is that two players that both propose to each other may not get matched. In Example 5.9, this may occur in matching μ_1 under level-1 foresight where players m_2 and w_2 propose to each other but the system has a positive probability to move to μ_5 where m_2 is matched to w_1. Thus, under the set-up defined so far, such pairs may still exist and in future research, it should be considered how to deal with the existence of such pairs in a realistic way.
In the last part of this section, I sum up the newly introduced notation and concepts that are used in Section 5.2.2. The only new notation introduced is $A_{i\mu_x,K}$ that is the set of matchings that player i can create by deviating from μ_x under level-K foresight for which it holds that this is a utility improvement for him and by the opposite player if the deviation involves adding a link. Hence, for this player $j \neq i$ that i proposes to it must be that $V_{j\mu_y,K,U} > U^j(\mu_x(j))$. As well, I have added subscripts to indicate the utility maximisation setting. The recursion that is implied by the overview presented next works exactly the same as described in Section 4.2.

Definition 5.10. Let $K \geq 1$. The deviation $\mu_x \to_S \mu_y$ is a *stochastic level-K deviation for player $i \in S$ under utility maximisation* if $V_{i\mu_y,K,U} > V_{i\mu,K,U}$ for all $\mu \in A_{i\mu_x,K} \setminus \{\mu_y\}$ and $\mu_y \in A_{i\mu_x,K}$.

Equation 5.11. The probability that the system goes from matching $\mu_x \in \mathbb{M}$ to $\mu_y \in \mathbb{M}$ under utility maximisation and $K \geq 1$, when $|L_{K,\mu_x,U}| > 0$ is:
$P_{K,U}(\mu_y|\mu_x) = \frac{1}{|L_{K,\mu_x,U}|} \sum_{i \in L_{K,\mu_x,U}} I_{i\mu_x\mu_y,K,U}$.

5.2.2 Counterexample, proof and takeaways

Under level-1 foresight, it is clear that stochastic deviations are not always utility-maximising deviations. However, under level-1 foresight, the opposite is true: utility-maximising deviations are also always stochastic deviations. This raises the question of whether this property holds in general. In Example 5.12, I show that this property can not be said to hold for any K.

Example 5.12. In this example, I show that a utility-maximising level-2 stochastic deviation does not necessarily need to be a stochastic level-2 deviation under the assumptions as described in Section 4.2. I assume that the utilities are decreasing in equal steps and these are shown below for convenience:

$$m_1 : U^{m_1}(w_2) = 3; U^{m_1}(w_1) = 2; U^{m_1}(w_3) = 1; U^{m_1}(m_1) = 0$$

$$m_2 : U^{m_2}(w_1) = 3; U^{m_2}(w_2) = 2; U^{m_2}(m_2) = 1; U^{m_2}(w_3) = 0$$

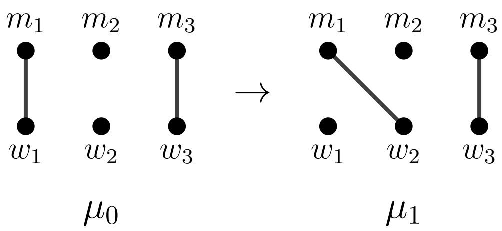
$$m_3 : U^{m_3}(w_1) = 3; U^{m_3}(w_2) = 2; U^{m_3}(m_3) = 1; U^{w_3}(w_3) = 0$$

$$w_1 : U^{w_1}(m_2) = 3; U^{w_1}(m_3) = 2; U^{w_1}(m_1) = 1; U^{w_1}(w_1) = 0$$

$$w_2 : U^{w_2}(m_2) = 3; U^{w_2}(m_3) = 2; U^{w_2}(m_1) = 1; U^{w_2}(w_2) = 0$$

$$w_3 : U^{w_3}(w_3) = 3; U^{w_3}(m_1) = 2; U^{w_3}(m_2) = 1; U^{w_3}(m_3) = 0$$

The deviation that is considered here is by m_1 and is as follows:



