Merton's portfolio problem with discrete trading and multiple risky assets.

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Abstract

This paper is an extension to Merton (1969) which examines the problem of optimal portfolio selection in continuous time. It is shown that it is optimal to have a constant fraction of wealth invested in risky assets. In practice it is often not possible to follow this strategy. To keep the relative weights of the assets in the portfolio at their optimal level, trading is required at every time the value of one of the assets in the portfolio changes, which is often close to continuously. This paper investigates the loss of not being able to trade continuously if multiple, up to 500, risky assets are available. The loss is expressed in terms of certainty equivalent loss instead of utility such that the magnitude of the loss can be interpreted in terms of value. Results are obtained via Monte Carlo simulation. When trading is done continuously an analytic solution exists. This continuous case, is used as control variate to reduce the variance of the Monte Carlo simulation. The simulations show that the certainty equivalent loss of infrequent trading is limited. For rebalancing the portfolio once per year the annual certainty equivalent loss is approximately 1.5 basis points for the used parameters in the base case where one risky asset is available. Furthermore a linear relation is found between the certainty equivalent loss and the length of the time period between rebalancing points. The simulation results also confirm that for the case with two available risky assets the CE loss is higher than in the base case with one risky asset. However, when adding more and more risky assets to the problem the CE loss converges towards the certainty equivalent loss of the single risky asset case.

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Contents

1 Introduction

This paper aims to extend the academic literature on optimal consumption and portfolio choice.

Merton (1969) examines the problem of optimal portfolio selection in continuous time where asset returns are stochastic. It is shown that it is optimal to have a constant fraction of wealth invested in risky assets. However in practice it is often not possible to follow this strategy. To keep the relative weights of the assets in the portfolio at their optimal level, trading is required at every time the value of one of the assets in the portfolio changes, which is often close to continuously.

Keeping the fraction of assets in a portfolio constant is also important in the context of passive funds with an equal weight strategy. Passive funds track a certain portfolio, often a market weighted index. They try to follow a certain index as closely as possible. Funds that follow an index but use other weighting than the market capitalization are known as "Smart Beta" funds. These funds try to outperform the market by exploiting market inefficiencies in a rule based way, which makes that these funds also have lower costs than active funds. The equal weight strategy is a simple and popular form of "Smart Beta". This gives every asset in the portfolio the same weight. A main advantages of this strategy is that the fund is better diversified. Furthermore it gives better exposure to the Fama and French factors (Plyakha, Uppal, and Vilkov (2012)). To keep costs low these types of funds typically do not rebalance as much as possible to follow the strategy as close as possible. Rebalancing may be done monthly or quarterly for example.

The potential loss in terms of utility can be obtained by simulation of the portfolio values with discrete rebalancing strategies. This is for example done by Holth (2011). Holth concluded that the mean loss of utility of the semi-annual strategy and the annual strategy were not far from zero in a case with one risky and one risk-free asset, no transaction costs and constant volatility. This thesis differs from Holth (2011) by taking into account the multi-asset case and by expressing the losses in certainty equivalents rather than in terms of utility. By expressing the loss in certainty equivalents the magnitude of losses can be interpreted in terms of value instead of utility. Utility is an artificial quantity and hard to interpret in real world examples.

The multi-asset case is interesting because the certainty equivalent loss in the multi-asset is expected to be higher than the single asset case. Not only the ratio of risk-free and risky assets will be suboptimal between two points of rebalancing, but also the weights within the risky portfolio will not be optimal anymore. This suboptimal diversification is expected to result in a higher variance for which the risk-averse individual is not compensated.

The main research question is:

What is the certainty equivalent loss of not being able to trade continuously in the multi-asset case of Merton's portfolio problem?

This thesis is built up as follows. First of all, more details about Merton's portfolio problem will be given in Chapter 2. Chapter 3 describes how the research question can be answered by simulation. The results of the simple case with one risky asset and cases with multiple risky assets are given in Chapter 4. This chapter also includes a sensitivity analysis. The conclusion and recommendations for future research are given in Chapter 5. The appendix includes a mathematical derivation and results tables.

2 Merton's Portfolio Problem

Merton (1969) examines the problem of optimal portfolio selection in continuous time for individuals where income is generated by returns on assets which are stochastic. The rates of returns are generated by a Wiener Brownian motion process. A particular case is examined in detail: the two-asset model with constant relative risk aversion. Here a solution of a simplified version of this problem will be provided. Bequests and intermediate consumption are ignored and interest rates are assumed to be constant.

The risky asset S is assumed to follow a geometric Brownian motion which satisfies the following stochastic differential equation (SDE):

$$dS(t) = (r + \lambda\sigma)S(t)dt + \sigma S(t)dZ_t \tag{2.1}$$

The drift of the risky asset is the sum of the risk free return (r) and a compensation (λ) for each unit of risk (σ). The volatility is given by σ and Z_t is a standard Wiener Brownian motion.

Hence it is assumed that asset returns are stationary and log-normally distributed. Note that this widely used assumption is criticized because it contradicts with a set of empirical stylized facts given by Cont (2001). The assumed stochastic differential equation does for example not take into account that return distributions tend to have fatter tails, gain loss asymmetry and volatility clustering.

The SDE of the risk-free asset is given by:

$$dB(t) = rB(t)dt \tag{2.2}$$

The portfolio (W) consists partly of the risky and partly of the risk-free asset. The weight of the risky asset at time t is given by w_t. The SDE of the portfolio is given by:

$$\begin{aligned} dW(t) &= dw_t W(t) + d(1 - w_t)W(t) \\ &= w_t(r + \lambda\sigma)W(t)dt + w_t\sigma W(t)dZ_t + (1 - w_t)rW(t)dt \\ &= (w_t(r + \lambda\sigma) + r(1 - w_t))W(t)dt + \sigma w_t W(t)dZ_t \end{aligned} \tag{2.3}$$

To obtain the optimal solution, the weight of the risky asset (w) in the portfolio (W) must be chosen such that the utility ($U(W)$) at maturity (T) is maximized.

Hence the problem of choosing optimal portfolio selection can be formulated as follows:

$$\max_{w} \mathbb{E}[U(W(T))] \tag{2.4}$$

subject to the budget constraint.

Here $U(W)$ is the utility function. The utility function is assumed to be strictly concave. Hence $U'(W) > 0$ and $U''(W) < 0$. This means that more wealth implies a higher utility and that decision makers are assumed to be risk-averse. A fixed loss is preferred to a random loss of the same expected value. More formally, in Gollier, Eeckhoudt, and Schlesinger (2005) risk aversion is defined as :

"*An agent is risk-averse if, at any wealth level W, he or she dislikes every lottery with an expected payoff of zero.*" Hence, $\forall W, \forall \tilde{z}$ with $\mathbb{E}\tilde{z} = 0$.

$$\mathbb{E}[U(W + \tilde{z})] \leq U(W) \tag{2.5}$$

It can be proven that a decision maker with utility function U is risk averse if and only if U is concave. See Gollier et al. (2005) Chapter 1 for this derivation.

Here the more specific assumption of Constant Relative Risk Aversion is used. This means that if a change in wealth is experienced, the optimal relative level of risk in the portfolio remains unchanged.

The utility function is given by:

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma \geq 0, \gamma \neq 1 \\ \ln(W) & \text{for } \gamma = 1. \end{cases} \tag{2.6}$$

The Arrow-Pratt measure of relative risk aversion is given by:

$$\frac{-WU(W)''}{U(W)'} = -W\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} = \gamma \tag{2.7}$$

The variable γ is called the coefficient of risk-aversion.

