Longevity risk hedging with longevity swaps

Ruizhe Xue 231200

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Tilburg School of Economics and Management
Tilburg University

Supervisor:
Professor. B. Melenberg
Second supervisor:
Dr. Mitja Stadje

Abstract

Longevity risk is risk caused by pension policy holders who on average live longer than anticipated. It is the negative effect of changes in mortality rates. New possibilities to help pension providers to cover at least part of the longevity risk are longevity derivatives: products that depend on mortality indices but otherwise are similar to other financial derivatives. One of these products is the longevity swap. This thesis examines the possibilities that longevity swaps might offer in hedging longevity risk in a portfolio that also includes fixed income and equities. The results indicate that investors might allocate some part of their asset in longevity swaps to largely reduce the longevity risk and preserve most of the returns.

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Chapter 1 Introduction

Over a very long period there are two main potential treats to a pension plan: one is equity risk which is due to failure of investment strategy and the other one is interest rate risk which is due to an unexpected drop in interest rates. However, the life expectancy has risen continuously since the second half of the last century, so the other type of risk that a pension provider is facing has becomes more and more important. This third type of risk is the longevity risk, the risk that a pension provider has to provide more benefit to policy holders who on average live longer than expected. And it becomes a major threat to any pension provider including defined benefit pension plans, life annuity providers, and pension plan sponsors. They all need to cover the cost caused by the increase of life expectancy.

The policyholders receive periodical payments from their pension provider from the date of retirement until the date of their death. And without changing the retirement date the average times between these two dates is increasing significantly over the last century and is tending to increase in the future. As indicated by JPMorgan (2009) (Life Metrics Index Netherlands), the expected life expectancy for a 65 year old Dutch male and female was 13.37 and 13.9 years respectively in 1951. These the numbers have increased to 16.77 years for male and 20 years for female, at the year 2009. This shows that the pension payments need to be paid for a longer period for the policy holders. Although the increased lifetime can be predicted through applying scientific models, according to some research results (Cairns, Blake and Dowd (2006), Delwarde et al (2007)) that use the forecasted survival rates based on the past data will cause an underestimation of present values of the pension payments.

Normally the pension providers can deal with longevity risk at least in three different ways: first one is doing nothing; this will cause the pension provider to be unable to meet its liabilities; the second one is to hedge the risk by reinsurance; and the third one is to hedge the risk by longevity derivatives. Before the introduction of longevity derivatives into the market, reinsurance was the only choice for pension providers to cover longevity risk. But only a limited numbers of pension and annuity providers prefer reinsurance, because the costs for reinsurance are relatively high. And from the other side the capacity of reinsurers is limited;

one reason is that the longevity risk cannot be diversified away. So, most reinsurers do not prefer to accept any longevity risk and the availability of reinsurance for longevity risk is limited, all of this will provide a market for the longevity derivatives.

Longevity derivatives have been under development by researchers since the early 2000s. And the longevity derivative contracts could be traded at exchanges similar to financial derivatives. Although these longevity derivatives all can be constructed in theory just like the derivatives in the financial market, their implementation in practice is still problematic. The first longevity derivative introduced in the financial markets in the year 2004 is the BNP Paribas longevity bond. This bond was issued by the European Investment Bank with a maturity of 25 years, and the coupon payments were linked to the number of survivors for 65 year-old males from England and Wales. But the BNP Paribas longevity bond was redrawn from the market because it did not generate enough demand and needed redesign.

In April 2007 the world's first publicly announced longevity swap, Friends' Provident longevity swap, was issued by Swiss Re. In March 2007, JPMorgan launched LifeMetrics as an attempt to boost the development of the market for longevity derivatives. LifeMetrics is a toolkit for managing and measuring longevity risk' and intending to standardize the longevity derivatives' trading (Coughlan et al, 2007a). It makes that the determination of mortality rates and longevity derivative pricing can be based on indices.

This thesis examines whether longevity swaps offer opportunities to reduce longevity risk. Form this thesis, the theory of longevity swaps is worked out and the mortality rates are forecasted by using the US data. The main objective is to find out whether longevity swaps could reduce this risk using mean variance portfolio theory. In this paper the study of hedging of longevity risk will be conducted empirically. And the results indicate that investors might allocate some part of their asset in longevity swaps to largely reduce the longevity risk and preserve most of the returns. If the pension provider is risk averse then it's better to invest some proportion of the assets in the longevity swaps. If, however, the pension plan has a risk aversion consistent with a strategy of 33% in equity shares and 67% in fixed income, then the risk can be reduced by 8.7% from 0.0826 to 0.0754 by putting 6.5% of the assets in the longevity swap. The expected return is largely preserved, but risk is reduced. Therefore, longevity swap can be considered as an instrument to hedge the longevity risk.

Outline

The thesis is organized as follows. In the remainder of this thesis, the theory of longevity swaps is worked out and the main objective is to find out whether longevity swaps could reduce this risk using mean variance portfolio allocation.

We examine the background of the subject in chapter 2. First, an overview is provided with the relevant literature on the main theoretical concepts that are applied in later chapters. I also use data to provide empirical evidence of longevity and use the Lee-Carter model to forecast mortality rates in chapter 3. Chapter 4 further rooms on the longevity swap, to determine the mechanics of the longevity swaps. It discusses this product in detail, where particular attention is paid to the pricing of the swap. In chapter 5 I use efficient frontiers to see the benefits of allocate some part of asset in longevity swaps. The final section concludes and discuses extensions for future research.

Chapter 2 Background

This chapter introduces the concepts that are applied in later chapters. Section 2.1 presents a short description of the longevity derivatives market. Section 2.2 discusses longevity derivatives. Both the existing products in the financial markets and the products that theoretically could exist in the future are discussed. In section 2.3 the idea of using longevity swaps to hedge the longevity risk is introduced.

2.1 The longevity derivatives market

Some possible potential buyers of longevity derivatives are:

- Pension funds and life insurance companies that want to hedge the longevity risk;
- All kinds of investors including investment banks and hedge funds, that choose the longevity derivatives because the correlation between risk involved in longevity derivatives with other investment risk might be very low;
- Speculators and arbitrageurs;
- Governments, one reason are that they are exposed to longevity risk and the other reason

is that large companies can reduce the probability of bankrupted by their pension funds through the longevity derivatives market.

But in the market the number of financial agents who want to sell their longevity risk is more than the number of agents who willing to buy it. So, the mortality rates in the derivatives need to be set below the expected mortality rate as a compensation for the investors.

2.2 Longevity derivatives in theory

Longevity derivatives are financial products whose payoff is related to a mortality rate or mortality index. These products are innovated in order to hedge the longevity risk of pension funds. And through investment in the longevity derivatives the pensions fund can cover at least part of the longevity risk.

After a description of the current longevity derivatives market, the products that can be constructed in theory are examined. In theory, many features from financial derivatives such like bonds, options, forwards, swaps and swaptions can be translated to the longevity derivative market. The longevity bonds could vary in structure such as principal-at-risk longevity bonds and coupons based longevity bonds, and vary in maturity such as fixed maturity and stochastic maturity longevity bonds. And most financial options can be translated into the corresponding longevity options. A longevity swap consists of exchanging a series of fixed payments for a series of variable payments that dependent on the realized longevity at predetermined maturity dates. For each payments at maturity date t = 1, 2, …, T, the fixed amount $K(t)$ which depends on the predicted mortality rates is swapped for a variable amount $S(t)$ that is depends on the realized mortality rates. For example firm A pays firm B amount $K(t)$-$S(t)$ when $K(t)$ is higher than $S(t)$ and B pays A amount $S(t)$ -$K(t)$ when $S(t)$ is higher than $K(t)$. Hence, firm A needs to pay a higher amount to firm B when more people are lives longer than expected.

2.3 Hedging with longevity swaps

A number of longevity derivatives have been developed to help pension providers protect the longevity risk. One such derivative is the longevity swap, which swaps the realized mortality rate for the estimated mortality rate. The cash flows of a longevity swap are depending on the difference between the realized survival rate in a specified age group, and the expected survival rate according to the corresponding mortality index. Such a longevity swap is intended to ensure that any unexpected increase in life expectancy for the policy holders will not threat the funding ratio of the pension provider. The questions are what exactly should be hedged, i.e., how much of the asset should be invested in the longevity swaps and at what price. We may answer these questions by using modern portfolio theory: try to find the optimal asset allocation by considering the trade-off between risk and return. We can treat longevity as an asset class, like swaps or other instruments traded in the financial market, in order to quantify the corresponding risk and return associated with a longevity swap.

