A comparison of Two-factor equilibrium models under negative interest rates

by
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Abstract

We calibrate and compared a variety of two-factor equilibrium short rate models using a constraint minimization problem with the aim to provide a fair comparison. In order to test their performance in times of negative rates we calibrate the model parameters to the market zero-coupon curve at 30/11/2020 and 29/10/2021. We obtain good results in the calibration exercise as the two-factors models reproduce the market term structure with high accuracy on both days. We then compared the poor capacity to fit the market zero-coupon curve of the one-factor models, the Vasicek model and CIR model, which encourage our choice of two-factor short rate models to better fit the market term structure with negative rates. We also ran some numerical experiments to test the two-factor equilibrium models given the calibrated parameters at 29/10/2021. We approximate the short rate by means of the Euler scheme and we compute various Monte Carlo experiments to ultimately drive a stylized option pricing problem.

Acknowledgements

My sincere thanks to my thesis supervisor dr. Nikolaus Schweizer. I really appreciate his assistance at every stage of the research and the constructive suggestions to improve the present thesis. Besides, it has been a great opportunity to attend the courses taught by dr. Nikolaus Schweizer which has risen my curiosity to learn more from such an interesting matter and get started in this stochastic modeling field. I would also like to thank my second reader dr. Serhan Sadikoglu for taking the time to go through all these pages.

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Chapter 1

Introduction

Before the global financial crisis, there was a common belief among academics and practitioners that interest rates have to be positive. Due to the zero lower bound framework, the well-known Vasicek [23] model was considered unrealistic since it assumes normally distributed interest rates. In other words, in the Vasicek model interest rates can reach negative values. The drawback of the mentioned Gaussian model led to some other alternatives, for instance Cox, Ingersoll and Ross [7] proposed a model where the instantaneous short rate remains positive.

However, in the aftermath of the global financial crisis the European Central Bank (ECB) had introduced a negative interest rate policy as a strategy to provide more credit to the economy. As a result, the CIR model is not a suitable short rate model in times of negative interest rates. To overcome its limitations new approaches were therefore needed. A simple and popular approach is to shift the model by subtracting a positive constant which enables the model to reach a negative-lower bound. Alternatively, Di Francesco and Kamm [8] have suggested a model based on the difference of two independent CIR processes.

Indeed, numerous challenges have arisen in relation to interest rate modelling. A substantial problem of one-factor models, namely the Vasicek and CIR model, is their likely poor ability to fit more complicated term structure shapes. Typically, the shape of the yield curve under negative interest rates is humped, which signals a higher degree of uncertainty in the economy. One-factor models are too rigid to properly function under the negative interest rate framework. It is reasonable, therefore, to consider more sources of uncertainty by increasing the number of factors to properly fit term structure shape. In particular, we focus our study on two-factor equilibrium models as they have an attractive economic interpretation.

To our knowledge, several papers have discussed the analytical features of the two-factor equilibrium models. However, little attention has been given to their empirical application under a framework of negative rates. Hence, we aim to contribute to the literature by testing the performance of the two-factor equilibrium models to the current market data.

The paper is organized as follows. In Chapter 2 we discuss the extensive literature on interest rate modelling. After that, in Chapter 3 we provide an overview of one-factor interest rate models and their main limitations. The poor capacity of one-factor models to reproduce the term structure leads to the introduction of two-factor equilibrium models in Chapter 4. We suggest that the two-factor approach is a suitable solution to match the yield curve accurately. For the aim of this thesis, we choose appealing extensions of the one-factor Vasicek and CIR model which are found in the literature. Then, we conduct some experiments to test the models. First, in Chapter 5 we calibrate all the proposed models to the market data at 30/11/2020 and 29/10/2021. The main goal in this Chapter is to provide a fair comparison between the models. Then, in Chapter 6 we choose the most suitable models based on the calibration analysis and we work out a numerical analysis to test their ability to price a stylized option.

Chapter 2

Literature Review

The term structure of interest rates plays a major role in monetary policy and it is a crucial macroeconomic indicator for financial and economic institutions. Hence, a better understanding of interest rates dynamics has caught a strong attention among academics and practitioners. Over the last decades, a vast literature has developed a theoretical and empirical framework of stochastic short rate models in a continuous time.

There are two approaches to model the short rate in continuous time. The equilibrium approach describes the underlying economy by assuming that interest rates revert towards a long-term mean rate. The second approach, the arbitrage-free model captures the current information of the market (e.g., investors expectations or risk premiums). The scope of this study will cover the equilibrium approach for two reasons: the economic interpretation of the models and the straight-forward closed-form solutions. However, we will briefly mention in this Chapter the main works that have been published regarding arbitrage-free models.

The pioneering works on equilibrium models were developed by Vasicek [23] in 1977 and Cox-Ingersoll-Ross [7] in 1985. Both models are time-homogeneous, meaning that the parameters of the models are constant. Before the European Central Bank had introduced its negative interest rate policy, the well-known Vasicek model was considered unrealistic since it assumes normally distributed interest rates. Thus, the CIR Model introduce a squared-root term $\sigma\sqrt{r(t)}$ in the diffusion coefficient forcing interest rates to be positive.

Additionally, the poor fitting of the initial term structure of interest rates in the Vasicek model had led to a popular arbitrage-free model proposed by Hull and White [11]. The model, also known as extended Vasicek model, is capable of exactly reproducing the initial term structure of interest rates by allowing the long-term mean parameter to be time-dependent. Another way to generalize this model is by also allowing a time-dependent volatility [10]. We notice that an extended CIR Model (ECIR) was proposed in [17]. However, the ECIR model was less successful as no analytical solutions for the long-term mean are available in the literature. Another important work in the arbitrage-free approach is the time-dependent log of the short rate model proposed by Black and Karasinski in [1].

Up to this point, the time-homogeneous models fail when we aim to fit the observed term structure of interest rates. Furthermore, a more general drawback of the one-factor models is that they are too rigid to fit a non-monotone shape of the term structure. Therefore, several papers have been proposed to solve this problem. One way to approach this is to add a deterministic-shift to fit the observed term structure. This follows another alternative which allows the short rate to depend on multiple factors. Hence, in order to include these new approaches, Brigo and Mercurio proposed the CIR++ Model in [4] and the Two-Additive-Factor Gaussian Model G2++ in [5]. Other notable contributions came from Hull and White in [9] with the Two-Factor version, Longstaff and Schwartz (LS) in [15] where the volatility of the short rate is defined as the second factor, and Brennan and Schwartz in [3] where the second factor is the long-term interest rate.

To a certain extent it is possible to add more factors at the expense of adding more complexity to the models. Langetieg in [14] proposed a theoretical framework of multi-factor models and Chen in [6] proposed a Three-Factor model. Nevertheless, some studies based on principal component analysis of the yield curve (cf. Jamshidian and Zhu [12]) have suggested that two factors explains the majority of the total variation in the yield curve. Thus, for the purpose of this thesis we are interested in two-factor models.

Lastly, most recent literature is produced under a negative interest rate environment. In 2014, the introduction of quantitative easing policies to boost the economy led to a paradigm shift for interest rate modelling. How the mentioned models perform under negative interest rates?

Orlando et al. [21] had proposed the CIR# Model. The model consists in shifting the market rates by adding an arbitrary positive constant $r_{shift}(t) = r(r) + \alpha$. This was followed by comparison analysis between the CIR# and the Hull-White Model in [20]. The referred paper concluded that the CIR# Model outperforms the Hull-White Model most frequently. In addition to that, a new model has been published this year by Di Francesco and Kamm [8] where there is no need to add a shift term to handle negative interest rates. The short rate dynamic in this model depends on the difference between two independent CIR processes. The authors have empirically tested the model for two different dates, 30/12/2019 and 30/11/2020. In short, the model presents some interesting results and works relatively well. Since the introduction of negative rates in June 2014 other papers proposed an extension of the CIR model ([24],[18] and [19]) to handle negative rates. However there is no empirical test of this models.

Regarding the performance of Gaussian models under negative interest rates there is no evidence in the literature, to our knowledge, on how these models behaves empirically. Is the G2++ Model a good candidate to handle negative interest rates? Could we obtain reasonable parameters estimates in the calibration procedure to the market data? In 2021, a paper published by Keller[13] had set the two-factor Vasicek Model with dependent factors and it was mathematically proved. Although, there is no numerical tests to the market data.

Due to the poor literature on testing the performance of equilibrium models under negative interest rates we would like to fill this gap by stating the following purposes: (i) we aim to conduct an empirical analysis of the Two-factor Gaussian and CIR models, (ii) we aim to provide a comparison between the mentioned models, and finally (iii) we question the performance of the classical models and test the accuracy of their two-factor extensions to fit the market data at two concrete days.

Chapter 3

Interest Rate Modelling

3.1 Term Structure of interest rate

It is a fact that interest rates change over time. If the European Central Bank (ECB) announces to buy a large amount of government bonds, then the demand increases followed by a rise of prices. As a result, the yields fall. On the contrary, if the ECB announces to issue a large amount of government bonds the yields tend to rise. The first case scenario is an example of *quantitative easing* policies which had led to erratic movements of interest rates. Hence, the stochastic behavior of interest rates leads to an increasing interest among academic and practitioners to model their evolution. In this chapter we will describe the general framework of interest rate modelling.

3.1.1 Definition

Interest rates have a term structure which means that short rates are typically different from long-term rates. The relation between interest rates and different maturities is represented by the so-called *yield curve*. This curve is a crucial tool for investors and policy makers as it provides information about expected growth of the economy. Resulting from the *Expectation Hypothesis* (see, Section 3.1.2) the shape of the yield curve reflect investors believes about future interest rates. Thus, the optimistic or pessimistic perspective of investors about future interest is an essential source of information for Central Banks and a driver of the health state of the economy.

This suggests that the large variety of the term structure shapes is an important matter to understand the current economy and its projections. The result of a *positive* slope of the yield curve means that yields strictly increase across maturities. In other words, short-term interest rates are lower than long-term interest rates. This is a typical shape of the yield curve as investors demand a higher compensation (higher rate) for taking more risk by means of lending money for a longer period. Hence, investors expect a future growth of the economy which could lead to a higher inflation. Conversely, if short-term interest rates are higher than long-term interest rates the slope of the curve is inverted. In that case, investors expect a slowdown in the economic growth. Therefore, an inverted yield curve is an indication of recession. Another possible shape is the flat curve which is very rare. This curve reflects a state of transition between a period of growth and a period of recession.

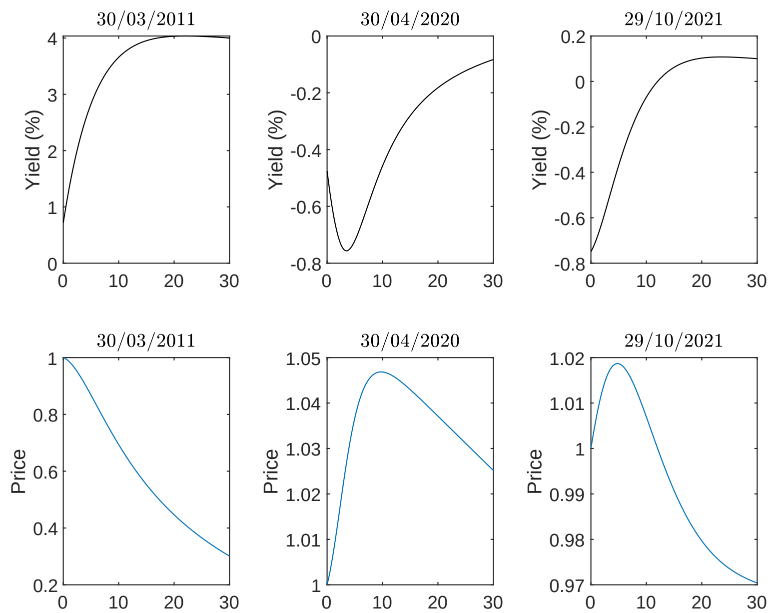


Figure 3.1: Euro area yield curve and Euro bond price curve (term to maturity in years). The figures in the top show the zero-coupon curve at three different dates and their respective bond price curve can be found at the bottom. The data is obtained from the statistics provided by the European Central Bank.

Moreover, it is important to note that a common characteristic of the referred three shapes is that they have a monotone evolution across maturities. Could the yield curve have a non-monotone shape? Do we observe it in the market?

The answer is yes. The yield curve could present, for instance, a hump. It is possible to find a local *maxima* and *minima* in the yield curve, although it is very rare to find a shape with multiple extreme values. A hump, as an inflexion point, shows some level of uncertainty in the economy. For instance, the Euro Area yield curve at 30/04/2020 has a local *minima* where mid-term interest rates are lower compared to short and long interest rates. The source of uncertainty shown by the humped yield curve is clearly caused by the COVID-19 pandemic which was declared by the World Health Organization (WHO) in March 2020. The pandemic had caused a severe shock on the global economy, and of course, had impacted the shape of the term structure of interest rates by warping the shape as is it shown in Figure 3.1. In short, interest rate models are an important matter for policy makers and investors.

Lastly, an accurate representation of a positive, inverted or flat yield curve is straightforward as the curve is smooth and monotonic. Nevertheless, modelling a humped curve will need a higher level of complexity as we require to add more sources of uncertainty to properly illustrate the observed shocks in the market. One possible way is by adding more factors to the general one-factor framework.

3.1.2 Expectation Hypothesis

In the previous section 3.1.1 we have seen that the term structure of interest rate reflects investors expectations about future interest rates. Accordingly, by the Expectation Hypothesis this means that the long-term rates are determined by the current and future expected short-term interest rates. Thus, the price at time 0 of a bond that pays one unit of currency at time T under a risk-neutral measure $\mathbb{Q}$ satisfies

$$P(0,T) = E^{\mathbb{Q}}\left[exp\left\{-\int_0^T r_t dt\right\}\right] \tag{3.1}$$

where r_t is the short rate at time t and by definition $P(T,T) = 1$. On the left hand of the equation we have the current term structure of the bond price and on the right hand we have the expectations under $\mathbb{Q}$ of the future evolution of the short rate. More generally, for a price at time $t > 0$ with maturity $T > t$ the relationship is given by

$$P(t,T) = E_t^{\mathbb{Q}}\left[exp\left\{-\int_t^T r_s ds\right\}\right] \tag{3.2}$$

Note that, the underlying short rate determines the stochastic properties of the zero-coupon bond price $P(t,T)$ with maturity T at time $t \in [0,T]$. For the purpose of this thesis, the bond price formula (3.2) will be used to derive all the models.

3.1.3 Fundamental Interest-Rate Curves

The term structure of interest rates can be described in different ways. For the purpose of this thesis, we will discuss the most fundamentals ones: the discount curve and the yield curve. Before describing the mentioned curves let us note that the *zero-coupon bond* is the type of product we will work with. The *zero-coupon bond* is a debt security that does not pay interest during their life. This bond is bought at a lower price than its face value, with the value repaid at the time of the maturity.

