Life-cycle investment, the benefits of inflation-linked bonds to Dutch pension schemes

by

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Abstract

Inflation-linked bonds protect investors during periods of inflation uncertainty. Although the total market for real debt securities is only a fraction of the nominal bond market, there are many benefits of real wealth preservation for investors and pension funds. Despite this, the Dutch government does not issue these instruments. This thesis investigates the welfare benefits of issuing Dutch inflation-linked bonds and the welfare loss of inflation in general. We analyze an intertemporal consumption and terminal wealth model under inflation when short positions in risky assets are precluded. The models are calibrated to the KNW(1.5%) parameter set. We find significant welfare benefits in certainty equivalent wealth when investors can invest in real debt securities.

Keywords: Portfolio and consumption choice; Inflation risk; Inflation-linked bonds; lifecycle investing, expected utility

Contents

1 Introduction

Inflation-linked bonds, bonds whose principal and coupon payments are linked to inflation, are anything but new financial instruments. The first inflation-linked bonds (ILB) issued date back to 1790 in the U.S. The economic reasoning around interest payments on debt paper in real rather than nominal terms is described well before the twenty-first century (see, for example, Jevons William (1875); Humphrey (1974); Shiller (2003)). A primary reason for real debt payments is that they better guarantee the value retention of debt payments than guaranteed payments in nominal terms. In addition, it is an efficient value-preserving instrument for investors to protect themselves during uncertain and volatile inflation. Despite all these arguments, the global market for real debt securities is only a fraction of its nominal counterpart. In the Netherlands, there is not even a market for inflation-linked bonds. Despite authors such as van Bilsen et al. (2020) and van Gastel et al. (2022) demonstrating the benefit of such derivatives to welfare effects.

The new Dutch pension contract identifies purchasing power as one of its concerns but does little to address inflation risk (van Gastel et al., 2022). Unjustifiably, there is a significant loss in real wealth even with relatively low inflation. For example, over a 50-year investment horizon, investors lose roughly 40% in real wealth if the inflation is at 2%. Furthermore, it seems that the stable and relatively low inflation rates over the past decade are something of the past. The times have changed, and high inflation is the new normal; the DNB expects core inflation in the Eurozone to be well above the desired 2% in the coming years (DNB, 2022). The sharp rise in inflation is only increasing investor demand for inflation-covering instruments. The same is valid for pension funds: an adjustment to steer towards indexed pensions does fit well in the context of the new pension contract. Issuance of index-linked bonds contributes to welfare by removing market failures. For pension funds, index-linked bonds are a welcome hedge against inflation risks. Thereby, ILBs contribute to the stability of the pension sector and indirectly to public finances (Werkgroep-Reële-Begroting, 2005). To that end, this thesis will contribute by looking at the welfare effects when pension funds, entirely or only partly, hedge both interest rate and inflation risk. To investigate this, we use the paper of van Bilsen et al. (2020), which describes both an intertemporal consumption and terminal wealth problem in which both risks are present.

For our simulations, we use the scenario set used by Metselaar et al. (2020), among others: the KNW(1.5%) set (see Koijen et al. (2010) for the KNW-model and Draper (2014) for a detailed description). This set is characterized by an expected long-term real interest rate of 1.5%, and the scenario set is calibrated with Dutch data. We apply a numerical sensitivity analysis, examining the welfare effects for more significant inflation shocks. In addition, we run the model for the scenario set of Zhou (2014), calibrated for U.S. data. A higher long-term inflation rate and considerable stochasticity characterize this dataset. We show that in the U.S., the need for inflation-covered portfolios is high, and the wealth loss is significantly higher for this dataset. With this result, Dutch investors have been warned, and there is a severe need for inflation-linked bonds in Dutch financial markets.

This thesis analyzes life cycle investment elements in the new Dutch pension system. We evaluate the portfolio weights for two models, one that maximizes the utility of consumption and another that maximizes terminal wealth at retirement. First, we examine the intertemporal consumption model from van Bilsen et al. (2020). We are using their intertemporal consumption problem, where premium payments are exogenous. We took their analytical solutions from the same paper but calibrated them to our scenario set. Dutch investors would allocate more than 50% of their total wealth into real debt securities. Afterward, we split the investors' total wealth into financial and human wealth to investigate the impact of human capital on the investors' optimal portfolio strategy. Through human capital, we understand the discounted value of future earnings. We find similar results; investors maintain a significant position of inflation-linked bonds in their portfolios in terms of financial wealth.

Next, we consider the terminal wealth problem of Brennan and Xia (2002) and use the optimal investment allocations found by van Bilsen et al. (2020). We look at two financial markets: a complete market in which inflation risk is tradable and an incomplete market in which it is not. In the incomplete market, the investor only has access to nominal bonds besides the risky stock. Complete real protection is only possible if the investor holds a portfolio of several long and short positions in nominal bonds with different maturities (Balter et al., 2021), as ILBs are unavailable. The disadvantage of this strategy is that it is susceptible to parameter assumptions. This this however, does not allow for short positions in risky assets, thereby following the model assumptions of of van Bilsen et al. (2020). Between the two financial markets, we will compare certainty equivalent and welfare effects. We find that moderate risk-averse investors ($\gamma = 5$) lose 75.40% of certainty equivalent wealth if they cannot hedge inflation risk perfectly. A comparison between Dutch and U.S. parameters shows that U.S. investors are better off, incurring a 44.06% welfare loss.

The following paragraphs summarize the theory behind life-cycle investing, documenting the foundational papers and noting the authors' main findings. Specifically, we look at what previous authors have found in situations considering both interest rate and inflation risk.

In the 1960s, Modigliani (1966) formulated the life-cycle hypothesis. It describes the behavior of consumption and savings over life cycles, mainly determined by expected income and life expectancy. According to the hypothesis, individuals at a specific age and wage could determine the optimal consumption and saving behavior. The goal is to smooth out their consumption so that it is better protected from future fluctuations in income. The life-cycle theory is built on the result that people accumulate their financial assets during their working years and then deplete them during retirement. It is, therefore, not surprising that the purpose of a significant part of an individual's savings is to finance her retirement. This behavior is empirically observed by Cocco et al. (2005), which leads to a hump-shaped savings pattern over time.

Merton (1969) created the first theoretical framework regarding this subject. In which, for the first time, a model in continuous time came to analytical answers. We will briefly explain how this model works and what we can learn from it. It contains only equity risk and takes a single risky asset and a risk-free bank account. Merton (1969) therefore assumes the well-known Black-Scholes model. The Black-Scholes model assumes interest rates, wages, and inflation to be constant, as are the price of equity risk and the volatility of assets. The individual maximizes the expected consumption utility, where the constant relative risk aversion (CRRA) class represents the investors' preferences.

Following the original hypothesis, an individual prefers stable savings and consumption. In addition to Merton, Samuelson (1969) also derived closed-form solutions. Both concluded that optimal consumption depends on the wealth and age of the investor. In addition, the fraction of consumption to total assets is time-dependent. On the other hand, the optimal fraction of investment in the risky asset is constant during the period of optimization and, thus, independent of age. Furthermore, the optimal allocation depends on the market price of equity risk, volatility of assets, and risk aversion. Risk aversion indicates the extent to which the investor desires a certain but perhaps low outcome over a higher but uncertain one.

Merton and Samuelson's first models did not include the split of total wealth into financial and human wealth. Human wealth is the discounted value of future labor earnings. When an individual starts their working life, he or she has a relatively large amount of human capital, which gradually decreases until retirement. After retirement, the human capital is depleted, and the remaining wealth equals the financial wealth. Human assets are not tradable, and the decreasing relationship over time changes the optimal constant exposure or assets relative to the risky asset. Especially when income is constant and deterministic, human wealth is more or less the same as an investor taking a position in a risk-free bank account (see Munk and Sørensen (2010) and Cairns et al. (2006)). It yields no financial risk, and the only existing risks come from outside financial markets and investment options.

This thesis builds upon several previous papers that examined dynamic asset allocation with inflation risk. Which Brennan and Xia (2002) and Campbell and Viceira (2001) are some of the works that are leading. Similar to both papers, we model inflation using an Ornstein-Uhlenbeck process, also called a mean-reversion process. Both papers show that hedge demands depend on the investor's time horizon and risk aversion. Koijen et al. (2010) and Sangvinatsos and Wachter (2005), on the other hand, relax that assumption; they furthermore assume risk premia on risky assets are stochastics. We, however, keep risk premia constant and known in advance. While the assumption of constant risk premia may seem restrictive, Bossaerts and Hillion (1999) find no evidence of out-of-sample excess return predictability in 14 countries using as potential predictors lagged excess returns, January dummies, bond and bill yields, and dividend yields. Munk et al. (2004) examines asset allocation using a power utility investor. Their model calibrates U.S. stock, bond, and inflation data and illustrates investment recommendations for various investment horizons. He furthers different appetites for risk parameters, and his results are, in line with other papers, that aggressive investors tend to allocate more to a stock index. In contrast, conservative investors hold larger allocations in cash and bonds. Another method is to include household prices as a risky investment; Van Hemert (2010) does this by investigating household interest rate risk management. His models not only describe real allocations but finds interesting results concerning preferences in housing mortgages for different cohorts.

Adding inflation risk, as mentioned earlier, has been the subject of numerous studies. On the other hand, studies adding inflation-linked bonds to hedge this risk has been much less so. The reason is that ILBs were only introduced in 1997 by the U.S. Treasury, and only a few other countries have followed suit. Although the liquidity of the ILB market has increased in recent years, it remains a mere fraction of the nominal government bond market (Viceira, 2013). Therefore, this security is not always available in desired quantities by portfolio managers globally. However, recent papers, such as Campbell and Viceira (2001) and Brennan and Xia (2002), show that an infinitely risk-averse investor would only invest in ILB to protect purchasing power. They are showing that there is indeed a desire for investors to have such products available to them. When these securities are not part of the asset menu, investors prefer to keep their capital in cash (Campbell & Viceira, 2001).

The structure of this thesis is as follows; Section (2) focuses on the intertemporal consumption model of van Bilsen et al. (2020). In which the investor has the option to invest in a 10-year nominal bond, 30-year inflation-linked bond, stock, and risk-free bank account. We evaluate the optimal asset allocations in Section (3) when the investor maximizes the utility of consumption. In the end, we split total wealth into human and financial capital and present the optimal allocations as a fraction of financial wealth.

Section (4) continues with the second model, in which the investor maximizes the utility of terminal wealth. In this setting, we evaluate the optimal asset allocations for two different financial markets. A complete market, in which we follow van Bilsen et al. (2020), in which the investor has all risky assets and thus can fully protect himself against all risks. Furthermore, an incomplete market, in which we remove the inflation-linked bond, but inflation risk remains present. For this market, we look at Brennan and Xia (2002). We describe the intuition behind the investor choices for our general scenario set. Section (4.4.1) numerically examines the impact of risk aversion and stochasticity of inflation on the certainty equivalent wealth of investors. The numerical analysis includes a comparison between Dutch and US investors.

2 Consumption model

This chapter presents the benchmark model. It describes the preferences, financial market, wealth dynamics, model calibration, and optimal strategies. As mentioned in the introduction, the benchmark model is based on the model of van Bilsen et al. (2020) and Brennan and Xia (2002).

2.1 Assumptions

Before describing the benchmark model, we discuss several investor and financial markets assumptions. We start with the assumptions for the investor. The investor starts working at time $t = 0$ and continues to work until retirement age $t = T_R$ and dies at $t = T_D$. These investors' assumptions align with van Bilsen et al. (2020). The income earned by each individual is assumed to be risk-free, and income from work is defined by $Y(t)$. The investors' income is equal to one during the working periods and zero after the retirement age is reached.

For the investors' personal preferences, we make the following assumptions. The investor maximizes the expected lifetime utility from real consumption. The optimal investment strategy is determined to maximize total wealth to achieve this. We assume that the investor leaves no inheritance to relatives; therefore, total wealth is fully consumed and completely depleted at the time of death T_D. The individual has CRRA preference over real consumption, implying that the expected lifetime utility is given by

$$U(c(t)) = \begin{cases} \frac{c(t)^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0,\infty) \setminus \{1\}, \\ \log(c(t)) & \text{if } \gamma = 1, \end{cases} \tag{1}$$

where γ is the parameter for relative risk aversion and $c(t)$ is the individual's consumption choice at age t. The relative risk aversion parameter describes an individual's tendency toward uncertain events. The following example explains this principle. Suppose an investor can choose between two investments; one will pay a fixed return with certainty, say 4%, for example, a guaranteed government. The second investment return, however, is uncertain, say the stock of a large tech company. The return on the stock investment must be greater than 4%; otherwise, investors will not buy the stock. Investors who are risk averse will elect the first (certain) alternative over the second (risky) alternative.

A rise in the inflation rate decreases purchasing power and, consequently, the individual expected lifetime utility. The process that describes this price increase is referred to as $\Pi(t)$, which denotes the Consumer Price Index. Finally, we assume that the investor is impatient with consumption in future periods. This is called time preference and reflects an individual's trade-off between consumption in the future and consumption today. Mathematically, time preference is denoted by the parameter $\delta \geq 0$ and can be interpreted using a so-called discount function. It determines whether the consumption of a unit today yields more than a unit in the future. Combing the time preference without assumed utility function allows us to define the individual's expected lifetime utility by

$$U = \mathbb{E}\left[\int_0^{T_D} \mathrm{e}^{-\delta t} U\Big(\frac{c(t)}{\Pi(t)}\Big) \mathrm{d}t\right], \tag{2}$$

This rounds out the section regarding the individual's preferences, work, and retirement period. The next section describes the assumptions of financial market processes. It explains the choices made that align with the relevant studies.

2.2 Financial market

This section defines the financial market in which the investor may invest. We start by describing risks central to this thesis, inflation and interest rate risk. After that, we introduce the stochastic processes for the three risky assets; a stock index, a nominal bond, and an inflation-linked bond. Besides the risky assets, the investor also has the option to take a risk-free cash position. We follow the notation of van Bilsen et al. (2020).