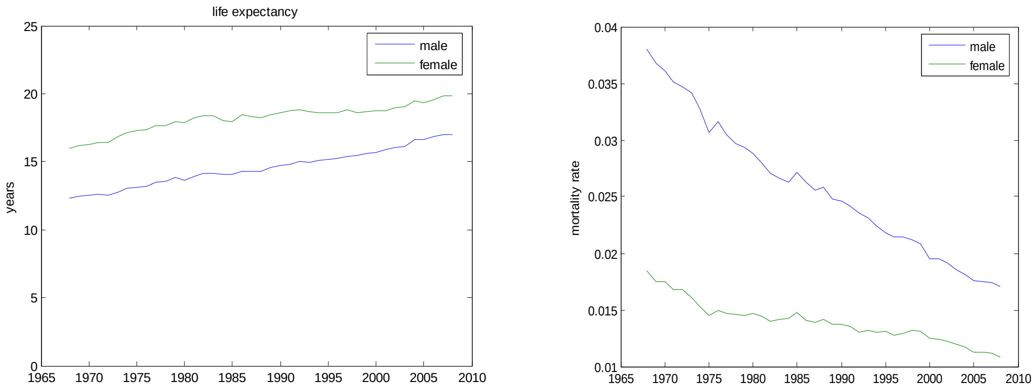
Chapter 3 Empirical evidence

This chapter begins with a short overview of longevity in the US, and provides review of the Lee-Carter model to forecasting mortality. Then we use the Lee-Carter model to forecast mortality rates, and compare a few alternative mortality forecast models.

3.1 Longevity in the US

As indicated by JPMorgan (2009) (Life Metrics Index US), the expected life expectancy for a 65 year old US male and female was 12.3 and 15.97 years respectively in 1966. And the numbers has increased to 17 years for male and 19.85 years for female, at the year 2009. In the figure 1 we illustrate the process of the improvements in longevity from two aspects: one is life expectancy and the other is mortality rate. The left figure shows the life expectancy for a 65 year old person. The figure has indicates that the life expectancy is increasing over the period 1966-2009 both for US male and female, where the increase was a little stronger for males as the life expectancy for males is steeper. The right figure shows the mortality rates over the same period. As we can see that the improvement in longevity is significant, especially for the males which the mortality rate in 2009 is only half of the mortality rate in year 1966.

Figure 1: Longevity improvements in the US, 1966-2009



The left figure shows the life expectancy in the US who are 65 years old for both men and women. The right figure shows the development of mortality rates over the same period.
Source: JP Morgan Life Metrics Index US

3.2 Mortality Forecasting

Although the models to formulate the mortality rates for pension provider and other life insurance product have been developed over hundreds years, these models are normally assumed to be deterministic or based on specified assumptions and scenarios. But it is very important from the pension provider to incorporate the improvements for future longevity in the determination of the premium policy to avoid potential losses. Since then some methods are derived for the extrapolation of past trends of the mortality rates into the future and among these models the most frequently used methods are the Lee-Carter model. The Lee-Carter model is introduced by Lee, R.D. and Carter, L. (1992) by considering the improvements in longevity to forecasting the future mortality rates, and their model is focused on incorporrate the history mortality data as good as possible. In this thesis I use the Lee-Carter (1992) model to forecast the future survival probabilities. Because Lee-Carter model is widely used and are considered as the leading statistical mortality model, and Lee-Carter model has been shown to fit the data for many countries relatively well. And most importantly, one advantage of the

Lee-Carter model is that it is a relatively simple model. The log-mortality rates are given by:

$ln(m_{x,t}) = a_x + b_x * k_t + \varepsilon_{x,t};$

Where $m_{x,t}$ is the central death rate for age *x* at time *t*. The a_x coefficients are a vector age-specified component which describes the average age profile. The b_x tell us which rates each age group changes in response to changes in the index k_t. The coefficients b_x are normalized to sum to one. The mortality index k_t tracks the general changes in mortality rates over time and is normalized to sum to zero. And as noted, the mortality index k_t has the same movement as the mortality rates. When the mortality index k_t is linear over time, the mortality rates at each age group will changes at its own exponential rate.

The most distinguishing feature of Lee-Carter model is the use of a stochastic process to model the uncertainty about the future. Lee and Carter (1992) use least squares to estimate the coefficients *a*, *b* and *k* with U.S. mortality data from 1933 to 1987 respectively. They first estimate coefficient *a* by averaging the log of mortality rates over time, and estimate coefficients *b* and *k* by using a singular value decomposition (SVD) method. The SVD method is a factorization of a matrix and here is a method to approximate a matrix as the product of two vectors. And in the Lee-Carter's original paper they use the data consisted of mortality rates up to ages 80-84 and 85+ year old.

The second distinctive feature of the Lee-Carter model is that it reduces the mortality indexes which are time dimension to a single index *k*. They model the estimated time-dependent mortality index *k* as a stochastic process, and then based on the estimated index they use statistical time series methods to forecast this index *k*. Lee and Carter notice that the index behaves like a random walk with drift through application to U.S. mortality data, except for the flu epidemic of 1918. Lee and Carter (1992) have described the evolution process of mortality index k_t over time is a random walk with drift:

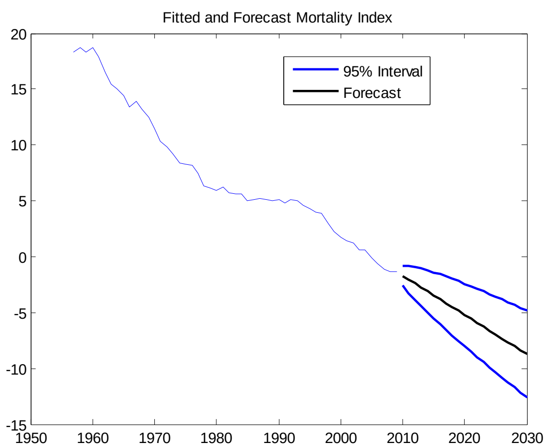
$k_t = u + k_{t-1} + \varepsilon_t$

where u is the drift parameter which captures the average change in *k* each time period in order to forecasts the changes in the mortality index in the long run, and the error term ε_t is normally distributed with mean zero and standard deviation σ. We can use this model to project the stochastic mortality index. When u is negative the mortality index is tend to improve over time and the variance of mortality index k_t increases with the forecast horizon.

I apply the Lee-Carter model to forecasting the US mortality rates. The mortality rates data for U.S. population from 1956 to 2009 using conventional five-year age groups is obtained from the Human Mortality Database.

1) I estimate coefficients a_x simply as the average value of the logarithm of mortality rates, averaged over all time *t* , for each age group *x*. $\hat{a}_x = \frac{1}{n} * ln(\prod_0^t \hat{m}_{x,t})$ and $\hat{m}_{x,t} = d_{x,t}/l_{x,t}$, where $d_{x,t}$ = total observed deaths; *2*) I use the singular value decomposition (SVD) method to decompose the vector $[ln(\hat{m}_{x,t}) - a_x]$ then obtain the estimated vectors $\hat{b}_x$ and $\hat{k}_t$ *3*) And then re-estimate k_t to let $\sum d_{x,t} = \sum l_{x,t} * \exp(\hat{a}_x + \hat{b}_x * \hat{k}_t)$ *4*) According to the estimated mortality index $\hat{k}_t$, and use the statistical time series model ARIMA(p,d,q) to forecast future mortality index $\hat{k}_{n+s}$, for s>0. And here I choose the ARIMA(0,1,0) model. The raw estimates of a_x, b_x are inserted in *Table 1* and the forecasted mortality index k_t is in the *Table 2* in *Appendix*. *5*) The final step in the forecast is to construct a life table from the forecasted age-specific mortality index $\hat{k}_t$ and once $\hat{k}_t$ is obtained I combine it with the estimated vectors $\hat{a}_x$ and $\hat{b}_x$ to obtain a forecast of age-specific mortality rate $\hat{m}_{x,t_{n+s}}$. That is $\hat{m}_{x,t_{n+s}} = \exp(\hat{a}_x + \hat{b}_x * \hat{k}_t)$ for $s > 0$. This can be done by using the standard techniques and here I list part of forecasted age-specific mortality rate for the selected years in *Table 3* in *Appendix*.

Figure 2 Forecasts of Male Mortality Index k with confidence interval



The mortality rates index for U.S. population from 1956 to 2009 using conventional five-year age groups is obtained from the Human Mortality Database. And the mortality forecast from 2009 to 2029, With 95% Probability Interval and the Model is (0,1,0).

Alternative models

Although Lee-Carter model has gained importance because it is relatively simple model and the quality of forecast is well, but it also has some disadvantages. As Cairns et al (2007) listed several drawbacks of the Lee-Carter model in their paper as follow:

- The Lee-Carter model is a one-factor model and the mortality improvements at all age groups are perfectly correlated.

- If a cohort effect is observed in the past then the model probably result in a poor fit to the historical data.

- The uncertainty in future mortality rates is proportional to coefficients b_x.

- The Lee-Carter model may result in not sufficient smoothness across ages or years.

After the original work by Lee and Carter (1992) there is a large literature extending the Lee-Carter model either on additions or modifications, such like Brouhns, Denuit and Vermunt (2002), Cairns, Blake and Dowd (2006), Renshaw and Haberman (2003,2006), De Jong and Tickle (2006) and Delwarde et al (2007). But as noted that most of these new models only focus on the one of the problems of the original Lee-Carter model and still have other disadvantages.