It is shown by Merton (1969) that with CRRA utility it is optimal to have a constant fraction of wealth invested in risky assets. The optimal weight (w^*) is given by:

$$w^* = \frac{\lambda}{\gamma\sigma}. \tag{2.8}$$

From Equation 2.3 it follows that the solution of the portfolio with the optimal fraction of wealth invested in the risky asset ($W(t)^{\text{opt}}$) is given by:

$$W^{\text{opt}}(t) = W(t_0) \exp\left(((r + \lambda\sigma)w^* + r(1 - w^*) - 0.5\sigma^2 {w^*}^2)t + \sigma w^* Z_t\right) \tag{2.9}$$

In this paper the case with multiple risky assets will be considered. As noted in Merton (1971), when log-normality is assumed, the multi-asset case can without loss of generality be regarded as a case with just two assets, one risk free and one risky. The risky assets are then a composite asset which is also log-normally distributed.

Note that some errors are found in Merton (1969). They are described in Merton (1973) and Sethi and Taksar (1988). Non of these remarks impact the conclusions that are used in this paper.

3 Methodology

This chapter describes the simulation methodology. Section 3.1, 3.2 and 3.3 describe the simulation of the individual assets, the composite asset and the total portfolio value. The use of the certainty equivalent loss is explained in Section 3.4. Section 3.5 gives more information about the use of a control variate to improve the accuracy of the simulation. In section 3.6 more details are provided about the construction of the confidence interval.

3.1 Simulation of the Asset Sample Path

Using Ito's formula the stochastic differential equation of the risky assets (2.1) can be solved analytically. The solution is given by:

$$S(t) = S(t_0)\exp\left(\left((r+\lambda\sigma) - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right) \tag{3.1}$$

The time space form time t_0 to T is discretized. Every trade day is a discretization point for which the value of the simulated asset is calculated. The distance in time between two discretization points is given by dt. The value of the risky asset at the next trading day in the single risky asset case is simulated as follows:

$$S(t+dt) = S(t)\exp\left(\left(r+\lambda\sigma - \frac{1}{2}\sigma^2\right)dt + \sigma Z_{t+dt}\right). \tag{3.2}$$

Here the value for Z_{t+dt} is generated by:

$$Z_{t+dt} = Z_t + \sqrt{dt}X. \tag{3.3}$$

Here X is a draw from the standard normal distribution.

$$X \sim N(0,1) \tag{3.4}$$

3.2 Simulation of Composite Asset

The weighted average of all underlying risky assets is called the composite asset (P) by Merton (1971). In this thesis it will be assumed that all risky assets have the same dynamics. Therefore the expression for the composite asset that is provided by Merton (1971) can be simplified. The drift of the composite asset is then equal to the drift of the individual asset.

The SDE of the composite asset is given by:

$$dP(t) = \left(\sum_{i=1}^{n} S^i \delta_i \right) = \mu_P P(t) dt + \sigma_P P(t) dZ_t^P \tag{3.5}$$

Where

$$\mu_P = r + \sigma_P \lambda \tag{3.6}$$

$$\sigma_P = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j \rho_{ij} \sigma_i \sigma_j} \tag{3.7}$$

$$dZ_t^P = \sum_{i=1}^{n} \delta_i \sigma_i dZ_t^i / \sigma_P \tag{3.8}$$

The weight of the i^{th} asset is given by δ_i with $\sum_{j=1}^{n} \delta_i = 1$. Note that all weights are positive. The correlation between two assets i and j is given by ρ_{ij}.

The single risky asset case is used as basis. In order to compare this case to portfolio's with multiple risky assets, the single risky asset is regarded as the composite asset. When a second risky asset is added, the dynamics of both risky assets is such that the average of the two has the same dynamics as the risky asset in the simple case were only one risky asset is available.

Intuitively the single risky asset can be seen as an index. In the case with two risky assets it is not possible to invest directly in the index. Instead both halves of the index are traded separately. In this way the optimal weight of the risk-free and risky part of the total portfolio remains equal between the two cases. The optimal total weight of the composite asset relative to the risk-free asset does not depend on the number of risky assets that are available.

Simulation of multiple correlated geometric Brownian motions can be done using the Cholesky decomposition. In Schumacher (2015, chap. 2) this is explained.

If $\mathbf{Z}_t$ is a vector Brownian motion with variance-covariance matrix $\sum$, we can think of $\mathbf{Z}_t$ as being generated by

$$\mathbf{Z}_t = \mathbf{M}\hat{\mathbf{Z}}_t \tag{3.9}$$

where $\hat{\mathbf{Z}}_t$ is a standard vector Brownian motion and $\mathbf{M}$ is any matrix such that that $\sum = \mathbf{M}\mathbf{M}'$ The decomposition of a positive definite matrix $\sum$ in the form $\mathbf{M}\mathbf{M}'$ where $\mathbf{M}$ is a lower triangular and has positive entries on the diagonal is known as the Cholesky decomposition.

It is assumed that all risky assets have the same dynamics and the same level of correlation towards each other.

$$\rho_{i,j} = \rho \quad \forall\, i, j \tag{3.10}$$

It is ignored here that in reality some stocks are more correlated towards the rest of the market than others. To measure the general impact of correlation this assumption is preferred. It is harder to compare cases with different amount of stocks when correlations are defined on the individual asset level. The underlying correlation matrix $\mathbf{C}$ for the covariance matrix $\sum$ can be described as given below. It is assumed that all assets have equal variance.

$$\mathbf{C} = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix} \tag{3.11}$$

$$\sum = \begin{pmatrix} 1 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & 1 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & 1 \end{pmatrix} \tag{3.12}$$

It is assumed that investors in the risky asset are only compensated for the systematic risk in the risky asset therefore the solution for the risky asset in the multi-asset environment is:

$$\mathbf{S}(t+dt) = \mathbf{S}(t) \exp\left(\left(r + \lambda\sigma_P - \frac{1}{2}\sigma^2\right) dt + \sigma \mathbf{Z}_{t+dt}\right) \tag{3.13}$$

The value for $\mathbf{Z}_{t+dt}$ is generated by:

$$\mathbf{Z}_{t+dt} = \mathbf{Z}_t + \sqrt{dt}\mathbf{x} \tag{3.14}$$

Here $\mathbf{Z}_{t+dt}, \mathbf{Z}_t$ and $\mathbf{x}$ are all vectors with a length equal to the number of risky assets N_s. The vector $\mathbf{x}$ represents random values from the multivariate normal distribution with mean zero and covariance matrix $\sum$.

In the optimal risky portfolio the weights of the risky assets are equal because their dynamics are equal.

$$\delta_i = \frac{1}{n} \qquad \forall i \tag{3.15}$$

The difference compared to Equation (3.2) is that σ in the drift term is replaced by the volatility of the return of the index of risky assets σ_P. Note that for uncorrelated risky assets $\sigma_P = \sigma$. For correlated assets the relation between σ_P and σ is given below. The derivation can be found in Appendix A.

$$\sigma = \sqrt{\sigma_P^2 / \left(\frac{1}{n} + \frac{n-1}{n}\rho\right)} \tag{3.16}$$

3.3 Simulation of the Portfolio Value

The solution for the optimal portfolio value in case of a single risky asset is given in Equation 3.17. Using the composite asset instead of the single risky asset the optimal portfolio value is given in Equation 3.17 and the optimal weight w^* is given in Equation 3.18.

$$W^{\text{opt}}(t) = W(0) \exp\left(((r + \lambda\sigma_P)w^* + r(1-w^*) - 0.5\sigma_P^2 {w^*}^2)t + \sigma_P w^* Z_t^P\right). \tag{3.17}$$

$$w^* = \frac{\lambda}{\gamma \sigma_P} \tag{3.18}$$

To compare different rebalancing strategies it is also needed to simulate the portfolio value if the weights are not continuously rebalanced. The portfolio value in the multi asset case depends on the value of the risk-free and risky assets, the weight of the risky assets in total and the weights of the risky assets within the index of risky assets.

