A dynamic version of the Vasicek model

the influence on capital requirements

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ABSTRACT

Under the internal ratings-based approach within the Basel II / III capital accords, banks are required to determine input parameters that are representative for the amount of risk the bank takes. Capital requirements are calculated based on the values of these input parameters. However, different philosophies underlying the determination of the parameters lead to differences in capital requirements. In this thesis, we investigate the effect of the through-the-cycle philosophy versus the point-in-time philosophy on capital requirements. We find that the chosen philosophy is of influence on the level and dynamics of capital requirements. Furthermore, we investigate the implications of the differences in the dynamics on the procyclicality of capital requirements.

CONTENTS

1. INTRODUCTION

Banks hold capital in order to absorb losses from, for example, defaulting loans. The more outstanding loans a bank has, and the more likely those loans are to go into default, the more capital the bank needs. As a consequence, it seems natural to base capital requirements on the amount of risk a bank takes. Capital requirements are based on a prescription by the Basel Committee on Banking Supervision (BCBS), part of the Bank for International Settlements (BIS). Over the past years, the Basel Committee on Banking Supervision aimed to make capital requirements more risk sensitive. They did so in particular with the introduction of the Basel II capital accord, and the accompanying internal ratings-based approach (IRB). Under IRB, banks are required to determine input parameters that are representative for the amount of risk the bank takes. Based on those input parameters, the risk-weighted assets (RWA) are determined, which in turn determine capital requirements. Although the improvement of risk sensitivity in the capital requirements is a sound objective in itself, it has led to some considerations.

First, differences can occur between the risk-weighted assets of different banks. As IRB is designed to be more risk-sensitive, differences in the risk-weighted assets will naturally occur due to a difference in the risk profile of banks. However, as banks are free to use their own models for the estimation of the input parameters, differences in risk-weighted assets might also occur due to different calculation methods. For example, the Basel Committee on Banking Supervision (2013) claims that up to three quarters of the variation in risk-weighted assets is explained by the underlying differences in the risk profile of banks. Hence, this part of variation is intended. The remaining variation is unintended, and is driven by diversity in both bank and supervisory practices.

Second, greater sensitivity to risk may lead to a higher degree of variation in risk-weighted assets over time. Presumably, risk is high when macroeconomic conditions are bad, while risk goes down when macroeconomic conditions improve. Hence, when risk-weighted assets are sensitive to risk, they swing along with the business cycle. This in turn may cause procyclicality, which amplifies the economic situation. For example, Daníelsson et al. (2001) expressed their concerns regarding procyclicality in capital requirements under the Basel II capital accords. Also, Heid (2007) finds that, indeed, pro-cyclical effects of capital requirements are to be expected. However, the effects might be mitigated by capital buffers.

Both of the aforementioned issues originate in the way banks determine the input parameters for the calculation of the capital requirements. In this thesis we will identify different philosophies underlying the determination of the capital requirements. Although extensive regulations regarding capital requirements are set in place, there are no explicit prescriptions regarding the philosophy under which capital requirements should be determined. Hence, we will investigate the differences in capital requirements between different philosophies, statically as well as over time. The issue of rating philosophy has been discussed in the literature to some extend. For example, Rosch (2005) found that the rating philosophy influences the level and volatility of capital requirements.

Chapter 2 gives an overview of the regulations set in place by the Basel Committee on Banking Supervision. In particular, the most important aspects of the different versions of the Basel capital accords are being discussed. In Chapter 3, we will discuss the Vasicek model, being the model underlying the formula for the calculation of capital requirements under the IRB approach. Furthermore, we will examine properties of the capital requirement formula, such as its concaveness as a function of PD. In Chapter 4, the Vasicek model will be extended to incorporate autocorrelation and regime switching, resulting in a regime switching version of the Vasicek model. Furthermore, we will examine the differences in capital requirements under the Vasicek model and under the regime switching version of the Vasicek model. Chapter 5 contains a methodology for modelling transitions between different buckets in the total portfolio. In particular, we develop a regime switching version of the transition matrix exhibiting cross-sectional dependence in a similar fashion as in Chapter 4. Finally, Chapter 6 concludes with an analysis of capital requirements under different philosophies. We make use of the aforementioned chapters to model capital requirements under the different philosophies.

2. BASEL COMMITTEE ON BANKING SUPERVISION

The core activity of banks is to attract savings and lend money. By doing so banks are subject to credit risk, i.e. the risk that a loan defaults causing the bank to lose its money. In order to assure that banks do not take too much credit risk, regulations are set in place. Financial regulators supervise banks to assure that regulations are correctly followed. Financial regulators usually operate on a national level. However, in order to enhance regulation, supervision and practices of banks around the world, there is the need for an international committee. Hence, the Basel Committee on Banking Supervision (BCBS) was introduced after the 1970's crisis, as part of the Bank for International Settlements (BIS). The committee has been responsible for guidelines on the supervision of banks worldwide, and consists of members from countries all over the world.