The price of zero-coupon bond (discount curve or zero-bond curve) is denoted by $P(t,T)$. The curve shows the relation between the discount factor and different maturities T of the zero-coupon bond.

$$T \to P(t,T), T > t$$

The continuously-compounded short rate (yield curve or zero-coupon curve). The curve shows the relation between the yields and different maturities T of the zero-coupon bond.

$$T \to R(t,T), T > t$$

We can derive the yield curve $R(t,T)$ with maturity T at time $t \in [0,T]$ from the zero-coupon bond price, also known as discount factor. Note that the discount curve and the yield curve have an inverse relation (when prices increase, the returns goes down and vice-versa).

$$R(t,T) = -\frac{1}{T-t}\log P(t,T) \tag{3.3}$$

3.1.4 One-Factor Short Rates Models

An interest rate model is a probabilistic description of how interest rates can change over time. The stochastic state variable of the model is the short rate (or instantaneous spot rate) which is the interest rate for a infinitesimally short time $r(t) = \lim_{T\to t} R(t,T) = R(t,t)$. Therefore, the continuous-time process of the short rate is based on Stochastic Differential Equations (SDE) driven by a Brownian motion denoted by W_t. The SDE has two components, the drift and the volatility term. Each one of the components depends on the specification of the model.

The general form of the short rate under the risk-neutral measure $\mathbb{Q}$ is as follows,

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \tag{3.4}$$

The evolution of the short rate is described by the drift term $\mu(t, r(t))$ and a diffusion term $\sigma(t, r(t))$. The random component W_t are increments of normally distributed Brownian Motions $dW(t) \sim \mathcal{N}(0, dt)$. Note that the drift and the volatility term can be also defined as time-homogeneous components of the SDE, such as $\mu = \mu(t, r(t))$ and $\sigma = \sigma(t, r(t))$.

It is important to point out that the short-rate dynamics could be defined under an equivalent measure, called the real-world measure $\mathbb{P}$. For the purpose of this thesis, we are mainly concerned by computing the expectations under the $\mathbb{Q}$-dynamics. Thus, the models in Section 3.2 and Chapter 4 are defined under the risk-neutral measure $\mathbb{Q}$. In case we are interested to define the models under the real-world measure we need to specify the market price of risk denoted by λ as this parameter connects both measures. To do so we need to compute a change of measure based on the Girsanov Theorem. Given that we are not describing the short rate dynamics under $\mathbb{P}$ we are not explaining the mathematics behind the change of measure by the Girsanov Theorem neither the analytical closed-form solutions under $\mathbb{P}$. But we will rather give a brief derivation of the Classical models under the real-world measure as an illustration.

One of the most striking questions is how interest rates evolve over time. Do they follow a specific trend? Or do interest rate evolve in a similar way as stocks do? The answer for equilibrium models and economic theory is that interest rates have a mean reverting behavior. This assumption implies that shocks on interest rates are transitory, for instance a jump of inflation or a financial crisis cause a deviation from the long-term mean which will revert to its level in the long run.

To represent the mean reversion behavior in the SDE of the short rate it requires to modify the drift term in (3.4). We also generalize the diffusion term such as:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma r(t)^{\gamma} dW(t) \tag{3.5}$$

Where κ is the speed of reversion, θ is the long-term mean and σ is the volatility. All these three parameters are positive constants as far as we stay in the time-homogeneous models. We decide to also generalize the volatility term by including the parameter γ and the short rate $r(t)$. As we will see in the following Sections, for the Vasicek model we have that $\gamma = 0$ and for the Cox-Ingersoll-Roos model we have that $\gamma = \frac{1}{2}$. The same values are applicable for their two-factor version.

3.2 Classical Short-rate Models

A benchmark in the interest rate model literature is the work done by Vasicek in 1977 and and Cox, Ingresoll and Ross in 1985. These models are time-homogeneous and are also known as *endogenous models* as the term structure of interest rates is an output of the model.

3.2.1 The Vasicek Model

Vasicek (1977) is the simplest mean reversion model and follows the Ornstein–Uhlenbeck process [1]. The model is one-dimensional as the short rate is the unique state variable. Therefore, we are in the scope of one-factor models with normally distributed interest rate changes.

The Stochastic Differential Equation under the risk-neutral measure $\mathbb{Q}$ is as follows:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t), r(0) = r_0 \tag{3.6}$$

where κ, θ, σ and r_0 are positive constants. Note that if $r(t) > \theta$, then the drift term is negative so that r is pushed downwards to get closer on average to the long-term mean θ. Whereas if $r(t) < \theta$ the drift term becomes positive so that r is pushed upwards getting again closer in average to the long-term mean level. Besides, in case κ take higher values the short rate will reach sooner the mean level.

The short rate $r(t)$ is normally distributed and by integrating the SDE (3.6) we obtain the following process for each $s \leq t$,

$$r(t) = r(s)e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}dW(u) \tag{3.7}$$

The $\mathcal{F}_s$ conditional mean and variance of $r(t)$ are given by

$$E\{r(t)|\mathcal{F}_s\} = r(s)e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) \tag{3.8}$$

$$Var\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-s)}\right] \tag{3.9}$$

The price of the zero-coupon bond. We can obtain the zero-coupon bond price formula by computing the t-conditional expectation under the $\mathbb{Q}$-dynamics in (3.2). Thus, we obtain a more convenient expression of the bond price which just depends on two deterministic functions $A(t,T)$ and $B(t,T)$ and the short rate $r(t)$:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{3.10}$$

where,

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4\kappa}B(t,T)^2\right\}$$
$$B(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]$$

[1]The Ornstein-Uhlenbeck process is a stochastic process that satisfies a stochastic differential equation with the form $dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t)$

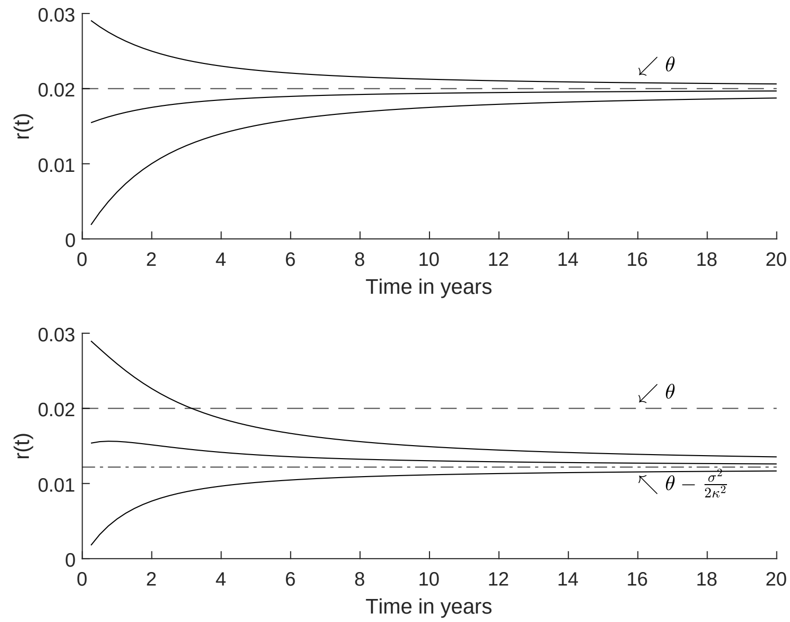


Figure 3.2: Yield curve shape of the Vasicek model for different initial conditions. The figure on the top shows the behavior of the short rate with low volatility $\sigma = 0.001$ and the figure on the bottom shows the behavior of the short rate with high volatility $\sigma = 0.1$. The speed of reversion and long-tern mean are $\kappa = 0.8$ and $\theta = 0.02$, respectively. The initial conditions for the short rate are given by $x_0 = \{0.00, 0.015, 0.030\}$.

To obtain the continuously-compounded short rate we just have to replace the equation (3.10) in (3.3).

Objective measure dynamics. For the purpose of this research, we will not cover the dynamics of the interest models under the real-world measure $\mathbb{P}$. However, we had derived the process under $\mathbb{P}$ if further research is needed regarding this matter. The derivation can be found in Appendix A.1.

Shape of the yield curve and parameters contribution. When the time to maturity goes to infinity, the yield curve converges to $R(0, \infty) = \theta - \frac{\sigma^2}{2\kappa^2}$. Depending on the values of the volatility σ and the speed of reversion κ the yields will converge or deviate from the long-term mean θ.

In Figure 3.2 we show the sensitivity of the Vasicek model to the volatility term. For a high value of the volatility $\sigma = 0.1$ the yield curve completely deviates from the long-term mean. In other words, we observe a persistent shock. We can compensate this high volatility by setting a higher value of the speed reversion κ. Nevertheless, a too high value of the speed could be unrealistic.

3.2.2 The Cox, Ingersoll and Ross Model (CIR)

The Cox-Ingersoll-Ross Model (1985) has the same mean reversion process as the Vasicek Model with the exception that the volatility term is no longer constant. The squared-root of the short rate is added in the diffusion term to preserve non-negative interest rates.

The Stochastic Differential Equation under the risk-neutral measure $\mathbb{Q}$ is as follows

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), r(0) = r_0 \tag{3.11}$$

where κ, θ, σ and r_0 are positive constants. Moreover, the condition $2\theta\kappa > \sigma^2$ ensures that the origin is not reach. Besides, this condition guarantees a stationary diffusion process (cf. [22]). Though, we can relax the condition by setting $2\theta\kappa \geq \sigma^2$ so we don't need to exclude the origin.

The short rate $r(t)$ has a noncentral χ^2 distribution and the $\mathcal{F}_s$ conditional mean and variance are as follows

$$E\{r(t)|\mathcal{F}_s\} = r(s)e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right)$$

$$Var\{r(t)|\mathcal{F}_s\} = r(s)\frac{\sigma^2}{\kappa}\left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \theta\frac{\sigma^2}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2$$

The price of the zero-coupon bond with maturity T at time t is as follows:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{3.12}$$

where,

$$A(t,T) = \left[\frac{\phi_1 e^{\phi_2(T-t)}}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1}\right]^{\phi_3}$$

$$B(t,T) = \left[\frac{e^{\phi_1(T-t)} - 1}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1}\right]$$

with $\phi_i \geq 0$ and $i = 1, 2, 3$,

$$\phi_1 = \sqrt{\kappa^2 + 2\sigma^2}, \quad \phi_2 = \frac{\kappa + \phi_1}{2}, \quad \phi_3 = \frac{2\kappa\theta}{\sigma^2}$$

To obtain the continuously-compounded short rate we just have to replace the equation (3.12) in (3.3).

Shape of the yield curve and parameters contribution. The CIR process converges to $R(0,\infty) = \frac{2\theta\kappa}{\kappa+\sqrt{\kappa^2+2\sigma^2}}$. If the volatility σ is closed to zero, then the rate will converge to the long-term mean θ. We can observe in the formula that the speed of reversion has a stronger impact on the yield curve compared to the volatility.

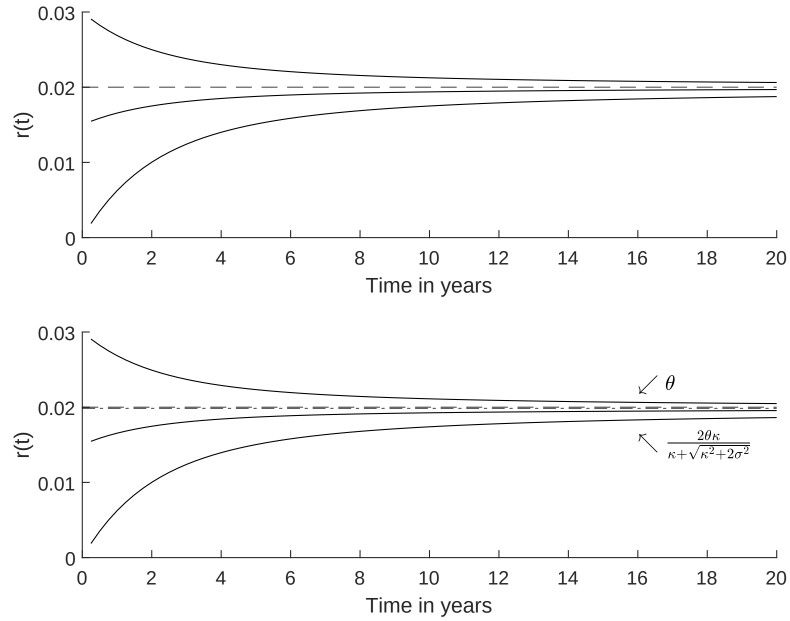


Figure 3.3: Yield curve shape of the CIR model for different initial conditions. The figure on the top shows the behavior of the short rate with low volatility $\sigma = 0.001$ and the figure on the bottom shows the behavior of the short rate with high volatility $\sigma = 0.1$. The speed of reversion and long-tern mean are $\kappa = 0.8$ and $\theta = 0.02$, respectively. The initial conditions for the short rate are given by $x_0 = \{0.00, 0.015, 0.030\}$.

We can observe that in Figure 3.3 that a high volatility do not significantly impact the yield curve. Due to the fact that the diffusion process of the CIR model is shrink $\sigma\sqrt{r(t)}$ the deviation from the long-term mean is less visible compared to the Vasicek model.

3.3 How to deal with negative interest rates?

Several key issues arise from modelling the term structure under negative interest rates. One can distinguish three main challenges: (i) the possibility to reach negative interest, (ii) the further capability of the model to fit the market data, and (iii) the representation of realistic parameters estimates.

Regarding the Vasicek Model (i) is not a major problem as the model can have negative interest rates with positive probability. However, as interest rates are normally distributed, they can reach extremely negative values which is not observable in the market data. Moreover, the CIR Model also fall through. By construction, the CIR model is not able to deal with negative interest rates as the diffusion term is defined as a squared root process. Thus, it is not allowed to have negative interest rates.

With reference to (ii) neither of both models are flexible enough to fit the term structure under negative rates. As it was previously analyzed, in Figure 3.1 we can observe that the yield curve in the recent years, for instance at 30/04/2020, shows a hump together with negative interest rates. Thus, it is needed to implement a model able to capture a non-monotonic shape of the term structure. Recall that the presence of a hump in the yield curve reflects a shock in the market. A reasonable solution is then to increase the number of sources of uncertainty to capture different financial shocks.

We are also concerned by the fact that the parameters of the short rate models need to be in some extent realistic. The advantage of equilibrium models is that they can describe the behavior of interest rates. Hence, the parameters calibrated to the market data are required to satisfy the premise of the exposed expectation hypothesis.

Finally, Brigo and Mercurio explained in [5] that one-factor models assume perfectly correlated interest rates for different maturities $\mathrm{Corr}(R(t,T_1),R(t,T_2)) = 1$. This feature of one-factor models is not realistic. It is then natural to argue that a model which allows for some decorrelation between rates is a suitable improvement to better fit the term structure of interest rates.

Altogether, there are several reasons motivating the multi-factor models. For the purpose of this thesis, we consider potential two-factor equilibrium models to solve the problems previously mentioned. We have decided to focus on two-factor models for their better implementability and their capacity to explain most of the variations in the yield curve (cf.[12]). This leads to a model of the short rate with the form:

$$r(t) = x(t) + y(t)$$

Note that other expressions could be computed (by adding a deterministic shift to the process or working a difference between the two factors).