At the end of the first period the value is given by:

$$W(t_0 + dt) = w^* W(t_0) \frac{1}{N} \sum_{i=1}^{N} \left(\frac{S_i(t_0 + dt)}{S_i(t_0)} \right) + (1 - w^*) W(t_0) \frac{B(dt)}{B(t_0)} \tag{3.19}$$

$$= w^* W(t_0) \frac{1}{N} \sum_{i=1}^{N} \left(\frac{S_i(t_0 + dt)}{S_i(t_0)} \right) + (1 - w^*) W(t_0)\, \mathrm{e}^{r \cdot dt} \tag{3.20}$$

At t_0 the optimal weight between the risk-free and the risky asset is chosen as defined in Equation 2.8. The optimal weights within the index of risky assets are equal for all individual assets, because the dynamics and correlation for all assets are equal. Hence these weights are equal to $1/N_s$.

Without rebalancing, the portfolio value at the end of the second period is given by:

$$W(t_0 + 2dt) = w^* W(t_0) \frac{1}{N} \sum_{i=1}^{N} \left(\frac{S_i(2dt)}{S_i(t_0)} \right) + (1 - w^*) W(t_0)\, \mathrm{e}^{r2dt} \tag{3.21}$$

With rebalancing after the first period the value at the end of the second period is given by:

$$W(t_0 + 2dt) = w^* W(t_0 + dt) \frac{1}{N} \sum_{i=1}^{N} \left(\frac{S_i(2dt)}{S_i(dt)} \right) + (1 - w^*) W(dt)\, \mathrm{e}^{rdt} \tag{3.22}$$

Formulated in a more general way, the portfolio value at time t can be defined as:

$$W(t) = w^* W(t^*) \frac{1}{N} \sum_{i=1}^{N} \left(\frac{S_i(t)}{S_i(t*)} \right) + (1 - w^*) W(t^*)\, \mathrm{e}^{r(t - t^*)} \tag{3.23}$$

were t^* is the last rebalancing point in the past ($t > t^*$).

3.4 Certainty Equivalent Loss

The certainty equivalent is the lowest certain amount of money that someone would be willing to accept instead of a risky payoff. This means that the utility of the certainty equivalent is equal tot the expected utility of the risky payoff, which is the portfolio value at maturity in this case. Interpreting a loss expressed in utility is difficult, because it is not clear what one unit of utility means. Furthermore it is harder to compare results between utility functions. Von Neumann-Morgenstern utility functions can be replaced by their linear transformation. The chosen scale of the utility function determines the level of utility loss. By expressing the loss in certainty equivalents this drawback is avoided.

The certainty equivalent (C) is the fixed amount of money at time T that results in the same utility as the expected utility from the portfolio at end date T.

$$U(C) = E[U(W(T))] \tag{3.24}$$

The certainty equivalent for the CRRA utility function as given in Equation 2.6 is:

$$C = U^{-1}(E[U(W(T))]) = (E[U(W(T))](1-\gamma))^{\frac{1}{1-\gamma}} \tag{3.25}$$

The difference of the obtained certainty equivalent using a specific rebalancing strategy (C) with the certainty equivalent of the optimal solution with continuous rebalancing (C^*) is called the certainty equivalent loss. The loss is given per amount of currency initial capital. The loss does not depend on the amount of initial capital because of the characteristics of the CRRA utility curve.

$$\text{CE loss} = C^* - C \tag{3.26}$$

3.5 Variance Reduction Using a Control Variate

The simulation method described in previous section can take significant time if the number of risky assets and the desired accuracy is high. Furthermore the memory of the computer is not large enough at some point which also makes it harder to get accurate results.

To obtain more accurate results with less computation time and memory usage a control variate is used. This is explained for example in Schumacher (2015). It works as follows:

Define variable of interest X and control variable Y with variance σ_X^2 and σ_Y^2 respectively and correlation coefficient ρ_{XY}. A new random variable Z can now be defined.

$$Z = X - \alpha(Y - \mathbb{E}Y) \tag{3.27}$$

Here α is a chosen constant. Note that $\mathbb{E}Z = \mathbb{E}X$ so instead of computing the expectation of X the expectation of Z can be computed. The variance of Z is:

$$\text{Var}(Z) = \sigma_X^2 + \sigma_Y^2 - 2 * \text{Cov}(X, Y). \tag{3.28}$$

Hence, the variance can be reduced if the control variate is correlated with the variable of interest. The variable α can be chosen such that variance is optimally reduced. Choosing

$$\alpha = \rho_{XY}\frac{\sigma_X}{\sigma_Y} \tag{3.29}$$

gives

$$\text{Var}(Z) = (1 - \rho_{XY}^2)\sigma_X^2. \tag{3.30}$$

The variable of interest on which the variance reduction technique is applied is the utility of the rebalancing strategy. The control variate is the utility of the optimal portfolio when rebalancing is done continuously. This variable has an analytic solution and is highly correlated to the utility of other trading strategies.

$$Z = U(W(T)) - \rho_{U(W(T))U(W^{\text{opt}}(T))} \frac{\sqrt{\text{Var}(U(W(T)))}}{\sqrt{\text{Var}(U(W^{\text{opt}}(T)))}} \left(U(W^{\text{opt}}(T)) - \mathbb{E}[U(W^{\text{opt}}(T))]\right) \tag{3.31}$$

The analytic solution for the expected utility with continuous rebalancing is given by:

$$\begin{aligned}
\mathbb{E}[U(W^{\text{opt}}(T))] &= \mathbb{E}\left(\frac{W^{\text{opt}}(T)^{1-\gamma}}{1-\gamma}\right) \qquad (3.32)\\
&= \frac{1}{1-\gamma}\mathbb{E}\left(\left(W(t_0)\exp\left((\mu_P - \frac{1}{2}(\sigma_P w^*)^2)T + \sigma_I w^* \sqrt{T} Z_t^P\right)\right)^{1-\gamma}\right)\\
&= \frac{1}{1-\gamma} W(t_0)^{1-\gamma} \exp\left((\mu_P - \frac{1}{2}(\sigma_P w^*)^2)T(1-\gamma)\right)\mathbb{E}\left(\exp\left(\sigma_I w^* \sqrt{T} Z_t^P (1-\gamma)\right)\right)\\
&= \frac{W(t_0)^{1-\gamma}}{1-\gamma}\exp\left((\mu_P - \frac{1}{2}(\sigma_P w^*)^2)T(1-\gamma)\right)\exp\left(\frac{1}{2}(\sigma_P w^*)^2 T(1-\gamma)^2\right)
\end{aligned}$$

To calculate the expectation in the derivation above it is used that Z_t^P has a standard normal distribution and that the expectation of a log-normal variable of which the logarithm has mean μ and variance σ is given by $\exp(\mu + \frac{1}{2}\sigma^2)$.

The variance of the control variate can also be calculated analytically. Note that the variance of a log-normal variable X ($X \sim \text{LN}(\mu, \sigma^2)$) is given by:

$$\text{Var}(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \tag{3.33}$$

The variance of the control variate is:

$$\begin{aligned}
\text{Var}(U(W^{\text{opt}}(T))) = \frac{W(t_0)^{2(1-\gamma)}}{(1-\gamma)^2} &\exp\left(2(\mu_P - \frac{1}{2}(\sigma_P w^*)^2)T(1-\gamma)\right)\\
&\exp\left((\sigma_P w^*)^2 T(1-\gamma)^2\right)\left(\exp\left((\sigma_P w^*)^2 T(1-\gamma)^2\right) - 1\right)
\end{aligned} \tag{3.34}$$

Even for most cases with no rebalancing at all for one year the estimated correlation coefficients between the simulated utility and the utility corresponding to the Merton solution are higher than 0.98. This means that the variance can be reduced with more than a factor 25. Hence, the confidence intervals of the estimations are more than five times smaller.

Preferably the variance of the variable of interest and the correlation with the control variate is known, but they can be estimated based on their sample. The expectation and variance of $U(W(T))$ are not known for sub-optimal trading strategies. These are estimated based on their sample.