2.1 Basel I

The first guideline, referred to as Basel I, was formed in 1988. The accord consists of a set of minimum capital requirements for banks. The main requirement focusses on credit risk and states that, in case of risky loans, a minimum of 8% of assets has to be kept by the bank as capital. The requirement is relaxed in case the asset falls in a low-risk category. In particular, assets are classified in one of five categories, each categorie is assigned a weight. Basel I prescribes that banks must hold a minimum capital of 8% of risk-weighted-assets.

2.2 Basel II

In 2004 the second Basel accord was introduced, referred to as Basel II. Under Basel II, capital requirements can be calculated by means of two different approaches, the standardized approach (SA) and the internal ratings-based approach (IRB). Under the standardized approach banks use external ratings in order to calculate capital requirements. Different ratings correspond to different weights. As under Basel I, capital requirements equal 8% of risk-weighted-assets. However, under the IRB approach banks use their own knowledge about the credit risk they are subject to. Hence, IRB allows banks to estimate both the probability of default (PD), the loss given default (LGD) and the exposure at default (EAD) of outstanding loans. Capital requirements are calculated based on estimates of these parameters. Basel II gives an explicit formula for the calculation of capital requirements under the IRB approach. The underlying idea is to impose a level of capital such that there is only a very low, fixed probability that losses exceed this level of capital. Hence, in theory, if the minimum capital requirement is satisfied the probability that a bank becomes insolvent is very low.

2.3 Basel III

In 2010, after the credit crisis of the late 2000's, the Basel II capital requirements proved to be insufficient. Hence, additions to Basel II were introduced . The resulting set of capital requirements is referred to as Basel III. Under Basel III, the minimum capital requirements stay at the same level as under Basel II, which is 8%. However, a capital conservation buffer of 2.5% is introduced in order to absorb losses during periods of financial and economic stress. Moreover, a countercyclical buffer is introduced. The countercyclical buffer ranges from 0% to 2.5% depending on national conditions. We will discuss the reasoning behind the countercyclical buffer extensively in Chapter 5. In effect, banks are required to hold 10.5% of risk-weighted-assets as capital, and even up to 13% in times with good national conditions. Next to increased capital requirements, the aim is to decrease leverage and increase liquidity. Hence, a minimum leverage ratio of 3% and two liquidity requirements were introduced. All of the changes will be gradually introduced and Basel III will be fully effective as of January, 2019.

3. VASICEK SINGLE FACTOR MODEL

Merton (1974) proposed a framework to assess credit risk. It is assumed that a company cannot repay its debt when, at a certain point in time, its assets fall short on its debt. Merton-type models are based on the capital structure of the firm, hence they are also referred to as structural models. They are an important class of models in the literature of assessing credit risk. In line with the Merton-type models, Vasicek (1987) proposed a model for generating the loss distribution of a credit portfolio. Again, Vasicek (1987) assumes that a borrower defaults when the value of its assets falls below a certain threshold. However, interest lies not only in individual loans but rather in a portfolio of loans. In the so-called single factor Vasicek model, it is assumed that loans in a portfolio are subject to a single common risk factor. Hence, part of the credit risk in a portfolio of loans is systematic. Note that in the case of corporate loans, the threshold can be interpreted as the value of the liabilities. However, it is difficult to determine the assets and liabilities in case of retail loans. Although the framework can still be applied to retail loans, the interpretation of the different variables is not as intuitively clear.

Vasicek's framework can be written as follows:

$$D_{it}^* = \sqrt{\rho} M_t + \sqrt{1-\rho}\,\epsilon_{it}, \tag{3.1}$$

where ϵ_{it} are *iid* $N(0,1)$ over i and t, independent of M_t *iid* $N(0,1)$. Next, define

$$D_{it} = \begin{cases} 1 & \text{if} \quad D_{it}^* \leq c_i, \\ 0 & \text{if} \quad D_{it}^* > c_i, \end{cases} \tag{3.2}$$

where D_{it} equals 1 in case of default of obligor i during $[t, t+1)$. Hence, D_{it}^* can be seen as a latent variable. Default occurs when the latent variable drops below the treshold c_i. From Equation (3.1) we can see that the default events of loans are correlated to each other only through M_t. Thus, M_t represents the systematic part of the credit risk. In fact,

$$\begin{aligned} Cov[D_{1,t}^*, D_{2,t}^*] &= Cov[\sqrt{\rho} M_t + \sqrt{1-\rho}\,\epsilon_{1,t}, \sqrt{\rho} M_t + \sqrt{1-\rho}\,\epsilon_{2,t}], \\ &= Var[\sqrt{\rho} M_t] = \rho. \end{aligned} \tag{3.3}$$

Hence, ρ can be seen as a measure of the strength of cross-sectional dependence in D_{it}^*. In order to define D_{it}^*, we must have that $\rho \geq 0$. Therefore, negative correlation between D_{it}^*'s is excluded, which is a reasonable restriction in empirical applications. Furthermore, note that D_{it} follows a Bernoulli distribution, as can be seen from Equation (3.2).