First, I show that the deviation from μ_0 is not a stochastic level-2 deviation for m_1 without assuming utility-maximising players. In matching μ_0, player m_1 has a utility equal to 2. Therefore, to check whether $\mu_0 \to \mu_1$ is a stochastic level-2 deviation by m_1, I check m_1's expected utility following a level-2 deviation to μ_1. In μ_1, under level-1 foresight, deviations exist by players all players in $N \setminus \{m_1\}$. As defined in Section 4.2, out of all deviators, a deviator is randomly picked and of this deviator's possible deviations a deviation is randomly picked. Hence, each player in $N \setminus \{m_1\}$ has a probability of 1/5 to be chosen as the player to deviate.

In μ_1, player m_2 deviates by matching either w_1 or w_2, resulting in an expected utility for m_1 when m_2 deviates equal to $0.5 * 3 + 0.5 * 0 = 1.5$. Player m_3 deviates by matching either w_1, by matching w_2, or by becoming single, resulting in an expected utility for m_1, when m_3 deviates, equal to $0.33 * 3 + 0.33 * 0 + 0.33 * 3 = 2$. Player w_1 deviates by matching either m_2 or m_3, resulting in an expected utility for m_1, when w_1 deviates, equal to 3. Player w_2 deviates by matching either m_2 or m_3, resulting in an expected utility for m_1, when m_2 deviates, equal to 0. Player w_3 deviates by unmatching m_3, resulting in an expected utility for m_1, when w_3 deviates, equal to 3. Now, taking all these utilities together with their probabilities, the expected utility of a stochastic level-2 deviation to μ_1 equals $0.2 * (1.5 + 2 + 3 + 0 + 3) = 1.9$. This is lower than the utility that m_1 gets in μ_0 and therefore, this is not a stochastic level-2 deviation for m_1 without the assumption of utility maximisation.

Now, I show that the deviation from μ_0 to μ_1 is a stochastic level-2 deviation for m_1 when assuming utility-maximising players. Again, in μ_1, each player in $N \setminus \{m_1\}$ has a stochastic level-1 deviation. However, now each of these deviators deviates and proposes to his top preferred among the opposite players that would accept him, and each player accepts a proposal if this means a utility increase for him. Each deviator in $N \setminus \{m_1\}$ has a probability equal to 1/5 to be chosen the player to deviate.
In μ_1, player m_2 deviates by matching w_1, resulting in an expected utility for m_1, when m_2 deviates, equal to 3. Player m_3 deviates by matching w_1, resulting in an expected utility for m_1, when m_3 deviates, equal to 3. Player w_1 deviates by matching m_2, resulting in an expected utility for m_1, when m_2 deviates, equal to 3. Player w_2 deviates by matching m_2, resulting in an expected utility for m_1, when w_2 deviates, equal to 0. Player w_3 deviates by unmatching m_3, resulting in an expected utility for m_1, when w_3 deviates, equal to 3. Now, taking all these utilities together with their probabilities, the expected utility of a stochastic level-2 deviation under utility maximisation to μ_1 equals $0.2 * (3 + 3 + 3 + 0 + 3) = 2.4$. This is higher than the utility that m_1 gets in μ_0 and therefore, this is a stochastic level-2 deviation for m_1 under the assumption of utility maximisation.
It is easily observable that $\mu_0 \to \mu_1$ is also a stochastic level-2 deviation under utility-maximising players for w_2 because $U^{w_2}(\mu_0(w_2)) = 0$ and there exists a positive probability of getting a partner in an induced matching after μ_1 while being worse off than in μ_0 is impossible. ▲

Now the difference has been established between stochastic deviations with and without the assumption of utility maximisation, I show in the next Theorem 4.20 that stochastic deviations under utility maximisation must also be optimistic deviations for any $K = 2$. This proof is very similar to the proof of Theorem 4.20 because the proof relies on the interaction of preferences in the optimistic setting and the utilities in the stochastic setting, while the same induction argument is used as in Theorem 4.20.