3.6 Confidence Interval

Confidence interval for Certainty Equivalent loss is approximated via simulation and by applying the Central Limit Theorem. The expected utility of a rebalancing strategy $\mathbb{E}[U(W(T))]$ is approximated by the sample mean $\bar{Z}$.

$$\bar{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i \tag{3.35}$$

The standard deviation is approximated using the sample standard deviation S_Z.

$$S_Z = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} |Z_i - \bar{Z}|^2} \tag{3.36}$$

The 95% confidence interval for $\mathbb{E}[U(W(T))]$ is approximated as given in Equation 3.37. Here Φ^{-1} is the inverse CDF of the standard normal distribution.

$$\left[\bar{Z} - \frac{S_Z}{\sqrt{N}}\Phi^{-1}(0.975); \bar{Z} + \frac{S_Z}{\sqrt{N}}\Phi^{-1}(0.975)\right] \tag{3.37}$$

The upper and lower bound of the CE loss are obtained by mapping the upper and lower bound of $\mathbb{E}[U(W(T))]$ to the certainty equivalent via Equation 3.25. The confidence interval for the CE loss is obtained using Equation 3.26. Note that the upper and lower bound swap by defining the CE loss instead of the CE gain.

4 Results

4.1 Single Risky Asset

Simulation results for different rebalancing strategies in case of one risky asset are given in Table 4.1. The table shows the annual certainty equivalent loss in basis points and the 95 percent confidence interval in basis points. The used model parameter values are given in Table 4.2.

The estimated yearly certainty equivalent loss is found to be approximately 1.1 basis points. The CE loss increases if the time between two rebalancing points becomes larger. This confirms that with less rebalancing points the difference to the optimal continuous rebalancing solution is larger.

TABLE 4.1: One risky asset. The number of simulations is 35 million

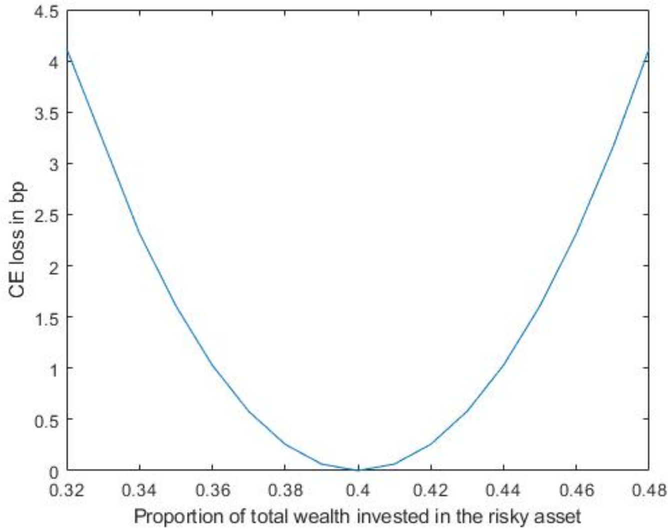
Strategy	CE loss in bp	CI of CE loss in bp
Daily	0.0045	(0.0022; 0.0068)
Every Other Day	0.0083	(0.0051; 0.0116)
Weekly	0.0214	(0.0164; 0.0264)
Monthly	0.106	(0.0955; 0.1165)
Quarterly	0.2835	(0.2655; 0.3016)
Semiannually	0.5468	(0.5214; 0.5722)
Annually	1.1222	(1.0867; 1.1576)

To give an idea of the size of the CE loss due to not continuous rebalancing, compared to another related deviation of the optimal solution, an example is shown in Figure 4.1. Here rebalancing is done optimally, but a sub-optimal percentage of wealth is invested in the risky asset. Given the used parameters given in Table 4.2 the optimal percentage of wealth invested in the risky asset can be calculated using Equation 2.8. This is 40 percent. It is assumed that rebalancing is done continuously, but the chosen percentage of the risky asset is chosen unequal to 40. Figure 4.1 shows the resulting yearly CE loss in basis points. Choosing 36 percent or 44

TABLE 4.2: Chosen parameter values

Parameter	Value
r	0.02
λ	0.2
σ	0.25
ρ	0.5
γ	2
T	1
dt	1/250

FIGURE 4.1: Yearly certainty equivalent loss in basis points for different proportions of total wealth invested in the risky asset with continuous rebalancing using parameter values as given in Table 4.2.



percent of wealth invested in the risky asset instead of the optimal 40 percent leads to a yearly CE loss of around 1 basis point.

4.2 Multiple Risky Assets

4.2.1 Two Risky Assets

Simulating the risky assets as two separate correlated risky assets instead of one leads as expected to a higher CE loss. The estimated yearly CE loss is approximately 1.5 basis points if rebalancing is done only once a year and less than a basis point for semiannually rebalancing. The additional CE loss of the case with two risky assets compared to that of one risky asset comes from the weights within the risky part of the portfolio that diverge from the optimum. Both risky assets have the same dynamics and hence the diversification within the portfolio is optimal if both risky assets have the same relative weight. By not rebalancing continuously, the relative weight of the risky assets will diverge from the optimum due to differences in return between both assets.

TABLE 4.3: Two risky assets. The number of simulations is 25 million

Strategy	CE loss in bp	CI of CE loss in bp
Daily	0.0075	(0.0043; 0.0106)
Every Other Day	0.0117	(0.0073; 0.0161)
Weekly	0.0309	(0.0240; 0.0377)
Monthly	0.1309	(0.1167; 0.1451)
Quarterly	0.3733	(0.3489; 0.3978)
Semiannually	0.75	(0.7156; 0.7844)
Annually	1.5056	(1.4576; 1.5536)

The estimated CE loss for the case with two risky assets differs statistically significant from the single asset case for rebalancing periods that are larger than or equal to a week. However in practice a loss of less than two basis points may still be negligible compared to other factors such as transaction costs.

4.2.2 Ten Risky Assets

Table 4.4 shows simulation results for the case with ten risky assets. The table shows the annual certainty equivalent loss in basis points and the 95% confidence interval in basis points. The used model parameter values are given in Table 4.2. The estimated CE loss is lower compared to the case with two risky assets. The estimated yearly CE loss is approximately 1.4 basis points if rebalancing is done only once a year.

TABLE 4.4: Ten risky assets. The number of simulations is 15 million

Strategy	CE loss in bp	CI of CE loss in bp
Daily	0.0146	(0.0107; 0.0185)
Every Other Day	0.0182	(0.0127; 0.0237)
Weekly	0.0318	(0.0233; 0.0403)
Monthly	0.1269	(0.1093; 0.1444)
Quarterly	0.364	(0.3337; 0.3944)
Semiannually	0.6876	(0.6449; 0.7303)
Annually	1.4146	(1.3549; 1.4744)

4.2.3 Five Hundred Risky Assets

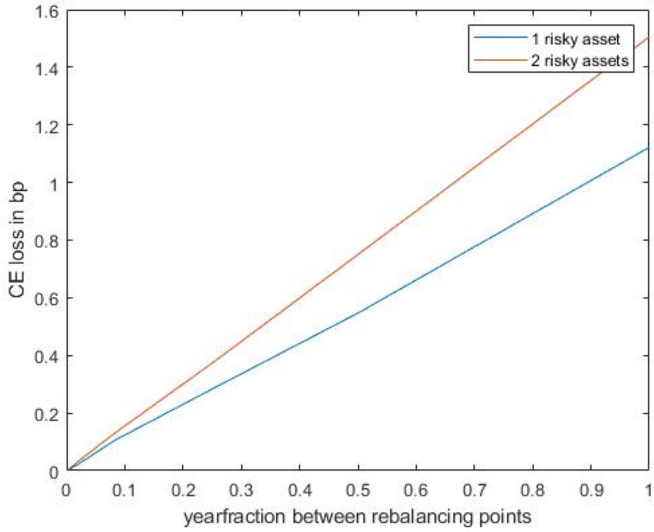
For the case with 500 risky assets shown in Table 4.5 the estimated CE loss is lower compared to the case with ten risky assets. The estimated yearly CE loss is approximately 1 basis point if rebalancing is done only once a year.