Brouhns et al (2002) has developed a model by allowing for the Poisson error assumption and use maximum likelihood estimation. They apply their model to Belgian data. Renshaw and Haberman (2003) extended the Lee-Carter model by including an additional non-linear age factor and uses the England and Wales data. Their analysis suggests that even with a non-linear age factor the Lee-Carter model does not fit the data well and the same with a Poisson error assumption. After their effort in 2003 Renshaw and Haberman (2006) proposed the first model that incorporated the cohort effect by including a new factor -year of birth that has an impact on the improvement rate of longevity and this cohort factor is found to be significant by applying the mortality data of UK. All these models developed after Lee and Carter (1992) have many parameters and can be improved. Cairns, A.J.G., Blake, D., Dowd, K.(2006b) proposed a simpler model by applying a two- factor model, similar to the Lee-Carter model . Cairns et al(2007) have conducted a quantitative comparison of stochastic longevity models using England and Wales and US mortality data. And by adding an age effect, a quadratic age effects and a cohort effect they have demonstrated that these models can fit the data well. And all these empirical studies have suggested that the model for longevity improvement need more than one factor.

Chapter 4 Longevity Swaps

After a short introduction in chapter 2, longevity swaps are discussed more extensively in this chapter. The chapter starts off with a detailed explanation of the mechanics of the swap. In section 4.2, one pricing method is chosen that is found in the literature to pricing the longevity swaps which will be used later.

4.1 Longevity swaps mechanics

The most important feature of a longevity swap is that at maturity time t both fixed and random payment depend on the mortality index which either includes the mortality rates of the entire population or a certain group of the population. At the beginning of the contract, two parties agree to exchange a series of fixed payments linked to a predicted mortality rate for a series of variable payments that are dependent on the realized mortality rate at predetermined maturity dates. The transactions at each maturity time t is the difference between the fixed amount $K(t)$ and the variable amount $S(t)$, and paid by the party with the higher amount at time t. For example, a pension provider with a certain number w of policyholders from the same age group of 65 year old can enter the fixed side of a longevity swap contract. The policyholders receive 1 euro from the pension provider each year if he still alive at year t, so $S(0)$ is w and $S(t)$ will decrease with the number of policyholders still alive at each time t. And the pension provider need to pay an amount $K(t)$ at each maturity date that the number of policyholders is expected to be alive at time t which is set at the beginning of the contract regardless of the realized number. The counterparty in the variable side needs pay the difference between the fixed and the variable amount when the realized mortality rate is lower than expected, and the pension provider needs to pay to this counterparty when the expected mortality rate appear higher than realized one.

4.2 Longevity swaps pricing

In order to determine the return of the investment in longevity swaps I need to know the price of it. In this section I consider the determination of the price of the swap, which is a premium added to the fixed side of longevity premium. The premium (price) is proportional to the fixed side payments and is set such that the present value of two parties of the swap is equal at the initiation of the swap, so the value of the contract is 0 at time *t*=0.

According to Heleen Westland (2009) there are two different procedures widely used to determine the value of the premium (price):

1) The first one is the Wang transform method, which transfers the current expected variable payments into the risk neutral equivalents and determines the premium (price) by solving the equation that both sides of the swap contract are equal. Dowd et al (2005) has presented the Wang transform procedure as a pricing method in the determination of the prices for several longevity swaps in practice.
2) The other one is the Sharpe Ratio method that is based on similarities with the capital market, which imposes a Sharpe Ratio and determines the premium (price) of longevity swap by solving that the present value of the fixed leg payments and premiums (prices) equal the present value of the floating leg payments. Loeys et al (2007 JPMorgan) has proposed the Sharpe Ratio method, which does not need to use a conversion of mortality rates and a specific insurance market price of risk. The basic idea of their method is the Sharpe Ratio, which theoretically is equivalent to the parameter for the market price of risk. The corresponding risk premium should be higher when the correlation between longevity swaps and other assets is higher or when the volatility of the longevity swaps is higher. But the required Sharpe Ratio for longevity swaps should be lower than in equity

market, as the correlations between the longevity risk with other risks are relatively low. JP Morgan (June 5,2009) shows that the correlation between the changes of mortality rates in UK and Japanese to equities is low and the correlation between changes of the mortality rates in US and Dow Jones Industrial Average (DIJA) returns is also very low that are close to zero, which ranging between -0.2 and 0.2.

4.2.1 Wang transform

The Wang transform method proposed by Wang (2002) is essentially focused on not normally distributed variables and determines of a series of standardized probabilities, and these probabilities are shifted by an amount ¸which is known as the market price of risk. In the last by using the standard normal distribution the shifted probabilities are transformed back.

The basic idea of Wang transform in Wang (2000) is to consider a financial asset over a time horizon $[0,T]$ and $X=\{X_t\}$ is denote as this financial asset's value at future time $t=T$. The corresponding cumulative distribution function of X is $F(x)=\Pr\{X\leq x\}$ and Wang (2002) has proposed the following transform which can be used as a pricing method:

$F*(x) = \Phi(\Phi^{-1}(F(x)) + \lambda);$

where Φ is the standard normal cumulative distribution, λ is the market price of risk. Te market price of risk reflects the level of systematic risk and Hansen and Jagannathan (1991) have indicated that the market price of risk can be defined as the ratio of the standard deviation of a stochastic discount factor and the mean of this stochastic discount factor. The Wang transform will transform cumulative distribution function $F(x)$ of an asset X to a risk-adjusted cdf $F*(x)$.

The transformation can be applied here for pricing the longevity swaps and the distribution of the estimated survival rates (p) is substituted for the original X:

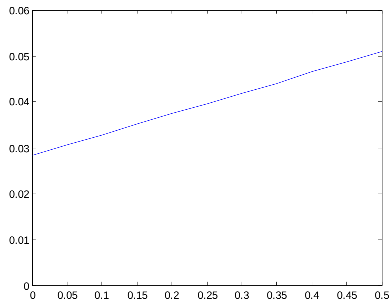
$F*(p) = \Phi(\Phi^{-1}(F(p)) + \lambda)$

The first procedure is to determine the appropriate market price of risk, and then is the price of a longevity swap which can be derived from the present value of the fixed and floating payments.

Market price of risk

The key point to determine the premium by using the Wang transform is to choose the appropriate market price of risk.

Figure 3 Premium for different level of market prices of risk



The figure shows the level of premium of the longevity swap, which corresponding to different values of the market price of risk (between 0% and 50%).

As we have calculated the mortality rates in chapter 3, we can get the corresponding survival rates: $p=1-m$. By inserting different value of λ (between 0% and 50%) into formula (1), we obtain the transformed survival probability, and the premium of longevity swap can be found by solving that the present value of the fixed leg payments and premiums equals the present value of the floating leg payments. For example when λ is 0 the premium $\pi= \frac{\text{PV } [S(t)]}{\text{PV } [k(t)]} =$ 2.83%. (Where PV $[S(t)]$ is present value of the floating payments and PV $[k(t)]$ is present value of the fixed payments and we simply use the expected survival rate as the fixed payment and use the transformed probability as floating payment). The corresponding premium when the market price of risk is changed between 0 and 0.5 is shown in Figure 3. And the figure indicates that increasing the market price of risk will cause a higher level of premium for the longevity swap. This is because that by the definition of transformed probability in the Wang transform, a higher level of market price of risk will cause the transformed probability to be higher. So the present value of the floating side of the longevity swap will be higher as the premium level need to make the present values of both sides equal.

4.2.2 Sharpe ratio method

In this paper I choose the Sharpe ratio method and I will use the method and assumptions derived by Loeys et al (2007 JPMorgan) to determine the price of the swaps. The method proposed by Loeys et al (2007 JPMorgan) determines the payments of the fixed leg based on the similarities with other financial investments. The annualized expected return equal to the difference between the expected mortality rate $(\mathrm{m_e})$ and the realized mortality rate $(\mathrm{m_r})$, divided by *t* (the number of years when the payment is realized). Theoretically, the Sharpe ratio (SR) is equal to $\frac{E(R-R_f)}{\sqrt{VAR\,(R)}}$, where R is the asset return and R_f is the return on a benchmark asset so in this case SR=$\frac{\text{Expected annual return}}{\text{annualized risk}}=\frac{(m_e-m_r)/t}{\sigma}$. The annualized risk of a forward equals the standard deviation of year-on-year percentage changes in the mortality rate (σ): the historical rate of changes in mortality rates (σ_r; the relative rate) times the expected mortality rate($\mathrm{m_e}$): $\sigma = \sigma_r * m_e$.

From $m_r - m_e$=-(SR*t*σ_r)* m_e and $\pi_t=(m_r - m_e)/m_e$,

So the resulting premium at maturity time *t* is: π_t =SR*t*σ_r;

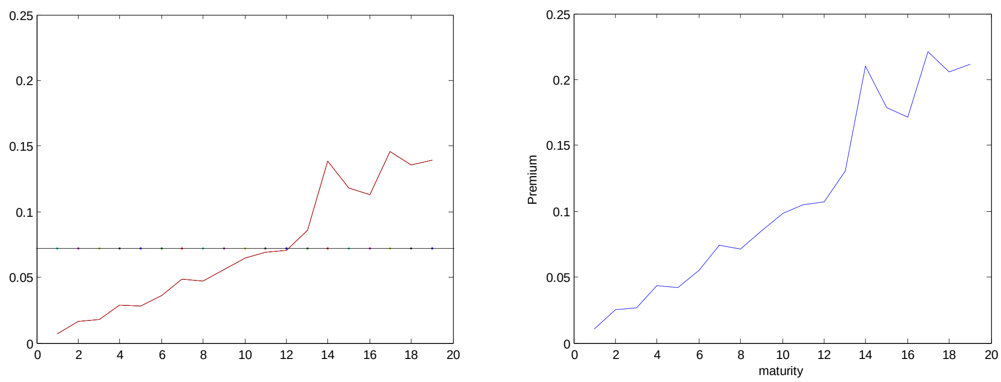
Loeys et al (2007 JPMorgan) assumed that the annualized Sharpe ratio is around 0.25.