First and foremost, in this thesis, we largely use the Black-Scholes financial market (Black and Scholes (1973)). Therefore, the financial market is complete, and arbitrage opportunities do not exist. We impose short-selling constraints on the investors for risky assets, but taking a short position bank account is possible. The financial market is complete if the number of linearly independent risky assets equals the number of risk factors. Thus any risk can be hedged by taking a position (long or short) in the corresponding asset. For this, investors expect compensation in the form of a risk premium, which determines the reward-to-risk of the portfolio. In the case of an incomplete market, the number of linearly independent is strictly smaller than the number of risk factors. Finally, the market is also assumed to be frictionless; investors do not incur any costs while trading assets.

As mentioned in Section 2.1, we adjust the consumption value to the price increase, denoted by $\Pi(t)$. Inflation, denoted by $\pi(t)$, is then measured as the percentage increase in the CPI in a given period compared to the same period of the previous year. We model both processes as follows:

$$\Pi(t) = \exp\left\{ \int_0^t \pi(s)\mathrm{d}s \right\}, \tag{3}$$

and

$$\mathrm{d}\pi(t) = \kappa_\pi \Big(\bar{\pi} - \pi(t) \Big)\mathrm{d}t + \sigma_\pi \mathrm{d}Z_\pi(t), \tag{4}$$

where $\bar{\pi}$ describes the long-run mean of the rate of inflation, κ_π the mean-reversion coefficient, and σ_π the volatility of the inflation rate. The second risk process we consider in this thesis is interest rate risk. The instantaneous real interest rate r_t is assumed to follow the one-factor Vasicek (1977) model. The following Ornstein-Uhlenbeck process describes its dynamics

$$\mathrm{d}r(t) = \kappa_r \Big(\bar{r} - r(t) \Big)\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t), \tag{5}$$

where $\bar{r}$ is the long-run mean real interest rate, κ_r the mean-reversion coefficient and σ_r is the diffusion coefficient for the real interest rate. The term $\mathrm{d}Z_r(t)$ is a Wiener process that drives the real interest rate process; we know that this process follows the following distribution: $\mathrm{d}Z_r(t) \sim \mathcal{N}(0, \mathrm{d}t)$.

Finally, it is essential to describe the process for the nominal interest rate. After all, this is used for the processes of risk-filled assets. We understand nominal interest rates as the cost of credit and the money investors earn in their savings accounts. The difference with real interest is that nominal interest does not consider the number of goods and services they can eventually buy from their savings account. We denote by $R(t)$ the process for the instantaneous nominal interest at age t; its formula is given by:

$$R(t) = r(t) + \pi(t). \tag{6}$$

The process $S(t)$ is assumed to follow a geometric Brownian motion for the stock index. To model its dynamics, we use the process described in the famous Black-Scholes model (Black

& Scholes, 1973); the stochastic differential equation is given by:

$$\frac{\mathrm{d}S(t)}{S(t)} = (R(t) + \sigma_S \lambda_S)\mathrm{d}t + \sigma_S \mathrm{d}Z_S(t), \tag{7}$$

where $R(t)$ is the nominal interest rate from (6) and σ_S is the diffusion coefficient. The term $\mathrm{d}Z_S(t)$ is a Wiener process that drives the stock-index process; we know that this process follows the following distribution: $\mathrm{d}Z_S(t) \sim \mathcal{N}(0, \mathrm{d}t)$. We have that λ_S is the market price of equity risk, also known as the Sharpe-Ratio Sharpe (1966); it describes the instantaneous excess rate of return per unit of risk. This ratio is frequently called the price of risk. It tells by how much the expected rate of return of an asset must increase if the standard deviation of that rate increases by one unit. As mentioned in the introduction, we keep the risk premia for all risky assets constant and known in advance.

So far, we have not mentioned anything about the linear correlation coefficients between the different Brownian increments. To formulate these, we use the constant local correlation assumption, i.e., $\mathrm{d}[D, N]_t = \rho \mathrm{d}t$ for Brownian motions D and N and constant correlation parameter $\rho \in \mathbb{R}$. We summarize in the correlation matrix ρ all the correlation coefficients between the three Wiener processes:

$$\rho = \begin{pmatrix} 1 & \rho_{rS} & \rho_{r\pi} \\ \rho_{rS} & 1 & \rho_{S\pi} \\ \rho_{r\pi} & \rho_{S\pi} & 1 \end{pmatrix}, \tag{8}$$

where ρ_{ij} $(i, j \in \{r, S, \pi\}, i \neq j)$ denotes the correlation coefficient between $\mathrm{d}Z_i(t)$ and $\mathrm{d}Z_j(t)$. We will describe the corresponding coefficient values in Section 2.5.

Given that in this thesis, we solve the intertemporal consumption problem using the Martingale Method, the stochastic discount factor (SDF) must first be described for this purpose. First proved by Harrison and Kreps (1979) and later generalized by Harrison and Pliska (1981) resulted in the famous first fundamental theorem of asset pricing (FTAP). The theory shows that if the no-arbitrage assumption holds, then a strictly positive SDF exists such that the price of an asset equals the expected discounted value of the asset's future stochastic cash flows. Furthermore, in the second fundamental theorem of asset pricing, if the market is complete, then a unique SDF, or pricing kernel, exists. A definition for an arbitrary time step size $s \geq 0$ given by:

$$\frac{M(t+s)}{M(t)} = \exp\left\{ -\int_t^{t+s} \left(R(u) + \frac{1}{2}\|\phi\|^2 \right) \mathrm{d}u - \int_t^{t+s} \phi^\top \mathrm{d}Z(u) \right\}, \tag{9}$$

where

$$\mathrm{d}Z(t) = \begin{pmatrix} \mathrm{d}Z_r(t) \\ \mathrm{d}Z_S(t) \\ \mathrm{d}Z_\pi(t) \end{pmatrix} \text{ and } \phi = \begin{pmatrix} \phi_r \\ \phi_S \\ \phi_\pi \end{pmatrix}.$$

We have that $\|\cdot\|$ is the Euclidean norm of ϕ where $\phi = (\phi_r, \phi_S, \phi_\pi)^\top$ represents a vector of constant loadings on stochastic innovations in the economy. These loadings determine the market prices of risk, $\lambda_r, \lambda_S, \lambda_\pi$, which are connected with developments in $\mathrm{d}Z(t)$. More specifically, we can find each market price of risk following Brennan and Xia (2002) from its corresponding vector loading to give the following set of equations:

$$\begin{aligned} \lambda_r &= -\phi_r - \rho_{rS}\phi_S - \rho_{r\pi}\phi_\pi, \\ \lambda_S &= -\phi_S - \rho_{rS}\phi_r - \rho_{S\pi}\phi_\pi, \\ \lambda_\pi &= -\phi_\pi - \rho_{r\pi}\phi_r - \rho_{S\pi}\phi_S. \end{aligned} \tag{10}$$

Because of the completeness of the financial market and the absence of arbitrage opportunities, the pricing kernel is uniquely defined. Using Itô's Lemma, we can establish a closed analytical formula. It is given by:

$$\frac{\mathrm{d}M(t)}{M(t)} = -R(t)\mathrm{d}t + \phi^\top \mathrm{d}Z(t). \tag{11}$$

Using the pricing kernel, nominal- and inflation-linked bond dynamics can be made. For complete proof, we would like to refer to appendix (A.1) in van Bilsen et al. (2020), but in this section, some steps are mentioned to construct both bond dynamics. Starting with the price of the nominal zero-coupon bond $P_N(t, h_N)$ is the discounted payoff at maturity h_N, which is equal to one at that time. This implies that the price of the bond at time t is given by:

$$\begin{aligned} P_N(t, h_N) &= \mathbb{E}_t\left[\frac{M(t+h)}{M(t)}\right] \\ &= \mathbb{E}_t\left[\exp\left\{-\int_0^h \left(r(t+v) + \pi(t+v) + \frac{1}{2}\phi^\top\rho\rho\right)\mathrm{d}v + \phi^\top \int_0^h \mathrm{d}Z(t+v)\right\}\right] \end{aligned} \tag{12}$$

Rewriting and solving the conditional expectation would allow us to find the following expression for the second equation of 12:

$$P_N(t, h_N) = \exp\{-r(t)B_r(h_N) - \pi(t)B_\pi(h_N) - M(h_N)\}, \tag{13}$$

where $M(h_N)$ collects all terms independent of time but dependent on the maturity h_N (see appendix (A.1) of van Bilsen et al. (2020) for a derivation of $M(h_N)$). Terms $B_r(h_N)$ and $B_\pi(h_N)$ denote the bond price sensitivity for real interest rate and inflation, respectively. Unpredictable changes in both risks will affect the bond prices. In the case of interest rate risk, bonds with different characteristics - different maturities - will respond differently to changes in the nominal interest rate. Specifically, longer-term bonds will react more strongly than short-term bonds, providing more risk to the investor. Formally we can define the interest rate sensitivity of the nominal bond by

$$\frac{1}{P_N(t, h_N)}\frac{\partial P_N(t, h_N)}{\partial R(t)} = -B_r(h_N), \tag{14}$$

where $B_r(h_N) = (1 - \mathrm{e}^{-\kappa_r h_N})/\kappa_r \in [0, h_N]$. In our setting, nominal zero-coupon bonds are perfectly correlated with nominal interest rate shocks; this allows investors to use them to hedge nominal interest rate risk. Furthermore, we observe that the function $B_r(h_N)$ becomes smaller for a larger mean reversion coefficient κ_r; this is because a larger coefficient ensures that the stochastic process returns to the long-run mean faster. With this, the effect of interest rate shocks over longer periods becomes smaller, and so does sensitivity.

There is a second sensitivity term in equation 13, namely the sensitivity of the zero-coupon nominal bond with respect to the inflation rate. A formal definition is given by:

$$\frac{1}{P_N(t, h_N)}\frac{\partial P_N(t, h_N)}{\partial \pi(t)} = -B_\pi(h_N), \tag{15}$$

where $B_\pi(h_N) = (1 - \mathrm{e}^{-\kappa_\pi h_N})/\kappa_\pi \in [0, h_N]$. Similar to the nominal interest rate sensitivity, we see that the function of inflation rate sensitivity is decreasing with the inflation mean-reversion coefficient κ_π. Using both sensitivity and Itô's Lemma, we can find the dynamics for the nominal zero-coupon bond by:

$$\frac{\mathrm{d}P_N(t, h_N)}{P_N(t, h_N)} = \left(R(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N)\right)\mathrm{d}t - B_r(h_N)\sigma_r\mathrm{d}Z_r(t) - B_\pi(h_N)\sigma_\pi\mathrm{d}Z_\pi(t), \tag{16}$$

where one can observe that the diffusion coefficients for interest rate risk and inflation risk are given by $-\sigma_r B_r(h_N) < 0$ and $-\sigma_\pi B_\pi(h_N) < 0$ respectively.

Inflation-linked bonds are financial securities, mainly issued by central banks of countries with highly developed economies. These products are designed to protect investors against inflation. They work as follows: the principal and interest payments are contractually linked to the inflation measure of the nation issuing the bond. We can find the price process of the inflation-linked bond with maturity h_I rather simple as it is a special case of the price process for the nominal bond. I.e. if we set $\pi(0) = \sigma_\pi = 0$ we can define the ILB price as follows:

$$\frac{\mathrm{d}P_I(t,h_I)}{P_I(t,h_I)} = \Big(R(t) - \lambda_r\sigma_r B_r(h_I)\Big)\mathrm{d}t - B_r(h_I)\sigma_r \mathrm{d}Z_r(t), \tag{17}$$

where the market price of interest rate risk λ_r is the same as in the dynamics of the nominal zero-coupon bond. Now that both bond dynamics are defined, we can find their respective risk premiums by the following equations:

$$\mathbb{E}_t\left[\frac{\mathrm{d}P_N(t,h_N)}{P_N(t,h_N)}\right] - R(t)\mathrm{d}t = -\lambda_r\sigma_r B_r(h_N)\mathrm{d}t - \lambda_\pi\sigma_\pi B_\pi(h_N)\mathrm{d}t > 0, \tag{18}$$

and

$$\mathbb{E}_t\left[\frac{\mathrm{d}P_I(t,h_I)}{P_I(t,h_I)}\right] - R(t)\mathrm{d}t = -\lambda_r\sigma_r B_r(h_I)\mathrm{d}t > 0. \tag{19}$$

If λ_r, λ_π are not smaller than zero, we would have negative bond risk premiums. The nominal bond risk premium is larger than the inflation-linked bond risk premium. Furthermore, when $B_r(h)$ and $B_\pi(h)$ are increasing functions for maturity h, we have that the long-term bond risk premium exceeds the short-term bond risk premium. Also, both $B_r(h)$ and $B_\pi(h)$ depend on their respective mean-reversion coefficients. More specifically, both functions are decreasing with their respective coefficients. The ILB risk premiums are small if real interest rates are predictable, and nominal-bond risk premiums are small if inflation rates are predictable.

For simplicity we define the vector $X(t)$ containing the three risky assets, i.e. $X(t) = [S(t), P_I(t,h_I), P_N(t,h_N)]^\top$. This vector satisfies the following dynamic equation:

$$\frac{\mathrm{d}X(t)}{X(t)} = \mu(t)\mathrm{d}t + \Sigma\mathrm{d}Z(t), \tag{20}$$

where

$$\mu(t) = \begin{pmatrix} R(t) + \lambda_S\sigma_S \\ R(t) - \lambda_r\sigma_r B_r(h_I) \\ R(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N) \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 0 & \sigma_S & 0 \\ -\sigma_r B_r(h_I) & 0 & 0 \\ -\sigma_r B_r(h_N) & 0 & -\sigma_\pi B_\pi(h_N) \end{pmatrix}. \tag{21}$$

We have that $\mathrm{d}Z(t)$ is the same as in equation 9. We will need the process $X(t)$ for the next section. Where we will describe the wealth process that the investors require to be able to invest in the risky stock, nominal bond with maturity h_N, ILB with maturity h_I, and cash. Because we use the dynamics of nominal returns, we also simulate nominal wealth. We deviate from van Bilsen et al. (2020), which expresses returns in real terms. Note that in our objective function, nominal wealth is corrected for the price index, with which the results are ultimately the same.