For the scope of this thesis we disregard the analysis of the two-factor performance regarding the correlation patterns. We focus our analysis on the capacity of the models to fit the term structure of interest rates and their flexibility to capture a non-monotonic shape. Besides, the value of the parameters, for instance the volatility, are important to be well calibrated. A practical implementation of stochastic interest rates is to price derivatives. Therefore, a better understanding of the role of the parameters and its values is important.

Chapter 4

Two-Factor Interest Rate Models

The Two-factor model is a better alternative to handle complicated yield curve shapes and a step-forth to represent non-perfect correlation between yields. Moreover, from an empirical point of view is it safe to say that the term structure of interest rates could be influenced by different factors of risk.

4.1 Multi-factor Gaussian Models

The models proposed in this Chapter are based on the benchmark work by Brigo and Mercurio [5]. To handle negative interest rates, we will implement some novelties and adjustments to the models.

4.1.1 The Two-factor Gaussian Model (G2++)

The G2++ model is introduced in [5] by Brigo and Mercurio. In this thesis, we propose a new version of the two-factor gaussian model. We have decided to maintain the short dynamics defined as the sum of two dependent gaussian processes and a deterministic shift φ. However, instead of constructing φ to fit exactly the initial zero-coupon curve we set φ as the expected drift term of the short rate.

The instantaneous short-rate process is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0 \tag{4.1}$$

With x and y being two dependent gaussian defined as follows

$$dx(t) = -\kappa_x x(t)dt + \sigma_x dW_x(t), \quad x(0) = 0 \tag{4.2}$$

$$dy(t) = -\kappa_y y(t)dt + \sigma_y dW_y(t), \quad y(0) = 0 \tag{4.3}$$

The parameters of the model $\kappa_x, \sigma_x, \kappa_y$, and σ_y are positive constants. Note that $r_0 = \varphi(0)$ and we allow the initial condition r_0 to be negative. The two-dimensional Brownian motion $(W_x(t), W_y(t))$ under the risk-neutral measure have instantaneous correlation denoted by ρ.

Correlated Brownian Motions. Given two independent Brownian motions $(\tilde{W}_1, \tilde{W}_2)$, W_x and W_y can be expressed as follows:

$$dW_x(t) = d\tilde{W}_1(t)$$
$$dW_y(t) = \rho d\tilde{W}_1(t) + \sqrt{1-\rho^2}d\tilde{W}_2(t)$$

Deterministic shift. We define the deterministic function φ as the mean of the short rate of the one-factor model given in equation (3.8):

$$\begin{aligned}\varphi(t) &= r_0 e^{-\kappa_x t} + \int_0^t \theta e^{-\kappa_x(t-v)} dv \\ &= r_0 e^{-\kappa_x t} + \frac{\theta}{\kappa_x}(1 - e^{-\kappa_x t})\end{aligned}$$

Note that we allow the mean-reversion parameter θ to reach negative values.

By doing a simple integration of equation (4.2) and (4.3) we obtain the following expression for the short rate process in (4.1) for each $s \leq t$,

$$r(t) = x(s)e^{-\kappa_x(t-s)} + y(s)e^{-\kappa_y(t-s)} + \sigma_x \int_s^t e^{-\kappa_x(t-u)} dW_x(u) + \sigma_y \int_s^t e^{-\kappa_y(t-u)} dW_y(u) + \varphi(t)$$

The mean and the variance of $r(t)$ conditional on $\mathcal{F}_s$ is as follows

$$\begin{aligned}E\{r(t)|\mathcal{F}_s\} &= x(s)e^{-\kappa_x(t-s)} + y(s)e^{-\kappa_y(t-s)} + \varphi(t) \\ Var\{r(t)|\mathcal{F}_s\} &= \frac{\sigma_x^2}{2\kappa_x}\left[1 - e^{-2\kappa_x(t-s)}\right] + \frac{\sigma_y^2}{2\kappa_y}\left[1 - e^{-2\kappa_y(t-s)}\right] + 2\rho\frac{\sigma_x\sigma_y}{\kappa_x+\kappa_y}\left[1 - e^{-(\kappa_x+\kappa_y)(t-s)}\right]\end{aligned}$$

The price of the zero-coupon bond with maturity T at time t is as follows (Brigo Mercurio):

$$P(t,T) = exp\left\{-\int_t^T \varphi(u)du - B_x(t,T)x(t) - B_y(t,T)y(t) + \frac{1}{2}V(t,T)\right\}$$

where,

$$B_z(t,T) = \frac{1 - e^{-\kappa_z(T-t)}}{\kappa_z}, z \in \{x, y\}$$

To calculate the variance $V(t,T)$ the following formulas have to be used:

$$\begin{aligned}V(t,T) = &\frac{\sigma_x^2}{\kappa_x^2}\left[T - t + \frac{2}{\kappa_x}e^{-\kappa_x(T-t)} - \frac{1}{2\kappa_x}e^{-2\kappa_x(T-t)} - \frac{3}{2\kappa_x}\right] \\ &+ \frac{\sigma_y^2}{\kappa_y^2}\left[T - t + \frac{2}{\kappa_y}e^{-\kappa_y(T-t)} - \frac{1}{2\kappa_y}e^{-2\kappa_y(T-t)} - \frac{3}{2\kappa_y}\right] \\ &+ 2\rho\frac{\sigma_x\sigma_y}{\kappa_x\kappa_y}\left[T - t + \frac{e^{-\kappa_x(T-t)} - 1}{\kappa_x} + \frac{e^{-\kappa_y(T-t)} - 1}{\kappa_y} - \frac{e^{-(\kappa_x+\kappa_y)(T-t)} - 1}{\kappa_x+\kappa_y}\right]\end{aligned}$$

In particular, we have

$$r(t) = \sigma_x \int_0^t e^{-\kappa_x(t-u)} dW_x(u) + \sigma_y \int_0^t e^{-\kappa_y(t-u)} dW_y(u) + \varphi(t)$$

The price at time 0 of the zero-coupon bond with maturity T is as follows,

$$P(0,T) = exp\left\{ -\int_0^T \varphi(u)du + \frac{1}{2}V(0,T) \right\}$$

Therefore, we obtain

$$\begin{aligned}\int_0^T \varphi(u)du &= \int_0^T r_0 e^{-\kappa_x u} + \frac{\theta}{\kappa_x}(1 - e^{-\kappa_x u})du \\ &= r_0 \int_0^T e^{-\kappa_x u} du + \frac{\theta}{\kappa_x} \int_0^T (1 - e^{-\kappa_x u})du \\ &= \frac{r_0}{\kappa_x}(1 - e^{-\kappa_x T}) + \frac{\theta}{\kappa_x}\left(T + \frac{e^{-\kappa_x T}}{\kappa_x} - \frac{1}{\kappa_x} \right)\end{aligned}$$

The zero-coupon bond price at time 0 with maturity T is given by,

$$P(0,T) = exp\left\{ -\frac{r_0}{\kappa_x}(1 - e^{-\kappa_x T}) - \frac{\theta}{\kappa_x}\left(T + \frac{e^{-\kappa_x T}}{\kappa_x} - \frac{1}{\kappa_x} \right) + \frac{1}{2}V(0,T) \right\}$$

4.1.2 The Two-factor Vasicek Model

The Two-factor version of the Vasicek model describes the evolution of the instantaneous short rate according to the sum of two independent Gaussian processes x_t and y_t. Both factors follow a mean-reverting process. We will set that x_t is the *fast* spot rate and y_t is the *slow* spot rate where $k_y < k_x$.

The instantaneous short-rate process be given by

$$r(t) = x(t) + y(t), \quad r(0) = r_0$$

The two independent factors are given by

$$\begin{aligned} dx(t) &= \kappa_x(\theta_x - x(t))dt + \sigma_x dW_x(t), \quad x(0) = x_0, \\ dy(t) &= \kappa_y(\theta_y - y(t))dt + \sigma_y dW_y(t), \quad y(0) = y_0, \end{aligned}$$

Where $(\kappa_z, \theta_z, \sigma_z)$,$z \in \{x, y\}$, are positive constants and the two Brownian Motions are not correlated $dW_1 dW_2 = 0$. Besides, we allow the initial conditions x_0 and y_0 to reach negative values.

The short rate $r(t)$ is normally distributed as it is the sum of two independent gaussian processes. We obtain the following process of the short rate for each $s \le t$,

$$\begin{aligned} r(t) = x(s)e^{-\kappa_x(t-s)} + \theta\left(1 - e^{-\kappa_x(t-s)}\right) + y(s)e^{-\kappa_y(t-s)} + \theta_{yx}\left(1 - e^{-\kappa_y(t-s)}\right) \\ + \sigma_x \int_s^t e^{-\kappa_x(t-u)} dW_x(u) + \sigma_y \int_s^t e^{-\kappa_y(t-u)} dW_y(u) \end{aligned}$$

The $\mathcal{F}_s$ conditional mean and variance of $r(t)$ are given by

$$E\{r(t)|\mathcal{F}_s\} = E\{x(t)|\mathcal{F}_s\} + E\{y(t)|\mathcal{F}_s\}$$
$$Var\{r(t)|\mathcal{F}_s\} = Var\{x(t)|\mathcal{F}_s\} + Var\{x(t)|\mathcal{F}_s\}$$

The price of the zero-coupon bond with maturity T at time t is as follows:

$$P(t, T; x(t), y(t), \alpha) = P^1(t, T; x(t), \alpha_x)P^1(t, T; y(t), \alpha_y) \tag{4.4}$$

With $\alpha_z = (\kappa_z, \theta_z, \sigma_z)$ for $z \in \{x, y\}$. Due to the independence of the factors we can derive the formula of the bond price based on the analytical solutions of the one-factor Vasicek Model in equation 3.10. For instance, for $P^1(t, T; x(t), \alpha_x)$ we have to replace $r(t)$ by $x(t)$ and the parameters κ,θ and σ by α_x.

4.1.3 Illustration

Under the one-factor Vasicek model interest rates can become negative . Nevertheless, one source of uncertainty is too restrictive to properly fit the term structure under a recession context or unconventional monetary policies. Based on these considerations, we now illustrate the role of the two-factor processes parameters and how they are able to capture different fluctuations of the market, e.g. different uncertainty scenarios.

Firstly, we would like to highlight the main similarities between the one-factor Vasicek model and its two-factor versions. As it was mentioned earlier (see Section 3.2) the Vasicek model is remarkably sensitive to the volatility parameter σ. A reasonable high value of the volatility forces the short rate process to fall from the long-term mean. Hence, long-term interest rates $r(\infty)$ converge to $\theta - \frac{\sigma^2}{2\kappa^2}$ instead of θ. One can notice that the sensitivity of the short rate to σ is an interesting pattern to reach negative rates. This feature is also shared by the two-factors extensions. As it is shown in Figure 4.1, the two-factor Vasicek and G2++ short rate deviates from the long-term mean, which is the sum of the two factors mean parameter θ_x and θ_y, when the volatility is increased. Suppose that a financial shock occurs in the market, therefore, the volatility of both factors would be remarkably high. In this scenario, the speed of reversion of both factors κ_x and κ_y plays an important role. A high value of the speed of reversion could compensate the impact of the shock and diminish the ratio between σ_z and κ_z, $z \in (x, y)$, the towards zero. Hence, the short rate process will converge to the long-term mean avoiding a persistent shock in the market. Furthermore, let us note that the correlation between the two factors plays an important role regarding the long-term rates of the G2++ model. With $\rho > 0$ the long-term rates reach a lower level, with positive probability of reaching negative rates as it is shown in in Figure 4.1 for a given correlation of $\rho = 0.8$. On the contrary, with $\rho < 0$ the long-term rate is pushed towards the initial long-term mean.

Based on this considerations, one can argue that the parameters of the two-factor models have appealing features to be used in a negative interest rates environment. Nevertheless, as the number of parameters is higher compared to the one-factor Vasicek model it is risky to draw any conclusion without understanding better their role. The multi-dimensional space of the short rate gain to some extent flexibility with the cost of increasing the complexity of the analytical formulas.

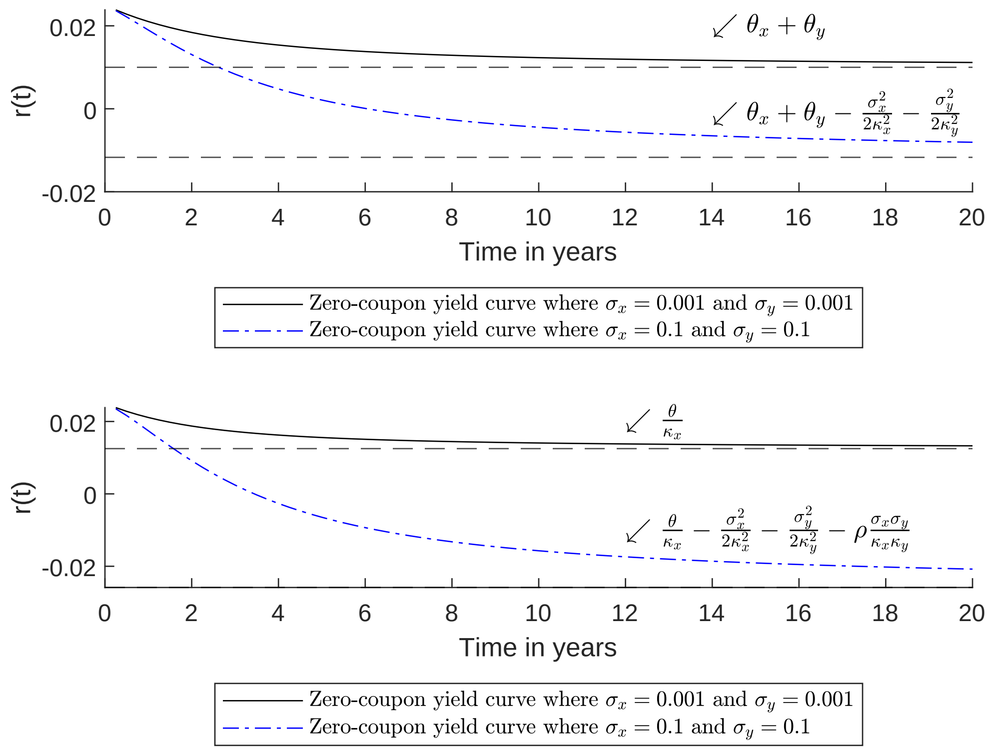


Figure 4.1: Sensitivity of the two-factor Vasicek model and G2++ model to the volatility of both factors $x(t)$ and $y(t)$. The figure in the top shows the zero-coupon yield curve of the two-factor Vasicek model with parameters $\kappa_x = 0.8, \kappa_y = 0.6, \theta_x = 0.006, \theta_y = 0.004$ and initial conditions $x_0 = 0.01$ and $y_0 = 0.015$. The figure in the bottom shows the zero-coupon yield curve of the G2++ model with parameters $\kappa_x = 0.8, \kappa_y = 0.6, \rho = 0.8, \theta = 0.01$ and initial condition $r_0 = 0.025$. For each model we can find the long-term convergence of interest rates.