Because the longevity swap consists of a series of longevity forwards, the premium (price) of the longevity swap is set separately for each maturity date. The present value of fixed payments of the longevity forwards is assumed to equal the present value of the future floating payments. The payment at the maturity date is determined by the difference between the realized survival rate and (1 + π_t) times the expected survival rate. When the realized rate is higher than (1 + π_t) times the expectation, the short counterparty pays the difference times the notional amount; the long counterparty pays when the realized rate is lower.

As an illustration I use a longevity swap with a maturity of 10 years with only one future payment at maturity date for the age group of 65-year old US males. The standard deviation of the historical rate of changes in mortality rates for age at maturity date is 75 year olds has been 2.5907% of the mortality rate. The corresponding price (risk premium) is determined as follows:

π_{10} = *0.25 * 10 * 0.0*25907 = *0.064766=6.4766%.*

***Figure 4:Swap prices for each maturity dates*(*Left: SR=0.25, Right: SR=0.38*)**



The left figure shows the level of premiums of a 20 year longevity swap determined by using the Sharpe Ratio method with the corresponding Sharpe ratio equal to 0.25 and its average premium, and right figure shows the premiums with the Sharpe ratio equal to 0.38.

In this paper I only consider the longevity swaps with maturity of 20 years. A longevity swaps with 20 year maturity is considered as a series of one-period swaps with different maturities from 1 to 20 years. The resulting premiums per period I calculated are based on the data from the JP Morgan LifeMetrics database and use the formula from Loeys et al (2007). The corresponding results are shown in Figure 4. The average premium is 7.1876% indicated as horizontal line in the figure. As show in the Figure 3 the premiums (price) are increasing with the maturity years, one reason is that the uncertainty in mortality rates will increase as well.

The procedure proposed by Loeys et al (2007) generates some problems. One of the main problems of the Sharpe Ratio method is that it is hard to determine the premium (price) of a longevity swap on people older than 90 years, because the volatility σ of the historical data for the higher ages is large and the corresponding premiums for old ages will grow rapidly with maturity years. So, the Sharpe Ratio method cannot be used to determine the premium of the longevity swaps for age group of 65 with a maturity longer than 34 years as there is no historical data to determine the volatility σ available from the database. Here, I only consider longevity swaps with maturity of 20 years for the age group of 65, because this enables to price the longevity swaps at the older ages and the longevity swaps whit the 20 year maturity horizon will probably be more liquid than the maturity with 40 year horizon.

4.3 Comparison

From the results of two presented pricing methods we can see different premiums (price) for the longevity swap. Because the premium obtained by Sharpe ratio method is a single payment swap for each time *t* during the maturity year, so the premium is increasing over the maturity. And the Wang transform method results in a constant premium over the maturity year of the longevity swap because it use only one market price of risk parameter. For a longevity swap on the 65-years-old US policyholder with a maturity of 20 years, the premium resulted by the Sharpe ratio method on average is 7.1876% and the premium resulted by the Wang transform method ranging from 2.83% to 5.1% depends on the different level of market price of risk. From the results of both premiums we can see that the premium level from the Wang transform method can never reach the average of the premium from Sharpe ratio method. It is only possible when the market price of risk is unrealistically high for the premium form Wang transform method to reach the Sharpe ratio method. And the results from the Sharpe ratio method and the Wang transform method will equal when we assume a Sharpe ratio was 0.0697, which is quite far from the Sharpe ratio we used in this paper.

Chapter 5 Longevity Risk Hedging With Longevity Swaps

Although pandemics influenza, wars or other serious illnesses are the major concern of unexpected changes for the improvements in mortality rates, the decreasing of the mortality rates has been a continuous process over the past several decades and is still a long term decreasing trend of mortality development in the future. These improvements in the long term trend of the survival probabilities have highlighted the relevance of longevity risk hedging. Normally the expected improvements in longevity can be forecasted either based on historical data or consulting expert opinions. Longevity swaps can be seen as a valid instrument to hedge the longevity risk and there exist a large number of potential buyers of the longevity swaps such as pension providers and government as they are exposures to the longevity risk.

This chapter applies the mean variance portfolio theory to portfolio allocation, including longevity swaps. The mean-variance efficient frontier is introduced in section 5.1. Section 5.2 discusses the effect of applying longevity swaps to the asset allocation, starting without longevity swaps, and then a particular portfolio of swaps. The results are presented in section 5.3.

5.1 Mean-variance efficient frontier

The mean-variance optimization (MVO) method as developed by Markowitz (1952) opened a new era for modern finance. The framework is the basis of modern investment theory which aims to find the set of optimal portfolios. The analysis of the Mean-variance optimization theory is important for both investors and researchers in finance. The MVO theory suggests that mean-variance efficient portfolios play an important role in portfolio allocation that can minimize the portfolio risk for a level of acceptable return for investors.

In order to compute the mean-variance efficient frontier and use the information it provides to select the unique optimal portfolio for a given level of risk (or return), we have to know the stochastic mechanism generating the returns for a given set of assets. In its standard formulation, the mean-variance efficient frontier model makes the assumption that the assets' returns are independently over time and identically distributed (i.i.d.) as a multivariate normal distribution $N(\mu,\Sigma)$, where μ is the vector of the securities' mean returns and Σ is the covariance matrix of the securities' returns. However, the uniqueness of the mean variance efficient frontier (MVEF) solution (i.e., the uniqueness of the optimal weights of the securities in the portfolio given a specific level of risk or return) depends on the implicit assumption that the inputs μ and Σ are known, whereas they must be estimated. In practice however, the multivariate normal i.i.d. model is widely used.

Consider an investor facing the decision what proportion of his or her wealth should be allocated to the different types of available investments on the market, such as bonds and equities. Each of these types of investments exhibit different risk/return characteristics under different economic conditions. Another complexity of the problem is that the above investments also exhibit correlation to others. Mean variance optimization can help to answer the above question to minimize the portfolio risk for a level of acceptable return. The three inputs are: returns on each investment, standard deviation of returns on each investments, and covariance between the investments.

The following is the MVO problem:

Minimize the portfolio variance $\min_{\omega}\ \omega' \sum \omega$

Subject to $\begin{cases} \omega'\mu = \mu_T \\ \sum_{i=1}^{n} \omega_i = 1 \\ \omega_i > 0 \end{cases}$

Where ω is a vector of portfolio weights, $\sum$ is the covariance matrix of investment returns, u is vector of investment returns, μ_T is the total portfolio return and n is the number of different investments in the optimization algorithm.

5.2 Mean variance portfolio and longevity hedging

Data

First, I focus only on fixed income and equity. For the fixed income we consider bond and bills and for the equity we consider both U.S. equity and international equity for which we choose the S&P 500 and the MSCI EAFE, respectively. MSCI EAFE Index consists of the 22 developed country market indices in Europe, Australasia and Far East which is a free market capitalization index that is used to measure the performance of equity markets from these developed markets, but excluding the equity market for US and Canada. For the data of these four assets I consider a sample period of 20 years which from 1991 to 2010. And the corresponding data is shown in *Table 5* in *Appendix.*

Secondly, I add the longevity swap to the portfolio. And I employ the following methodology to obtain the correlation between different assets.

Methodology

In this paper I use vector-autoregressive (VAR) models to analyze the correlation between different assets (S&P 500, MSCI, Bond and Bill) because the VAR model describes the evolution of the including variables from their own history dynamically. A general VAR(p) model for a k-dimensional vector Y_t with lag p is given by

$Y_t = \delta + \theta_1 Y_{t-1} + \ldots + \theta_p Y_{t-p} + \varepsilon_t$

Where θ_j is k*k matrix and ε_t is a k-dimensional vector of error terms with conditional mean zero and conditional covariance matrix Σ, conditional upon information available at time t-1. We estimate a vector autoregressive model using the ordinary least squares method. For the general case from residuals of each k equations, $e_{1t},\ldots,e_{jt}$, we can estimate the (i,j)-element in Σ as $\hat{\sigma}_{ij} = \frac{1}{T-p}\sum_{t=p+1}^{T} e_{it}\, e_{jt}$,

so that $\hat{\Sigma} = \frac{1}{T-p}\sum_{t=p+1}^{T} e_t\, e_t'$, where $e_t = (e_{1t}\ \ldots e_{jt})'$, T is sample period and p is the lag length.

Empirical results

We want to know whether the longevity swaps can hedge the longevity risk, and we can combine the mean-variance portfolio theory with longevity hedging. The efficient frontier shows the association of risk and return with the optimal allocations of investment assets. So, it provides which asset allocation can generate the greatest return for the corresponding level of risk. Because in an empirical application of is difficult to determine which value of lag p is most appropriate, one reasonable way to solve this problem is first to use different values of p to estimate a VAR model and then choose the most appropriate values for lag length p based on the Akaike information criteria (AIC).