2.3 Wealth processes

In the previous section, we described the financial market with its various assets and risks. With these dynamics we can describe the wealth process of the investor. We start with a general description of the total wealth. Then we break this dynamic down into financial and human value. As mentioned before, the investor has to opportunity to invest in several different assets: a stock, an inflation-linked bond, and a nominal bond. The remainder is invested into a risk-free bank account. Since the number of linearly independent assets span the number of risks, the market is complete by definition. Using the processes of all assets, we can define the dynamics of total wealth by:

$$\mathrm{d}W(t) = (R(t) + \omega(t)^\top[\mu(t) - R(t)])W(t)\mathrm{d}t + \omega(t)^\top \Sigma W(t)\mathrm{d}Z_t - c(t)\mathrm{d}t, \tag{22}$$

where $\omega(t) = (\omega_S(t), \omega_P(t), \omega_p(t))'$ is the vector of portfolio weights, with $\omega_S(t)$ the share of wealth invested in the risky stock index, $\omega_P(t)$ in the inflation-linked bond and $\omega_p(t)$ the share of wealth invested in the nominal bond. The remainder, $1 - \omega(t)^\top \iota$, is invested in the bank account. The consumption on time t is given by $c(t)$. The returns and volatility of the risky assets, $\mu(t)$ and Σ, are given in equation (45).

The life-cycle principle describes that investors own human and financial wealth. With this, we can break down total wealth described in (22), similar to Munk and Sørensen (2010) and Bodie et al. (1992), as

$$W(t) = H(t) + F(t). \tag{23}$$

Where H_t equals human wealth, and F_t equals financial wealth. At our starting point, the moment an individual starts their working life and earns a wage, we assume that the financial wealth is equal to zero ($F(0) = 0$). Therefore, the total wealth at $t = 0$ only consists of human capital, $W(0) = H(0)$. During the working years, human wealth gradually converts to financial wealth and is completely depleted when the retirement age is reached.

In our case, we assume that future labor earnings can be seen as a traded asset. Therefore, we can treat labor income as bond-like. More specifically, it can be seen as a risk-free inflation-linked zero-coupon bond portfolio consisting of bonds with different maturities for each year t. Therefore, we can define human capital as follows:

$$H(t) = \int_0^{T_D - t} \mathbb{E}_t\left[\frac{M(t+h)}{M(t)} Y(t+h)\right]\mathrm{d}h = \int_0^{T_D - t} H(t,h)\mathrm{d}h, \tag{24}$$

where $M(t)$ is the Pricing Kernel defined in 11. We use the SDF to obtain the function $H(t, h)$, which represents the discounted income on time t that is received in future period $t+h$. The individual income $Y(t+h)$ is assumed to be known in advance and fixed at time $t+h$. For each working year, the individual will have one unit of labor. However, inflation affects income; therefore, in this thesis, we assume income is expressed as a proportion of the consumer price index. In retirement, the individual income will reduce to the state old age pension benefit s, which is a fraction of previously earned income. For simplicity, we will set the parameter equal to zero. Formally $Y(t+h)$ can can be written as

$$Y(t+h) = \begin{cases} \Pi(t+h) & \text{for } t+h \in [0, T_R) \\ s \cdot \Pi(t+h) & \text{for } t+h \in [T_R, T_D] \end{cases}, \tag{25}$$

where T_R and T_D denote the age at which the investor retires and dies. From van Bilsen et al. (2020), we can derive the dynamics of the differential equation for human wealth using equations (24) and (25). The result is given by

$$\mathrm{d}H(t) = \Big(R(t) - \lambda_r \sigma_r D_H(t)\Big)H(t)\mathrm{d}t - D_H(t)\sigma_r H(t)\mathrm{d}Z_r(t) - Y(t)\mathrm{d}t, \tag{26}$$

where

$$D_H(t) = \int_0^{T_D - t} \frac{H(t,h)}{H(t)} B_r(h)\mathrm{d}h, \tag{27}$$

is the duration, the interest rate sensitivity, of human capital. The definitions for human capital and the discounted value from future labor, $H(t)$ and $H(t,h)$ are given in equation (24). Although financial welfare is implicitly defined by equation (23), the differential equation is presented below for completeness:

$$\mathrm{d}F(t) = \Big(r(t) + \hat{\omega}(t)^{\top}(\mu(t) - r(t)\Big)F(t)\mathrm{d}t + \hat{\omega}(t)^{\top}\Sigma(t)F(t)\mathrm{d}Z(t) + \big(Y(t) - c(t)\big)\mathrm{d}t. \quad (28)$$

Here, $\hat{\omega}(t) = \big(\hat{\omega}_S(t), \hat{\omega}_P(t), \hat{\omega}_p(t)\big)'$ is the vector of weights invested into the risky assets as part of the financial wealth at time t. The remainder, $1 - \hat{\omega}(t)^{\top}\iota$, is invested in the risk-free bank account. In the next section, we define the objective function and corresponding constraints.

2.4 Maximization problem

The investor maximizes the real expected utility from consumption over the lifetime

$$\max_{c(t),\omega(t)} \quad \mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{1}{1-\gamma}\left(\frac{c(t)}{\Pi(t)}\right)^{1-\gamma}\mathrm{d}t\right], \quad (29)$$

$$\text{s.t. } \mathbb{E}\left[\int_0^{T_D} M(s)c(s)\mathrm{d}s\right] \leq W_0, \quad (30)$$

where the objective follows from the assumed utiltiy function. The starting value for total wealth is, as described earlier, equal to maximum human wealth. $M(t)$ is the pricing Kernel from equation 11.

2.5 Parameter values

This section elaborates on this thesis's parameter choices. All tables and graphs created in this thesis use these values unless otherwise stated. A complete overview of all parameters can be found in the Appendix, table (4).

As mentioned earlier, an individual starts working when he or she is 20 years old ($t = 0$) up until retirement age of 68 ($T_R = 68-20 = 48$). We assume the time of death is deterministic and that pass away at the age of 90, which implies $T_D = 70$. We assume that a fixed and constant income is earned during the investors working life. This wage $Y(t) = \Pi(t)$ for all working years and decreases to $Y(t) = 0$ in retirement, implying that individuals do not earn Old age pension benefits from the state of from own savings. This is a departure from papers such as van Bilsen et al. (2020) that show simulations for different values of pension benefit but is outside the focus of this thesis. Note that labor income is prone to inflation, as depicted in equation (25).

Regarding the choice of time preference and relative risk aversion parameters. We set the time preference parameter δ equal to 3%. This is in correspondence with Samwick (1998), who find values between 3% and 4% for American households. The default relative risk aversion parameter γ is equal to 5. The default value implies that the investor is moderately risk-averse. However, in Section (4.4.2), we consider a range of different risk aversions. In doing so, we investigate what choices will be made by investors with either a large or small risk aversion. This choice of default risk aversion is in line with studies such as Brennan and Xia (2002) and Koijen et al. (2010).

Next, the parameter selection for both risk processes is outlined. Since both processes follow an Ornstein-Uhlenbeck equation, mean version parameters exist for both processes. For example, for real interest $r(t)$, when the process for time t is above the long-term-average, i.e. $r(t) > \bar{r}$, then the short rate is expected to increase in the next instant. Conversely, when $r(t) < \bar{r}$, then the short rate is expected to increase in the next period. A realistic feature of the model is that short-term rates follow a normal distribution because they can take on any value, whether positive or negative. The mean reversion coefficients are found using the half-time of interest rate η; this is the time it takes for a stochastic process to recover to the long-term average from the current point in the absence of unexpected shocks. The resulting coefficient can be computed through $\kappa_r = \log(2)/\eta$. We followed the estimations of Metselaar and Zwaneveld (2020), Their values for the mean reversion parameters of real interest rate and expected inflation rate are $\kappa_r = 7.63\%$ and $\kappa_\pi = 35.25\%$. This implies that the processes' half-lives are approximately 9.1 and 2 years.

Economic scenarios for Dutch CPB pension models are often generated with the so-called KNW model. The KNW-model originates from Koijen et al. (2010), and for a detailed description, we refer to Draper (2014). The predominant scenario, which the CPB has used in recent years, is the KNW(1.5%) set, for which we refer to Metselaar and Zwaneveld (2020). It has as characteristic parameters an expected real long-term interest rate, $\bar{r}$, of 1.5%, which we also adopt. From the same set, we also use that expected long-term inflation equals 1.29%. We assume that both risk processes start from their long-term assumed average. From van Gastel et al. (2022), we take the volatility parameters of both risk factors and the stock index, i.e., σ_r, σ_π and σ_S are assumed to be equal to 2.4%, 0.9% and $17,68\%$ respectively. Similar to van Bilsen et al. (2018), we choose a maturity of 30 years for the nominal bond and 10 years for the inflation-linked bond.

The following assumptions are made for the risk premiums. First and foremost, we assume that all premiums are constant. We take the risk premium for equity from van Gastel et al. (2022), that is based on the KNW(1.5%) set. The resulting equity premium λ_S equals 22.06%. The risk premia for real interest and inflation are estimated using data from CRSP and the Federal Reserve Bank; we refer to Zhou (2014) for a complete description of how the calibrated parameters are found. The market price of interest λ_r equals -21.55% and the inflation premium λ_π equals -10.23%.

Finally, the correlation matrix ρ describes the correlations between the several stochastic processes. In this, we again follow the results found by Zhou (2014), which is based on the paper by Brennan and Xia (2002). The result is found in Table (4). The following Sections will describe the maximization problem and derive the analytical solutions given this set of parameters.

3 Optimal life-cycle allocations

The previous Section described the benchmark model, which is based mainly on van Bilsen et al. (2020) and Brennan and Xia (2002). We have described the stochastic discount factor and risky assets and explained the parameters we chose. This Section will elaborate on these, notably solving the maximization problem and the associated optimal consumption flow. We visualize how the wealth processes evolve and look at the optimal asset allocations, first as a fraction of total wealth and second as a fraction of financial wealth.

3.1 Optimal consumption strategy

We have the following menu of risky assets, a 30-year nominal bond, a 10-year ILB bond, and a stock. The result for the optimal real consumption $c^*(t)$ is derived by van Bilsen et al. (2020). The result is found by using the martingale method (Pliska (1986),Karatzas et al. (1987)), a formal proof can be found in their paper (see Appendix (A.3) for a derivation). The optimal consumption choice at time t is then given by

$$c^*(t) = c^*(0)\exp\left\{ \int_0^t \pi(s)\mathrm{d}s + \frac{1}{\gamma}\int_0^t \left(r(s) + \frac{1}{2}\phi^\top \rho \phi - \delta \right)\mathrm{d}s - \frac{1}{\gamma}\phi^\top \int_0^t \mathrm{d}Z(s) \right\}, \tag{31}$$

where $c^*(0)$ denotes the optimal consumption decision at the start of the life cycle. We set the consumption value at $t = 0$ so that the investor's total wealth equals the market-consistent value of the optimal consumption stream, i.e.

$$\mathbb{E}\left[\int_0^{T_D} M(t)c^*(t)\mathrm{d}s \right] = W(0). \tag{32}$$

Note that we can think of consumption as a liability and that for any period in the future, the investor gets the optimal consumption payout. Let $V^*(t)$ be the market-consistent value of optimal consumption at time t. We price this market-consistent consumption as follows

$$\begin{aligned} V^*(t) &= \mathbb{E}_t\left[\int_0^{T_D-t} \frac{M(t+h)}{M(t)} c^*(t+h)\mathrm{d}h \right] \\ &= c^*(t)\int_0^{T_D-t} \mathbb{E}_t\left[\frac{M(t+h)}{M(t)} \frac{c^*(t+h)}{c^*(t)} \right]\mathrm{d}h \\ &= c^*(t)A^*(t), \end{aligned} \tag{33}$$

where the optimal annuity factor $A^*(t)$ is defined as

$$A^*(t) = \int_0^{T_D-t} \mathbb{E}_t\left[\frac{M(t+h)}{M(t)} \frac{c^*(t+h)}{c^*(t)} \right]\mathrm{d}h = \int_0^{T_D} \exp\{-d^*(t,h)h\}. \tag{34}$$

The function $d^*(t,h)$ denotes the market-consistent discount rate at time t for all future periods $h \geq 0$. After some rewriting (for the complete derivation, we refer to Appendix (A.3) in van Bilsen et al. (2020)), we can show that the market-consistent discount rate is equal to

$$d^*(t,h) = \frac{1}{h}\Bigg[\left(1-\frac{1}{\gamma}\int_0^h\left(r(t)+\kappa_r B_r(h)(\bar{r}-r(t))+\frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}h\right.$$
$$-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\int_0^h(\phi_r-B_r(h)\sigma_r)^2\mathrm{d}h$$
$$-\left(1-\frac{1}{\gamma}\right)^2\rho_{r\pi}\int_0^h(\phi_r-B(h)\sigma_r)\phi_\pi\mathrm{d}h \tag{35}$$
$$\left.-\left(1-\frac{1}{\gamma}\right)^2\rho_{rS}\int_0^h(\phi_r-B(h)\sigma_r)\phi_S\mathrm{d}h\right]$$
$$-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\phi_\pi^2-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\phi_S^2-\left(1-\frac{1}{\gamma}\right)^2\rho_{\pi S}\phi_\pi\phi_S.$$

3.2 Optimal asset allocation in terms of total wealth

Using the optimal consumption decision, we can derive the optimal portfolio weights. We aim to optimize total wealth, as the split between human and financial wealth will be made in the next Section. We denote by $\omega^*(t)$ the vector of optimal weights; we find it using the replication argument. The results are given by (see Appendix (A.3) of van Bilsen et al. (2020) for derivation)

$$\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_S}{\sigma_S}, \tag{36}$$

$$\omega_I^*(t) = \frac{1}{B_r(h_I)}\left[\frac{1}{\gamma}\frac{\phi_r}{\sigma_r}-\omega_N^*(t)B_r(h_N)\right]+\frac{D_A^*(t)}{B_r(h_I)}, \tag{37}$$

$$\omega_N^*(t) = \frac{1}{\gamma}\frac{\phi_\pi}{B_\pi(h_N)\sigma_\pi}. \tag{38}$$

The remainder of the investor's capital is invested in a cash position $\omega_C^*(t)$. Throughout this thesis, we do not allow for short positions in risky assets. The investor is, however, able to obtain a short position on his bank account, i.e., a loan. $D_A(t)$ denotes the interest rate sensitivity of the optimal annuity factor, which equals $A^*(t) = W^*(t)/c^*(t)$. Where $W^*(t)$ is the optimal total level we achieve by implementing the optimal investment choices. The formal expression for $D_{A^*}(t)$ is given by

$$D_{A^*}(t) = \left(1-\frac{1}{\gamma}\right)\int_0^{T-t}\frac{V^*(t,h)}{V^*(t)}B_r(h)\mathrm{d}h, \tag{39}$$

where $V^*(t,h)$ is the market-consistent value at adult age t of current and future optimal consumption choices.