Theorem 5.13. If $\mu_2 \to_S \mu_1$ is a utility-maximising stochastic level-2 deviation, then it must also be an optimistic level-2 deviation.

Proof. Let's assume that there exists a utility-maximising stochastic level-2 deviation $\mu_2 \to_S \mu_1$ that is not an optimistic level-2 deviation, and $|S| \in \{1, 2\}$. Now, all players in S deviate to μ_1, and thus, for all players in S, their expected utility in μ_1 following an optimistic level-2 deviation to μ_1 is higher than their utility in μ_2. Hence, for all $i \in S$ $V_{i\mu_1,2} > U^i(\mu_2(i))$, which is equivalent with $\sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) > U^i(\mu_2(i))$. Now there are two situations to consider following the stochastic level-2 deviation to μ_1:
Situation 1: no more optimistic level-1 deviations exist from μ_1 and hence no more utility-maximising level-1 deviations exist. In that case $P_1(\mu_1|\mu_1) = 1$. Now to let this be a valid utility-maximising level-2 deviation, it must hold for all $i \in S$ that $V_{i\mu_1,2} = U^i(\mu_1(i)) * P_1(\mu_1|\mu_1) = U^i(\mu_1(i)) > U^i(\mu_2(i))$, equivalent with $\mu_1(i) \succ_i \mu_2(i)$. However, this leads to a contradiction because I assumed that $\mu_2 \to_S \mu_1$ is not an optimistic level-2 deviation.
Situation 2: there exists at least one utility-maximising level-1 deviation in μ_1. Now, following the definition of a utility-maximising level-2 deviation, for all $i \in S$, $V_{i\mu_1,2} = \sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) > U^i(\mu_2(i))$. Because $\mu_2 \to_S \mu_1$ is not an optimistic level-2

deviation, there does not exist for each $i \in S$ a matching μ_0 such that $\mu_0 \in f_1(\mu_1)$ and $\mu_0(i) \succ_i \mu_2(i)$. Now I know that utility-maximising level-1 deviations are also always optimistic level-1 deviations, meaning that if $P_1(\mu_0|\mu_1) > 0$, it must be that $\mu_0 \in f_1(\mu_1)$. Now for each matching $\mu_0 \in \mathbb{M}$ for which $P_1(\mu_0|\mu_1) > 0$ it must be that for at least one player $i \in S$: $U^i(\mu_0(i)) \leq U^i(\mu_2(i))$. Namely, if this does not hold then $\mu_2 \rightarrow_S \mu_1$ is an optimistic level-2 deviation because then, for all $i \in S$, there would exist a $\mu_0 \in f_1(\mu_1)$ with $\mu_1 \in f_2(\mu_2)$ and $\mu_0(i) \succ_i \mu_2(i)$. However, this leads to a contradiction because now $\sum_{\mu_0 \in \mathbb{M}} P_1(\mu_0|\mu_1) * U^i(\mu_0(i)) \leq U^i(\mu_2(i))$ for at least one i because for all μ_0 for which it holds that $P(\mu_0|\mu_1) > 0$, it holds $U^i(\mu_0(i)) \leq U^i(\mu_2(i))$. Therefore, this cannot be a utility-maximising level-2 deviation for i if this is not an optimistic level-2 deviation and thus leads to a contradiction with the assumption that $\mu_2 \rightarrow_S \mu_1$ is not an optimistic level-2 deviation.
Now it can be concluded that each utility-maximising level-2 deviation must also be an optimistic level-2 deviation because assuming the existence of a utility-maximising level-2 deviation that is not an optimistic level-2 deviation leads in each case to a contradiction.
□

Now it has been shown that utility-maximising stochastic level-2 deviations must also be optimistic level-2 deviations in Theorem 4.20. Furthermore, it has also been shown that utility-maximising stochastic level-2 deviations do not need to be stochastic level-2 deviations in Example 5.12. Hence, the following Venn diagram in Figure 2 is drawn about the relation of these different types of deviations:

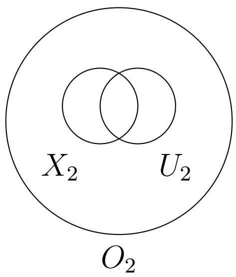


Figure 2: Relation of the different level-2 deviations: optimistic (O_2), stochastic without utility maximisation (X_2) and stochastic with utility maximisation (U_2).