Without longevity swaps

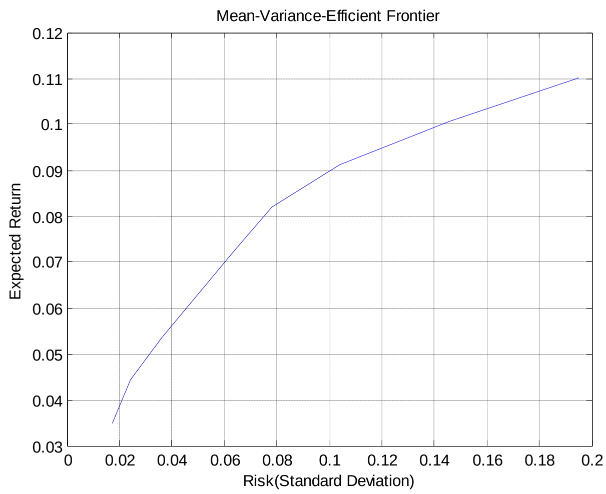
Firstly I choose general case; the VAR model consists of four equations with p lags. The returns are from S&P 500, MSCI, Bond and Bill. Then I use AIC to determine the appropriate values for p, where the AIC statistic is: AIC = (–2*LLF) + (2*$NParams$). Where LLF is values of optimized loglikelihood objective function that associated with the corresponding estimates of parameters, and *NParams* is Number of estimated parameters and the loglikelihood function (LLF) with the maximum likelihood estimates of the model parameters is used here to calculate the AIC value. Usually, the model with the smallest AIC value is preferred, so I found that a value of 3 for *p* will be chosen. In order to get the corresponding residuals I estimate each equation by applying the different estimator: the OLS estimator, then I obtained the covariance between four assets and the output is shown in *Appendix* where Output 1.1 contains the parameter estimates and covariance matrix. Output 1.2 contains the corresponding standard errors of the parameter estimates which are maximum likelihood estimates. The efficient frontier constructed without longevity swaps is shown in the Figure 5. I consider that a portfolio is efficient if we cannot obtain a higher return without increasing the associated risk (standard deviation).

Table 1 Annualized Statistics

				Covariance	**Matrix**	
	Return	Stdev	Bill	Bond	SP	MSCI
Bill	0.0342	0.0185	0.0009396	-0.0002497	0.001058	0.000598
Bond	0.0709	0.0962	-0.0002497	0.005524	-0.001401	-0.003015
SP	0.1101	0.1951	0.001058	-0.001401	0.01999	0.016249
MSCI	0.0803	0.2076	0.000598	-0.003015	0.016249	0.02038

The annualized statistics of the four assets, including returns and standard deviations from S&P 500, MSCI, Bond and Bill for the sample period of 20 years. And the correlation between each of these four assets is obtained by VAR model with a leg value of 3.

Figure 5: Efficient frontier without longevity risk



The efficient frontier constructed without longevity swaps by including SP, MSCI, Bond and Bill.

The Table 1 has illustrates the covariance between each asset, and indicates that the bill and bond have negative correlation and have positive correlation with SP and MSCI, but the correlation is higher for Bill and SP (0.001058) than Bill and MSCI (0.000598). The results are as expected only except that the Bonds have demonstrated negative correlation with the other three assets. Several studies have shown that the correlation between stock and bond returns exhibits considerable time-variation such as (Cappiello, Engle, Sheppard, 2003; and Ilmanen, 2003). Although stock and bond prices tend to move in the same direction, some recent studies (Magnus Andersson, Elizaveta Krylova, and Sami Vähämaa 2004) have also documented sustained periods of negative correlation to support the results I derived here. One macroeconomic variable may affect the stock-bond return correlation is inflation. An increase in expected inflation tends to raise discount rates, and hence, is inevitably bad news for the bond markets. However, the impact of increasing inflation on stock prices is ambiguous, as both the expected future cash flows and the discount rates are likely to be affected. The other one is that the financial market dynamics have an important impact on the relationship between stock and bond returns. For instance, in periods of financial market turbulence, the equity risk premium demanded by investors to hold stock may increase relative to the term premium for bonds. Stocks and bonds tend to move in the same direction during periods of high inflation expectations, and the negative stock bond return correlation seem to coincide with the lowest levels of inflation expectations. Magnus Andersson Elizaveta Krylova, and Sami Vähämaa , (2004) find that expected stock market uncertainty as measured by implied volatility is negatively related to the correlation between stock and bond returns. Figure 5 indicates that when the risk of the portfolio is 0.06 the corresponding expected return is 0.07. When the portfolio's risk level is 0.1 the corresponding expected return is 0.09.

With longevity swaps

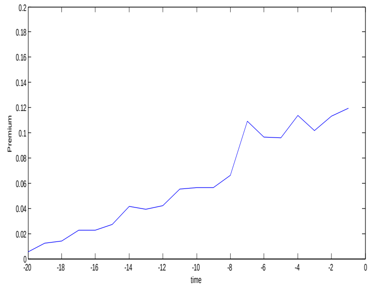
The key point for employing the mean-variance theory to analyze the asset returns by including investment in longevity swaps is how to determine the return and volatility of the longevity swaps. In order to determine the return we need to know the price of the longevity swaps, also both the fixed and random variable payment from both sides of the contract. As the price for longevity swaps in section 4, and to analyze the effect of including the longevity swaps that with a maturity of 20 years is applied to the standard swap contract. It concerns a swap on US males aged 65 in 2009.

The fixed leg payments of the longevity swap is determined using the life tables estimated from the Lee-Carter model and subtracting the premium. This premium is proportioned to the expected mortality rate. The cash flows arising from the longevity swap contract is from the difference between the estimated mortality rates based on the Lee-Carter model minus the premium and the realized mortality rates. I assume a premium of 7.019%, as the price is determined by the Sharpe Ratio method in chapter 4. This premium indicates that the counterparty only pays when the realized mortality rates are more than 7.019% under the estimated mortality rates, and the pension provider needs to pay when the realized mortality rates are more than the estimated mortality rates.

After the determination of the price, the fixed payment and the random payments of the longevity swaps, we can derive the return and volatility of the longevity swaps investment. In order to construct the mean variance efficient frontiers for portfolios with longevity swaps we need the corresponding covariance with other investments. I use the same methodology to calculate the correlation between longevity swaps and the other four different assets. But in order to construct a VAR model by including the return of longevity swaps we need to know the past 20 years' returns. Since we know the realized and forecasted mortality rates for the past, we use the Sharpe ratio method discussed in Chapter 4 to determine the premium for the past 20 years as shown in Figure 7. So the corresponding returns are obtained as the difference between the expected survival rates and longevity premiums and the realized survival rates which are shown in Table 2. Once the data for longevity swaps returns is calculated we can formulate the VAR model with 5 equations by including longevity swaps together with the other four assets. The variables included in Y_t are now returns from S&P

500, MSCI, Bond, Bill, and Longevity swap. The VAR model with two lags is chosen according to the AIC. The parameter estimates and covariance matrix are shown in Output 2.1 and Output 2.2 in the *Appendix* contains the corresponding standard errors of the parameter estimates. The corresponding annualized results are shown in Table 3.

Figure 7: Longevity Swap prices



The figure shows the level of premiums of a 20 year longevity swap determined by using the Sharpe Ratio method with the corresponding Sharpe ratio equal to 0.25 for the period from 1991 to 2010.

Table 2 Forecasted survival rate, realized survival rate and Swap returns (period 1991-2010)

Forecasted	Realized	Swap returns
0.97985	0.981	0.00027
0.97966	0.98142	0.00029
0.97725	0.97957	0.00027
0.97468	0.97778	0.00027
0.97402	0.97605	-0.00041
0.97249	0.97442	-0.000205
0.96731	0.97236	0.000117
0.96421	0.96987	0.00036
0.96196	0.96789	0.00035
0.95963	0.96613	0.00065
0.95714	0.96369	-0.00071
0.95094	0.96084	0.001737
0.94816	0.95777	0.00102
0.94549	0.95544	0.00089
0.94034	0.95108	0.00079
0.95323	0.95246	-0.01131
0.93625	0.94833	0.00079
0.92459	0.93741	0.00075
0.91957	0.93337	0.00087
0.91472	0.92563	0.0008

The corresponding returns are obtained by the difference between the expected survival rates and *longevity premiums and the realized survival rates. For example, the Forecasted survival rate and realized survival rate in first year are 0.97985 and 0.981 respectively. And the corresponding premium is approximate 0.0009. So the swap return is obtained by 0.981-0.97985*(1+0.0009), which is 0.00027.*