A common parameter the two-factor Vasicek and the G2++ model share is the speed of reversion. One may argue that the effect of κ_z to the short rate of both models do not differ substantially. Thus, the sensitivity of κ_z to the two-factor Vasicek short rate volatility serves as illustration. In Figure 4.2 we can observe the volatility structure of the short rate at one-year maturity for different values of κ_z together with two scenarios where we combine different volatilities for the state variables $x(t)$ and $y(t)$. When the volatility of one of the factors, in this case σ_y, is remarkably low the contribution of κ_y is negligible. On the other side, if the volatility of both factors is high, then, the contribution of κ_x and κ_y is important and helps to reduce the volatility of the short rate if we increase the value of this parameters. Likewise, a suitable values for κ_z and σ_z could generate different shapes of the volatility structure, from a flat to a curved surface. Therefore, we can improve the flexibility of the models by increasing the number of factors.

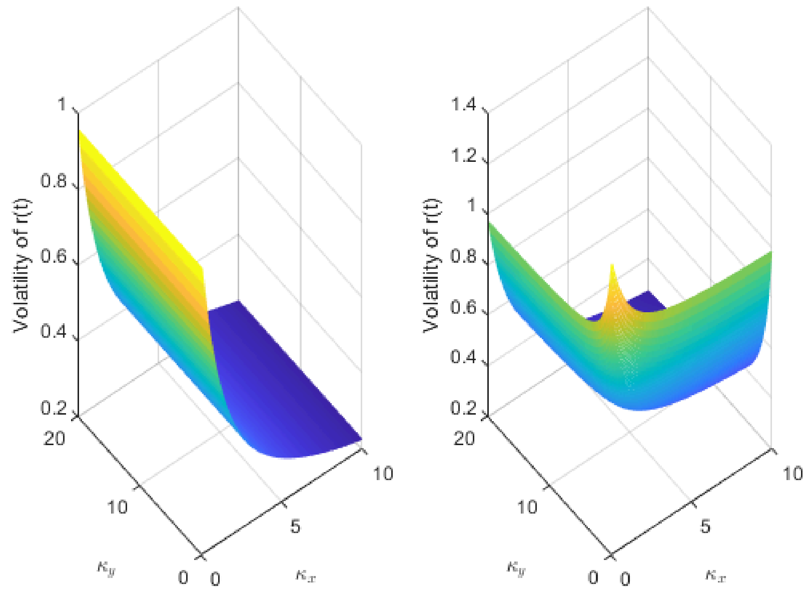


Figure 4.2: Sensitivity of the short rate volatility of the two-factor Vasicek model to the speed of reversion of both factors. The volatility has been computed by the squared root of the variance of the current short at one-year maturity $R(0,1)$ (the variance of the short rate is the sum of the two factor variance defined in equation 3.9). The left figure shows the short rate volatility where $\sigma_x = 1$ and $\sigma_y = 0.001$ and the right figure shows the short rate volatility where $\sigma_x = 1$ and $\sigma_y = 1$.

As it was pointed out previously the correlation parameter ρ plays an important role in the G2++ model. To illustrate this point, we have computed multiple yield curves, and their respective bond price curve, according to a set of correlation values. The impact of ρ is shown in Figure 4.3 where we can observe that the short rates with positive correlation, such as $\rho > 0$, are below the short rate with negative correlation. Thus, depending on the sign of the correlation the short rate evolves differently. We also notice that ρ has a higher impact on longer maturities. Lastly, we briefly remark that if the speed of reversion of the factors is substantially high, then, the impact of ρ became negligible (for illustration we refer to Appendix A.4).

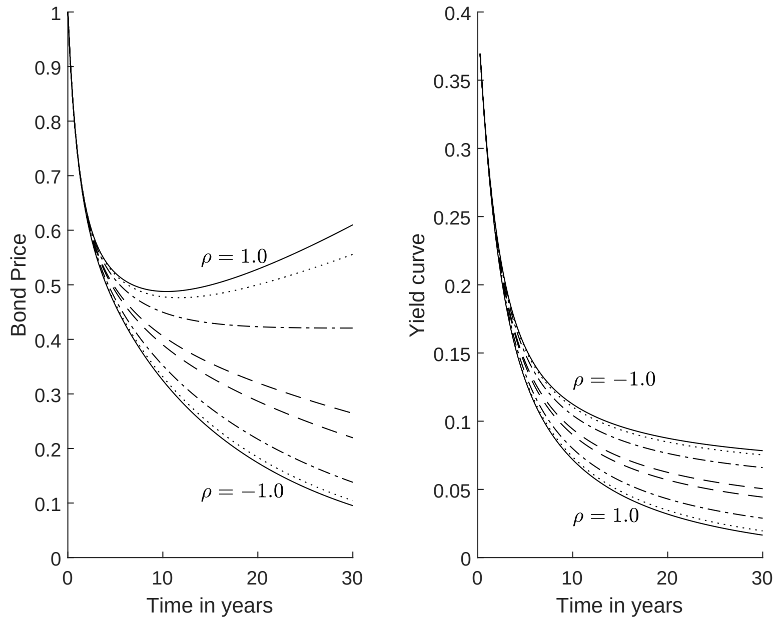


Figure 4.3: Sensitivity of the bond price and zero-coupon yield curve to the correlation between the factors for the G2++ model. The short rate parameters are: $\kappa_x = 0.8, \sigma_x = 0.1, \kappa_y = 0.2, \sigma_y = 0.06, \theta = 0.06$ and the initial condition $r_0 = 0.4$. The correlation between $x(t)$ and $y(t)$ is conditioned to the following set of values $\rho = \{-1.0, -0.9, -0.6, -0.1, 0.1, 0.6, 0.9, 1.0\}$.

It has been shown that the two-factors models are able to reproduce different volatilities surfaces by *playing* with different combinations of parameters values. Moreover, the introduction of the correlation between two state variables has an appealing economic interpretation if we assume y and x to represent macroeconomic factors. The correlation enriches the model flexibility for suitable values of the speed of reversion and volatilities. One is then tempted to say that the properties of the referred models are potential solutions to handle negative interest rate framework and fit different shapes of the term structure of interest rates.

4.2 Multi-factor CIR Models

In this Section we propose two models which are able to handle negative interest rates while preserving the analytical tractability of the original CIR model.

4.2.1 The Two-factor CIR Model (CIR2++)

Brigo and Mercurio [5] had proposed the Two-Additive-Factor CIR model (CIR2++). This model consists in the sum of a deterministic shift and two independent squared root processes. The deterministic shift is constructed by the difference of the current market instantaneous forward curve and the model forward curve $\varphi(t) = f^{\mathrm{M}}(0, t) - f^{x}(0, t)$. On the other hand, another model was proposed by Orlando et al., [21] where the short rate depend on one-factor and a constant shift $\varphi(t,\alpha) = c$. For the aim of this thesis, we proposed a combination of both models which have the following advantages: we keep the model simple as the deterministic shift is a constant while increasing the flexibility of the model by adding one more factor.

The short rate dynamics under the risk-neutral measure is as follows: The sort-rate dynamics is given by:

$$r(t) = x(t) + y(t) + c \tag{4.5}$$

Where the two independent CIR processes are given by

$$\begin{aligned} dx(t) &= \kappa_x(\theta_x - x(t))dt + \sigma_x\sqrt{x(t)}dW_x(t), \quad x(0) = x_0 \\ dy(t) &= \kappa_y(\theta_y - y(t))dt + \sigma_y\sqrt{y(t)}dW_y(t), \quad y(0) = y_0 \end{aligned}$$

Where $(\kappa_z, \theta_z, \sigma_z)$, $z \in \{x, y\}$, are positive constants such that $2\kappa_z\theta_z \geq \sigma_z^2$ is satisfied. We allow the constant shift c to be negative. However, we require the initial conditions to be non-negative $x_0 \geq 0$ and $y_0 \geq 0$. Note that W_x and W_y are independent Brownian motions.

The $\mathcal{F}_s$ conditional mean and variance of $r(t)$ are given by

$$\begin{aligned} E\{r(t)|\mathcal{F}_s\} &= E\{x(t) + y(t)|\mathcal{F}_s\} = E\{x(t)|\mathcal{F}_s\} + E\{y(t)|\mathcal{F}_s\} + c \\ Var\{r(t)|\mathcal{F}_s\} &= Var\{x(t)|\mathcal{F}_s\} + Var\{y(t)|\mathcal{F}_s\} \end{aligned}$$

Derivation of the bond price analytical solution: The price at time t of a zero-coupon bond maturing at T is:

$$\begin{aligned} P(t,T;x(t),y(t),\alpha) &= E\left\{exp\left[-\int_t^T (x_s + y_s + c)ds\right]|\mathcal{F}_t\right\} \\ &= exp\left[-\int_t^T cds\right] E\left\{exp\left[-\int_t^T (x_s + y_s)ds\right]|\mathcal{F}_t\right\} \\ &= exp[-c(T-t)]\left\{exp\left[-\int_t^T x_s ds\right]|\mathcal{F}_t\right\} E\left\{exp\left[-\int_t^T y_s ds\right]|\mathcal{F}_t\right\} \\ &= exp[-c(T-t)]P^1(t,T,x_t;\alpha_x)P^1(t,T,y_t;\alpha_y) \end{aligned}$$

Note that the bond price analytical solution for each factor, $P^1(t,T,x_t;\alpha_x)$ and $P^1(t,T,y_t;\alpha_y)$ is exactly the same as in the One-factor CIR Model in (3.12). Let us point out that the parameters of this model are determined by $\alpha_x = (\phi_1^x, \phi_2^x, \phi_3^x)$ and $\alpha_y = (\phi_1^y, \phi_2^y, \phi_3^y)$.

4.2.2 The differenced Two-factor CIR Model

Francesco and Kamm propose a new model which is an extension of the CIR Model. For further detail we refer to [8]. For the purpose of this thesis, we will briefly describe the main features concerning the dynamics of the short rate.

The instantaneous short-rate process depends on the difference between two-independent CIR processes

$$r(t) = x(t) - y(t), \quad r(0) = r_0$$

The two independent factors are given by

$$dx(t) = \kappa_x(\theta_x - x(t))dt + \sigma_x dW_x(t), \quad x(0) = x_0,$$
$$dy(t) = \kappa_y(\theta_y - y(t))dt + \sigma_y dW_y(t), \quad y(0) = y_0,$$

Where $(\kappa_z, \theta_z, \sigma_z)$,$z \in \{x, y\}$, are positive constants such that $2\kappa_z\theta_z \geq \sigma_z^2$ is satisfied. We require the initial conditions to be non-negative $x_0 \geq 0$ and $y_0 \geq 0$ and the two Brownian Motions are not correlated $dW_1 dW_2 = 0$.

The $\mathcal{F}_s$ conditional mean and variance of $r(t)$ are given by

$$E\{r(t)|\mathcal{F}_s\} = E\{x(t) - y(t)|\mathcal{F}_s\} = E\{x(t)|\mathcal{F}_s\} - E\{y(t)|\mathcal{F}_s\}$$
$$Var\{r(t)|\mathcal{F}_s\} = Var\{x(t)|\mathcal{F}_s\} + Var\{y(t)|\mathcal{F}_s\}$$

The price of the zero-coupon bond with maturity T at time t is as follows:

$$P(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right] = E_t^{\mathbb{Q}}\left[e^{-\int_t^T (x(s)-y(s))ds}\right] = E_t^{\mathbb{Q}}\left[e^{-\int_t^T x(s)ds}\right] E_t^{\mathbb{Q}}\left[e^{\int_t^T y(s)ds}\right]$$

Thus,

$$P(t,T) = A_x(t,T)e^{-B_x(t,T)x(t)}A_y(t,T)e^{B_y(t,T)y(t)}$$

where,

$$A_z(t,T) = \left[\frac{\phi_1^z e^{\phi_2^z(T-t)}}{\phi_2^z(e^{\phi_1^z(T-t)} - 1) + \phi_1^z}\right]^{\phi_3^z}$$
$$B_z(t,T) = \left[\frac{e^{\phi_1^z(T-t)} - 1}{\phi_2^z(e^{\phi_1^z(T-t)} - 1) + \phi_1^z}\right]$$

with $\phi_i \geq 0$ and $i = 1, 2, 3$,

$$\phi_1^x = \sqrt{\kappa_x^2 + 2\sigma_x^2}, \quad \phi_2^x = \frac{\kappa_x \phi_1^x}{2}, \quad \phi_3^x = \frac{2\kappa_x\theta_x}{\sigma_x^2}$$
$$\phi_1^y = \sqrt{\kappa_y^2 - 2\sigma_y^2}, \quad \phi_2^y = \frac{\kappa_y + \phi_1^y}{2}, \quad \phi_3^y = \frac{2\kappa_y\theta_y}{\sigma_y^2}$$

4.2.3 Illustration

The one-factor CIR model has been considered a suitable model to represent the dynamics of the term structure of interest rates. However, in times of negative interest rates the CIR model is no longer a suitable solution. Recall that, the interest rates under the two-factor

Vasicek model can become negative due to its Gaussian distribution. However, this does not apply for the two-factor CIR model. The sum of two independent CIR processes do not perform well under negative rates. Indeed, the diffusion process of both CIR processes contains the squared root of the short rate which avoids negative rates. Hence, to enable the model to overcome the zero lower bound two solutions have been covered in the previous sections: the two-factor CIR model with shift and the model proposed by [8].

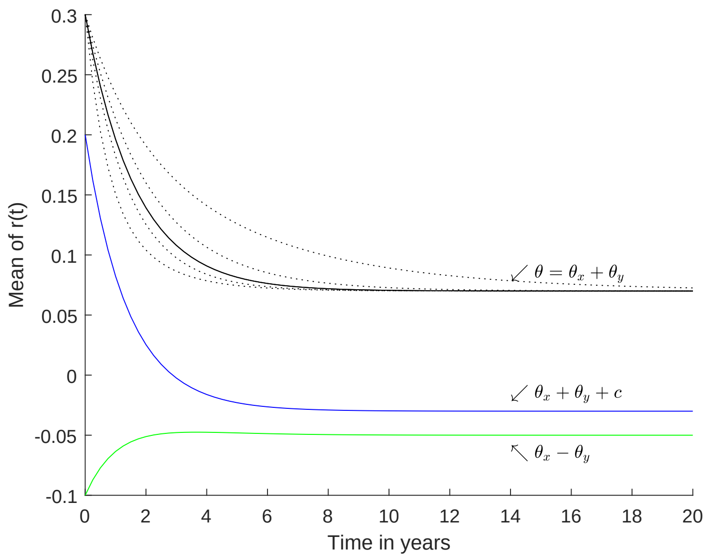


Figure 4.4: A comparison of the short rate mean $\mathbb{E}\{r(t)|\mathcal{F}_0\}$ under the CIR framework. The common parameters values for the two-factor models are the initial condition $x_0 = 0.1$, $y_0 = 0.2$ and the long-term mean $\theta_x = 0.01$, $\theta_y = 0.06$. Notice that the parameters left differ from one model to the other. The blue solid line illustrates the CIR2++ model with $\kappa_x = 0.6$, $\kappa_y = 0.8$ and $c = -0.1$. The green solid line shows the differenced-two-factor model with $\kappa_x = 0.6$, $\kappa_y = 0.8$. The dotted lines show the two-factor Vasicek or CIR2, where the short rate is the sum of two independent factors, in this case the speed of reversion of one of the factors takes different values such as $\kappa_y = \{0.2, 0.4, 0.8, 1.5\}$. Lastly, the black solid line represents the one-factor CIR model with $\kappa = 0.6$, $\theta = \theta_x + \theta_y$ and $r_0 = x_0 + y_0$.