Because the utility-maximising stochastic level-2 deviation is a restricted version of the optimistic level-2 deviation as has been shown, Theorem 4.21 about the relation of the stochastic stable sets and the stable sets in the optimistic setting must also hold for utility-maximising deviations. Namely, that result fully depends on the fact that stochastic deviations are restricted optimistic deviations.
Since it might be more realistic to assume that players deviate to the most preferred partner as is done in the utility-maximising set-up, this set-up might be more preferred. Nonetheless, I have shown that utility-maximising deviations significantly differ from non-utility-maximising stochastic deviations, while both are also always optimistic deviations for any $K > 0$. However, in this paper, analytical derivations about the differences of the

approaches have not been given and these are suggested for future research. Additionally, any concrete example of the benefit of these approaches in future research would have to involve at least 6 players to get a proper view of the dynamics of each of the approaches, while under levels of foresight larger than 1, the derivations under stochasticity become very extensive.

5.3 Coalitional deviations

One of the key assumptions in both the optimistic and the stochastic description of the marriage market was that individuals were given the chance to deviate and that they proposed to other individuals when forming a link, or could solitarily deviate when a link was to be cut. The deviators anticipated further deviations also by individuals that proposed to other individuals. In the literature, however, in network games, or more specifically in marriage market games, deviations by coalitions have also been considered. For instance, Herings and Khan (2022) also consider coalitional deviations in networks under limited foresight and Chwe (1994) studies coalitional moves by players under full foresight where players have preferences over the outcomes of the game. Therefore, to also show that this research can be applied in the context of coalitional deviations, I give an example of deviations by coalitions in both stochastic and optimistic settings. Further derivation of theorems and definitions on this subtopic is suggested for future research.

In this paper so far, definitions, examples, and theorems were given both in the optimistic and stochastic setting with the key underlying assumption of level-K foresight for each player in the marriage market. In this section, I define the concept of coalitional foresight which considers the possibility of a coalition of players deviating. Including foresight, this means that a group of players in the game may jointly deviate to some new matching in anticipation of further deviations by coalitions. In Example 5.14, I show how a marriage market can evolve by optimistic level-2 deviations where coalitions $S \subseteq N$ can deviate without the restriction that $|S| \leq 2$. The key underlying assumption is that deviating coalitions can expect deviations by other coalitions in induced deviations. The members of the deviating coalition simultaneously deviate and the links that they connect or break form one step. A link formed must be between two members of the coalition and a link deleted must involve only one member of the coalition. Players can be part of a coalition in the optimistic setting when they see the path of induced coalitional deviations to a more preferred partner, while players within the coalition can have different beliefs about the formation of subsequent coalitions. In the stochastic setting, players can be part of a coalition when the expected utility in the matching that the coalition deviates to is higher. In Example 5.14, I show optimistic level-2 deviations and in Example 5.15, I show a stochastic deviation under level-2 foresight.