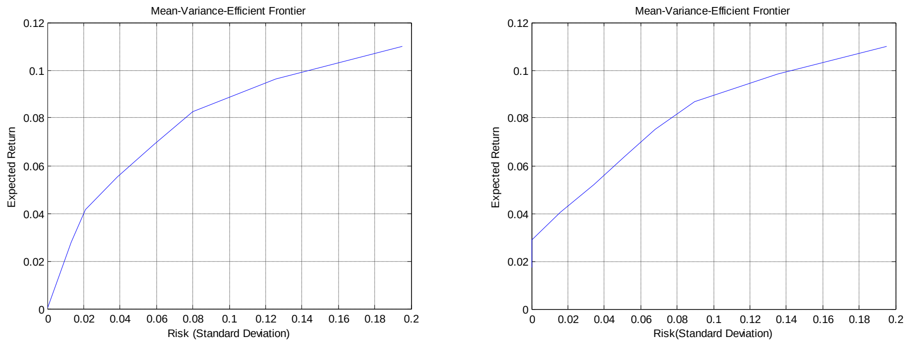
Table 3 Annualized Statistics

				Covariance	Matrix		
	Return	Stdev	Bill	Bond	SP	MSCI	SWAP
Bill	0.0342	0.0185	0.0009396	-0.0002497	0.001058	0.000598	-0.002665
Bond	0.0709	0.0962	-0.0002497	0.005524	-0.001401	-0.003015	-0.00578
SP	0.1101	0.1951	0.001058	-0.001401	0.01999	0.016249	-0.00565
MSCI	0.0803	0.2076	0.000598	-0.003015	0.016249	0.02038	-0.00652
Swaps	0.000588	0.0002519	-0.002665	-0.00578	-0.00565	-0.00652	0.000527

The annualized statistics of the five assets, including returns and standard deviations from S&P 500, MSCI, Bond, Bill and longevity swap for the sample period of 20 years. And the correlation between each of these four assets is obtained by VAR model with a leg value of 2.

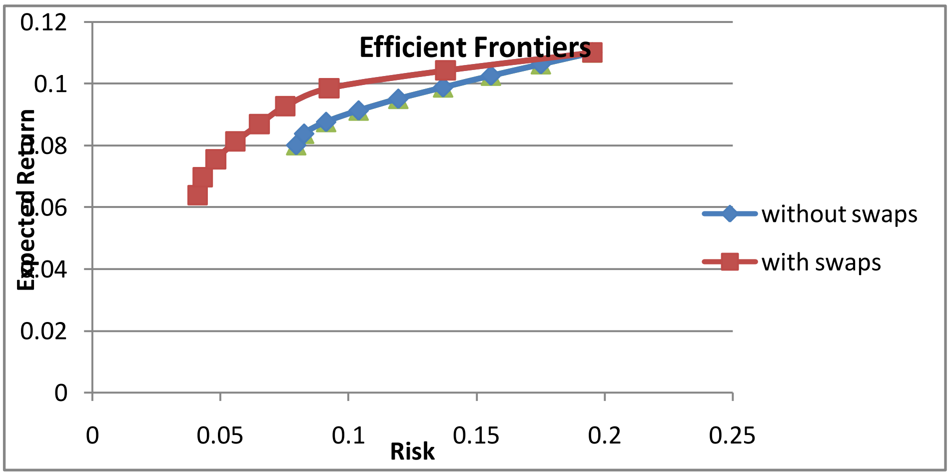
The results in Table 3 show that the longevity swaps has negative correlations with the other four assets, which have highest correlation with MSCI (-0.00652) and have lowest correlation with Bill (-0.002665). Based on the empirical results we find that there are very low and negative correlations between the longevity swaps with equities and the fixed income. There are some factors can be considered as suggesting a negative correlation between longevity swaps and equities and fixed incomes. Such as increasing in economic activity result in an increase of life expectancy because of more resources can available for people's health care and people will have more ability to affording the health care. There are also some factors suggesting a positive correlation between being longevity and equities, but less obvious as compared to the negative factors. Such as the increase of life expectancy, defined benefit pension providers' liabilities will increase, and this will negatively affects these firms' stock prices. Based on the data I used in this paper it shows that the risk associated in longevity swaps has very low correlations with equities and fixed income, which is consider as a good diversify of risk for investors. Even if the expected excess returns from the longevity swaps investment is close to zero, the investors' portfolios can add value through investment in longevity swaps. Then I construct the following mean variance efficient frontier in Figure 8, since the corresponding covariance matrix is obtained.

***Figure 8: Efficient frontier with longevity risk*(*Left: SR=0.25, Right: SR=0.38*)**



The efficient frontier constructed with longevity swaps by including SP, MSCI, Bond, Bill and longevity swap. The left figure is obtained by using the Sharpe ratio equal to 0.25 and right figure by using the Sharpe ratio equal to 0.38.

Figure 9: Efficient frontier with and without longevity risk



The comparison of efficient frontiers constructed with and without longevity swaps use the same portfolio allocation.

As I compare the efficient frontier from both portfolios in figure 9, one is the line which focuses only on fixed income and equity shares and the other line includes the longevity swap in the portfolio. If, however, the pension plan has a risk aversion consistent with a strategy of 33% in equity shares and 67% in fixed income (second point on the line without swaps), then the risk can be reduced by 8.7% from 0.0826 to 0.0754 by putting 6.5% of the assets in the longevity swap. The expected return is largely preserved, but risk is reduced (moving from second point on the line without swaps to the fourth point on the line with swaps). The figure also indicates that when we use the asset portfolios both with and without longevity swaps to get the same expected returns such as 0.08, the risk associated with portfolio including longevity swaps (0.05) is much more lower than risk without longevity swaps (0.75).

Figure 10: Portfolio allocation of second point on the efficient frontier without a longevity swap

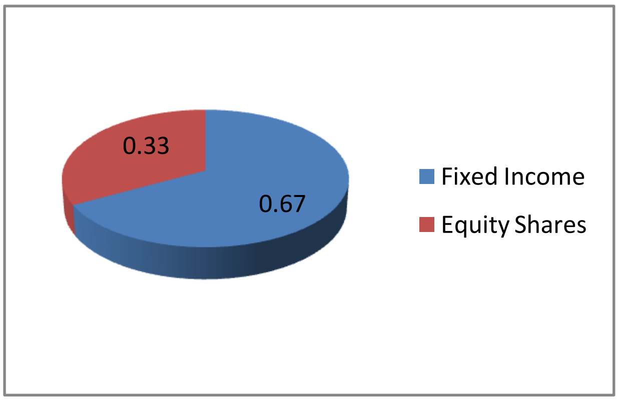
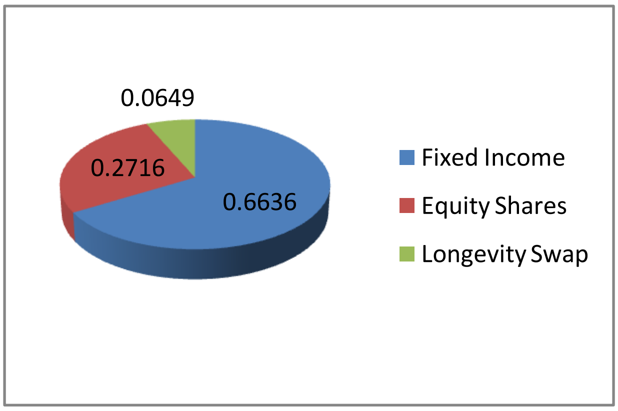


Figure 11: Portfolio allocation of fourth point on the efficient frontier with a longevity swap



Conclusion

The optimal asset allocation for that 100% investment in equity and with no allocation to the longevity swap and fixed income if the pension provide has a sufficient high level of risk tolerance. If the pension provider is risk averse then it's better to invest some proportion of the assets in the longevity swaps. A pension provider can protect itself from one of its three major risks as I pointed out earlier, and a pension provider with a typical level of risk tolerance is expected to benefit from allocating some part of its asset to longevity swaps in its portfolio.

Chapter 6 Discussion and Future Research

I study the potential role of longevity swaps in allowing pension provider to hedge longevity risk through mean variance portfolio theory. I find that the benefits from investing in this asset are significant, the pension plan has a risk aversion consistent with a strategy of 33% in equity shares and 67% in fixed income, and then the risk can be reduced by 8.7% from 0.0826 to 0.0754 by putting 6.5% of the assets in the longevity swap. The expected return is largely preserved, but risk is reduced. When the market for longevity derivatives is getting more mature and more information is available on the pricing for these longevity derivatives, it will be very interesting to repeat this analysis with longevity swaps for longer maturity years. And there are still some open questions: one is for the Sharp Ratio method to choose an appropriate Sharpe Ratio as it been assumed equal to 0.25 in the paper and the other is to adjust the length of period used to determine the standard deviation (volatility). Moreover, with a more mature market there also have some longevity derivatives other than longevity swaps which could be more attractive. Such like longevity swaptions as their benefits equal of the longevity swaps but the costs associated with longevity swaptions are limited.

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Appendix

Table 1 Fitted values of a and b

Age group	a_x	b_x
0-1	-4.34257	0.095400202
(1-4)	-7.4511	0.10800106
(5-9)	-8.19447	0.095160878
(10-14)	-8.12587	0.080645162
(15-19)	-7.06497	0.045321496
(20-24)	-6.75902	0.04785042
(25-29)	-6.72627	0.050977747
(30-34)	-6.54758	0.051763269
(35-39)	-6.23298	0.051797809
(40-44)	-5.83762	0.048207154
(45-49)	-5.4083	0.044798846
(50-54)	-4.97698	0.042876566
(55-59)	-4.56108	0.038376706
(60-64)	-4.13145	0.034698292
(65-69)	-3.73788	0.033840936
(70-74)	-3.32438	0.034143594
(75-79)	-2.90396	0.034781262
(80-84)	-2.44323	0.02961174
(85-89)	-1.98353	0.025365249
(90-94)	-1.55265	0.016671434
(95-99)	-1.178	-1.78E-05
(100-104)	-0.93709	-0.001737148
(105-109)	-0.88368	-0.008534881

Use least squares to estimate the coefficients a and b with U.S. mortality data from 1956 to 2009 respectively. First estimate coefficient a by averaging the log of mortality rates over time, and estimate coefficients b by using a singular value decomposition (SVD) method.