With the aim of analyzing the features of both models we would take a look at the distribution of $r(t)$, for instance the mean and the variance of the short rate. In Figure 4.4 it is illustrated the mean of the short rate $\mathbb{E}\{r(t)\}$ for the one-factor CIR model and its two-factor extensions. One can see from Figure 4.4 that thanks to the introduction of a constant shift the short rate can reach negatives values and long-term rates evolves through a new long-term mean below the zero boundary. Regarding the model proposed by Di Franceso and Kamm in [8], instead of shifting the short rate to reach negative rates they suggest to implement the difference of two CIR processes. Hence, both models are able to reach negative interest rates. Besides, due to the independence of the factors the analytical formulas and the distribution of the short rate are immediately derived from the one-factor CIR model.

As hinted in Section 3.2.2, the one-factor CIR process is too rigid in terms of its volatility.

Even if we set a sufficiently high σ to the short rate the change is relatively poor. For that reason, the parameter θ has an important role in this setting. Notice that if we increase the long-term mean, then the variance of the short rate $Var\{r(t)\}$ also increases. Based on this observation, the two-factor extensions of the CIR model could have interesting patterns regarding θ_x and θ_y. Figure 4.5 shows the relation between different combinations of the long-term mean for both factors and the volatility of the short rate. We observe that given suitable values of κ_x and κ_y the long-term mean of both factors led to different volatilities surfaces.

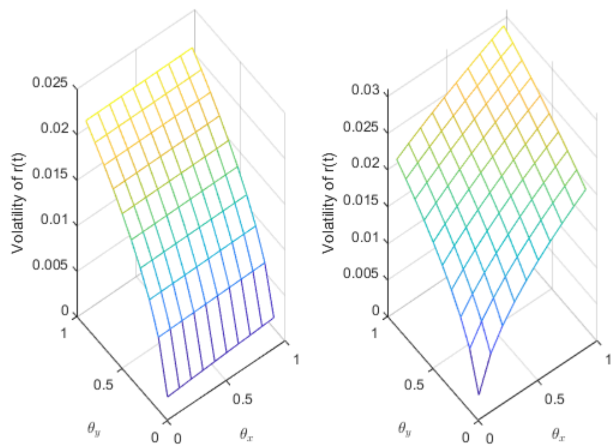


Figure 4.5: Sensitivity of the short rate volatility for the two-factor CIR extensions to the long-term mean of both factors. The volatility has been computed by the squared root of the variance of the current short at one-year maturity. The left figure shows the short rate volatility where $\kappa_x = 9$ and $\kappa_y = 0.001$ and the right figure shows the short rate volatility where $\kappa_x = 9$ and $\kappa_y = 10$. The initial conditions are $x_0 = 0.04$ and $y_0 = 0.02$, $\sigma_x = \sigma_y = 0.1$. (Note that if kappa is lower for a factor, then the impact of theta is big to increase the volatility.

In brief, the CIR2++ model and the differenced two-factor CIR are able to address the problem of negative interest under the CIR framework. Moreover, the referred models are flexible to handle complex volatility surfaces while preserving the analytical formulas of the one-factor mode.

Chapter 5

Empirical Analysis

5.1 Market Data

The market zero-coupon curve is obtained from the data published by the ECB. The data market consists in EURO rates of 3,6 and 9 months and 1 up to 30 years. We first build the yield curve by using the `Matlab`'s function `spline` which calculates the cubic spline interpolation to construct the market data. Then, we obtain the zero-coupon bond by means of the yield curve build from the data market.

As we are interested in a negative interest rate environment, we test the models at two different dates, 30/11/2020 and 29/10/2021. For the date first date the entire structure of the zero-coupon curve is negative. We notice that the model proposed by Franceso and Kamm [8] poorly matches the shape of the market curve at this date. The aim of this thesis is to test the performance of the mentioned CIR extension and compare it to alternative two-factor models. Regarding, the second date the rates are negative up to the twelve year and then it switches to positive values. It is important to test the models on two different days to verify their ability to fit different shapes of term structure of interest rates.

5.2 Calibration

We calibrate the model to the market term structure for two different shapes of the zero-coupon yield curve. We examine the behavior of the mentioned models at 30/11/2020 and 29/10/2021. Both dates are under a negative interest rate framework and have non-monotonic curves of the term structure.

5.2.1 Objective function

To check how well the models fits the term structure we aim to compare the market zero-coupon prices (see, Table A.1 and A.2) to the model zero-coupon prices. The objective function consists in minimizing the squared difference between the market data and the model zero-coupon price. Note that we have to calibrate four models.

The minimization problem is given by

$$\min_{\Pi \in \mathcal{A}} f(\Pi) \tag{5.1}$$

where

$$f(\Pi) := \sum_{i=1}^{n} \left(\frac{P^M(0,T_i)}{P(\Pi;0,T_i)} - 1 \right)^2$$

Where $n \in \mathbb{N}$ is the number of time point which represent the different maturities T_i, $i = 0.08, ..., 30$. The current zero-coupon bond market curve is denoted by P^M. The initial zero-coupon bond of the model is P where the vector of parameters that need to be calibrate is denoted by Π.

To avoid abuse of notation we had exposed the previous formulas in a general setting. Nevertheless, note that (5.1) has to be applied independently for each model. Therefore, the dimension of Π depends on the model specification. The mean relative error (MRE) is given by

$$\text{MRE} = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{P^M(0,T_i)}{P(\hat{\Pi};0,T_i)} - 1 \right|$$

To provide a fair comparison between the models we will compute the same objective function to calibrate the parameters. Despite, in order to consider the particular features of each model we will set a vector or matrix of constraints $\mathcal{A}$ to properly accommodate the models.

5.2.2 Parameter Constraints

An extensive setting of the parameter constraint is of key importance for our empirical analysis. By construction, each model has its own constraints and we need to search a suitable space for the parameters to deal with negative interest rates. To properly compute the models under the constraints that are defined in the following lines we have decided to use the minimization function `fmincon` in `Matlab` where we can easily implement the corresponding constraints. In this section we focus on the two-factor models, thus, we refer to the Appendix A.1 for further knowledge about the constraints of the one-factor Vasicek and CIR model together with some additional subtleties about two-factor models .

According to the models defined in Chapter 4 we aim to define the number of parameters that are calibrated to the market data and the different constraints to properly implement each model.

Parameter constraints of the G2++ model

i. Parameter vector: $\Pi_{\text{G2++}} = [\kappa_x, \sigma_x, \kappa_y, \sigma_y, r_0, \theta, \rho]$.

ii. Parameter search space: As we are interest in calibrating the parameters to the market data with negative interest rates, it is reasonable, therefore, to allow the long-term mean and the initial conditions take values in the following intervals $\theta \in (-1, 1)$ and $r_0 \in (-1, 1)$. We also set the speed of reversion and the volatility of the factor such as $\kappa_z \in (0, 10)$ and $\sigma_z \in (0, 1)$, $z \in (x, y)$. Lastly, the correlation between the two factors y and x is given by $\rho \in (-1, 1)$.

Parameter constraints of the two-factor Vasicek model

i. Parameter vector: $\Pi_{\text{Vasicek2}} = [\kappa_x, \theta_x, \sigma_x, \kappa_y, \theta_y, \sigma_y, x_0, y_0]$.

ii. Parameter search space: Likewise in the G2++ model the initial conditions are as follows: $x_0 \in (-1, 1)$, $y_0 \in (-1, 1)$. We define $\sigma_z \in (0, 1)$ and $\theta_z \in (0, 1)$,$z \in (x, y)$.

iii. We are concerned in this model to draw the state variable x as being the fast factor and y being the slow factor. Hence, $\kappa_y < \kappa_x$ leads to the following specification of the speed of reversion parameter: $\kappa_x \in (0, 20)$ and $\kappa_y \in (0, 1)$.

Parameter constraints of the CIR2++ model

i. Parameter vector: $\Pi_{\text{CIR2++}} = [\phi_1^x, \phi_2^x, \phi_3^x, \phi_1^y, \phi_2^y, \phi_3^y, x_0, y_0, c]$.

ii. We require $\sigma_z \geq 0$ and $\kappa_z \geq 0$, $z \in \{x, y\}$, which is equivalent to set that $\phi_1^z \geq \phi_2^z$ and $2\phi_2^z \geq \phi_1^z$ respectively.

iii. Regarding the condition $2\kappa_z\theta_z \geq \sigma_x^2$ we set that $\phi_3^z \geq 1$.

iv. We assume for this model that the initial condition of x is larger than the initial condition of y. Thus, we establish the following condition: $x_0 \geq y_0 + c$. Note that $x_0 \geq 0$ and $y_0 \geq 0$.

Parameter constraints of the differenced two-factor CIR model

i. Parameter vector: $\Pi_{\text{CIR2-difference}} = [\phi_1^x, \phi_2^x, \phi_3^x, \phi_1^y, \phi_2^y, \phi_3^y, x_0, y_0]$.

ii. The constraints **ii.** and **iii.** of the CIR2++ model are equivalent for the differenced two-factor CIR model.

iii. We require $x_0 \geq 0$ and $y_0 \geq 0$.

5.2.3 Calibration Results

In this Section we calibrate the different models to the market data given in Table A.1 and Table A.2. For each model we compute the mean relative errors (MRE) and the value of the optimization function $f(\hat{\Pi})$ given in equation 5.1. Due to the nature of our goals, we aim to show how these models behave empirically and test the technical features discussed in the previous chapter.

We have seen that the Gaussian models are a potential solution to deal with more complex term structure shapes and negative interest rates. One may wonder if the already mentioned flexibility of these models is beneficial and improves the model performance to fit the term structure of interest rates. We calibrated the model at two different dates as illustration.

Parameter	30/11/2020	29/10/2021
κ_x	0.768	0.964
θ_x	0.018	0.065
σ_x	0.111	0.284
κ_y	0.067	0.132
θ_y	0.027	0.033
σ_y	0.018	0.044
x_0	0.019	0.031
y_0	-0.026	-0.049
$f(\hat{\Pi}_{\text{Vasicek2}})$	1.154e-05	1.328e-05
MRE	0.026 %	0.021 %

(a) Calibration parameters of the Two-Additive-Factor Vasicek Model.

Parameter	30/11/2020	29/10/2021
κ_x	0.221	0.186
σ_x	0.061	0.152
κ_y	0.833	0.297
σ_y	0.227	0.216
ρ	0.678	-0.960
θ	0.028	0.005
r_0	-0.015	-0.010
$f(\hat{\Pi}_{\text{G2++}})$	1.473e-05	6.459e-06
MRE	0.026 %	0.019 %

(b) Calibration parameters of the G2++ Model.

Table 5.1: Calibration to the market data.

The results of the calibration to the market data at 30/11/2020 and 29/10/2021 are given in Table 5.1. Regarding the two-factor Vasicek model one can notice that the volatility of the *fast* spot rate $x(t)$ is higher than the volatility of the *slow* spot rate $y(t)$. Typically, short rates are generally associated with higher volatility levels and higher speed of reversion in contrast to the long-term rates. Thus, based on the calibration results we can consider $x(t)$ as a representation of short-rates and $y(t)$ as a representation of long-term rates. However, this is just a conjecture. Regarding the value of the parameters, we notice that the long-term mean values seem reasonable and the initial conditions of the short rate on both dates is negative, for instance $r_0 = -0.7\%$ at 30/11/2020, that makes perfect sense as the initial rate of the yield curve is negative.

The analysis of the calibrated results in the G2++ model is not trivial. With the introduction of the correlation between the factors the solely impact of the speed of reversion and volatility is less clear. Nevertheless, we can immediately notice that the sign of the correlation between the factors differs. At 30/11/2020 the correlation is 0.678 and at 29/10/2021 the correlations in -0.960. In Chapter 4 we showed that a negative correlation together with suitable values of κ_x and κ_y enables the model to reach higher interest rates. If we take a look to the term structure at 29/10/2021 (see Table A.2) we observe the following: the curve starts with negative rates and then reaches positives rates. In contrast to the term structure at 30/11/2020 where the entire curve is under negative rates. Thus, the calibration results suggest that a negative correlation between the factors contribute to some extent to reproduce the shape of the term structure of interest rates at 29/10/2021. Regarding the mean relative error, we notice a slightly improvement of the G2++ model in comparison to the two-factor Vasicek model at 29/10/2021. At the referred date the humped yield curve clearly suggests that the model needs to acquire enough flexibility to capture the hump. The G2++ model with correlated factors captures better the particular shape of the curve. In brief, both models perform similarly taking in account that the G2++ fit to the market data is partly improved due to an extra flexibility parameter, the correlation between factors, and gives a slightly better result at the day 29/10/2021. However, the improvement is negligible in terms of the MRE.

Now we concentrate the analysis on the calibration results under the CIR framework. By construction the model avoids negative interest rates. However, thanks to the introduction of a constant shift or computing the difference of two factors some interesting results are obtained.

The results of the calibration to the market data for the CIR2++ and the differenced two-factor CIR model are given in Table 5.2. Both models can reproduce the term structure of interest rates at both dates while keeping non-negative parameters. It should be noted the previous point is correct without taking in account the shift parameter of the CIR2++. A negative constant shift, $c = -0.0474$ at 30/11/2020 and $c = -0236$ at 29/10/2021, enables the model to reach negative rates. The entire curve is shifted downwards. Regarding the mean relative error, the differenced two-factor CIR model slightly outperforms the CIR2++ model at 29/10/2021. One can notice that the difference between two factors gives enough flexibility to the model to properly fit the term structure under negative interest rates. Moreover, similarly as it was mentioned for the two-factor Vasicek, it is possible to get negative rates due to the subtractions between the factors. Based on the calibration results the initial short rate of the yield curve at 29/10/2021 is -0.9%. We notice that the parameters estimate in both models are not terribly realistic. The volatility parameters appear to be remarkably high for each factor.

Parameter	30/11/2020	29/10/2021	Parameter	30/11/2020	29/10/2021
ϕ_1^x	0.638	0.221	κ_x	0.373	0.166
ϕ_2^x	0.506	0.193	θ_x	0.292	0.050
ϕ_3^x	1.623	1.563	σ_x	0.366	0.103
ϕ_1^y	0.451	0.258	κ_y	0.132	0.027
ϕ_2^y	0.292	0.142	θ_y	0.573	1.031
ϕ_3^y	1.624	1.670	σ_y	0.305	0.182
x_0	0.366	0.211			
y_0	0.087	0.018			
c	-0.474	-0.236			
$f(\hat{\Pi}_{\text{CIR2++}})$	8.888e-05	1.235e-05			
MRE	0.059 %	0.028 %			

(a) Calibration parameters of the CIR2++ Model.