The table shows the annual certainty equivalent loss in basis points and the 95% confidence interval in basis points. The used model parameter values are given in Table 4.2.

TABLE 4.5: 500 risky assets. The number of simulations is 500 thousand.

Strategy	CE loss in bp	CI of CE loss in bp
Daily	0.0051	(-0.0142; 0.0244)
Every Other Day	0.0381	(0.0108; 0.0654)
Weekly	0.0206	(-0.0217; 0.0629)
Monthly	0.1276	(0.0396; 0.2156)
Quarterly	0.2694	(0.1175; 0.4214)
Semiannually	0.6308	(0.4175; 0.8442)
Annually	0.9539	(0.6572; 1.2506)

FIGURE 4.2: Yearly certainty equivalent loss in basis points for different proportions of total wealth invested in the risky asset with continuous rebalancing using parameter values as given in Table 4.2.



4.3 Rebalancing Period and CE Loss

From the tables in this chapter it can be observed that there is a linear relation between the rebalancing strategy and the yearly CE loss. Doubling the size of the time interval between rebalancing points approximately doubles the annual CE loss. This is shown in Figure 4.2 where this relation is plotted for one and two risky assets.

4.4 Convergence of CE Loss when Adding Risky Assets

In a portfolio with many risky assets the relative weight of one risky assets is relatively low. Therefore it's relative weight will remain close to the optimum even if the return on this asset is extremely high or low. Simulations with up to 500 risky assets show that by adding more and more risky assets the CE loss will decrease at some point. The relation between the CE loss and the number of risky assets depends on the correlation. For the used correlation of 0.5 between all risky assets, the CE loss already decreases when adding a fifth risky asset. However, if correlation is low the CE loss may still increase when adding some more risky assets. For example for a correlation of 0.05 the CE for 20 risky assets is higher than the CE loss for 10 risky assets. This is shown in Table 4.6. The table shows the yearly CE loss in basis points when rebalancing is done only once a year. Note that correlation has no impact in the case where only one risky asset is available. The difference between a correlation of 5% and 50% for the single risky asset case is purely caused by the randomness of the Monte Carlo simulation.

The modeling choice made in Section 3.2 is important. The risky assets are modeled such that the continuous rebalanced portfolio of risky assets has the same distribution as the single risky asset. This makes sure that the optimal wealth invested in risky assets is equal no matter how many risky assets are available. Hence, if the correlation between the risky assets is less then one, the volatility of a single risky

TABLE 4.6: The annual certainty equivalent loss in basis points and the corresponding 95% confidence interval in basis points for correlation levels of 5% and 50%. The other parameter values are given in Table 4.2 and the number of simulations are given in Table C.3

Risky assets	CE loss, $\rho = 5\%$	CE loss, $\rho = 50\%$	CI, $\rho = 5\%$	CI, $\rho = 50\%$
1	1.1321	1.1173	(1.1111; 1.1531)	(1.0963; 1.1382)
2	3.762	1.5099	(3.7092; 3.8148)	(1.4759; 1.5438)
3	5.8612	1.5281	(5.7818; 5.9406)	(1.4868; 1.5694)
5	9.2731	1.5205	(9.1549; 9.3914)	(1.4724; 1.5687)
10	14.4229	1.3627	(14.1625; 14.6832)	(1.2809; 1.4446)
20	18.3725	1.2801	(18.0825; 18.6624)	(1.2013; 1.3589)
50	16.6325	1.0063	(16.0329; 17.2322)	(0.8434; 1.1692)
100	12.489	1.0896	(11.9677; 13.0102)	(0.9288; 1.2504)
150	10.2521	1.3606	(9.2120; 11.2921)	(1.0017; 1.7194)
200	8.6084	0.9667	(7.6581; 9.5586)	(0.6088; 1.3246)
250	7.3442	1.279	(6.4620; 8.2264)	(0.9225; 1.6356)
500	3.4842	1.224	(2.7846; 4.1838)	(0.8690; 1.5791)

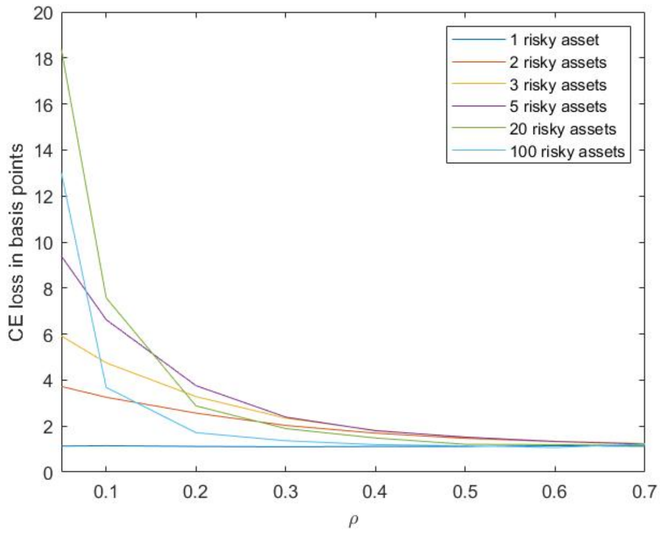
asset is higher than the volatility of the average of the risky assets. The volatility of the average is kept constant, so the volatility of a single risky asset increases when a higher number of risky assets is available. This leads to a high individual volatility. For example, for the case with 500 assets with a correlation of 50% the individual volatility is 35.3%.

When adding an additional risky asset, the higher individual variance causes the value of the risky assets and hence their relative weight to diverge more from each other. However, adding an additional asset also has a stabilizing effect. In a portfolio with more assets, the amount of money invested in an individual assets is smaller. Therefore changes in the value of one asset have a smaller impact on the value of the portfolio and also a smaller impact on the relative weights. Which of the two effects is larger depends strongly on the amount of risky assets that is already in the portfolio and the correlation between the risky assets. When the risky portfolio exists of a larger number of assets, adding an additional asset will more likely cause the CE loss to decrease. The correlation between the risky assets has an opposite effect. For a higher level of correlation, adding an additional asset will more likely cause the CE loss to increase. Table 4.6 shows some evidence that by adding more and more risky assets the CE loss converges to the single risky asset case.

4.5 Sensitivity Analysis

In this section the impact of the main parameters is further examined. These parameters are correlation, the risk aversion parameter, volatility, price of risk, and the risk-free rate. It is shown that level of CE loss is especially sensitive to volatility and correlation. However, even for extreme parameter values the observed yearly CE loss levels are still below 20 basis points. In Appendix B more simulation results for bumped parameter values are given for the case with yearly rebalancing.

FIGURE 4.3: Yearly CE loss in basis points for different levels of correlation with yearly rebalancing using parameter values as given in Table 4.2. The used number of simulations is given in Table C.3.