Figure 1 Fitted a_x *and* b_x

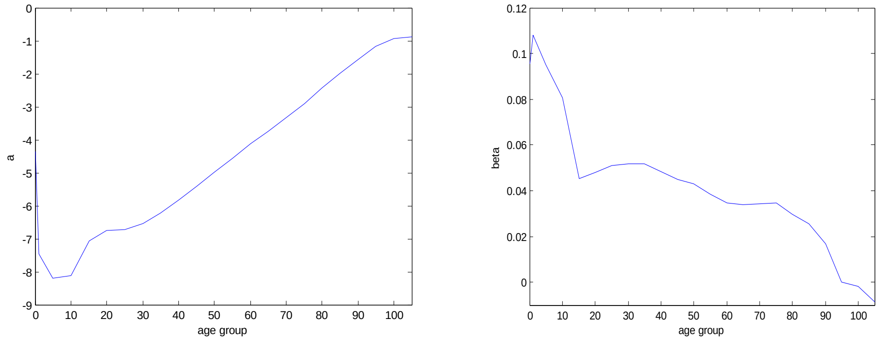

Table 2 forecasts of Mortality Index k over 2010-2029

	forecasted *k*
2010	-1.70717
2011	-2.05834
2012	-2.4095
2013	-2.76066
2014	-3.11182
2015	-3.46298
2016	-3.81414
2017	-4.1653
2018	-4.51647
2019	-4.86763
2020	-5.21879
2021	-5.56995
2022	-5.92111
2023	-6.27227
2024	-6.62343
2025	-6.9746
2026	-7.32576
2027	-7.67692
2028	-8.02808
2029	-8.37924

Table 3 Forecasts of age-specific mortality rates

			date		
Age group	2010	2015	2020	2025	2030
0-1	0.011049	0.009345	0.007904	0.006685	0.005654
(1-4)	0.000483	0.0004	0.000331	0.000273	0.000226
(5-9)	0.000235	0.000199	0.000168	0.000142	0.00012
(10-14)	0.000258	0.000224	0.000194	0.000169	0.000146
(15-19)	0.000791	0.00073	0.000675	0.000623	0.000575
(20-24)	0.001069	0.000983	0.000904	0.000831	0.000764
(25-29)	0.001099	0.001005	0.000919	0.00084	0.000768
(30-34)	0.001312	0.001198	0.001094	0.000999	0.000912
(35-39)	0.001797	0.001641	0.001498	0.001368	0.001249
(40-44)	0.002685	0.002467	0.002267	0.002083	0.001914
(45-49)	0.004149	0.003836	0.003545	0.003277	0.003029
(50-54)	0.006408	0.005943	0.005512	0.005113	0.004742
(55-59)	0.009788	0.00915	0.008554	0.007997	0.007475
(60-64)	0.015136	0.014241	0.0134	0.012608	0.011862
(65-69)	0.022468	0.021172	0.019951	0.0188	0.017715
(70-74)	0.033957	0.031981	0.03012	0.028367	0.026717
(75-79)	0.051646	0.048587	0.045708	0.043	0.040453
(80-84)	0.082597	0.078412	0.07444	0.070668	0.067088
(85-89)	0.131752	0.126013	0.120524	0.115274	0.110253
(90-94)	0.205746	0.199811	0.194047	0.188449	0.183013
(95-99)	0.307904	0.307914	0.307923	0.307933	0.307943
(100-104)	0.39293	0.394131	0.395335	0.396542	0.397754
(105-109)	0.419326	0.425657	0.432083	0.438607	0.44523

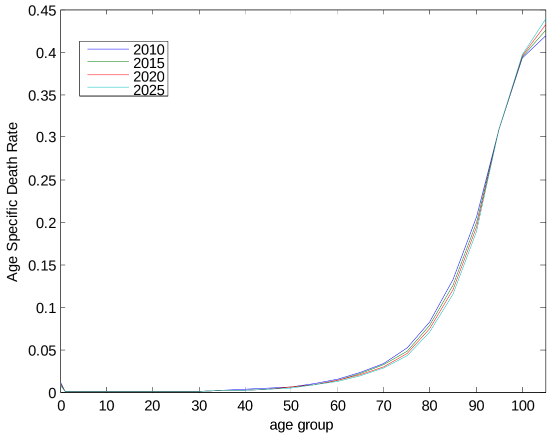

Table 4 Simulated and forecast Mortality Index

	Mortality Index	
	Forecast	Simulated
2010	-1.70717	-1.65586
2011	-2.05834	-2.0032
2012	-2.4095	-2.35647
2013	-2.76066	-2.70645
2014	-3.11182	-3.05714
2015	-3.46298	-3.40846
2016	-3.81414	-3.75925
2017	-4.1653	-4.11377
2018	-4.51647	-4.46556
2019	-4.86763	-4.80522
2020	-5.21879	-5.15014
2021	-5.56995	-5.49398
2022	-5.92111	-5.84612
2023	-6.27227	-6.20152
2024	-6.62343	-6.55586
2025	-6.9746	-6.9149
2026	-7.32576	-7.27263
2027	-7.67692	-7.62063
2028	-8.02808	-7.96808
2029	-8.37924	-8.32297

Table 5 Data

SP	**MSCI**	**Bill**	**Bond**
30.47%	30.23%	5.61%	15.00%
7.62%	7.49%	3.41%	9.36%
10.08%	9.97%	2.98%	14.21%
1.32%	1.33%	3.99%	-8.04%
37.58%	37.20%	5.52%	23.48%
22.96%	23.82%	5.02%	1.43%
33.36%	31.86%	5.05%	9.94%
28.58%	28.34%	4.73%	14.92%
21.04%	20.89%	4.51%	-8.25%
−9.10%	-9.03%	5.76%	16.66%
−11.89%	-11.85%	3.67%	5.57%
−22.10%	-21.97%	1.66%	15.12%

28.69%	28.36%	1.03%	0.38%
10.88%	10.74%	1.23%	4.49%
4.91%	4.83%	3.01%	2.87%
15.79%	15.61%	4.68%	1.96%
5.49%	5.48%	4.64%	10.21%
−37.00%	-36.55%	1.59%	20.10%
26.46%	25.94%	0.14%	-11.12%
15.06%	14.82%	0.13%	8.46%

Sample period from 1991 to 2010.

*1. For the annual data of S&P 500 and MSCI are obtained from financeandinvestments.blogspot.com/ And for the data of Bonds and Bills are obtained from pages.stern.nyu.edu/~adamodar/New_Home.../**data**file/histretSP.html*

0utput 1(Without longevity swaps)

Output 1.1 Parameter estimates and covariance matrix

logL - Optimized loglikelihood objective function value associated with
the parameter estimates.

W - Fit residuals and has the same size as input data.

Spec - A model specification structure for a multidimensional VAR time series process, and should contain model dimensions, a lag structure, if any, and parameter estimation indicators, if any.

u - Response data. u is a numObs-by-numDims matrix representing numObs observations of a single path of a numDims-dimensional time series. The last observation is assumed to be the most recent.

EstSE - VAR model specification structure containing the standard errors of the parameter estimates. Standard errors are maximum likelihood estimates, so a degree- of-freedom adjustment is necessary to form ordinary least squares estimates. To adjust for numObs observations and numActive unrestricted parameters, multiply by sqrt(numObs/(numObs-numActive-1))

```
Spec = vgxset('n',4,'nAR',3,'Constant',true)
[EstSpec,EstStdErrors,LLF,W] = vgxvarx(Spec,u)
 vgxdisp(EstSpec)
NumParam = vgxcount(Spec)
Spec =
     Model: 4-D VAR(3) with Additive Constant
         n: 4
       nAR: 3
       nMA: 0
        nX: 0
         a: []
        AR: {}
         Q: []
EstSpec =
     Model: 4-D VAR(3) with Additive Constant
```

n: 4
nAR: 3
nMA: 0
nX: 0
a: [0.0412828 0.116946 0.179569 0.191482] additive constants
asolve: [1 1 1 1 logical] additive constant indicators
AR: {3x1 cell} stable autoregressive process
ARsolve: {3x1 cell of logicals} autoregressive lag indicators
Q: [4x4] covariance matrix
Qsolve: [4x4 logical] covariance matrix indicators

EstStdErrors =

Model: 4-D VAR(3) with Additive Constant: Standard Errors
n: 4
nAR: 3
nMA: 0
nX: 0
a: [0.00834827 0.0316389 0.121796 0.122947] additive constants
asolve: [1 1 1 1 logical] additive constant indicators
AR: {3x1 cell}
ARsolve: {3x1 cell of logicals} autoregressive lag indicators
Q: []
Qsolve: [4x4 logical] covariance matrix indicators