The first element of $\omega^*(t)$ is the fraction of total wealth invested in the risky stock $\omega_S^*(t)$. From equation 36, we derive that demand for this risky asset is constant throughout the investor life-cycle, as we do not allow for stochastic risk premiums. The fraction depends on the fraction between the market price of equity and its volatility and is decreasing for the investor's risk aversion. The agent invests in the stock option to obtain the preferred consumption exposure to stock return risk. This speculative demand ensures the investor picks up the equity risk premium $-\lambda_S\sigma_S \geq 0$.

We have that $\omega_N^*(t)$ denotes the optimal fraction of total wealth invested in the nominal bond with maturity h_N. As mentioned earlier, there is one key reason an investor invests a

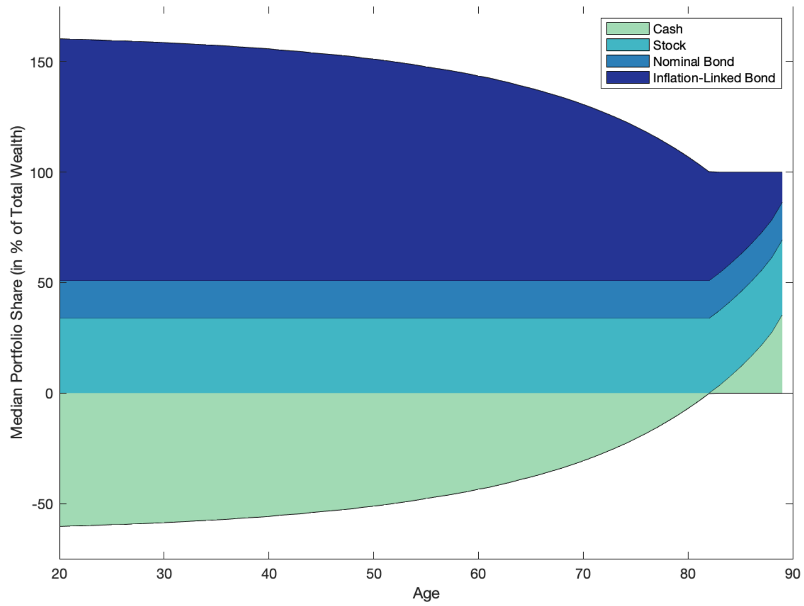


Figure 1: Median portfolio shares over the life cycle. The figure shows the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where the parameters defined in Section (2.5) are used to obtain the results.

portion in this type of bond. It is speculative, as one can see in the first term of equation **??**, the investor wants to benefit from the risk premium of inflation rates $-\lambda_\pi \sigma_\pi B_\pi(h_N) \geq 0$. This speculative demand increases with the unit factor ϕ_π/σ_π loading but is negatively affected when the coefficient of relative risk γ increases. This makes sense; if this coefficient increases, our investor becomes more risk-averse, which leads to a smaller appetite for speculative demand. In the case that the investor possesses an infinitely large risk aversion ($\gamma \to \infty$), the speculative demand drops out.

The fraction of total wealth invested in the inflation-linked bond with maturity h_I is denoted by $\omega_I^*(t)$. There are two main reasons the investor wants to invest in this fixed-income asset. The first, too, is speculative. The investor wants to benefit from the premium he can obtain from the interest rate risk premium, i.e., $-\lambda_r \sigma_r B_r(h_I) \geq 0$. However, investors also receive the premium for real interest rate risk $-\lambda_r \sigma_r B_r(h_N) \geq 0$ by investing a share of their total wealth $\omega_N^*(t)$ in the nominal bond with maturity h_N. Therefore, the speculative demand for the ILB could occasionally turn negative when either ϕ_π/σ_π is too large or ϕ_r/σ_r is too small.

In addition, the agent wants to invest in the ILB as a hedge against changes in the real interest rate in the annuity factor $A^*(t)$. This is the so-called intertemporal hedging demand. This demand depends on the duration of the annuity factor $D_{A^*}(t)$ defined by equation (39). This definition shows that the duration of the optimal annuity factor is decreasing for an increase in relative risk aversion. Meaning that in the case that the investor is infinitely risk averse, i.e., ($\gamma \to \infty$), the speculative demands drop out as investors want to be sure about their real consumption. Losing speculative demand negatively affects the growth of consumption. However, investors remain protected from real interest rate shocks.

Figure (1) shows the median of all fractions invested in risky assets; the two different bonds, stock, and cash over the investor's life cycle. We use the median for the invested fractions is risky assets because of the optimal annuity factor $D_{A^*}(t)$. This factor varies with the state of the economy for each simulation; therefore, we take the medians. The figure shows that during the working years, the investor borrows money (short) and buys risky assets (long) from them. At a young age, the agent has a high demand for inflation-linked bonds. However, we find in equation (37) that the fraction invested in ILBs depends on $D_{A^*}(t)$. The optimal annuity factor is a decreasing function over time because the impact of a change in the real interest rates becomes smaller for future periods. Therefore, the share of total wealth invested in the inflation-linked bond decreases with time. The intuition behind this is that an ILB is a hedge against both real interest rate and inflation risk. This is particularly interesting when the investor is young, so the retirement assets are well protected against both risks. The hedge component is financed by taking a short position in a risk-free bank account. As the hedging demand for ILB decreases, the cash position gradually increases and is even positive for high ages. The fractions invested in the nominal bond and the stock index are constant over the life cycle. Because for both risky assets, we assumed that the market prices of risk are constant. Hence, the resulting optimal fractions invested are constant.

3.3 Optimal asset allocation in terms of financial wealth

Now that we have defined the optimal allocations for the situation where total wealth is optimized, we are ready to split the problem into financial and human wealth. We assume that labor income is unexposed to stock market risk and risk-free in real terms (van Bilsen et al., 2020). Essentially, human capital can therefore be treated as an inflation-linked bond. The investor is physically unable to invest parts of its human capital into the financial assets, meaning that only financial wealth can be invested. The investor, however, does consider the monetary value of human capital for its investment decisions. For that reason, we create this split of total wealth. An additional benefit is that this allows us to draw an overall picture of optimal consumption and wealth processes. Once again, using the paper by van Bilsen et al. (2020) we have that the new optimal vector of allocations $\hat{\omega}^*(t) = \big(\hat{\omega}_S^*(t), \hat{\omega}_I^*(t), \hat{\omega}_N^*(t)\big)$, after splitting between human and financial wealth, can be expressed as

$$\hat{\omega}^*(t) = \left(1 + \frac{H(t)}{F(t)}\right)\omega^*(t) - \left(0, \frac{D_H(t)}{B_r(h_I)}\frac{H(t)}{F(t)}, 0\right) \tag{40}$$

Equation (40) shows that the allocations in terms of financial wealth are equal to the allocations in terms of total wealth when the investor retired. Then human wealth is completely depleted, equating both vectors of allocations. The resulting allocation in the risky stock is now given by

$$\hat{\omega}_S^*(t) = -\frac{1}{\gamma}\frac{\phi_S}{\sigma_S}\left(1 + \frac{H(t)}{F(t)}\right), \tag{41}$$

where financial wealth $F(t)$ and human wealth $H(t)$ are defined by dynamics (28) and (26), respectively. In line with such as Merton (1975), we see in our case that the allocation in the stock decreases as the investor's age increases. One reason is that human welfare has no risk over the stock market and inflation. As a result, the expected return on total wealth is too low. To compensate for these returns, the investor has to invest a larger portion of the stock since that asset has a large expected return. Furthermore, human wealth decreases while financial wealth increases with age. This gradually reduces the fraction between the two, resulting in a smaller fraction invested in the stock over time. This is intuitive when looking at the figure (3a), which shows both wealth processes. Explaining the fraction of financial wealth invested in the inflation-linked requires more details. Rewriting equation

(40) yields

$$\hat{\omega}_I^*(t) = \omega_I^*(t)\left(1 + \frac{H(t)}{F(t)}\right) - \frac{H(t)}{F(t)}\frac{D_H(t)}{B_r(h_I)}, \tag{42}$$

The consequence of the wealth split for allocation in the ILB is less intuitive. Because human capital depends on real interest rate risk, the investor wants to take a large enough position in the ILB to hedge the exposure against this risk. This partly depends on the duration of human wealth because if the duration is too low, the investor has not hedged enough against real interest rate risk. Conversely, when $D_H(t)$ becomes too large, the investor is overprotected. This is suboptimal as it depresses the return on assets. The following equation gives the optimal fraction of financial wealth invested in the nominal bond

$$\hat{\omega}_N^*(t) = \frac{1}{\gamma}\frac{\phi_\pi}{B_\pi(h_N)\sigma_\pi}\left(1 + \frac{H(t)}{F(t)}\right), \tag{43}$$

The investors' motive to invest in this bond is the same as before: a speculative reason. Investing in the nominal bond yields an investor the inflation risk premium $-\lambda_\pi \sigma_\pi B_\pi(h_N) \geq 0$. Again, the demand for this asset is negatively related to the risk aversion parameter γ while positive to the fraction of factor loading divided by the volatility of inflation rate ϕ_r/σ_r. The fraction invested decreases over time for the same reason as it is for the fraction invested in the stock: the low expected return on human wealth because it is unaffected by inflation and equity risk.

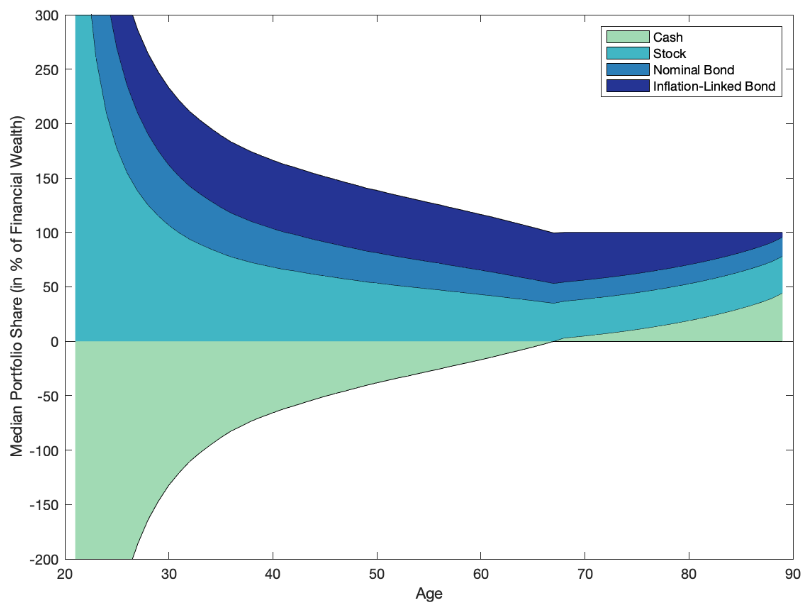
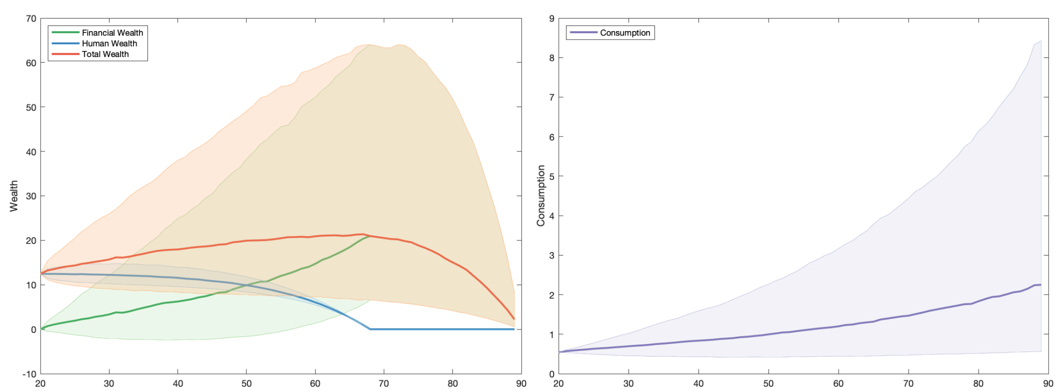


Figure 2: Median portfolio shares over the life cycle. The figure shows the median fractions of financial wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where the parameters defined in Section (2.5) are used to obtain the results.