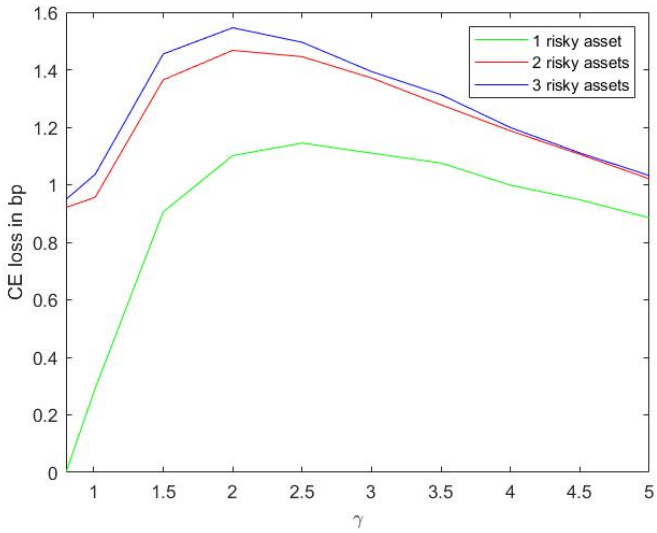
4.5.1 Correlation

The impact from the correlation parameter ρ is also displayed in Table C.2 in Appendix B. For one risky asset ρ cannot have any impact by definition, which is confirmed in the table. For two or more risky assets it can be seen that for low correlation the CE loss is higher than for high correlation. For perfectly correlated risky assets the multi-asset case would be the same as the single asset case. Rebalancing becomes especially important if the value of risky assets move in opposite directions. Then the relative weight of the asset diverges from the optimal solution. Hence, in case of lower correlation there is more need to rebalance positions. Figure 4.3 shows that CE loss levels can spike for low correlation levels. The Figure shows a CE loss of more than 18 basis points for a correlation of 5% and twenty risky assets. This is significantly higher than the CE loss found using the parameter values as given in Table 4.2. Note that the variance of each of the twenty individual risky assets is 80.6% in this case with $\rho = 0.05$ as explained in Appendix A.

4.5.2 Risk aversion

The chosen level for this parameter, γ, is 2. In Figure 4.4 the results are shown for different risk aversion levels with for one, two, or three risky asset available. The figure shown that the CE loss is more or less maximized for a risk aversion level of 2. For higher risk aversion levels, a bigger percentage of total wealth will be invested in the risk-free asset, for which no rebalancing is needed. Therefore the loss of not rebalancing is lower. For low risk aversion levels, a bigger percentage of wealth will be invested in risky assets. For $\gamma = 0.8$, 100% of total wealth is invested in the risky asset. In this case the CE loss comes only from sub-optimal rebalancing within the risky portfolio. Sub-optimal rebalancing between the risky portfolio and risky-free asset is not possible and therefore the CE loss is zero for the single risky asset case.

FIGURE 4.4: Yearly CE loss in basis points for different levels of γ with yearly rebalancing using parameter values for other parameters as given in Table 4.2. The used number of simulations is given in Table C.3.



4.5.3 Volatility of Risky Portfolio

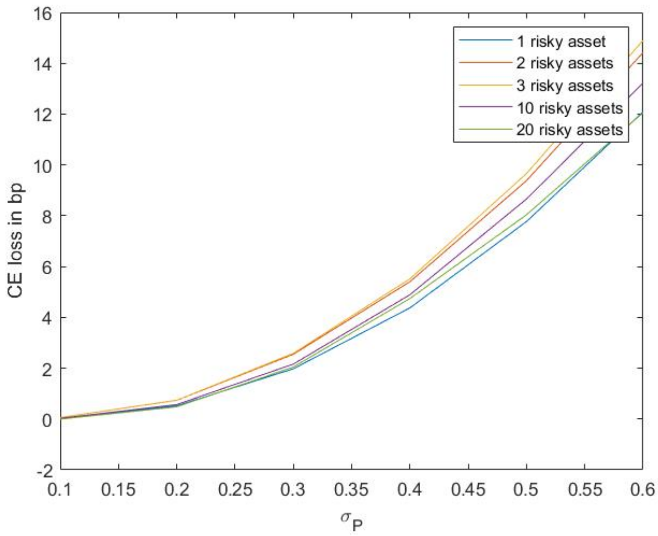
More risk in the risky portfolio will lower the relative weight of the risky portfolio. Via the lambda coefficient it also results in a higher drift for the risky assets. The economic logic behind this higher drift is that investors will demand a higher expected return for investments that bear a higher risk. Figure 4.5 shows that the annual CE loss is higher for higher volatility levels of the risky portfolio. For a volatility of 10 percent all wealth is invested in the risky-asset. Furthermore there is not much need to trade frequently between risky assets, because prices are relatively stable. For high levels of volatility, a part of the portfolio is invested in the risk-free asset, so rebalancing between risky assets and the risky free asset is needed in the optimal solution. Furthermore the high volatility causes big deviations from the optimal weights if rebalancing is not done infrequently. Figure 4.5 shows annual CE loss values up to 15 basis points.

4.5.4 Price of Risk

Market participants tend to be risk averse therefore a risk premium must be paid to compensate investors for taking risk. How much compensation is paid for taking extra risk is determined by the parameter λ. The drift of the risky asset is the sum of the risk-free rate (r) and a compensation (λ) for each unit of risk (σ) as shown in Equation 2.1. Figure 4.6 shows the relation between λ and the CE loss of rebalancing yearly instead of continously.

For the single asset case the CE loss is zero when λ is zero and for λ equal to 0.5. For λ equal to zero the whole portfolio is risk-free. All wealth is invested in the risk-free asset. No rebalancing is needed. For λ equal to 0.5 the total wealth is fully invested in the risky-asset. Therefore also no rebalancing is needed. For values between 0 and 0.5 the portfolio consists partly of risky assets and partly of the risk-free asset.

FIGURE 4.5: Yearly CE loss in basis points for different levels of volatility of the risky portfolio, σ_P, with yearly rebalancing. Parameter values for other parameters are given in Table 4.2. The used number of simulations is given in Table C.3.



Then continuous rebalancing is required to obtain the optimal solution. Infrequent rebalancing results in an annual certainty equivalent loss up to approximately 1 basis point.

In the case with multiple risky assets the situation is the same for λ equal to zero. Then the CE loss is zero, because all wealth is invested in the risk-free asset. However the situation for λ equal to 0.5 is different from the single risky asset case. If all wealth is invested in risky assets there is still a need to rebalance between the different risky assets. For the used parameter values the maximum annual CE loss is approximately 2.6 basis points.

4.5.5 Risk-free Rate

A higher risk-free rate leads to a higher compensation of both the risk-free and the risky asset. The optimal weight invested in the risky asset remains unchanged. The higher drift terms cause a slightly higher annual CE loss. This is shown in Figure 4.7.

FIGURE 4.6: Yearly CE loss in basis points for different levels λ, with yearly rebalancing. Parameter values for other parameters are given in Table 4.2. The used number of simulations is given in Table C.3.

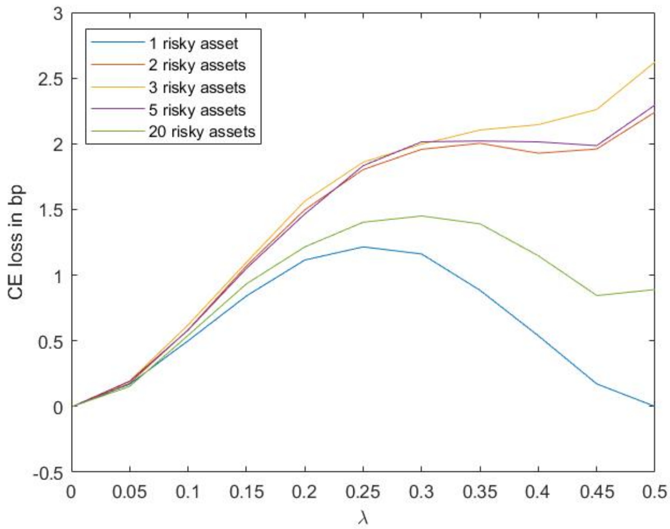
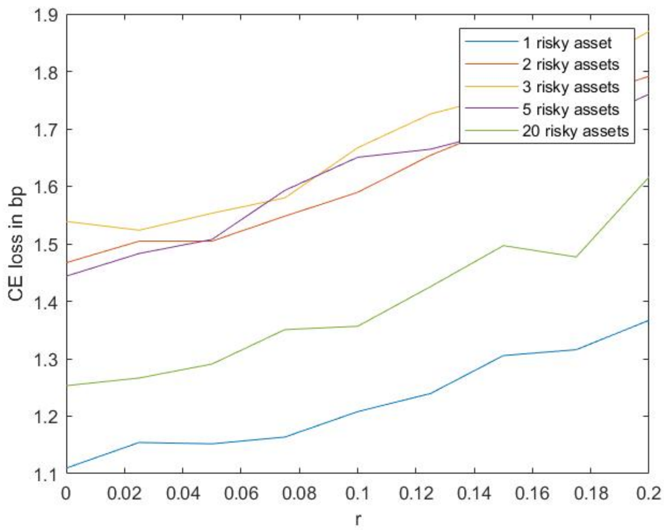


FIGURE 4.7: Yearly CE loss in basis points for different levels of the risk-free rate r, with yearly rebalancing. Parameter values for other parameters are given in Table 4.2. The used number of simulations is given in Table C.3.