Parameter	30/11/2020	29/10/2021	Parameter	30/11/2020	29/10/2021
ϕ_1^x	0.722	0.181	κ_x	0.523	0.049
ϕ_2^x	0.623	0.115	θ_x	0.178	0.234
ϕ_3^x	1.498	1.514	σ_x	0.352	0.123
ϕ_1^y	0.261	0.116	κ_y	0.488	0.258
ϕ_2^y	0.374	0.187	θ_y	0.114	0.073
ϕ_3^y	1.303	1.417	σ_y	0.292	0.163
x_0	0.250	0.037			
y_0	0.271	0.046			
$f(\hat{\Pi}_{\text{CIR2-difference}})$	6.356e-05	1.296e-05			
MRE	0.046 %	0.028 %			

(b) Calibration parameters of the Two-Difference-Factor CIR Model (Di Franceso and Kamm [8].

Table 5.2: Calibration to the market data.

5.3 How much can we gain with increasing the model complexity?

In this section we compare the performance of one-factor models and their two-factor extensions. Based on the results of the calibration we aim to show the improvement of the two-factor models in fitting the term structure of interest rates at both dates compared to the one-factor models. Besides, one may wonder how much complexity we accept to improve the accuracy of models in the match to the market data. With the purpose of answering this question we will briefly discuss the balance between implementing a model which can be used for practical purposes, increasing its complexity and the capacity to reproduce realistic patterns of the market.

5.3.1 One-factor vs. Multi-factor models

We have suggested that Two-factor models have flexible features to properly match the term structure of interest rates. Thus, we will explain how much flexibility is gained by adding more factors to the short rate dynamics which leads to complicated closed form solutions. Alternatively, we can look at the problem from the opposite point of view and also consider the advantages of a more simplified version of the model.

Let us start with the performance of the one-factor Vasicek model compared to its multi-factors extension. The results of the calibration to the market data for the parameters of the Vasicek model at both dates are displayed in Table 5.3. Notice that we are allowing the initial condition to reach negatives rates. As we are calibrating the models to a market data where interest rates are negative it seems reasonable to extent the space of the initial rate r_0. Besides, this adjustment is in line with the model dynamics. As it was previously mentioned, the mean relative errors at 29/10/2021 for the two-factor Vasicek model and G2++ model are 0.021% and 0.019% respectively. On the other side, the mean relative error at the same day for the one-factor models in 0.31%. Thus, we observe a substantial improvement on the behalf of the two-factor extensions. At 30/11/2020 we also observe that the two-factors models outperform the one-factor Vasicek short rate model.

Paramters	Vasicek Model	CIR Model
κ	0.063	0.549
θ	0.017	0.000
σ	0.011	0.002
x_0	-0.011	0.000
$f(\Pi)$	0.000	0.296
MRE (in %)	0.120	4.75

(a) Calibration results under positive 30/11/2020.

Paramters	Vasicek Model	CIR Model
κ	0.577	0.000
θ	0.007	6.191
σ	0.054	0.009
x_0	-0.016	0.000
$f(\Pi)$	0.002	0.011
MRE (in %)	0.31	0.72

(b) Calibration results under positive 29/10/2021.

Table 5.3: Calibration results for the Classical models.

Indeed, the one-factor CIR model does not perform properly under negative interest rates. The mentioned model is constructed to guarantee positive interest rates. Thus, the mean relative error is the highest among all the models that are discussed in this thesis. Furthermore, the parameters are not realistic, for instance at 29/10/2021 the value of the parameter κ is towards zero meaning that the drift term of the short rate process is completely vanished. Based on these results, the CIR2++ and the models proposed by Di Franceso and Kamm are a meaningful improvement to fit the market data under negatives interest rates.

As has been noted, the two-factor extensions outperform their one-factor peer at both dates. Hence, another question arises naturally: How much complexity we are willing to handle for practical purposes? How much we can gain in fitting the term structure of interest rates by increasing the complexity of the models?

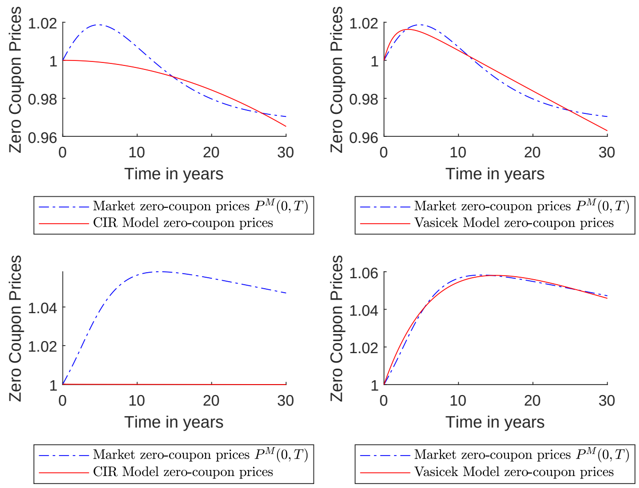
5.3.2 Comparison with Multi-factors models

A major advantage of one-factor models is their appealing analytical formulas. Accordingly, if we are interested in implementing the models for practical purposes, for instance pricing derivatives, it is desirable to work with a model able accurately fit the term structure of interest rates while preserving the analytical tractability of the short rate process and its closed-form solutions.

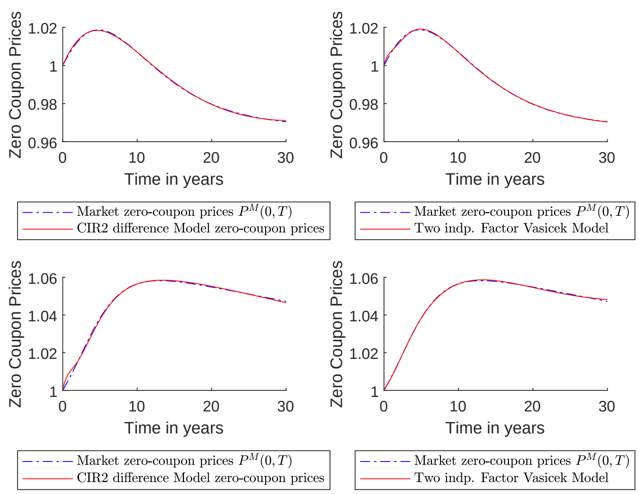
Regarding the Gaussian models one can claim that the fit to the term structure of interest rates at 30/11/2020 and 29/10/2021 was successfully achieved. However, the degree of complexity differs from one model to another. The G2++ model adds to the short rate dynamics a deterministic shift, denoted by $\varphi(t)$, and the correlation between factors, denoted by ρ, which increases the complexity of the model. Besides, the closed-form solutions of the the zero-coupon bond (in Section 4.1.1) are arduous and less appealing for practical purposes. By contrast, the two-factor Vasicek model is easier to handle as it is the sum of two independent Ornstein–Uhlenbeck processes. In other words, we can immediately derive the bond price and the continuously-compounded short rate from the one-factor analytical formula. In addition, as it was shown in the calibration analysis the mean residual error on both days do not differ substantially. The slightly difference at 29/10/2021 between the mean relative error of the G2++ and the two-factor Vasicek model is 0.002 percentage points on behalf of the model with correlated factors. Notice that the MRE of the one-factor Vasicek model is almost 1.0 percentage points above the two-factor models. Indeed, even if the G2++ model has desirable features and the improvement is negligible.

Under the two-factor CIR framework, we also notice that the model improvement regarding the MRE is not high. We suggest that the differenced two-factor CIR model is an interesting model for practical purposes at it contains less parameters compared to the CIR2++. The model proposed by Di Francesco and Kamm does not need to introduce a shift to the short rate process to address the problem of negative interest rates. Therefore, we avoid the problem of a bad calibration of the shift parameter which could lead to some problems during the simulation and pricing exercise.

Under the scope of this thesis, let us name the two models that based on the previous considerations are capable to accomplish the balance between realism and simplicity : the two-factor Vasicek model and the difference Two-factor CIR model. As it is shown in Figure A.3 the match between the market zero-bond curve and the model zero-bond curve at 29/10/2021 is nearly perfect. As it was previously mentioned and in line with the representation in Figure A.3 the match between the market poor and the one-factor models is poor. The one-factor Vasicek models is capable to better represent the term structure, however, at 29/10/2021 the zero-coupon price curve of the Vasicek model is too rigid to match the market term structure.



(a) one-factor models



(b) two-factor models

Figure 5.1: A comparison between the market zero-bond curve and the model zero-bond curve. The figure (a) shows the market fit at 29/10/2021 in the top and the market fit at 30/11/2020 in the bottom for the one-factor models. The figure (b) shows the market fit at 29/10/2021 in the top and the market fit at 30/11/2020 in the bottom for the two-factor models.

Chapter 6

Numerical Analysis

In the previous chapter we have highlighted the appealing features of the two-factor Vasicek model and the differenced two-factor CIR model. Both models are capable to represent the market term structure at the already mentioned days and they have simple analytical expressions. Hence, in this chapter we aim to test both models to see how the perform in the following practical approach: pricing derivatives. For this purpose, we will first introduce the simulation method and test the quality of the parameters obtained at the most recent available data 29/10/2021. Then, we propose a simple pricing exercise *stylized option pricing problem* to test the models. This simple approach will allow us to identify the main problems which could be arise in a real option pricing problem, for instance when pricing caps or swaptions.

6.1 Euler-Monte-Carlo Simulation

To price interest rates derivatives more accurately we need to provide a good fit to the market term structure of interest rates. Thus, we make use of the parameters obtained in Table 5.1a for the two-factor Vasicek model and Table 5.2b at 29/10/2021 as the error between the market zero-coupon bond and the model zero-coupon bond is sufficiently low. Note that we will not price an existing derivative, however, if in further research we aim to price swaptions, for instance, the simplify approach we conduct in this chapter will give us some preliminary results.

There are several reasons motivating the computation of simulations methods to price interest rates derivatives. In many cases it is not possible to obtain analytical solutions of the pricing equations. Hence, numerical algorithms must be used. For the two-factors short rate case we need to use numerical methods as there are barely closed-form solutions to price derivatives. In particular, we use an Euler discretization of the short rate SDE to compute a Monte Carlo approximation of the *stylized option price* that we will explain in more detail through the next section.

Let us first derived the discretization schemes that had been used for both models. To turn a SDE into a discrete recursion we have decided to compute an Euler approximation with the following time grid $0 = t_0, ..., t_n = T$ and interval $[0, T]$. The number of time steps is denoted by N and the length of the time interval is $t_{i+1} - t_i = \Delta t_i = T/N$. Additionally, we define the time increments of the Brownian Motions as $\Delta W_z(t_i) = W_z(t_{i+1}) - W_z(t_i)$, $z \in \{x, y\}$. We notice that in the proposed two-factor interest models we require to set independent increments of the Brownian Motions.

The approximated solution of the discretized short rate under the two-factor Vasicek models has the following form $r(t_{i+1}) = x(t_{i+1}) + y(t_{i+1})$. In such a case

$$x(t_{i+1}) = x(t_i) + \kappa_x(\theta_x - x(t_i))\Delta t_i + \sigma_x \Delta W_x(t_i)$$
$$y(t_{i+1}) = y(t_i) + \kappa_y(\theta_y - y(t_i))\Delta t_i + \sigma_y \Delta W_y(t_i)$$

Regarding the differenced two-factor CIR model we implement a *full truncation* method to the Euler scheme. This method is proposed by Lord, Koekkoek and Van Dijk in [16]. Therefore, the discretized short rate is given by $r(t_{i+1}) = x(t_{i+1}) - y(t_{i+1})$. In such a case

$$x(t_{i+1}) = x(t_i) + \kappa_x(\theta_x - x(t_i))\Delta t_i + \sigma_x \sqrt{\max(x(t_i), 0)}\Delta W_x(t_i)$$
$$y(t_{i+1}) = y(t_i) + \kappa_y(\theta_y - y(t_i))\Delta t_i + \sigma_y \sqrt{\max(y(t_i), 0)}\Delta W_y(t_i)$$

The number of trajectories for this simulation exercise is given by $n = 10^6$ and the number of time steps $N = 120$. Let us briefly explain the algorithm for the computation of the Monte Carlo estimate of the bond price:

Step 1. Discretize the period $[0, T]$ into N intervals og length $\Delta t_i = T/N$.

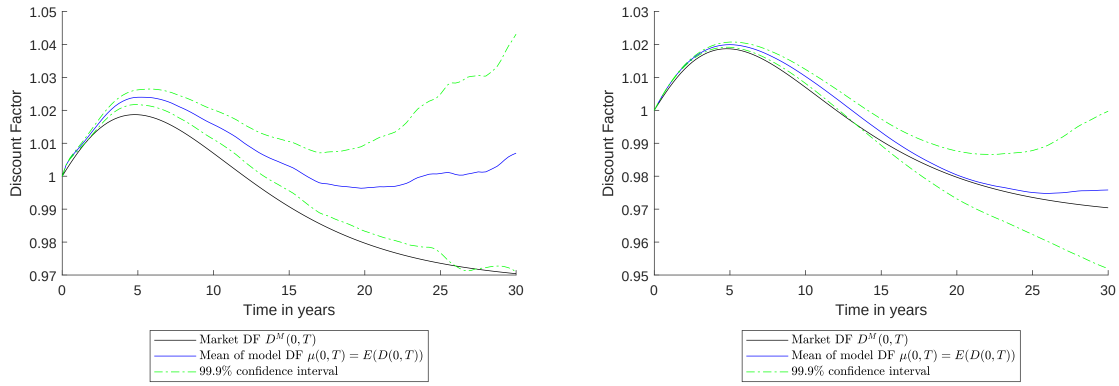
Step 2. Calculate the simulated path i $(i = 1, ..., n)$ of $r^{(i)}(t_j)$ at each time step j $(j = 0, ..., N)$.

Step 3. Compute the the price of the bond $P^{(i)}(t_j) = exp(-\int_0^N r^{(i)} dt)$ by calculating the numerical value of the integral of the short rate for each simulation.

Step 4. Compute the average over all the trajectories for each time step to obtain the Monte-Carlo estimator $E[\bar{P}(t_j)]$.

The mean over all simulations for the two-factor Vasicek model and the differenced two-factor CIR model is shown in Figure 6.1. The mean over all the trajectories for the two-factor Vasicek model is very noisy. Recall that 10^6 trajectories have been drawn for the simulation of the short rate. We claim that this amount of trajectories should be enough to give accurate results. Hence, the parameters obtained in the calibration exercise do not work well in the simulation. Suppose that we decrease the parameter value σ_x. Then, we obtain a smoother curve with fewer trajectories ($n = 10^4$). For illustration we refer to Appendix A.5. Regarding the variance of both figures we observe that the confidence interval is relatively high from 15 years onward.

Let us recall that the Monte Carlo method has two types of errors: the simulation error and the discretization error. The Monte Carlo error creates variances whereas the discretization error creates bias. Typically, the discretization error is due to the approximation of the integral of the discretized short rate to the sum. This is not our case as we have computed



(a) Two-Additive-Factor Vasicek Model (b) Difference Two-factor CIR Model

Figure 6.1: Comparison of the market discount factor to the Monte Carlo estimate of the discount factor with parameters given in Table 5.1a for the Two-Vasicek model and the parameters in Table 5.2b for the difference Two-factor CIR model at the last available date 29/10/2021. Note that $\Delta = \frac{30}{120}$ and $n = 10^6$.

the integral by means of the `Matlab`'s function `cumtrapz`. Then the discretization error left is the following one: the trajectories are taken from an approximate distribution and not from the actual distribution of $r(t)$. Regarding the simulation error we can increase the number of paths. However, we will then increase the bias in the approximation. This dilemma between of the variance and the bias is one of the problems we have to confront when using Monte Carlo methods.