Figure (2) shows how the medians of allocations as a fraction of financial wealth evolve with the agent's age. Again, investing in a 10-year inflation-linked zero-coupon bond, a 30-year nominal zero-coupon bond, equity, and a risk-free bank account is possible. Figure (2) shows that even investors invest a large fraction of financial wealth in nominal and inflation-linked bonds. In fact, between the ages of 20 and 30, they retain roughly 70% of real and nominal bonds in their portfolios. However, this is mostly not reflected in practice. Indeed, bonds are usually absent in younger investors' portfolios; due to their high-risk appetite, they more often choose assets with higher returns, such as equities, but nowadays also crypto and ETFs and, in our model, a risky stock. Furthermore, the negative position in the bank account is notable. This means that investors, mostly at a young age, borrow huge amounts relative to their human wealth to then invest in risky assets. Again, this is not much reflected in practice. However, young people are increasingly opening investment accounts or investing fractions of their student loans; we do not see massive borrowing for speculative purposes. Furthermore, it is also unclear how young people get liquidity; lenders often require collateral or proof of periodic income, which is mostly lacking among young people and just-starting workers. If they manage to take out a loan, perhaps partly financed by a benefactor, there is often there to purchase real estate.

Now that we have determined the optimal consumption flow, we can simulate the wealth processes. The result for 10,000 simulations are shown in figure (3). This clearly shows how, in addition to the process of consumption, the processes of human, financial, and total wealth develop. We see that total wealth W_0 for $t = 0$ equals human wealth, i.e., $W_0 = H_0$, where human capital is the economic value of generated labor income up to retirement. After the retirement date, the human capital is depleted. In order to be able to consume after retirement, the agent borrows money against its human capital and invests this into risky assets. Over time the agents human capital is converted into financial wealth. At T_R the financial wealth will be near its maximum. Afterward, financial wealth decreases until it is completely depleted at death. In reality, financial wealth often increases further after retirement. A key driver could be uncertainty about the future; the investor could keep a 'rainy day fund' for that purpose. Another driver could be healthcare costs; these are rather uncertain and potentially high in less developed economies. Agents want to retain some financial wealth to ensure their ability to cover healthcare bills. A third driver could be a desire to leave a bequest to any children or donate a sum to charity. For our simulations,



(a) Wealth processes

(b) Consumption process

Figure 3: Optimal median wealth and consumption. The figures are based on $n = 10,000$ scenarios, where the shaded area is the interquartile range respecting the same color. Figure (a) clearly shows that human capital is converted into financial wealth during the working years. After retirement, the financial wealth is completely depleted. Figure (b) shows the optimal consumption flow, again with the shaded area as the interquartile range. The agent determines the optimal consumption choice and the optimal share of total wealth invested in the risky assets such as to maximize expected lifetime utility

especially for financial wealth we find significant spreads in Figure (3). This spread is partly due to the various stochastic risk factors, namely the nominal interest rate and stock index, and to inflation because of the objective function and optimal consumption strategy. Equation (31) shows that the investor would like to preserve purchasing power before and during retirement. We see that the quantile range is maximum at the time of retirement. On the contrary, the range for human wealth is much less volatile because only interest rates cause this volatility.

4 Terminal wealth model

The following section examines the different welfare effects of adding inflation-linked bonds. To investigate welfare effects, we look at a complete and incomplete market. In the complete market, in addition to a nominal bond, stock, and risk-free bank account, the option to buy an ILB is also available. In the incomplete market, the reverse applies, in which the investor can only invest in a nominal bond, stock index, and bank account. The problem is, however, that in the case of an incomplete market, no analytical solution is possible Liu (2007). Therefore, we look at a terminal wealth problem. The investor is only interested in maximizing his real wealth at retirement. To achieve this, the investor invests in optimal but different investment strategies in both situations.

The structure of this section is as follows: we start with the exposition of the dynamic maximization problem, in which the investor maximizes the utility of total real wealth at retirement age. We then describe the method used by van Bilsen et al. (2020) for finding the optimal asset allocations for both the complete and incomplete market scenario. We assume that the dynamics of the various risky assets remain unchanged. This is followed by analyzing the optimal solutions to provide context for the investor's requirements. Finally, a sensitivity analysis, in which we create simulations of wealth loss for different parameter values.

4.1 Assumptions

We leave labor income completely out of this model and look only at the individual's investments. The timeline is partly the same; for $t = 0$ the investor starts investing, at which point the real age is 20 years. The investments aim to maximize real wealth at retirement age, $T_R = 67$. So we also no longer consider the death age of each individual in this model. For the investors' personal preferences, we make the following assumptions. The investor maximizes the expected terminal wealth on retirement. The optimal investment strategy is determined to maximize total wealth. We again assume that the individual has CRRA preference over real wealth. Implying that the expected lifetime utility is found by equation (1).

The processes for inflation and CPI are the same as in the previous model; this also applies to nominal and real interest rates. Furthermore, the dynamics for risk-filled assets remain the same as in Section (2.1). One difference with the consumption model is that the correlations between different risks are now set to zero in the terminal wealth model, i.e., $\rho = 0$. We use the values from table (4) for other parameters.

Although the different assets' dynamics are similar in both models, the incomplete market differs from the complete market because the investor does not have access to an inflation-linked bond. This means that the vector with risky assets $X_I(t) = [S(t), P_N(t, h_N)]^\top$ is given as follows:

$$\frac{\mathrm{d}X_I(t)}{X_I(t)} = \mu_I(t)\mathrm{d}t + \Sigma_I \mathrm{d}Z(t), \tag{44}$$

where

$$\mu_I(t) = \begin{pmatrix} R(t) + \lambda_S\sigma_S \\ R(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N) \end{pmatrix} \text{ and } \Sigma_I = \begin{pmatrix} 0 & \sigma_S & 0 \\ -\sigma_r B_r(h_N) & 0 & -\sigma_\pi B_\pi(h_N) \end{pmatrix}. \tag{45}$$

4.2 Dynamic maximization problem

The first model discussed in this thesis examined the optimal consumption choice over the life cycle, where we assumed that the investor maximizes CRRA utility. We used the optimal asset allocations from van Bilsen et al. (2020) found using the martingale method, in which they replaced the dynamic budget constraint with a static budget constraint. Thus allowing the value of the optimal consumption stream to be smaller than the initial wealth. The problem with this method is that it is only applicable in complete financial markets, i.e., we can hedge every possible risk. As mentioned before, this section considers a situation where we cannot hedge all risks. More specifically, we consider a situation where we aim to provide an inflation-linked pension. However, the investor is not able to invest in an inflation-linked bond. The obtain this inflation-linked pension; the investor maximizes the real expected utility from terminal wealth by

$$\max_{\omega(t)} \quad \mathbb{E}\left[u\left(\frac{W_{T_R}}{\Pi_{T_R}}\right)\right] = \mathbb{E}\left[\frac{1}{1-\gamma}\left(\frac{W_{T_R}}{\Pi_{T_R}}\right)^{1-\gamma}\right], \tag{46}$$

$$\text{s.t. } \mathrm{d}\widehat{W}(t) = \big(r(t) + \omega(t)^\top[\mu(t) - R(t)]\big)\widehat{W}(t)\mathrm{d}t + \omega(t)^\top \Sigma \widehat{W}(t)\mathrm{d}Z(t). \tag{47}$$

The objection function follows the assumed utility function. Only real terminal wealth $\widehat{W}(t) = W(t)/\Pi(t)$ matters, where the dynamics of real found can easily be found by applying Itô's Lemma. The dynamics of the price index can be found in equation (3). Throughout this chapter, we simulate the real wealth process via a log-transformation derived in Appendix (C). We assume that the initial real starting wealth equals 10, i.e., $\widehat{W}(0) = 10$. The following section will analyze the optimal asset allocation for both financial markets, providing insight into investors' choices over time.

4.3 Optimal asset allocations complete market

Now that we have named the two financial markets' different assumptions, we can describe the optimal asset allocations for the dynamic maximization problem. We start with the complete market, where investors can access interest-rate and inflation-covering assets. The resulting optimal allocations for the complete market originate from van Bilsen et al., 2020 (see their Appendix A.5 for proof) and are given by the vector $\omega_C^*(t)$, defined by:

$$\omega_C^*(t) = \frac{1}{\gamma}\begin{pmatrix} -\dfrac{\phi_S}{\sigma_S} \\ \dfrac{\phi_r}{B_r(h_I)\sigma_r} - \dfrac{\phi_\pi}{B_\pi(h_N)\sigma_\pi}\dfrac{B_r(h_N)}{B_r(h_I)} \\ \dfrac{\phi_\pi}{B_\pi(h_N)\sigma_\pi} \end{pmatrix} + \frac{\gamma-1}{\gamma}\begin{pmatrix} 0 \\ \dfrac{B_r(T_R-t)}{B_r(h_I)} \\ 0 \end{pmatrix} \tag{48}$$

The optimal investment strategy can be decomposed into two parts. The first part is the first term on the right-hand side of the equation (48). This term represents the speculative portfolio demand; it depends on the expected excess return on the financial assets. The second part is the so-called hedging portfolio demand. This term expressed the investor's demand to hedge against low real interest rates.

The first term of $\omega_C^*(t)$ denotes the share of total wealth invested in the risky stock, which is constant. It depends on the investor's risk aversion and the market price of equity risk, the extra expected return per unit of volatility. The optimal fraction invested is the same as in the intertemporal consumption problem. The same applies to the fraction invested in the zero-coupon nominal bond with maturity h_N. The nominal bond is exposed to inflation rate risk, which will affect the bond's price if the inflation rate changes. From an investor's point of view, the nominal bond is the only asset that will be exposed to inflation. Investors, however, still want to allocate wealth into nominal bonds to benefit from the inflation rate risk premium. The reason why investors expose themselves to inflation rate risk is because of this additional return they receive.

The optimal fraction invested in the inflation-linked bond differs from the intertemporal consumption problem. By allocating wealth to ILBs, investors can pick up the real interest rate risk premium. In the speculative demand for ILBs, we find an additional term. This is a correction term for real interest rate risk exposure of the nominal bond allocation; this correction is depicted by the second term of the speculative demand. With the hedging term, investors can protect themselves against real interest rate risk. In the case that the investor possesses an infinitely large risk aversion ($\gamma \to \infty$), the speculative demand drops out, and investors only want to invest in inflation-linked bonds.

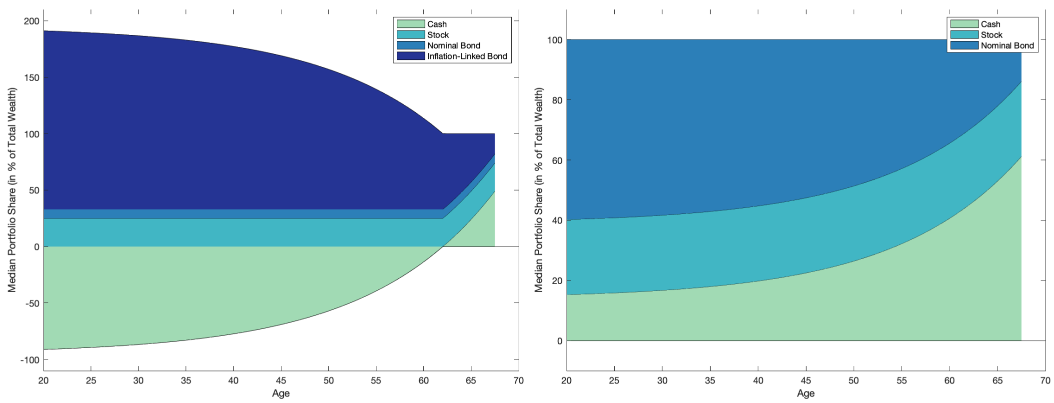
4.4 Optimal asset allocations incomplete market

Next, we turn to the incomplete market solution. In this market, investors can invest in a zero-coupon nominal bond with maturity h_N, risky stock, and a risk-free bank account. ILBs are only sometimes available to investors and are typically very illiquid. The question is how portfolio managers allocate their real wealth in an incomplete market. The answer is given by the vector $\omega_I^*(t)$:

$$\omega_I^*(t) = \frac{1}{\gamma} \begin{pmatrix} -\dfrac{\phi_S}{\sigma_S} \\ \dfrac{\phi_r B_r(h_N)\sigma_r + \phi_\pi B_\pi(h_N)\sigma_\pi}{B_r^2(h_N)\sigma_r^2 + B_\pi^2(h_N)\sigma_\pi^2} \end{pmatrix} + \frac{\gamma - 1}{\gamma} \begin{pmatrix} 0 \\ \dfrac{B_r(T_R - t)}{B_r(h_N)} \left(1 + \dfrac{B_\pi^2(h_N)\sigma_\pi^2}{B_r^2(h_N)\sigma_r^2}\right)^{-1} \end{pmatrix} \tag{49}$$

We see that the share of total wealth invested in the risky stock remains constant and equals the solution found in (48). Only the fraction invested in the nominal bond has changed. This change is because the investor now uses these nominal bonds for multiple goals. Investors want to hedge against low real interest rate risk while benefiting from the speculative risk demand. As both goals can not be uniquely identified, i.e., the investor cannot perfectly hedge real interest rate risk with an ILB anymore, investors typically purchase nominal bonds with low maturities to partially hedge inflation. The demand for the nominal bond shows an intuitive pattern. If we compare the speculative demands for both financial markets, the numerator contains the expected excess return, while the denominator contains a "volatility" term. However, the investor considers this fraction of risk premiums and volatility terms together in the incomplete market. According to the theory, the relationship between an asset's risk premium and variance determines its speculative demand.

In the incomplete market case, we find the additional term $\left(1 + \frac{B_\pi^2(h_N)\sigma_\pi^2}{B_r^2(h_N)\sigma_r^2}\right)^{-1}$ in the investors hedging demand. It is clear that this additional term drops out without inflation risk, i.e., $\sigma_\pi = 0$, and the hedging demand is equal to that in the complete market case. If the volatility of inflation is larger than zero, the hedging demand decreases. The problem that arises whenever the investor only has excess to a nominal bond is that his portfolio allocation is not optimal when his risk aversion is infinitely large ($\gamma \to \infty$). This is because the nominal bond is a suboptimal hedging instrument, as it does not protect against inflation risk. To compensate for the suboptimality, the investor chooses a bond duration lower than the complete market case. Generally, the higher the uncertainty in the inflation rate, the lower the nominal bond duration. This also explains why Dutch pension funds do not match the duration of their liabilities. The duration of the bond portfolio of a pension fund is generally smaller than the liabilities. If pension funds were to hedge their liabilities fully, they are completely hedged against real interest rate risk. However, in such a scenario, they can only pay out nominal money to the policyholders, resulting in unindexed pensions.