5 Conclusion

Merton (1969) examines the problem of optimal portfolio selection in continuous time where asset returns are stochastic. It is shown that it is optimal to have a constant fraction of wealth invested in risky assets. Continuous trading is required to achieve this. When trading is done at discrete times instead of continuously the risky return trade off is suboptimal. The loss in utility due to this discrete trading can be expressed in a certainty equivalent loss. This is done using a utility function with constant relative risk aversion. For the used parameter values the certainty equivalent loss for trading at discrete times with intervals up to one year is found to be less than two basis points.

If two risky assets are available instead of one the CE loss is as expected larger than in the single asset case. However, for rebalancing periods of less then half a year the difference in CE loss is less than a basis point for the used parameter values and approximately 1.5 basis points if rebalancing is done only once a year.

The additional CE loss of the case with two risky assets compared to that of one risky asset comes from the weights within the risky part of the portfolio that diverge from the optimum. By not rebalancing continuously the relative weight of the risky assets will diverge from the optimum due to differences in return between both assets. In practice a loss of a basis point due to infrequent rebalancing may be negligible compared to other factors such as transaction costs.

Simulation results show that there is a linear relation between the rebalancing strategy and the yearly CE loss. Doubling the size of the time interval between rebalancing points approximately doubles the annual CE loss.

Simulations indicate that the CE loss converges to a value corresponding to that of the case with only one risky asset when more and more assets are added to the problem. In a portfolio with many risky assets the relative weight of one risky assets is relatively low. Therefore it's relative weight will remain close to the optimum even if the return on this asset is extremely high or low. With many risky assets there is also a diversification effect. It is more likely that a big increase in value is offset by big decrease in value of another risky asset if the portfolio consists of many assets. This keeps the total weight of the risky assets close to the optimum.

The level of correlation plays a role in how fast the CE loss converges to the single risky asset case. For a correlation of 0.5 between all risky assets, the CE loss already decreases when adding a third risky asset. However if correlation is low the CE loss may still increases when adding more risky assets. This is a result of the way the risky assets are modeled. They are generated such that the continuous rebalanced portfolio of risky assets has the same distribution as the single risky asset in the base case. The intuition behind this is that the single risky asset is regarded as an index. In the multi-asset case it is not possible to invest directly in this index. Parts of the index are available instead. The total amount invested in risky assets does therefore not depend on the amount of risky assets available. A result of this choice is that the volatility of an individual risky asset is higher for cases with more risky assets. Higher volatilities cause bigger deviations from the optimal weights.

The analysis can be extended in many ways in future research. For example, transactions costs could be added to the model, different dynamics of the risky assets could be assumed, different utility curves can be used and heterogeneous assets with different correlation matrices could be assumed.

A Volatility of Individual Risky Asset

The variance of a portfolio σ_p of n assets can be expressed in terms of the volatility of the individual risky asset. If the assumption is made that all risky assets have the same volatility. The portfolio volatility is given by:

$$\sigma = \sqrt{\sigma_P^2 / \left(\frac{1}{n} + \frac{n-1}{n} \rho \right)} \tag{A.1}$$

In order to obtain this relation the variance of a sum of n correlated variables will be given first. If the variables are correlated, then the variance of their sum is the sum of their covariances.

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^{n} X_i \right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{N} \text{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \text{Cov}(X_i, X_j) \end{aligned}$$

If all variables have equal variance σ^2 and the average correlation between two different variables is given by ρ, then the variance of the mean is given by:

$$\begin{aligned} \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^{n} X_i \right) \\ &= \frac{1}{n^2} \left(n\sigma^2 + n(n-1)\sigma^2 \rho \right) \\ &= \frac{\sigma^2}{n} + \frac{n-1}{n} \sigma^2 \rho \\ &= \sigma^2 \left(\frac{1}{n} + \frac{n-1}{n} \rho \right) \end{aligned}$$

Rewriting gives the volatility of a single variable in terms of the variance of the average, which is denoted by σ_I here.

$$\sigma = \sqrt{\sigma_I^2 / \left(\frac{1}{n} + \frac{n-1}{n} \rho \right)}$$

B Result Tables

In this section the annual certainty equivalent return of different rebalancing strategies is given. This is done for the single risky asset case, and for 2, 3, 10, 100 and 500 risky assets. The tables show the annual certainty equivalent loss in basis points, the 95% confidence interval in basis points and the certainty equivalent return in percentage. The used parameter values are given in Table 4.2.

TABLE B.1: One risky asset. Number of simulations = 35 million

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0045	(0.0022; 0.0068)	3.0454
Every Other Day	0.0083	(0.0051; 0.0116)	3.0454
Weekly	0.0214	(0.0164; 0.0264)	3.0452
Monthly	0.106	(0.0955; 0.1165)	3.0444
Quarterly	0.2835	(0.2655; 0.3016)	3.0426
Semiannually	0.5468	(0.5214; 0.5722)	3.04
Annually	1.1222	(1.0867; 1.1576)	3.0342

TABLE B.2: Two risky assets. Number of simulations = 25 million

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0075	(0.0043; 0.0106)	3.0454
Every Other Day	0.0117	(0.0073; 0.0161)	3.0453
Weekly	0.0309	(0.0240; 0.0377)	3.0451
Monthly	0.1309	(0.1167; 0.1451)	3.0441
Quarterly	0.3733	(0.3489; 0.3978)	3.0417
Semiannually	0.75	(0.7156; 0.7844)	3.038
Annually	1.5056	(1.4576; 1.5536)	3.0304

TABLE B.3: Three risky assets. Number of simulations = 20 million

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0065	(0.0030; 0.0101)	3.0454
Every Other Day	0.0123	(0.0073; 0.0173)	3.0453
Weekly	0.0322	(0.0245; 0.0400)	3.0451
Monthly	0.1302	(0.1141; 0.1463)	3.0442
Quarterly	0.3796	(0.3518; 0.4074)	3.0417
Semiannually	0.7485	(0.7094; 0.7875)	3.038
Annually	1.5251	(1.4705; 1.5797)	3.0302

TABLE B.4: Ten risky assets. Number of simulations = 15 million

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0146	(0.0107; 0.0185)	3.0453
Every Other Day	0.0182	(0.0127; 0.0237)	3.0453
Weekly	0.0318	(0.0233; 0.0403)	3.0451
Monthly	0.1269	(0.1093; 0.1444)	3.0442
Quarterly	0.364	(0.3337; 0.3944)	3.0418
Semiannually	0.6876	(0.6449; 0.7303)	3.0386
Annually	1.4146	(1.3549; 1.4744)	3.0313

TABLE B.5: 100 risky assets. Number of simulations = 1 million

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0093	(-0.0045; 0.0231)	3.0454
Every Other Day	0.0207	(0.0012; 0.0403)	3.0452
Weekly	0.0248	(-0.0055; 0.0551)	3.0452
Monthly	0.0966	(0.0337; 0.1595)	3.0445
Quarterly	0.3724	(0.2642; 0.4806)	3.0417
Semiannually	0.6474	(0.4951; 0.7997)	3.039
Annually	1.1414	(0.9285; 1.3543)	3.034

TABLE B.6: 500 risky assets. Number of simulations = 500 thousand.