For this analysis we have calculated the root mean squared error between the mean of the model discount factor and the market discount factor at 29/10/2021. The RSME for the two-factor Vasicek model is 0.0173 and the RMSE for the differenced two-factor CIR model is 0.0024. One can see that the mean for the differenced two-factor model does not differ substantially from the market discount factor. We cannot claim any further conclusions apart from suggesting that the Monte Carlo estimates of both models are not terribly accurate given the parameters values calibrated at 29/10/2021.

6.2 Stylized option pricing problem

In the previous section we have shown some preparatory work to price a simplified call option. Modelling the term structure of interest rates plays a key role in pricing options. Several research has been done on this topic, for instance the paper published by P.P Doyle in [2] about the option valuation using Monte Carlo simulation. Another important source of knowledge regarding this matter is the work done by M. Schulmerich in [22] were we can found a variety of tools to price options with stochastic interest rates. For the scope of this thesis, we propose to solve a simplified pricing problem. As a consequence, we can identify the major obstacles that arise from the stylized valuation problem. This allows us to have an initial contact for future experiments with more sophisticated options. Before starting with the problem setting let us note briefly that the parameters are calibrated to the market data under the risk-neutral measure $\mathbb{Q}$. Thus, we can use the values of the parameters obtained in the calibration section to price the stylized option at 29/10/2021.

The setting of the stylized pricing problem is as follows: Let's denote $r(t)$ the stochastic short rate, k the strike rate and T the time of the contract expiration. We define the payoff of the interest rate call option as follows:

$$f(k) = (r(T) - k)^+ \tag{6.1}$$

where $r(T)$ is the rate at time T and $(r(T)-k)^+$ is the shorthand notation of the optimisation problem $\max(r(T) - k, 0)$.

The Monte Carlo estimate of the call option payoff is simply

$$\bar{f}(k) = \frac{1}{n}\sum_{i=1}^{n}(r^{(i)}(T) - k)^+$$

and the variance of the MC estimates is as follows

$$\text{std}\left(\bar{f}(k)\right) = \frac{1}{n}\text{std}\left(f(k)\right)$$

Notice that we have first to disctetized the short rate process by means of the Euler scheme. Thus, for the computation of the Monte Carlo simulation we can replicate **Step 1.** and **Step 2.** of the algorithm explained in the previous section. However, for the following steps we proceed differently. To price the interest rate call option we evaluate the short rate at each simulation i ($i = 1, ..., n$). Then, we calculate the mean over all the simulations that solves the maximization problem $(r^{(i)}(T) - k)^+$ just at the terminal date. We compute the MC estimates for the call option with the short rate under the two-factor Vasicek model and with the short rate under the differeced two-factor CIR model. For the simulation we compute $n = 10^4$ trajectories, as it is computationally costly to draw $n = 10^6$ trajectories.

To compare the performance of the proposed two-factor equilibrium models in this pricing exercise we require to set a suitable interval of strike price k. A reasonable interval for the strike rate is inside the support of $r(t)$. So far, the distribution of the two-factor Vasicek short rate (see Appendix A.6) shows that a suitable interval for the strike price is $k \in (-0.3, 0.3)$.Now, the question that naturally arise is if this interval is also appropriate to compute the payoff of the differenced two-factor CIR model. The results of the MC estimates as a function of the strike price are shown in Figure 6.2. Regarding the confidence interval of the MC estimate we refer to the Appendix A.3.

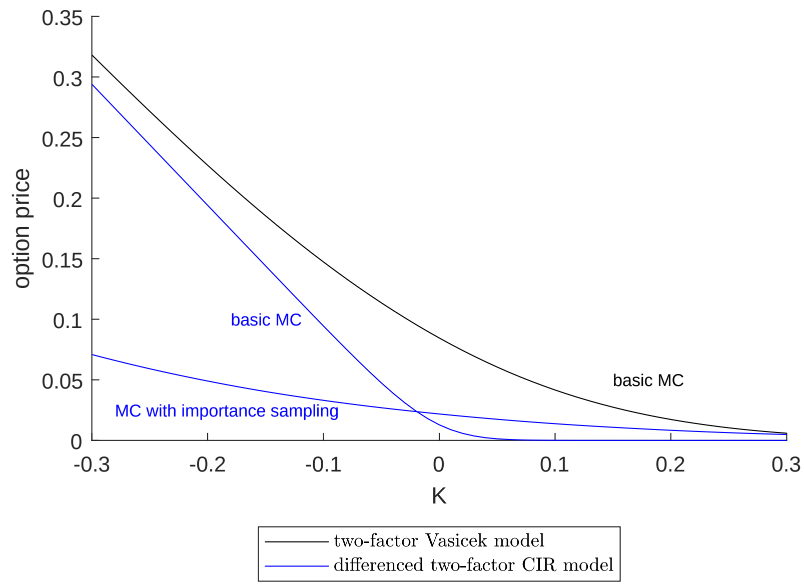


Figure 6.2: Interest rate call option payoff by using Monte Carlo simulation. We have computed $n = 10^4$ samples and the time grade $\Delta = \frac{1}{250}$.

We observe that the payoff of the option under the differenced two-factor CIR model is *out of the money* meaning that the short rate at T is below the strike rate. Hence the option has no intrinsic value. To solve this problem, we can distort the probability distribution from which random samples are taken and increase the probability of having paths where the interest rate is above the strike rate. This method is called Importance Sampling. By computing this technique, the payoff of the option with Importance sampling is smaller compared to the basic Monte Carlo method for the differenced two-factor CIR model. We also notice that the confidence interval of the Monte Carlo estimate in the Importance sampling method is narrower compared to the basic Monte Carlo.

Chapter 7

Conclusion

We reached the following conclusions. Without loss of generality, the two-factor equilibrium short rate models have shown some interesting features. With the aim to represent the term structure under negative interest rates we show that the one-factor models are too rigid in comparison to their two-factor extensions. Indeed, the increased number of parameters enables the two-factor short rate models to fit the market term structure very well. To be more precise, we show a comparison between the market zero-bond curve for the euro area and the model zero-bond curve using the parameters calibrated at 30/11/2020 and 29/10/2021. For both days we show that the proposed models are capable to fit the term structure of interest rates with high accuracy. However, we show that this led to a certain cost. On the one hand, the interpretation of the parameters values is less straightforward due to the high dimension of the short rate space. As we show in the technical analysis, the volatility structure is quite complex. So far, we draw some conclusions about the sensitivity of the short rate volatility to the model parameters. On the other hand, the calibration of the models demands more sophisticated computational techniques. To properly implement the models, we need to set the parameters constraints in an efficient manner which is not always trivial. Stated briefly, the two-factor equilibrium models can represent exactly the term structure of interest rates at 30/11/2020 and 29/10/2021, even though we are faced with calibrating more parameters.

A question arises naturally when one might choose a model to represent some economic or financial behavior. The balance between simplicity and realism is a dilemma that we have to confront up to some point. Regarding the interest rate modelling we proposed the two-factor Vasicek model and the model proposed by Di Francesco and Kamm, named in this thesis differenced two-factor CIR model, as suitable short rate models to solve the mentioned dilemma. Both models fit the term structure accurately on 30/11/2020 and 29/10/2021. Besides, due to the independence of the factor, the analytical solutions of the bond price are straightforward. Note that we cannot claim any further conclusions regarding their capacity of predicting interest rates. For the scope of this thesis, we show their performance at two specific days and not to the entire time series of the short rates.

Lastly, we are interested in testing the above mentioned models in a numerical analysis approach. Typically, the simulation methods are used when a closed-form solution does not exist to price a certain derivative. We show that the Monte Carlo estimate for the two-factor

Vasicek model and the model proposed by Di Francesco and Kamm with the calibrated parameters to the market data at 29/10/2021 have a large confidence interval from 15 years onwards. Moreover, the average over all the simulations for the two-factor Vasicek model is substantially noisy in contrast to the differenced two-factor model which is relatively smooth. The ultimate goal of modelling interest rates is to price a derivative. Hence, we proposed a stylized option price problem to test the ability of these two models to price a simplified interest rate call option. The major problem that has arisen in the procedure of the Monte-Carlo simulation is the search of a reasonable interval for the strike rate to properly compare the payoff estimates of both models. We have proposed the importance sampling method to have an initial contact on the search of a desirable solution. Notice that a similar problem could arise for more sophisticated derivatives, for instance, caps or swaptions.

We would like to point out that a two-factor equilibrium models may be considered as a promising interest rate model under the negative interest rates framework. Thus, future research on the empirical applications might be extended. A following step is to test their capacity to forecast interest rates. For that, we propose to use some time series techniques. We suggest the implementation of a Vector Autoregressive model (VAR) to work out the mentioned analysis. Other possible studies can be addressed to solve the main limitations we have found over this thesis. We show that the match to the market bond price at both dates is very accurate. However, we need a relatively high number of parameters and the calibrated parameters values dramatically change over time. As it is shown in the previous chapter, we have obtained substantially different parameters values at 30/11/2020 and 29/10/2021. Hence, together with the forecasting analysis we can continue the search of suitable models to reproduce the term structure under the current negative rates framework.

Appendix

A.1 Additional analysis for the short rate models

Objective measure dynamics for the Vasicek Model. We can obtain the objective measure or real-world measure from 3.6 and the specification of the market price of risk $\lambda(t, r(t)) = \lambda$ (constant) (see, Girsanov theorem, change of measure). Let's denote $dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} + \lambda dt$.

$$\begin{aligned} dr(t) &= \kappa(\theta - r(t))dt + \sigma dW_t^{\mathbb{Q}} \\ &= \kappa(\theta - r(t))dt + \sigma(dW_t^{\mathbb{P}} - \lambda dt) \\ &= \kappa(\theta - \frac{\sigma\lambda}{\kappa} - r(t))dt + \sigma dW_t^{\mathbb{P}} \\ &= \kappa(\bar{\theta} - r(t))dt + \sigma dW_t^{\mathbb{P}} \end{aligned}$$

The process under the risk-neutral measure has the same expression compared to the original measure. We have just to replace θ by $\bar{\theta} = \theta - \frac{\sigma\lambda}{\kappa}$.

Objective measure dynamics for the CIR Model.We can obtain the objective measure or real-world measure from 3.11 and the specification of the market price of risk $\lambda(t, r(t)) = \lambda$ (constant) (see, Girsanov theorem, change of measure). Let's denote $dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} + \lambda\sqrt{r(t)}dt$.

$$\begin{aligned} dr(t) &= \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t^{\mathbb{Q}} \\ &= \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}(dW_t^{\mathbb{P}} - \lambda\sqrt{r(t)}dt) \\ &= \kappa(\theta - r(t))dt - \lambda\sqrt{r(t)}dt\sigma\sqrt{r(t)} + \sigma\sqrt{r(t)}dW_t^{\mathbb{P}} \\ &= \bar{\kappa}(\bar{\theta} - r(t))dt + \sigma dW_t^{\mathbb{P}} \end{aligned}$$

Where $\bar{\kappa} = \kappa + \lambda\sigma$ and $\bar{\theta} = \frac{\kappa\theta}{\kappa + \lambda\sigma}$.

Parameter constraints of the one-factor Vasicek Model
According to the model proposed in Section 3.2.1 and the well-defined formulas the constrains are as follows:

i. Parameter vector: $\Pi_{\text{Vasicek}} = [\kappa, \theta, \sigma, r_0]$.

ii. Parameter search space: All the parameters are positive constants. Thus, $\theta \in (0, 1)$, $\sigma \in (0, 1)$, $r_0 \in (-1, 1)$ and $\kappa \in (0, 10)$.

Parameter constraints of the one-factor CIR Model
According to the model proposed in Section 3.2.2 and the well-defined formulas the constrains are as follows:

i. Parameter vector: $\Pi_{\text{CIR}} = [\phi_1, \phi_2, \phi_3, r_0]$.

ii. Parameter search space: All the parameters are positive constants. Thus, $\theta \in (0,1)$, $\sigma \in (0,1)$, $r_0 \in (0,1)$ and $\kappa \in (0,10)$.

iii. We require the following condition: $\phi_3 \geq 1$.

Constraints Matrix for two-factor short rates models
To set the constraint discussed in the calibration we construct a vector A which contains a system of linear inequality constraints.

- Constraints matrix for the two-factor Vasicek model:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The admissible set of parameters is given by

$$\mathcal{A}_{\text{Vasicek2}} := \left\{ \Pi_{\text{Vasicek2}} \in \mathbb{R}^8 : A \cdot \Pi_{\text{Vasicek2}} \leq 0 \right\}$$

- Constraints matrix for the two-factor CIR model (CIR2++):

$$A := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

The admissible set of parameters

$$\mathcal{A}_{\text{CIR2++}} := \left\{ \Pi_{\text{CIR2++}} \in \mathbb{R}^9 : A \cdot \Pi_{\text{CIR2++}} \leq 0 \right\}$$

- Constraints matrix for the differenced two-factor CIR model:

$$A := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Where the admissible set of parameters is as follows:

$$\mathcal{A}_{\text{CIR2-difference}} := \left\{ \Pi_{\text{CIR2-difference}} \in \mathbb{R}^8 : A \cdot \Pi_{\text{CIR2-difference}} \leq 0 \right\}$$

A.2 Tables and Figures

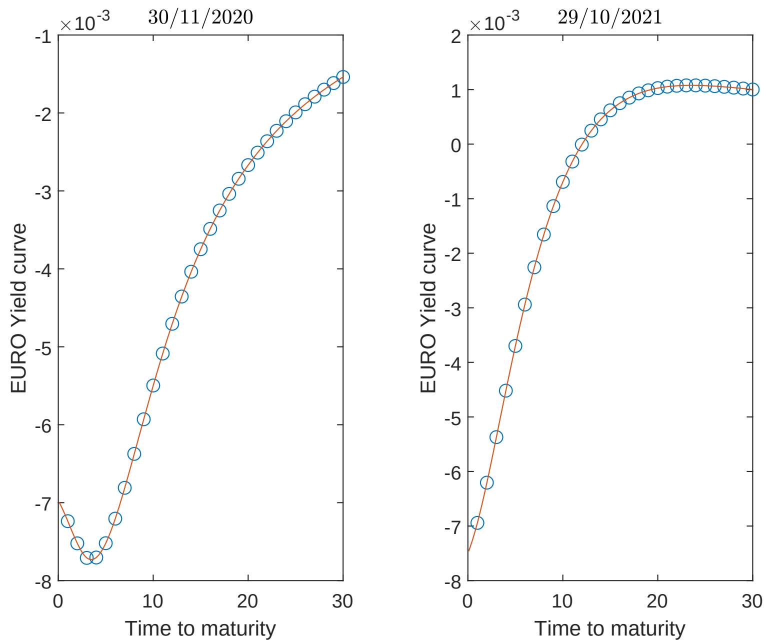


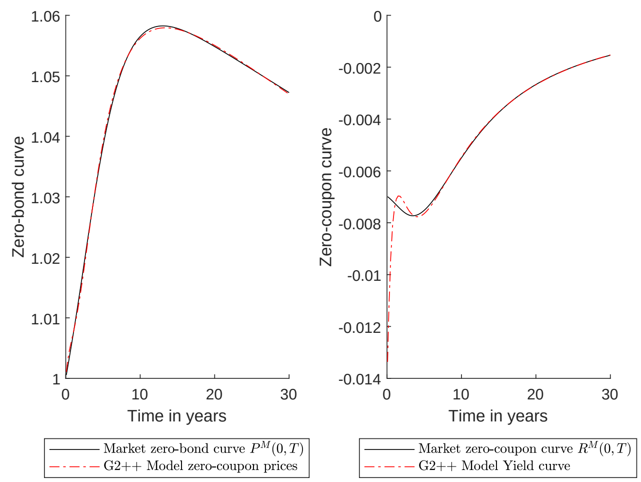
Figure A.1: Cubic Spline Interpolation of the market yield curve (from annual data to quarterly data).