(a) Complete market

(b) Incomplete market

Figure 4: Median portfolio shares over the life cycle. The figure shows the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where the parameters defined in Section (2.5) are used to obtain the results. Figure (a) shows the optimal allocation in the complete market, whereas figure (b) depicts the optimal allocation for the incomplete market.

Figure (4) shows the median of the fraction invested from total wealth over the investor's life-cycle. Figure (a) shows the optimal allocation in the complete market; the investor borrows money for a significant duration and buys risky assets from them. Like the intertemporal consumption problem, investors have a high demand for inflation-linked bonds. The same intuition applies; investors, especially young, want to protect themselves from inflation and real interest rate risk and problem. In the incomplete market, the nominal bond serves both inflation and real interest rate hedging but is a sub-optimal instrument. Nonetheless, they invested a substantial proportion in it at a young age, and this gradually decreases as speculative demand decreases with age. In both markets, the fraction invested in the risky stock is constant for all ages.

4.4.1 Impact on certainty equivalent wealth

The following section describes how to measure the wealth effect between an investor who can hedge inflation risk completely and another investor who cannot. We measure the certainty equivalent of welfare for several numerical simulations, which are calculated using the utility function defined in (1). Subsequently, we calculate the welfare effect as the percentage increase or decrease in the certainty equivalence between the two financial scenarios.

We can describe certainty equivalence using the following example. Imagine owning a ticket for the lottery on which there is a 10% chance of winning €100,000 and a 90% chance of winning nothing. The player is given the option to sell the ticket before the draw. When the owner is risk neutral, the lowest amount he wants to sell the ticket is equal to the expected value, i.e., €10,000. However, when the player is risk averse, the minimum resale amount is lower, say €8,000. It follows that the player's certainty equivalent of this lottery is the lowest amount for which he wants to sell the ticket rather than enter the draw. The degree of risk aversion determines the minimum amount for which the player is indifferent between the two options.

The investor invests in the optimal allocations described in equations (48) and (49). The certainty equivalent, i.e., the wealth for which the investor becomes indifferent between receiving CE with certainty or investing using θ_C^* and θ_I^* with an uncertain outcome, is given by

$$\mathrm{CE} = \mathrm{U}^{-1}\left(\mathbb{E}\left[u\left(\frac{W_{T_R}}{\Pi_{T_R}}\right)\right]\right), \tag{50}$$

where total real wealth is found using the dynamics in equation (47) and the price index using equation (3). The welfare effect (Δ), in terms of certainty equivalent terminal wealth, is calculated as follows:

$$\Delta = \frac{\mathrm{CE}_I - \mathrm{CE}_C}{\mathrm{CE}_C}, \tag{51}$$

where CE_C is the certainty equivalent terminal wealth in the complete financial market, and CE_I for the incomplete financial market.

4.4.2 Impact γ on certainty equivalent wealth

Table (1) shows the welfare effects when we numerically adjust the risk aversion parameter. The corresponding optimal allocations for each scenario are shown in figure (6). We see that for the default setting, i.e., table (4) parameters, an investor loses 75.40% wealth in certainty equivalent when the option to perfectly hedge inflation is no longer possible.

γ	2	5	6	10	40
CE_C	114.40	70.48	66.60	58.88	45.29
CE_I	105.42	17.34	12.90	6.69	2.60
Δ	-7.85%	-75.40%	-80.63%	-88.63%	-94.26%

Table 1: Welfare effects for investors following optimal investment strategies for both markets. We have that CE_C is the certainty equivalent for an investor in the complete market, and CE_I is the certainty equivalent for an investor in the incomplete market. The welfare effect of not being able to invest in an inflation-linked bond is given by Δ for different values of the risk aversion parameter, γ. Other parameter values are given by table (4).

For a smaller risk aversion parameter ($\gamma = 2$), the certainty equivalent of wealth increases significantly. We explain the increase in CE using the top two figures in figure (6). We see that for both financial scenarios, the investor borrows a large share of total wealth in cash. The cash obtained from the short position is then invested mainly in risky stock in both scenarios, with a high expected return. The remainder is divided among the available bonds. The investor in the complete market takes a larger short position in the risk-free bank account; therefore, the risky-asset portfolio is also more extensive. This leads to a larger certainty equivalent in real terminal wealth. The same holds for investors in the incomplete market; we find an increase in risky assets for a lower risk aversion which results in a larger CE. Especially the share of total wealth invested in equity changes greatly to benefit from the large expected return. However, this strategy is suboptimal due to the inability to hedge inflation. Therefore, the investor's welfare loss is 7.85% relative to CE wealth, meaning that the investor is worse off in the incomplete market scenario.

We obtained the following results for higher values of relative risk aversion γ. If we increase γ, the investors' speculative demand decreases, leading to less equity investment in both financial markets. The speculative demand for the nominal bond differs for each scenario;

in the complete market, the demand for this security decreases. However, in the incomplete market, the investor invests partially in the nominal bond regardless of risk aversion value. The investors' hedging demand causes this preference, which increases for larger values of γ.

The same is true for inflation-linked bonds in the complete market; the investors' speculative demand decreases with γ. At the same time, the investors' hedging demand increases. Thus, a significant fraction of the total wealth will be invested in the ILB. In the incomplete market, the fraction of total wealth invested in the nominal bond increases. However, investing in both bonds gives the investor less return than equity. The certainty equivalent of terminal wealth reflects this CE effect; for both financial markets, an increase in relative risk aversion decreases the certainty equivalent.

We find that the CE of terminal wealth in the complete market is significantly larger than the CE in the incomplete market for values of $\gamma = \{6, 10\}$. The welfare loss, too, is significant; an investor not having the option to invest in an ILB loses $\{80.63\%, 88.63\%\}$ in certainty equivalent terminal wealth. The welfare loss increases for larger values of γ; because, for a considerable risk aversion, investors are primarily interested in hedging risks. However, investors cannot hedge inflation risk in an incomplete market, resulting in welfare loss. If the risk aversion is equal to 40, we see that the investors hardly have any speculative demand left and only want to hedge their real wealth against real interest rate and inflation risk. Investors incur a significant welfare loss of 94.26%; thus, the investment strategy for highly risk-averse investors loses if we compare both markets. Even more so, we find that CE_I for $\gamma = 40$ is lower than the investor's initial wealth $\widehat{W}_I(0)$. Investors are indifferent between either receiving 2.60 now or investing $\widehat{W}_I(0)$ optimally.

4.4.3 Impact σ_π on certainty equivalent wealth

Next, we analyze the effect of inflation volatility on the certainty equivalent. We do this using numerical analysis. We look at the optimal portfolio's welfare effect and composition for each simulation when the investor maximizes real terminal wealth. The results are shown in table (2).

For the benchmark parameter set, we see the same result described in Section (4.4.2); investors incur a welfare loss of 75.40% when ILB is no longer part of the asset menu. When the volatility of inflation is lower, i.e., $\sigma_\pi \in \{0.0225, 0.045\}$, we see the following trend. The certainty equivalent in the incomplete market increases substantially, CE_I equals 115.70 and 65.40, respectively. For $\sigma_\pi = 0.0225$, we find a welfare gain of 64.05% for investors in the incomplete market. Figure (E) shows the optimal allocations, from which we derive that a large part of total welfare is borrowed in both financial markets. The wealth effect arises from the different portfolios purchased with the short position in the bank account. There is less investor demand for inflation-covering securities; therefore, investors are better off in CE wealth when investing only in nominal bonds and equity, yielding a larger expected return. We conclude that the premiums on inflation-linked securities are insufficient for investors in periods with low inflation. This carries a risk; when investors optimize their portfolio during low inflation, significant inflation shocks can result in disastrous adverse value effects in CE terminal wealth. Therefore, real debt paper must be introduced in Dutch financial markets to absorb the wealth loss of an inevitable future inflation shock.

Increasing the volatility yields the following results: for values σ_π of 0.135 and 0.18, the wealth losses are 90.56% and 94.95% if investors cannot hedge inflation completely. In the complete financial market, speculative demand decreases in nominal bonds, as investors want fewer nominal bonds when inflation risk increases, given that these securities do not provide real protection. On the other hand, speculative demand for the inflation-linked bond increases. ILB's coupon payments and principal increase with inflation, leading to a higher expected return. The fraction invested in equity remains constant in all scenarios. Ultimately, investors are better off when they can invest a fraction of their assets in real debt securities during inflation uncertainty.

σ_π	CE_C	CE_I	Δ
0.0225	70.53	115.70	64.05%
0.045	70.51	65.40	−7.25%
0.09	70.48	17.34	−75.40%
0.135	70.43	6.65	−90.56%
0.18	70.36	3.55	−94.95%

Table 2: Welfare effects for investors following optimal investment strategies for both markets when $\gamma = 5$. We have that CE_C is the certainty equivalent for an investor in the complete market, and CE_I is the certainty equivalent for an investor in the incomplete market. The welfare effect of not being able to invest in an inflation-linked bond is given by Δ for different values of the inflation rate volatility, σ_π. Other parameter values are given by table (4).

We observe an interesting pattern in the certainty equivalent wealth in the complete market. Namely, we find that for an increase in the volatility of the price index, CE_C remains fairly constant. The reason CE_C stays constant is due to the fraction of real debt securities in the portfolio whose coupon payments and principal rise along with the price index. These real debt products have comparatively excellent returns in market with high inflation. CE_I shows a reverse effect; for a low price index volatility, the investors thrive better in the incomplete market. However, CE_I plunges for a price inflation shock due to the inability of investors to hedge inflation.

4.4.4 Comparison certainty equivalent to U.S. data

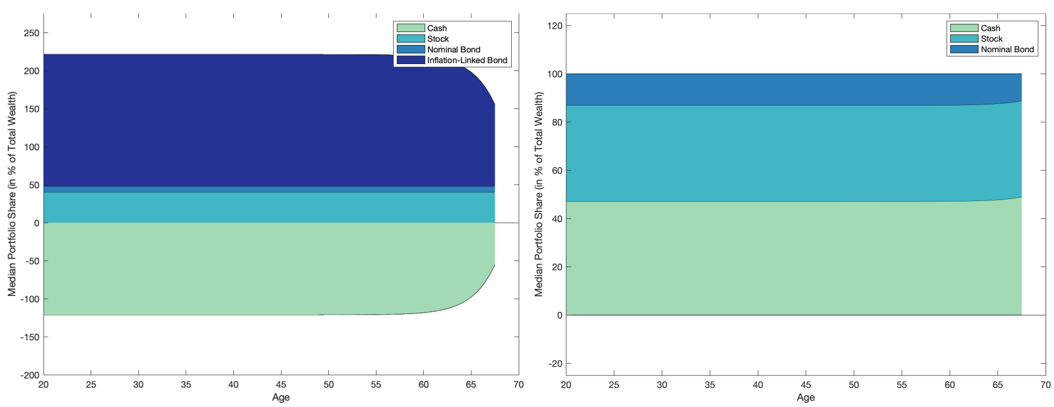
Finally, we reflect our results against another dataset. Namely, a parameter set is calibrated on U.S. data, where equity parameters are based on NYSE, AMEX, and NASDAQ data from 1952 to 2012. Real interest rates and inflation are based on U.S. treasury bonds over the same period. The dataset comes from Zhou (2014), which obtained the parameter set by entering the dataset into the B.X. model (Brennan & Xia, 2002). We chose U.S. parameters because this is an economy where central banks first released inflation-linked bonds and thus have been available to investors for an extended period.

γ	2	5	6	10	40
CE_C	132.92	46.53	41.43	32.82	25.21
CE_I	76.90	26.03	22.64	16.46	8.83
Δ	−42.15%	−44.06%	−45.34%	−49.87%	−64.97%

Table 3: Welfare effects for investors following optimal investment strategies for both markets. We have that CE_C is the certainty equivalent for an investor in the complete market, and CE_I is the certainty equivalent for an investor in the incomplete market. The welfare effect of not being able to invest in an inflation-linked bond is given by Δ for different values of the risk aversion parameter, γ. Other parameter values are given by table (5).

We note the main differences between the two parameter sets. The equity premium is higher for U.S. investors, but the long-term real interest rate is lower than the benchmark parameters. The low real interest rate is primarily due to high long-term inflation, which is almost two and a half times higher than in the KNW(1.5%) set. Volatility for inflation is also twice as high, with which we expect demand for real debt securities to be higher than in the Netherlands. Furthermore, we keep the maturities of both bonds the same in both parameter sets, i.e., the maturity of the nominal bond is 30 years, and the maturity of the ILB is 10 years.

The welfare effect for investors without access to real debt securities is shown in the table (3). The welfare loss in the U.S. is particularly less consequential than in the Netherlands. This difference is mainly due to differences in the real interest rate, inflation process and risk premiums. It is notable that the certainty equivalent in absolute terms is higher in the U.S. than in the Netherlands. We find the following explanations for the risk premiums on different assets. We see that the volatility for equity and inflation is higher in the U.S. This makes the equity risk premium and inflation risk premium higher, leading to higher returns. Consequently, the certainty equivalent of terminal wealth is also higher.



(a) Complete market (b) Incomplete market

Figure 5: Median portfolio shares over the life cycle. The figure shows the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where the parameters defined in table (5) are used to obtain the results. Figure (a) shows the optimal allocation in the complete market, whereas figure (b) depicts the optimal allocation for the incomplete market. The relative risk aversion is set $\gamma = 5$

In addition, the certainty equivalent of terminal wealth differs between the two countries. We see that for increasing risk aversion, the CE decreases sharply. This is because investors' speculative demand decreases, desiring a significant fraction of safe derivatives in their portfolio. Nevertheless, investors are worse off because the expected return of both nominal and inflation-linked bonds is lower in the U.S. The reverse applies to low-risk aversion; U.S. investors can benefit from the higher equity risk premium, resulting in higher expected returns. However, in all situations we find a negative welfare affect as a result of the incapability to hedge inflation in the incomplete market.