LLF =

151.6222

W =

0.0148	0.0331	0.1251	-0.0701
-0.0106	0.0172	-0.1837	-0.1492
-0.0042	-0.0171	-0.0537	-0.0224
0.0093	0.0448	-0.0455	-0.1603
0.0164	0.0080	0.2407	0.1311
0.0194	-0.0092	0.1454	0.0583
0.0101	0.0183	0.1264	0.0188
-0.0067	0.0374	-0.0095	0.0348
-0.0015	-0.0487	0.1268	0.1908
-0.0009	-0.0112	-0.1165	-0.0993
-0.0080	-0.0554	-0.1069	-0.1040
-0.0021	0.0020	-0.1863	-0.0749
0.0052	-0.0019	0.1545	0.2042
-0.0018	-0.0300	0.0168	0.0880
-0.0093	-0.0178	-0.2091	-0.0735
-0.0019	-0.0091	0.0855	0.2214
0.0037	-0.0516	0.2014	0.2394
-0.0170	0.1083	-0.2154	-0.3036
-0.0027	-0.0110	0.0147	-0.0030

-0.0121 -0.0060 -0.1107 -0.1266

Model : 4-D VAR(3) with Additive Constant
Conditional mean is AR-stable and is MA-invertible

Parameter estimates

a Constant:

0.0412828
0.116946
0.179569
0.191482

AR(1) Autoregression Matrix:

0.186407	-0.0810143	0.017778	-0.00271245
1.58339	-0.995878	0.13908	-0.184016
-1.25443	0.397411	0.653871	-0.890816
-6.66973	1.23849	0.186393	-0.270954

AR(2) Autoregression Matrix:

-0.228196	-0.077481	0.0602308	-0.01358
3.07532	-0.438597	-0.214877	0.0590045
-8.92506	1.48748	0.342586	-0.0408637
-1.08761	1.57912	0.194477	-0.109774

AR(3) Autoregression Matrix:

-0.102713	-0.0152494	0.0416214	-0.0121751
-2.40713	0.0166385	0.0605454	-0.229456
1.95694	0.935574	-0.141117	0.0995773
-4.34132	1.19194	0.261501	-0.124377

Covariance matrix:

Q Innovations Covariance:

9.39564e-05	-2.49744e-05	0.00105767	0.000597635
-2.49744e-05	0.00134951	-0.0014014	-0.00301454
0.00105767	-0.0014014	0.0199985	0.0162486
0.000597635	-0.00301454	0.0162486	0.0203783

NumParam =
4

```
>> AIC = aicbic(LLF,NumParam)
```

AIC =
-295.2445

Output 1.2 Standard errors of the parameter estimates.

vgxdisp(EstStdErrors)

Model : 4-D VAR(3) with Additive Constant: Standard Errors
Standard errors without DoF adjustment (maximum likelihood)

a Constant:

0.00834827
0.0316389
0.121796

0.122947

AR(1) Autoregression Matrix:

0.252347	0.0432072	0.0257171	0.0219597
0.956364	0.16375	0.0974644	0.0832245
3.68158	0.630365	0.375195	0.320378
3.71638	0.636322	0.378741	0.323406

AR(2) Autoregression Matrix:

0.272537	0.0596192	0.028293	0.0273375
1.03288	0.225949	0.107227	0.103606
3.97613	0.869805	0.412776	0.398836
4.01371	0.878025	0.416678	0.402606

AR(3) Autoregression Matrix:

0.228274	0.0492729	0.0284035	0.0251289
0.86513	0.186738	0.107646	0.0952353
3.33037	0.71886	0.414389	0.366614
3.36185	0.725654	0.418306	0.370079

Output 2 (With longevity swaps)

Output 2.1 Parameter estimates and covariance matrix

```
v=[e,b,c,f,g];
 Spec = vgxset('n',5,'nAR',2,'Constant',true)
[EstSpec,EstStdErrors,LLF,W] = vgxvarx(Spec,v)
 vgxdisp(EstSpec);
NumParam = vgxcount(Spec)
Spec =
      Model: 5-D VAR(2) with Additive Constant
          n: 5
        nAR: 2
        nMA: 0
         nX: 0
          a: []
         AR: {}
          Q: []
EstSpec =

      Model: 5-D VAR(2) with Additive Constant
          n: 5
        nAR: 2
        nMA: 0
         nX: 0
          a: [5x1] additive constants
     asolve: [5x1 logical] additive constant indicators
         AR: {2x1 cell} stable autoregressive process
    ARsolve: {2x1 cell of logicals} autoregressive lag indicators
          Q: [5x5] covariance matrix
```

```
     Qsolve: [5x5 logical] covariance matrix indicators
EstStdErrors =

      Model: 5-D VAR(2) with Additive Constant: Standard Errors
          n: 5
        nAR: 2
        nMA: 0
         nX: 0
          a: [5x1] additive constants
     asolve: [5x1 logical] additive constant indicators
         AR: {2x1 cell}
    ARsolve: {2x1 cell of logicals} autoregressive lag indicators
          Q: []
     Qsolve: [5x5 logical] covariance matrix indicators
LLF =
  311.3549
W =
    0.0074   -0.0033   -0.0394   -0.0892   -0.0000
   -0.0123    0.0304   -0.1784   -0.1214    0.0001
   -0.0072    0.0811   -0.1449    0.1043   -0.0001
    0.0028    0.0046   -0.0299   -0.0272   -0.0000
    0.0035   -0.0196    0.2952    0.2627   -0.0001
    0.0121   -0.0311    0.1152    0.0548    0.0000
    0.0068   -0.0041    0.1448   -0.1016   -0.0000
    0.0011    0.0042    0.1152    0.1247   -0.0001
    0.0021   -0.0591    0.0827    0.1070    0.0000
   -0.0029   -0.0170   -0.0996   -0.0613    0.0001
   -0.0096   -0.0304   -0.0617   -0.1323    0.0000
   -0.0003   -0.0001   -0.2724   -0.1578    0.0001
   -0.0001    0.0358    0.2303    0.2087   -0.0000
   -0.0018    0.0133   -0.0336    0.0282   -0.0000
   -0.0094   -0.0141   -0.1226   -0.1083    0.0001
    0.0145    0.0237    0.1035    0.1531   -0.0001
    0.0132   -0.0468    0.1413    0.2350   -0.0001
   -0.0157    0.1107   -0.3018   -0.3782    0.0000
    0.0043   -0.0627    0.0677    0.0100    0.0000
   -0.0083   -0.0156   -0.0115   -0.1109    0.0000

  Model    : 5-D VAR(2) with Additive Constant
             Conditional mean is AR-stable and is MA-invertible
```

Parameter estimates:

```
a Constant:
     0.0486677
      0.153278
```

0.344103
0.21064
0.000307766

AR(1) Autoregression Matrix:

0.242398	-0.0922172	-0.00743463	0.0230213	-9.53594
2.27528	-1.00022	0.17878	-0.306722	-314.971
-3.57553	0.344233	0.567603	-0.611728	-145.362
-8.15598	0.793582	0.384074	-0.301587	172.921
-0.00323705	-0.000561773	-0.000118692	3.31325e-05	0.729565

AR(2) Autoregression Matrix:

0.0139999	-0.0801311	0.0387593	-0.00123045	-20.6958
-0.139521	-0.315854	-0.206771	0.12732	243.099
-4.0187	0.606005	0.138899	-0.00398385	28.12
-1.28552	0.489888	0.460969	-0.176905	-75.7328
0.00129851	-0.000539944	4.06936e-05	0.000245893	0.0286296

Covariance Q Innovations Covariance:

6.92583e-05	-0.00016317	0.000859345	0.000793523	-2.66534e-07
-0.00016317	0.00172368	-0.00300294	-0.00204948	-5.86357e-07
0.000859345	-0.00300294	0.0240166	0.0193006	-5.49538e-06
0.000793523	-0.00204948	0.0193006	0.0240008	-6.41326e-06
-2.66534e-07	-5.86357e-07	-5.49538e-06	-6.41326e-06	5.27483e-09

NumParam =
5
>> AIC = aicbic(LLF,NumParam)
AIC =
-612.7097

Output 2.2 Standard errors of the parameter estimates.

vgxdisp(EstStdErrors)

Model : 5-D VAR(2) with Additive Constant: Standard Errors
Standard errors without DoF adjustment (maximum likelihood)

a Constant:

0.00724824
0.0361597
0.134975
0.13493
6.32559e-05

AR(1) Autoregression Matrix:

0.214883	0.0337173	0.0199644	0.0198072	21.9749
1.072	0.168207	0.0995976	0.0988133	109.627
4.0015	0.627873	0.371772	0.368844	409.21
4.00018	0.627667	0.371649	0.368723	409.076
0.0018753	0.000294253	0.000174231	0.000172859	0.191776

AR(2) Autoregression Matrix:

0.237081	0.0381824	0.0235912	0.0226866	22.963
1.18274	0.190483	0.117691	0.113178	114.557
4.41486	0.711022	0.439309	0.422463	427.61
4.41341	0.710788	0.439165	0.422324	427.47
0.00206902	0.00033322	0.000205882	0.000197987	0.200399