Strategy	CE loss in bp	CI of CE loss in bp	CE return in %
Daily	0.0051	(-0.0142; 0.0244)	3.0454
Every Other Day	0.0381	(0.0108; 0.0654)	3.0451
Weekly	0.0206	(-0.0217; 0.0629)	3.0452
Monthly	0.1276	(0.0396; 0.2156)	3.0442
Quarterly	0.2694	(0.1175; 0.4214)	3.0428
Semiannually	0.6308	(0.4175; 0.8442)	3.0391
Annually	0.9539	(0.6572; 1.2506)	3.0359

C Sensitivity Analysis

This section shows tables used in the sensitivity analysis. Results are recalculated using a higher and lower parameter value for each parameter and keeping all other parameters at the default value. This is done for the single risky asset case, and for 2, 3, 5, 20, 100 and 500 risky assets.

The default parameters are given in Table 4.2. Table C.1 also shows the chosen low and high values that are used for the sensitivity analysis.

TABLE C.1: The default parameter values together with the low and high parameter value used in the sensitivity analysis.

Parameter	Low value	Default Value	High Value
r	0.01	0.02	0.05
lambda	0.1	0.2	0.35
sigma	0.1	0.25	0.35
rho	0.1	0.5	0.9
gamma	1.01	2	5

First the case with no rebalancing at all for one year is analyzed. Therefore maturity (T) is one and also the time step (dt) is one. No values within a year are simulated. Tables show the annual certainty equivalent return, certainty equivalent loss and a 95% confidence interval of the certainty equivalent loss for different rebalancing strategies. All quantities are given in basis points.

TABLE C.2: A sample long table.

Sensitivity test	Risky assets	CE loss in bp	CI of CE loss in bp	CE return in %
default	1	1.1292	(1.1082; 1.1502)	3.0342
default	2	1.5094	(1.4754; 1.5434)	3.0304
default	3	1.5245	(1.4832; 1.5658)	3.0302
default	5	1.4788	(1.4306; 1.5269)	3.0307
default	20	1.2298	(1.1510; 1.3086)	3.0332
default	100	1.0728	(0.9120; 1.2336)	3.0347
default	500	1.3022	(0.9484; 1.6559)	3.0324
σ low	1	-0	(-0.0000; 0.0000)	3.0455
σ low	2	0.0584	(0.0517; 0.0651)	3.0449
σ low	3	0.0519	(0.0433; 0.0604)	3.0449
σ low	5	0.0536	(0.0441; 0.0631)	3.0449
σ low	20	0.0295	(0.0189; 0.0400)	3.0452
σ low	100	0.0083	(-0.0024; 0.0190)	3.0454
σ low	500	0.0101	(-0.0029; 0.0232)	3.0454
σ high	1	3.0864	(3.0523; 3.1205)	3.0146
σ high	2	3.7461	(3.6926; 3.7996)	3.008

Continued on next page

Table C.2 – continued from previous page

Sensitivity test	Risky assets	CE loss in bp	CI of CE loss in bp	CE return in %
σ high	3	3.8724	(3.8076; 3.9372)	3.0067
σ high	5	3.8514	(3.7755; 3.9273)	3.0069
σ high	20	3.4411	(3.3149; 3.5672)	3.011
σ high	100	3.2834	(3.0231; 3.5437)	3.0126
σ high	500	3.1175	(2.5380; 3.6968)	3.0143
ρ low	1	1.1253	(1.1043; 1.1463)	3.0342
ρ low	2	3.1936	(3.1440; 3.2431)	3.0135
ρ low	3	4.7098	(4.6383; 4.7812)	2.9984
ρ low	5	6.6208	(6.5213; 6.7202)	2.9792
ρ low	20	7.7864	(7.5947; 7.9782)	2.9676
ρ low	100	3.9935	(3.7061; 4.2809)	3.0055
ρ low	500	1.7251	(1.2846; 2.1655)	3.0282
ρ high	1	1.1267	(1.1057; 1.1476)	3.0342
ρ high	2	1.1282	(1.0984; 1.1579)	3.0342
ρ high	3	1.1352	(1.0996; 1.1708)	3.0341
ρ high	5	1.1467	(1.1046; 1.1888)	3.034
ρ high	20	1.1493	(1.0751; 1.2235)	3.034
ρ high	100	1.1531	(0.9946; 1.3116)	3.0339
ρ high	500	1.0931	(0.7399; 1.4463)	3.0345
λ low	1	0.5046	(0.4907; 0.5185)	2.2705
λ low	2	0.61	(0.5886; 0.6313)	2.2694
λ low	3	0.6047	(0.5789; 0.6304)	2.2695
λ low	5	0.5992	(0.5689; 0.6294)	2.2695
λ low	20	0.5487	(0.4979; 0.5996)	2.27
λ low	100	0.479	(0.3731; 0.5849)	2.2707
λ low	500	0.6608	(0.4260; 0.8955)	2.2689
λ high	1	1.1388	(1.1170; 1.1606)	4.3302
λ high	2	1.9448	(1.9051; 1.9844)	4.3222
λ high	3	2.0794	(2.0306; 2.1282)	4.3208
λ high	5	2.01	(1.9535; 2.0665)	4.3215
λ high	20	1.4735	(1.3867; 1.5603)	4.3269
λ high	100	1.3096	(1.1398; 1.4794)	4.3285
λ high	500	0.9778	(0.6062; 1.3493)	4.3318
r low	1	1.1155	(1.0948; 1.1363)	2.009
r low	2	1.4842	(1.4506; 1.5179)	2.0053
r low	3	1.5497	(1.5089; 1.5906)	2.0046
r low	5	1.4987	(1.4510; 1.5464)	2.0051
r low	20	1.2342	(1.1563; 1.3122)	2.0078
r low	100	1.1819	(1.0228; 1.3409)	2.0083
r low	500	0.9203	(0.5678; 1.2729)	2.0109
r high	1	1.1636	(1.1420; 1.1852)	6.172
r high	2	1.5504	(1.5154; 1.5854)	6.1682
r high	3	1.6239	(1.5814; 1.6664)	6.1674
r high	5	1.5108	(1.4611; 1.5605)	6.1685
r high	20	1.2942	(1.2131; 1.3753)	6.1707
r high	100	1.1213	(0.9556; 1.2871)	6.1724
r high	500	1.3125	(0.9469; 1.6781)	6.1705
γ low	1	0.2774	(0.2627; 0.2922)	4.0577

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Table C.2 – continued from previous page

Sensitivity test	Risky assets	CE loss in bp	CI of CE loss in bp	CE return in %
γ low	2	1.0128	(0.9737; 1.0520)	4.0503
γ low	3	1.0504	(1.0013; 1.0995)	4.05
γ low	5	0.9391	(0.8832; 0.9950)	4.0511
γ low	20	0.5554	(0.4812; 0.6296)	4.0549
γ low	100	0.4651	(0.3416; 0.5887)	4.0558
γ low	500	0.2985	(0.0440; 0.5529)	4.0575
γ high	1	0.8808	(0.8692; 0.8924)	2.4202
γ high	2	1.014	(0.9964; 1.0317)	2.4189
γ high	3	1.0371	(1.0158; 1.0584)	2.4187
γ high	5	1.0105	(0.9855; 1.0355)	2.4189
γ high	20	0.9763	(0.9341; 1.0185)	2.4193
γ high	100	0.8895	(0.8014; 0.9775)	2.4201
γ high	500	0.925	(0.7305; 1.1194)	2.4198

TABLE C.3: Number of simulations used for Table C.2

Number of risky assets	Number of simulations
1	100 million
2	50 million
3	35 million
5	25 million
20	8 million
100	1.75 million
500	350 thousand

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