The optimal allocations for ($\gamma = 5$) differ from the Dutch parameter set. What stands out is the cash position in both financial markets; in the complete market, despite the higher volatility in inflation, the investor still wants to borrow a large amount of total wealth to invest. However, investors in the incomplete market hold a long position in a risk-free bank account. With the result, the wealth loss of 44.06% becomes evident when we look at figure (5). Although investors in the incomplete market invest a more significant fraction of their wealth in equity and a nominal bond, this does not outweigh the loss in terminal wealth due to inflation.

5 Discussion

For ILPs applies that the principal of the bond rises with inflation. Therefore protecting the real return will assure investors retain future purchasing power. This does not mean that ILBs are without risk. Like nominal bonds, the price fluctuates with a change in real yields, therefore containing interest rate risk. Because the securities are designed to protect against inflation, they offer little protection if the underlying economy experiences a period of deflation. The principal may end up below the bond's par value during such a period. Fortunately, issuing countries often offer deflation floors at maturity. If the principle is below the par value, the investor will be restored for the difference. However, the coupon payments are not compensated, so deflation risk cannot be fully hedged.

There are more arguments against inflation-linked bonds. Issues of ILBs by central banks can lead to the segmentation of government debt into many illiquid entities. Real debt securities are less liquid than nominal bonds (Townend, 1997). This segmentation, in turn, leads to higher costs for the issuing governments, and the higher costs are passed on, reducing the inflation premium for investors. The question does arise as to whether illiquidity is a concern for governments. The approach behind real debt securities is that investors purchase them for the long term, making the likelihood of a fire sale on these financial products a secondary risk.

It has frequently been argued that if inflation-linked bonds were as desirable as economists claim, they would already be more widely available. Academics have thoroughly examined the reasons for the lack of issuance of inflation-linked bonds on multiple occasions (see, McCulloch (1980); Price (1997)). The outcome is simple; the inflation risk premium is too low for issuing inflation-linked bonds to generate significant gains for the issuer. If expected returns were larger, investors would automatically invest more in ILBs. Another argument against ILBs is money illusion, the tendency to view wealth in nominal and not real terms caused by a lack of financial education. Even if inflation is at the desired 2%, investors lose about 67% of their nominal income $1/(1+0.02)^{20}$ over 20 years.

Still, the demand for inflation-linked bonds will grow as a mechanism for pension liability matching. Demographic trends will increase this demand. Aging combined with higher average life expectancy will lead to higher liabilities for pension funds. To still be able to pay indexed pensions, funds will feel compelled to include real debt securities in larger quantities in their portfolios.

Including inflation-linked bonds in portfolios is more challenging than it seems. Although theoretically, ILBs are a perfect inflation hedge when the market is complete, i.e., For every financial risk, there is a linearly independent asset. Enabling an investor to create a portfolio with long and short positions in the assets with which we can hedge all risks. However, in practice, we find that not every ILB hedges an investor's inflation risk. An example of this is the differences in inflation rates in Europe. Imagine that there are Dutch ILBs for sale on the financial market; this instrument protects an investor against Dutch inflation. However, when average European inflation is higher or lower than Dutch inflation, for example, there is still a risk for the investor. The latter has to decide which inflation benchmark to protect against..

Then there is the question of who should issue ILBs. Currently, these securities are issued by central banks, but a more obvious candidate might be the government. After all, they have a kind of natural hedge against inflation. This is because governments receive tax revenues, which rise with inflation. More precisely, when inflation rises, government revenues from taxes also rise. It is also more than logical for governments to take on this task. Indeed, according to economist Milton Friedman, governments are the cause of price increases, "The government created inflation in the first place and therefore has the responsibility to provide means by which citizens can protect their wealth ."The downside, however, and a primary reason why governments so far do not venture into their spending, is the risk. The problem is that inflation is theoretically unlimited; thus, governments' payment obligations can seriously add up. Both bond principal and coupon payments are contractually linked to inflation, so the final amount is not fixed for the issuing party. Governments can even run into financial problems in severe market conditions.

It is interesting for governments to issue these instruments, providing better covering pension funds and personal pensions against inflation. This could have a wealth-increasing effect which is essential for governments, given that they must ensure healthy pensions for young and old. On the other hand, it solves a problem from the pension funds' perspective. Funds are eager to hedge inflation, but given that the ILB market is often illiquid, they can only occasionally purchase these securities in the desired quantity.

A final point that can be challenged is the assumption that income is risk-free. People can lose their jobs during economic decline or negative growth. Besides that, human capital can act as an equity holding in a particular line of work, such as large corporate managers and entrepreneurs. Campbell (1996) shows a high correlation between market returns and human capital. Economic theory depicts that when income risks are uninsurable and non-diversifiable, financial market participation depends on the correlation between returns and income risks. If this correlation between income-return is low, then allocation into the financial market can provide a solid hedge against income risks (Bonaparte et al., 2014). However, forces other than the financial markets can impact one's labor income, e.g., illness and natural disasters. Cocco et al. (2005) show that income growth rates are higher for the young and gradually slow down over time. It possibly ends up slightly negative in some cases. Herefore, it is common to assume that income is not deterministic but stochastic (see Munk and Sørensen (2010) and Zhou (2014)).

6 Conclusion

This thesis examined two life cycle models using the KNW(1.5%) parameter set for both cases. The first model looks at an intertemporal consumption model of van Bilsen et al. (2020) against inflation and interest rate risk. We have shown that Dutch investors invest a significant fraction of their wealth in index loans even when inflation volatility is low. In addition, within this model, we split total wealth into human and financial wealth. Again, when investors invest optimally, significant fractions of financial wealth are invested in inflation-linked bonds. Young investors, in particular, benefit most from index-linked bonds, but older investors retain a high demand for these securities.

Section (4) looks at a terminal wealth model. In it, investors maximize their real wealth at retirement age. We looked at two situations, the first being a complete financial market in which investors can hedge inflation and interest rate risk perfectly. Furthermore, an incomplete financial market in which inflation risk cannot be hedged completely. We describe the optimal allocations and the effects of parameter choice for both markets. Again, investors buy real debt when it is available to them.

Section (4.4.1) looks at the welfare effects between the two financial markets in the terminal wealth model. We show the effect of the risk aversion parameter on the investor's certainty equivalent wealth. Our findings are that a significant wealth loss results when investors cannot access inflation-linked bonds. The same result holds when we numerically examine the effect of inflation volatility on certainty equivalent wealth. When volatility is three times higher, investors lose over a quarter of certainty equivalent wealth.

Finally, we compared the KNW(1.5%) parameter set with U.S. parameters from Zhou (2014). This parameter set defines an economy where inflation uncertainty is larger than the Dutch economy but also consists of larger risk premiums. We again find that investors are significantly worse off when they cannot access inflation-linked bonds. Investors then hold large amounts of money in risk-free bank accounts, which are severely devalued yearly in real terms.

Despite the apparent advantages of inflation-linked bonds, these securities do not exist in the Netherlands. A relevant follow-up study would be for which risk premia investors and governments can no longer ignore these bonds. The expected return is evidently too low, but whether governments can administer higher returns on these products remains to be seen. The study should identify the risks between the two parties and whether an equilibrium can be found in the terminal wealth model. So that future investors and pension funds can protect their wealth in real terms.

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Appendix

A Benchmark parameter set

Parameter	Value
Stock Return Process: $\frac{\mathrm{d}S(t)}{S(t)} = (R(t) + \sigma_S\lambda_S)\mathrm{d}t + \sigma_S\mathrm{d}Z_S(t)$	
σ_S	0.1768
λ_S	0.2206
Interest Rate Process: $\mathrm{d}r(t) = \kappa_r\Big(\bar{r} - r(t)\Big)\mathrm{d}t + \sigma_r\mathrm{d}Z_r(t)$	
$\bar{r}$	0.0150
κ_r	0.0763
σ_r	0.0240
λ_r	-0.2155
Inflation Process: $\mathrm{d}\pi(t) = \kappa_\pi\Big(\bar{\pi} - \pi(t)\Big)\mathrm{d}t + \sigma_\pi\mathrm{d}Z_\pi(t)$	
$\bar{\pi}$	0.0129
κ_π	0.3525
σ_π	0.0900
λ_π	-0.1023
Correlations	
ρ_{rS}	0.0749
$\rho_{r\pi}$	-0.5355
$\rho_{S\pi}$	-0.2490
Other Parameters	
Maturity Nominal Bond (h_N)	30
Maturity ILB (h_I)	10
Risk aversion (γ)	5
Subjective discount rate (δ)	0.03
Number of simulations (n)	10,000

Table 4: This table presents the parameter values for asset price, risk dynamics, and other values that need to be set in addition to the process values. A complete description is found in Section (2.5). The parameters are prodominantly based on the KNW(1.5%) set.

B U.S. parameter set

Parameter	Value
Stock Return Process: $\frac{\mathrm{d}S(t)}{S(t)} = (R(t) + \sigma_S \lambda_S)\mathrm{d}t + \sigma_S \mathrm{d}Z_S(t)$	
σ_S	0.1640
λ_S	0.3269
Interest Rate Process: $\mathrm{d}r(t) = \kappa_r \Big(\bar{r} - r(t)\Big)\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t)$	
$\bar{r}$	0.0095
κ_r	0.4148
σ_r	0.0181
λ_r	-0.2155
Inflation Process: $\mathrm{d}\pi(t) = \kappa_\pi \Big(\bar{\pi} - \pi(t)\Big)\mathrm{d}t + \sigma_\pi \mathrm{d}Z_\pi(t)$	
$\bar{\pi}$	0.0387
κ_π	0.0544
σ_π	0.0171
λ_π	-0.1023
Other Parameters	
Maturity Nominal Bond (h_N)	30
Maturity ILB (h_I)	10
Risk aversion (γ)	5
Subjective discount rate (δ)	0.03
Number of simulations (n)	10,000
Initial real wealth ($W(0)$)	10

Table 5: This table presents the parameter values for asset price, risk dynamics, and other values that need to be set in addition to the process values. The parameter values for the are calibrated by Zhou (2014) based on NSYE, AMEX and NASDAQ data. Real interest and inflation rates are based quarterly U.S. data of Treasury bonds and inflation rates in the period 1952-2012.

C Derivation terminal wealth processes

The dynamics for terminal wealth in Section (4) are found using a log transformation. We show the derivation for terminal wealth for both the incomplete and complete financial markets, as each market has a different asset set. In the incomplete market consists of two independent assets, we find:

$$\mu_I(t) = \begin{pmatrix} R(t) + \lambda_S\sigma_S \\ R(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N) \end{pmatrix} \text{ and } \Sigma_I = \begin{pmatrix} 0 & \sigma_S & 0 \\ -\sigma_r B_r(h_N) & 0 & -\sigma_\pi B_\pi(h_N) \end{pmatrix}. \tag{52}$$

For the complete market, we have:

$$\mu_C(t) = \begin{pmatrix} R(t) + \lambda_S\sigma_S \\ R(t) - \lambda_r\sigma_r B_r(h_I) \\ R(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N) \end{pmatrix} \text{ and } \Sigma_C = \begin{pmatrix} 0 & \sigma_S & 0 \\ -\sigma_r B_r(h_I) & 0 & 0 \\ -\sigma_r B_r(h_N) & 0 & -\sigma_\pi B_\pi(h_N) \end{pmatrix}. \tag{53}$$

The dynamics of real wealth are found by:

$$\mathrm{d}\widehat{W}(t) = \big(r(t) + \omega(t)^\top[\mu(t) - R(t)]\big)\widehat{W}(t)\mathrm{d}t + \omega(t)^\top\Sigma\widehat{W}(t)\mathrm{d}Z(t), \tag{54}$$

where $\widehat{W}(t) = W(t)/\Pi(t)$. Rewriting the real wealth proces (54) using (52) or (53) gives the processes for the incompete $\widehat{W}_I(t)$ and complete market $\widehat{W}_C(t)$, respectively. Hence, we find:

$$\begin{aligned} \mathrm{d}\widehat{W}_I(t) &= \big(r(t) + \omega(t)^\top[\mu_I(t) - R(t)]\big)\widehat{W}_I(t)\mathrm{d}t + \omega(t)_I^\top\Sigma\widehat{W}_I(t)\mathrm{d}Z(t) \\ &= \big(r(t) + \omega_S(t)\lambda_S\sigma_S - \omega_N(t)\big[\lambda_r\sigma_r B_r(h_N) + \lambda_\pi\sigma_\pi B_\pi(h_N)\big]\big)\widehat{W}_I(t)\mathrm{d}t \\ &\quad + \big(\omega_S(t)\sigma_S\mathrm{d}Z_S(t) - \omega_N(t)B_r(h_N)\sigma_r\mathrm{d}Z_r(t) - \omega_N(t)B_\pi(h_N)\sigma_\pi\mathrm{d}Z_\pi(t)\big)\widehat{W}_I(t) \\ &\equiv \widehat{\mu}_I(t,\widehat{W}_I(t))\mathrm{d}t + \widehat{\sigma}_I(t,\widehat{W}_I(t))\mathrm{d}Z(t) \end{aligned} \tag{55}$$

and

$$\begin{aligned} \mathrm{d}\widehat{W}_C(t) &= \big(r(t) + \omega(t)^\top[\mu_C(t) - R(t)]\big)\widehat{W}_C(t)\mathrm{d}t + \omega(t)_C^\top\Sigma\widehat{W}_C(t)\mathrm{d}Z(t) \\ &= \big(r(t) + \omega_S(t)\lambda_S\sigma_S - \omega_N(t)\big[\lambda_r\sigma_r B_r(h_N) + \lambda_\pi\sigma_\pi B_\pi(h_N)\big] - \omega_I(t)\lambda_r\sigma_r B_r(h_I)\big)\widehat{W}_I(t)\mathrm{d}t \\ &\quad + \big(\omega_S(t)\sigma_S\mathrm{d}Z_S(t) - [\omega_I(t)B_r(h_I) + \omega_N(t)B_r(h_N)\sigma_r]\mathrm{d}Z_r(t) - \omega_N(t)B_\pi(h_N)\sigma_\pi\mathrm{d}Z_\pi(t)\big)\widehat{W}_I(t) \\ &\equiv \widehat{\mu}_C(t,\widehat{W}_C(t))\mathrm{d}t + \widehat{\sigma}_C(t,\widehat{W}_C(t))\mathrm{d}Z(t) \end{aligned} \tag{56}$$

We know have rewrtitten both real wealth processes in terms of a drift and volotaility term. Usings Eulers approximation we have:

$$\widehat{W}_I(j+1) = \widehat{W}_I(j) + \widehat{\mu}_I(j,\widehat{W}_I(j))\Delta t + \widehat{\sigma}_I(t,\widehat{W}_I(t))\sqrt{\Delta t}B(j), \tag{57}$$

and

$$\widehat{W}_C(j+1) = \widehat{W}_C(j) + \widehat{\mu}_C(j,\widehat{W}_C(j))\Delta t + \widehat{\sigma}_C(t,\widehat{W}_C(t))\sqrt{\Delta t}B(j), \tag{58}$$

with known $\widehat{W}_C(0) = \widehat{W}_I(0) = 10$ and $B(j) \sim \mathcal{N}(0,\mathrm{d}t)$.