Example 5.14. The purpose of this example is to show several sequential coalitional optimistic level-K deviations. Let me consider the following marriage market following marriage problem $(M, W, \succ)$ with men $M = \{m_1, m_2, m_3\}$ and women $W = \{w_1, w_2, w_3\}$. The following preferences are present by the players of the game:

$\succ_{m_1}$: w_1, w_2, w_3, m_1

$\succ_{m_2}$: w_2, w_1, w_3, m_2

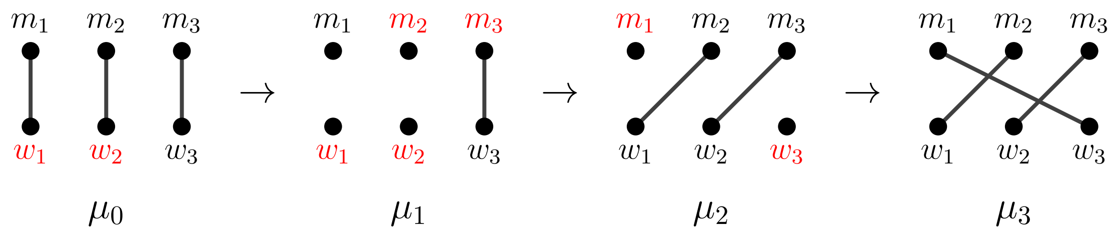
$\succ_{m_3}$: w_2, w_1, w_3, m_3

$\succ_{w_1}$: m_2, m_3, m_1, w_1

$\succ_{w_2}$: m_3, m_1, m_2, w_2

$\succ_{w_3}$: m_3, m_1, m_2, w_3

In the picture next, it is visible how the marriage market problem can evolve under level-2 foresight. In each step, the members of the coalition that initiate the deviation have been coloured red. In the next paragraphs, I illustrate the rationale behind each level-2 deviation.

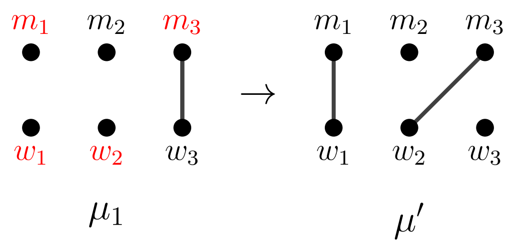


In matching μ_0, players w_1 and w_2 can form a coalition that both break their links with their respective partners. Player w_1 wants to end up with m_2 and she anticipates on the coalition $\{m_2, w_1\}$ that can perform a level-1 deviation in μ_1. Player w_2, forming a coalition with w_1, breaks with m_2 in μ_0 in anticipation of the coalition $\{m_3, w_2\}$ forming a match when performing a level-1 deviation in μ_1.
In μ_1, the players m_2, m_3, w_1 and w_2 can form a coalition that can perform a level-2 deviation by matching to each other. Players m_3 and w_2 match to each other through this coalitional level-2 deviation and both get their most preferred partner. Therefore, both players will no longer deviate and no level-1 deviations exist in μ_2 that affect them. Hence, for these players, this is a fruitful level-2 deviation. Player w_1 gets her most preferred partner and knows that after this coalitional level-2 deviation to μ_2, no coalitional level-1 deviations exist that let her end up with a less preferred partner than in μ_1. She knows this because m_2 can no longer match with his preferred w_2 and therefore for him no more level-1 deviations exist in μ_1 as w_1 is his second most preferred partner and he can no longer get w_2. In μ_2, also m_2 improves compared to μ_1 and no level-1 deviations exist anymore that let him end up with a less preferred partner than in μ_1. Hence, this is also for him a fruitful level-2 deviation.
In μ_2, players m_3, w_1, w_2 have their most preferred option. Hence, for them, no coalitional level-2 deviations exist. Player m_2 could only gain by matching w_2 which w_2 would not let happen, so also for m_2 no level-2 deviations exist. Therefore, the only players that can gain by deviating are players m_1 and w_3 that can match each other which is a coalitional level-2 deviation with incomplete support since from μ_3 no more deviations exist for these players. ▲

As can be seen in the previous example, coalitions can also exist of only two players of the opposite sex that decide to deviate, with one of the two proposing to the other. Coalitional deviations could even be defined to also include deviations by single players. Therefore, the optimistic level-K deviations can be viewed as a restriction on the coalitional level-K deviations in the optimistic setting.
The next example is about stochastic coalitional level-K deviations.