Table A.1: Market Data at 30/11/2020

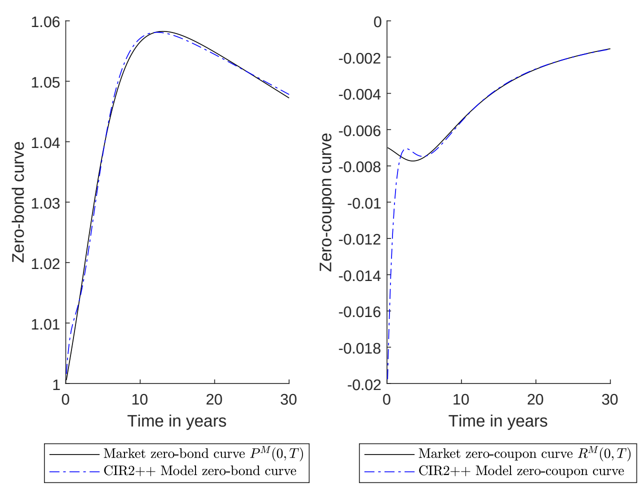
Maturity (in years)	Yield curve (in %)	Discount curve
0,08	-0,6993045	1,0005596
0,25	-0,703247622	1,001759665
0,5	-0,709622958	1,003554417
0,75	-0,716516781	1,005388341
1	-0,723741	1,007263663
1,25	-0,731107523	1,009180731
1,50	-0,738428258	1,011137995
1,75	-0,745515115	1,013131992
2,00	-0,75218	1,015157325
2,25	-0,758234823	1,017206641
2,50	-0,763491492	1,019270614
2,75	-0,767761915	1,021337919
3,00	-0,770858	1,023395213
3,25	-0,772632748	1,02542849
3,5	-0,773103525	1,027428032
3,75	-0,77232879	1,029385816
4,00	-0,770367	1,031294367
4,25	-0,767279529	1,033146893
4,50	-0,763139408	1,034937743
4,75	-0,758022583	1,036662142
5,00	-0,752005	1,038316083
5,25	-0,74516373	1,039896403
5,50	-0,737580344	1,041400997
5,75	-0,729337535	1,042828683
6	-0,720518	1,044179156
6,25	-0,711201503	1,045452801
6,50	-0,701456092	1,046650061
6,75	-0,691346885	1,047771905
7,00	-0,680939	1,048820008
7,25	-0,670294524	1,049796518
7,50	-0,659463414	1,050703307
7,75	-0,648492598	1,051542545
8,00	-0,637429	1,052316916
8,25	-0,626316824	1,053029384
8,5	-0,615189376	1,053682422
8,75	-0,60407724	1,054278618
9,00	-0,593011	1,054820901
9,25	-0,582019166	1,055312333
9,50	-0,571121958	1,055755459
9,75	-0,56033752	1,056152838
10,00	-0,549684	1,056507229
15,00	-0,374834	1,057835776
20,00	-0,266907	1,054831881
25,00	-0,199273	1,051080045
30,00	-0,153832	1,047231065

Table A.2: Market Data at 29/10/2021

Maturity (in years)	Yield curve (in %)	Discount curve
0,08	-0,746024146	1,000596997
0,25	-0,737917697	1,001846497
0,5	-0,724707122	1,003630109
0,75	-0,710081983	1,005339821
1	-0,694177	1,00696592
1,250	-0,677126891	1,008500008
1,50	-0,659066376	1,009935024
1,75	-0,640130172	1,011265258
2,00	-0,620453	1,012486372
2,25	-0,600169578	1,013595404
2,50	-0,579414624	1,014590787
2,75	-0,558322859	1,015472355
3,00	-0,537029	1,01624135
3,25	-0,515657251	1,016900078
3,5	-0,494289752	1,017450655
3,750	-0,472998127	1,017895672
4,00	-0,451854	1,018238486
4,25	-0,430924465	1,018483025
4,50	-0,410258493	1,018633102
4,75	-0,389900524	1,018692839
5,00	-0,369895	1,018666837
5,25	-0,350282577	1,01855997
5,50	-0,331088777	1,018376694
5,75	-0,312335338	1,01812152
6	-0,294044	1,017799191
6,250	-0,276234166	1,017414531
6,50	-0,2589159	1,016971948
6,75	-0,242096935	1,016475796
7,00	-0,225785	1,015930509
7,25	-0,209986292	1,015340482
7,50	-0,194700872	1,014709704
7,75	-0,179927266	1,014042039
8,00	-0,165664	1,013341332
8,25	-0,151908665	1,012611325
8,5	-0,138655111	1,011855409
8,750	-0,125896251	1,011076821
9,00	-0,113625	1,010278717
9,25	-0,101833766	1,009464128
9,50	-0,090512934	1,008635804
9,75	-0,079652386	1,007796342
10,00	-0,069242	1,006948228
15,00	0,061999	0,99074326
20,00	0,102682	0,979673036
25,00	0,107192	0,97355788
30,00	0,100134	0,970406522

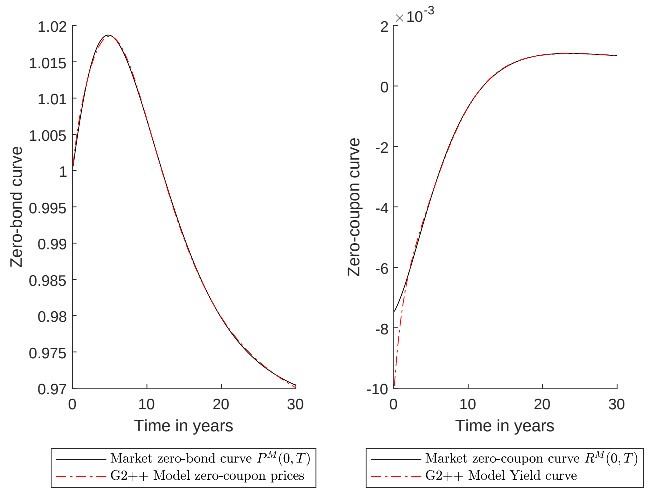


(a) two-factor Gaussian model

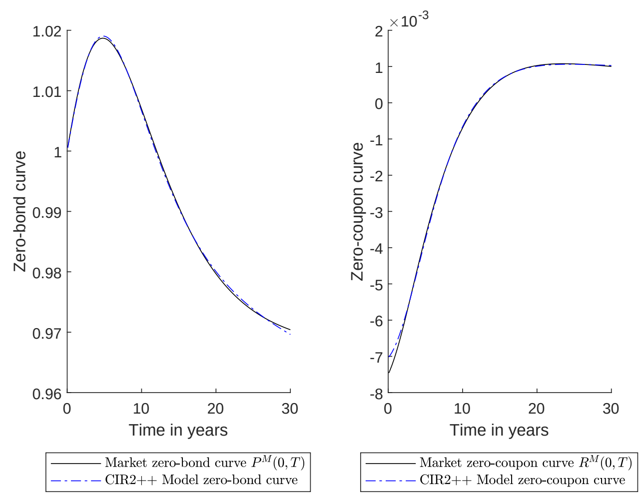


(b) two-factor CIR model

Figure A.2: A comparison between the market zero-bond curve/zero-coupon curve and the the two-factor short rate model zero-bond curve/zero-coupon curve (at 29/10/2021).



(a) two-factor Gaussian model



(b) two-factor CIR model

Figure A.3: A comparison between the market zero-bond curve/zero-coupon curve and the the two-factor short rate model zero-bond curve/zero-coupon curve (at 29/10/2021).

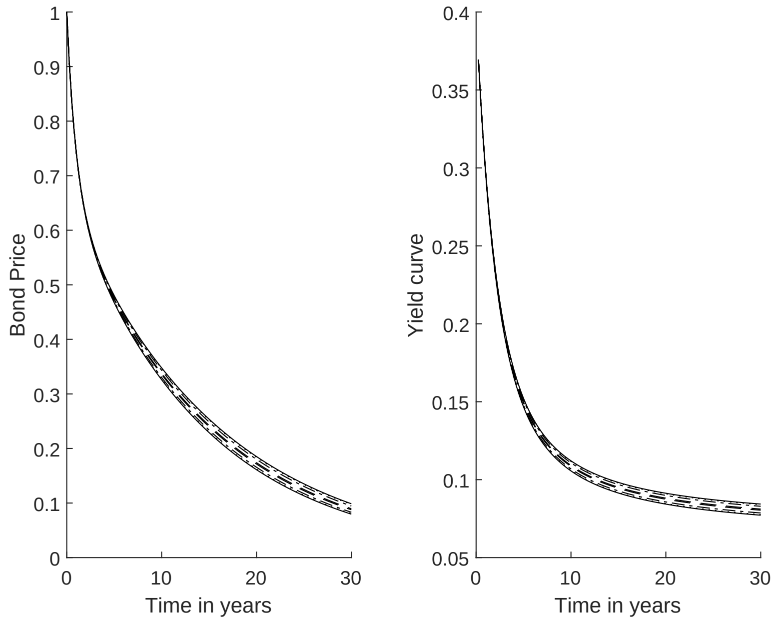


Figure A.4: Sensitivity of the bond price and zero-coupon yield curve to the correlation between the factors for the G2++ model. The short rate parameters are: $\kappa_x = 0.8, \sigma_x = 0.1, \kappa_y = 2, \sigma_y = 0.06, \theta = 0.06$ and the initial condition $r_0 = 0.4$. The correlation between $x(t)$ and $y(t)$ is conditioned to the following set of values $\rho = \{-1.0, -0.9, -0.6, -0.1, 0.1, 0.6, 0.9, 1.0\}$.

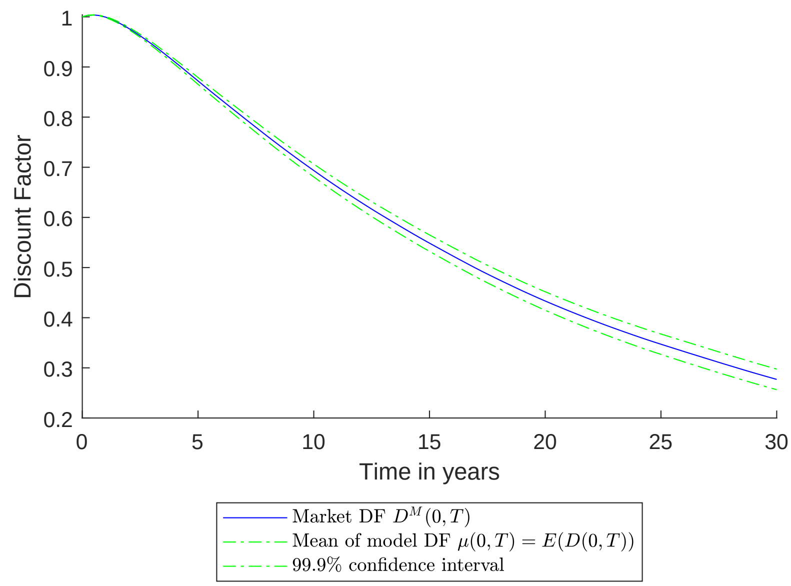


Figure A.5: Monte Carlo estimates of the discount factor for the two-factor Vasicek model. The parameters of the to model are given in Table 5.1a at 29/10/2021 with the following modification: we have decreased the value of σ_x from 0.284 to 0.0284. The number of trajectories for this computation is $n = 10^4$.

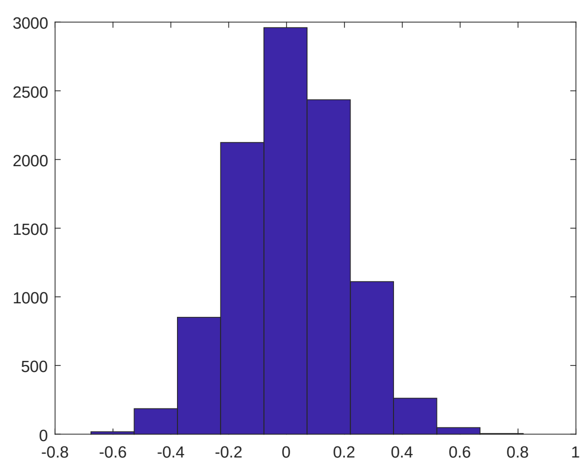


Figure A.6: Histogram of the short rate for the two-factor Vasicek model. The parameters values are given in Table 5.1a. Let us notice that the parameters values are calibrated to the market term structure at 29/10/2021.

	two-factor Vasicek		two-factor CIR basic MC		two-factor CIR MC-IS	
Strike Price	$\bar{x} - 1.96\frac{s}{\sqrt{n}}$	$\bar{x} + 1.96\frac{s}{\sqrt{n}}$	$\bar{x} - 1.96\frac{s}{\sqrt{n}}$	$\bar{x} + 1.96\frac{s}{\sqrt{n}}$	$\bar{x} - 1.96\frac{s}{\sqrt{n}}$	$\bar{x} + 1.96\frac{s}{\sqrt{n}}$
-0,3	0,315	0,322	0,293	0,295	0,072	0,079
-0,29	0,305	0,312	0,283	0,285	0,070	0,077
-0,28	0,296	0,303	0,273	0,275	0,068	0,074
-0,27	0,287	0,294	0,263	0,265	0,065	0,072
-0,26	0,277	0,284	0,253	0,255	0,063	0,069
-0,25	0,268	0,275	0,243	0,245	0,061	0,067
-0,24	0,259	0,266	0,233	0,235	0,059	0,065
-0,23	0,250	0,257	0,223	0,225	0,056	0,062
-0,22	0,241	0,248	0,213	0,215	0,054	0,060
-0,21	0,232	0,239	0,203	0,205	0,052	0,058
-0,11	0,152	0,158	0,104	0,105	0,035	0,040
-0,01	0,088	0,092	0,018	0,019	0,023	0,026
0,09	0,043	0,047	0,000	0,000	0,014	0,017
0,19	0,018	0,020	0,000	0,000	0,008	0,010
0,29	0,006	0,007	0,000	0,000	0,005	0,006
0,3	0,005	0,007	0,000	0,000	0,004	0,006

Table A.3: Confidence interval (95%) of the Monte Carlo estimator. Note that for the two-factor CIR model we have computed a basic MC simulation and importance sampling Monte Carlo.

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