D Simulations for risk aversion coefficient

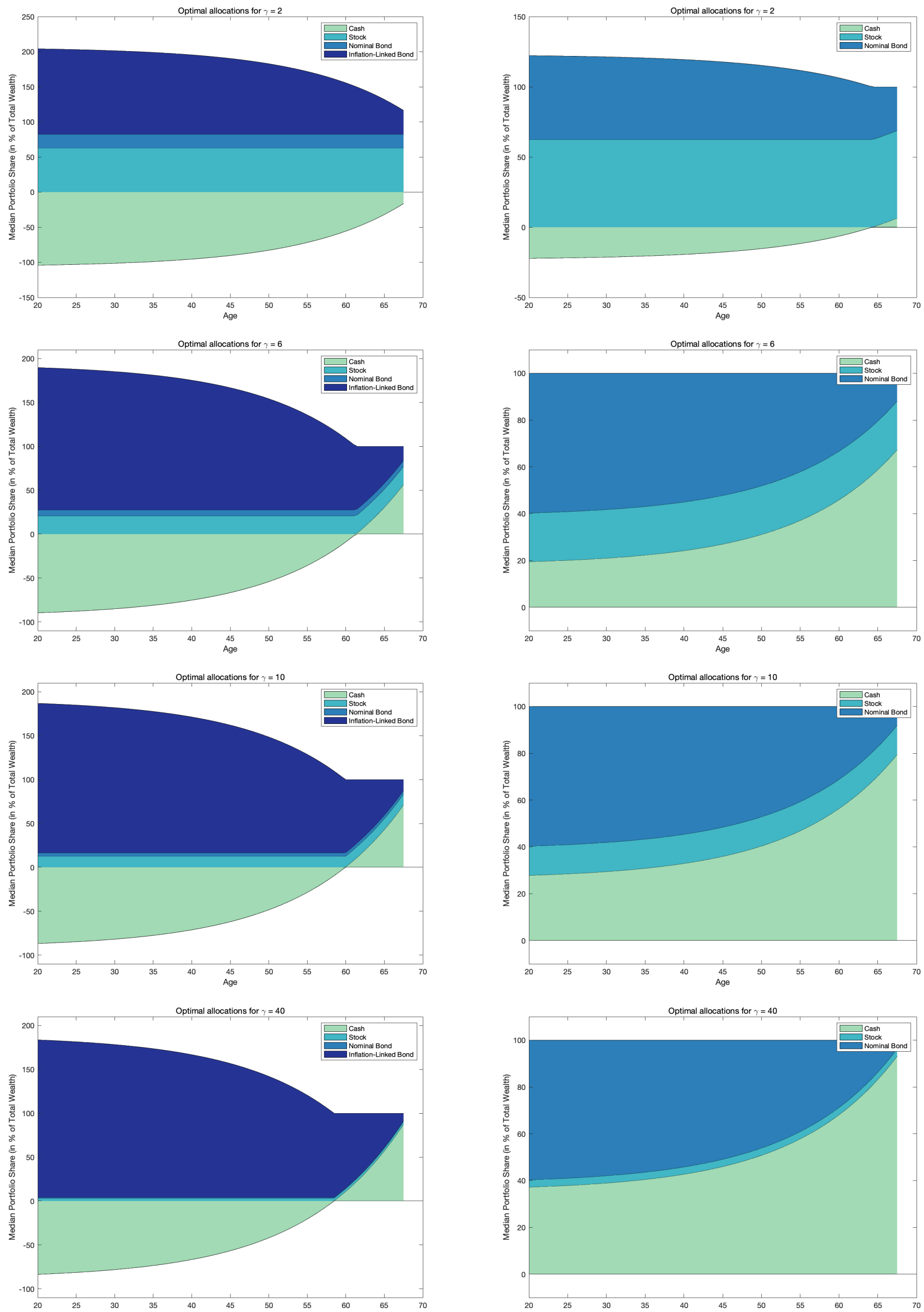

Figure 6: Median portfolio shares over the life cycle. These figures show the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where for each graph, we take a different $\gamma \in \{2, 6, 10, 40\}$. The left-hand side depicts the optimal allocation for the complete market setting and the right for the incomplete market.

E Simulations for volatility inflation rate

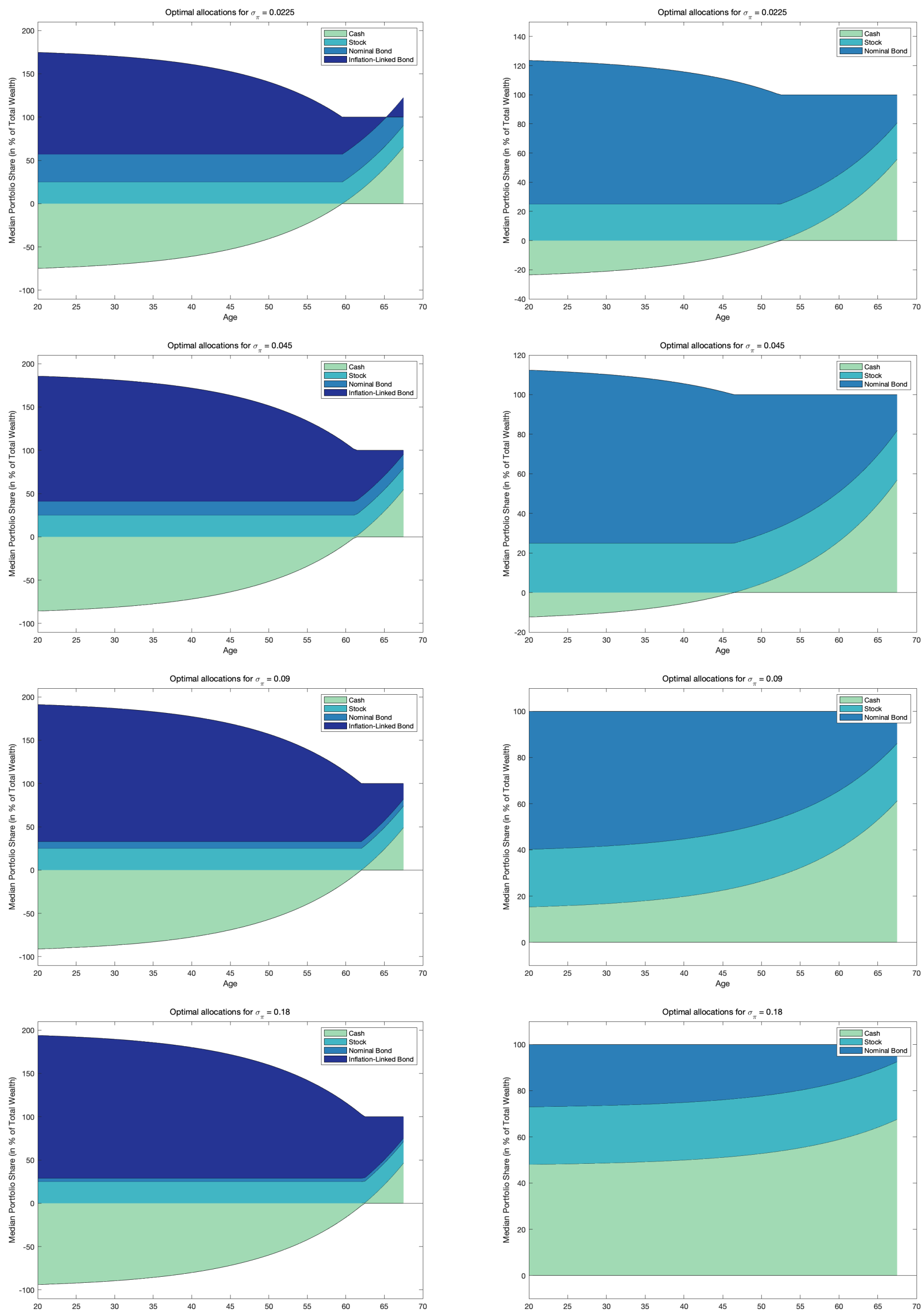

Figure 7: Median portfolio shares over the life cycle. These figures show the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where for each graph, we take a different $\sigma_\pi \in \{0.0225, 0.045, 0.09, 0.18\}$. The left-hand side depicts the optimal allocation for the complete market setting and the right for the incomplete market.

F Simulations for risk aversion coefficient U.S. parameterset

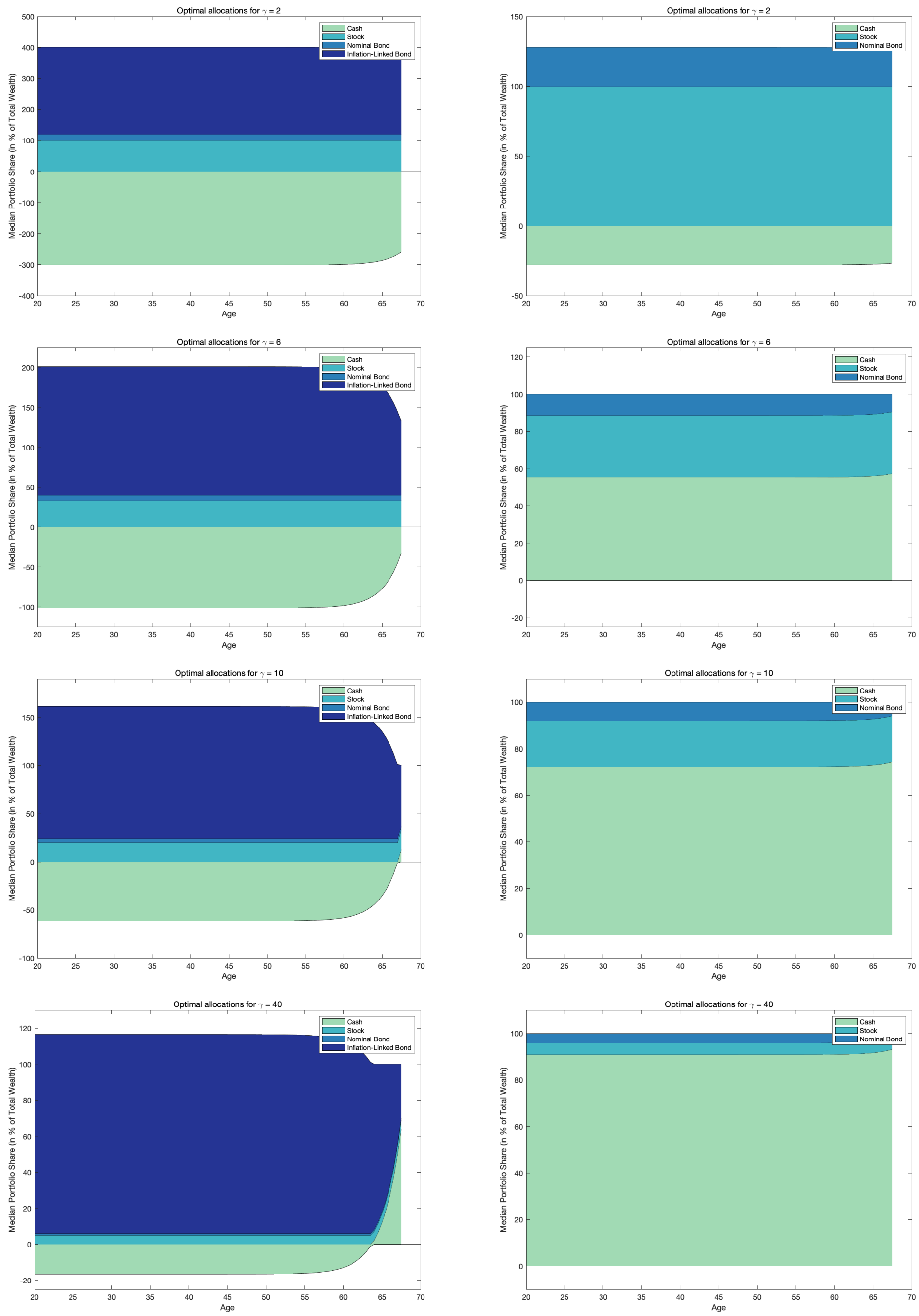

Figure 8: Median portfolio shares over the life cycle. These figures show the median fractions of total wealth invested in a 10-year inflation-linked zero-coupon bond, 30-year nominal zero-coupon bond, stock, and risk-free cash. The medians are based on $n = 10,000$ simulations, where for each graph, where for each graph, we take a different $\gamma \in \{2, 6, 10, 40\}$. The left-hand side depicts the optimal allocation for the complete market setting and the right for the incomplete market. The simulations are based on U.S. parameterset defined in the Appendix, Table (5).