Example 5.15. In this example, I show a coalitional stochastic level-2 deviation. First, however, I show how the probabilities regarding the evolution of the system are calculated in a level-1 setting in order to have this clarified before showing the coalitional stochastic level-2 deviation. The same set-up and preferences as in Example 5.14 are used.
I consider the possible coalitional stochastic level-1 deviations from the matching $\mu_0 \in \mathbb{M}$ where every player is single such that for every $i \in N$: $\mu_0(i) = i$. Because I consider level-1 foresight, no utilities need to be defined. Obviously, several players can improve in this matching as all players indicate that being single is their least preferred option. In fact, each set of players $S \subseteq N$ with an equal amount of men and women would be a possible coalition in which each player can a get more preferred partner in one step. Namely, in such a set, each player could just improve by matching any player of the opposite sex because this would always be an improvement on being single. However, some rule is needed to decide on the probability of each coalition being chosen. Thereafter, the probabilities of the system evolving from μ_0 to any matching $\mu \in \mathbb{M}$ can be established. The number of coalitions with two players is $\binom{3}{1}\binom{3}{1} = 9$, the number of coalitions with four players is $\binom{3}{2}\binom{3}{2} = 9$ and the number of coalitions with 6 players is $\binom{3}{3}\binom{3}{3} = 1$. This means $9 + 9 + 1 = 19$ possible coalitions exist. When this set-up was to be made formal, a realistic decision rule is to be chosen that decides on the right coalition that can deviate. For simplicity, I say that out of these 19 possible coalitions, one coalition is randomly chosen such that each coalition has a probability equal to $\frac{1}{19}$ to be chosen. Now, in μ_0, each coalition consisting of two players has only one possible matching to be formed, while each coalition of four players can form two different matchings and the coalition of all players could form six different matchings. For the sake of this example, I assume that each matching μ, given that a coalition that is chosen that can form the matching, has a chance of being chosen equal to 1 divided by the number of matchings that could follow a deviation by a coalition. As a consequence, each matching following a deviation by a coalition with cardinality two has probability $\frac{1}{19} * 1 = \frac{1}{19}$ to be the matching the system evolves to, while matchings formed by coalitions with cardinality four have probability $\frac{1}{19} * \frac{1}{2} = \frac{1}{38}$ and each matching following a deviation by a coalition with cardinality six has probability $\frac{1}{19} * \frac{1}{6} = \frac{1}{114}$. This decision rule is in line with the set-up from Section 4.2. In that set-up, over all possible deviators, a deviator was picked and then over all his possible deviations, a deviation was randomly picked. When considering the matching without links each matching can only be formed by a unique coalition. However, in general, this is not necessarily true because a link can be broken by two different players, and therefore a matching μ' resulting from a coalitional deviation in μ could be the consequence of deviations by two different coalitions. With even larger coalitions with deviators breaking links, matching μ' could be resulting from even more different coalitions. Nevertheless, for simplicity, I do not consider that possibility here.
Now, under level-2 foresight in some matching $\mu \in \mathbb{M}$, each player knows the probability

of the system evolving to each other matching by a level-1 deviation. Based on this notion, each player knows to which matching the system should evolve to improve on his expected utility. Namely, if the system evolves to μ', each player knows the probabilities of the system evolving through a level-1 deviation from μ' to any other matching in $\mathbb{M}$. Hence, each player decides to deviate from μ to that μ' such that from μ' the expected increase in utility is positive by a level-1 deviation by some coalition that is (randomly) picked according to a decision rule.
Let me now consider μ_1 from Example 5.14. I show that the deviation from μ_1 to μ' is a coalitional stochastic level-2 deviation. This can be shown without considering utilities. In the next figure, I have coloured the coalition red.



Clearly, in μ', players m_3 and w_2 are matched with each other and both are their best-preferred partners. Hence, in μ', both will not deviate and will therefore not be part of any coalition when in μ'. This means that for m_3 and w_2 deviating from μ_1 to μ' is a fruitful level-2 deviation. Also, m_1 has his best-preferred partner in μ' and hence will also not be part of a deviating coalition in μ'. This means that in μ' the possible coalitions are $\{m_2, w_1\}$ and $\{m_2, w_3\}$. Every player assumes that each coalition has a chance of 0.5 to be chosen and so that the following matching also has a probability of 0.5. For player w_1 it means that she has a probability of 50% to be matched with m_2 following a level-1 deviation from μ' resulting in a 50 % chance of improvement compared to μ_1 and a 50 % chance of keeping the same utility. Therefore, for w_1, $\mu_1 \to \mu'$ is a fruitful coalitional stochastic level-2 deviation. Player m_1 foresees that in μ' that either m_2 and w_3 form a link or m_2 and w_1 will do so. In the first instance, he improves on utility compared to μ_1 and in the latter, his utility remains the same. Therefore, the system evolving from μ_1 to μ' means also for him an expected improvement in utility. Consequently, the deviation from μ_1 to μ' is a valid coalitional stochastic level-2 deviation by $\{m_1, m_3, w_1, w_2\}$. ▲

From the examples that have been given in this section on coalitional deviations, certain conclusions can be drawn that should be kept in mind when writing formal definitions and drawing conclusions on this subtopic.
In both set-ups, it can be observed that the amount of possible coalitions that can be formed is generally large. As a consequence, the possible evolution in one single deviation of the marriage market can vary a lot. In fact, in the optimistic setting, it is obvious that coalitional moves are a relaxation of deviations by individuals and that therefore the amount of possible deviations is large. This does, however, not imply any conclusion about the outcome of the marriage market problem. When coalitions of any size are allowed to exist, then the system could even evolve from a matching without links to stable matchings in one step. Furthermore, in the stochastic set-up, when analysing possible deviations, the number of possible coalitions that may form induced deviations is also very high. This means that calculations about possible level-K deviations become very

extensive. To mitigate this a bit, a clear rule on how coalitions are picked that makes the system evolve should be formulated. In Example 5.15, I used a very general rule on how to formulate the beliefs of players on the evolution of the system. For certain, this rule should be made more realistic and restricted. Suggestions to improve this rule include restricting the coalition size or allowing people to choose to be part of a limited number of coalitions.

6 Conclusion

In this paper, I started off by introducing the concept of limited foresight in the one-to-one matching problem. This limited foresight comprises the feature that each player is aware of the fact that a change in the marriage market can induce further changes. The level of foresight determines the length of the horizon that players can foresee that is induced by their change of the system. Based on this notion of limited foresight, I define a stable set of matchings as an outcome of the marriage market problem. This initial set-up has been largely based on the approach of Herings and Khan (2022) that define limited foresight in network games. It must be possible to reach the stable set by consecutive deviations from all matchings outside the set and it must be impossible to leave the stable set by consecutive deviations once the system has evolved to the stable set. Furthermore, the stable set must meet minimality by assuring that no subset of the stable set exists that satisfies the two aforementioned criteria. The stable set has been proved to always exist and has been proved to be unique. The set depends on the level of foresight that is assumed in the matching game. The degree of foresight has always been assumed to be the same for all players in the game. Next to showing the existence and uniqueness of the stable set, I have shown that the stable set equals the union of cycles present in the game. Assuming myopic players is equivalent to assuming level-1 foresight and under that assumption, the stable sets equals the set of stable matchings and each of these matchings is a singleton cycle. Also, it has been shown that the level-1 stable set is a subset of the level-2 stable set. I have not shown that this result also holds vice versa, which is beyond the scope of this research. I suggest this to be investigated in future research. Last, under the assumption of α-reducibility, the level of foresight does not influence which matchings are part of the stable set.

In the approach of the previous paragraph, players were assumed to be optimistic about the outcomes of their deviations. This approach implies that players already deviate when there exists only one path out of many paths leading to some improvement compared to the matching that is deviated from. This optimism is no longer assumed in the so-called stochastic approach. Namely, under the stochasticity assumption, players incorporate the fact that many paths exist which influences their decision-making process when playing the game. Furthermore, I introduced the concept of utility that each player attaches to each partner of the opposite sex and to being single. Each player has a belief about the utility he expects to get when deciding on adding or deleting a link. It has been shown that deviations under the assumption of stochasticity must also be deviations under the assumption of optimism under level-2 foresight and that they are the same under level-1 foresight. As a result, under level-1 foresight, the stable sets are equal and under level-2 foresight, they might differ but the relationship between them is clear. For larger degrees of foresight, I have shown, under the additional assumption that players may have preferences over matchings instead of over players, that the stochastic deviations do not need to be optimistic deviations. Consequently, generally, the stable set under stochasticity differs from the stable set under optimistic players and none of the two sets is a subset of the other. Furthermore, also under stochasticity, in α-reducible matching problems, the level of foresight assumed does not influence which matchings are in the stable set. The new assumption of stochasticity also allows attaching probabilities to each stable set. Namely, from each matching in the marriage market, it is possible to calculate the

probability of ending in all existing stable sets.
In the last part of the paper, I change the assumptions that were made in the initial stochastic set-up and check, when possible, how this affects the results in the marriage market. I choose to investigate this because, when describing the initial stochastic set-up, I make assumptions for simplification purposes. Also, in the initial set-up, I stick close to the approach by Herings and Khan (2022). I change the initial assumptions in three different ways and I conclude which takeaways exist when each new assumption was to be made formal in possible future research.
In the decision-making process, under the initial assumptions of stochasticity, each player decides on additions or deletions by comparing the utility he gets from his deviation to the utility obtained from the partner matched to in the matching he considers to deviate from. Each player, however, knows that other players might deviate if he chooses to not deviate. Therefore, I assume that players decide on deviating by comparing to the expected utility of staying which I refer to as opportunity utility. I show that it is impossible for each player to know the true utility of staying because the deviation of some players affects the deviation of the other players and vice versa. I simplify this by assuming that all players assume that the other players deviate under the assumptions of the initial set-up. Under this simplification, it is shown that new deviations appear that are no deviations under the assumption of optimism or stochasticity. Therefore, the consequences for stability cannot easily be derived and are suggested to be derived in future research. Also, I show that this set-up could lead to an impasse that should be accounted for in possible future research.
In my initial framework, players are assumed to randomly deviate to any matching with a possible increase in expected utility. This is in line with the approach by Herings and Khan (2022). I change this assumption by letting players deviate only to the matching resulting in the highest expected utility. Players do nevertheless accept any utility-improving proposal. I show that such deviations must also be deviations under the optimism assumption under level-2 foresight. Furthermore, I show that deviations under utility maximisation do not necessarily need to be stochastic deviations. Because of this first result, the same conclusions can be drawn on the relation of the level-2 stable set under optimism and under stochasticity with utility-maximising players as on the relation of the stable set under optimism and under stochasticity with non-utility-maximising players. However, this does not say anything about what differences the utility maximisation assumption makes in relation to stability. Also, coming up with a proper example of a situation with different stable sets involves extensive calculations. Therefore, both the analytical derivation and the presentation of an example are suggested for future research.
Last, I also give several examples of coalitional deviations where coalitions consisting of more than two players have the opportunity to deviate as a coalition. These coalitional deviations in the optimistic set-up can in fact be viewed as less restricted deviations of deviations by couples or individuals. I show that this approach can also be implemented under limited foresight with optimistic players by giving an example of such a deviation. I give an example of coalitional deviations under stochasticity. Through that example, I show that it is possible to implement the assumptions on coalitional deviations but that it requires extensive calculations. Also, I illustrate that, if this set-up were to be applied in future research, many possibilities for the evolution of the system exist and that it is recommended to impose restrictions that mitigate this problem in possible future research.

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