3.1 Vasicek distribution

Banks evaluate every outstanding loan on the individual level. By doing so, banks are able to estimate the risk that is associated to the loan, such as the probability of default of the obligor. Hence, for retail loans like mortgages, individual characteristics such as income, age and residence as well as other factors, such as the nature of the loan and macroeconomic variables (e.g. houseprices and unemployment rates), determine the risk that is associated to a loan[1]. However, as the default events of loans are correlated to each other, the Basel capital accords require that capital requirements are calculated for a so-called bucket of loans. A bucket consists of multiple loans with similar characteristics. Hence, the input parameters for the calculation of capital requirements (PD, LGD and EAD) should be representative for the entire bucket. The PD of the entire bucket can be seen as the average of all individual PDs. The framework described in the previous section can be elaborated to arrive at the default fraction distribution of a bucket of loans, or credit portfolio. In order to do so, an important assumption has to be made. It is assumed that the credit portfolio is infinitely granular. This means that the portfolio consists of infinitely many loans and none of these loans represents a substantial part of the total portfolio exposure. A portfolio of infinite granularity has no idiosyncratic risk, as all idiosyncratic risk is diversified. The remaining risk is systematic risk, represented by the single common risk factor.

To arrive at the default fraction distribution of a credit portfolio, we will first determine the conditional probability of default of a single loan (where we condition on the common risk factor M_t):

$$\begin{aligned} P[D_{it} = 1 \mid M_t] &= P[D^*_{it} \leq c_i \mid M_t] \\ &= P[\sqrt{\rho} M_t + \sqrt{1-\rho}\epsilon_{it} \leq c_i \mid M_t] \\ &= P\left[\epsilon_{it} \leq \frac{c_i}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t \mid M_t\right] \\ &= \Phi\left(\frac{c_i}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right), \end{aligned} \tag{3.4}$$

where Φ denotes the cdf of the standard normal distribution. Since both M_t and ϵ_{it} are standard normally distributed and independent of each other, the latent variable D^*_{it} is standard normally distributed as well. Due to this observation, we can easily calculate the unconditional probability of default of a single loan as follows

$$P[D_{it} = 1] = P[D^*_{it} \leq c_i] = \Phi(c_i). \tag{3.5}$$

Hence,

$$c_i = \Phi^{-1}(PD_{it}), \tag{3.6}$$

[1] The determination of the riskiness of a loan, or creditworthiness of an applicant, is referred to as credit scoring. Complex statistical models, referred to as scorecards, are used to arrive at credit scores.

where $PD_{it} := P[D_{it} = 1]$. However, we will treat all loans in a bucket as having the same probability of default. Hence, PD_{it} equals PD for all i in the bucket, where PD is the average probability of default of a pool of loans with similar characteristics. Thus, inserting Equation (3.6) into Equation (3.4) yields

$$P[D_{it} = 1 \mid M_t] = \Phi\left(\frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right). \tag{3.7}$$

However, instead of looking at default probabilities of single loans, we are interested in the default fraction distribution of a bucket of loans. We can easily acquire the default fraction distribution in the case that $\rho = 0$. Indeed, if $\rho = 0$, we have that loans in the portfolio are independent and identically distributed. Hence, by the law of large numbers, the number of defaults in the infinitely granular portfolio converges to the default fraction as $n \to \infty$. However, In the case that $\rho > 0$, the loans are not independent. Loans depend on each other through M_t. Hence, conditional on M_t, we again have that the loans are independent and identically distributed. Thus, conditionally on M_t, we have, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} D_{it} \to \theta \ a.s., \tag{3.8}$$

with (see (3.7))

$$\theta = \Phi\left(\frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right). \tag{3.9}$$

The random variable θ can be interpreted as the default fraction in the infinitely granular portfolio, conditional on M_t. The cdf of the default fraction in the portfolio is defined by the probability that θ takes on a value less than a possible given value. Thus, the cdf is defined by the following equation, for $x \in [0,1]$,

$$\begin{aligned} P[\theta \leq x] &= P\left[\Phi\left(\frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right) \leq x\right] \\ &= P\left[M_t \geq -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\left(\Phi^{-1}(x) - \frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right] \\ &= 1 - P\left[M_t \leq -\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD)\right)\right]. \end{aligned} \tag{3.10}$$

Since M_t is normally distributed, we can rewrite Equation (3.10) as

$$\begin{aligned} P\left[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} D_{it} \leq x\right] &= P[\theta \leq x] \\ &= 1 - \Phi\left(-\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD)\right)\right) \\ &= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD)\right)\right). \end{aligned} \tag{3.11}$$

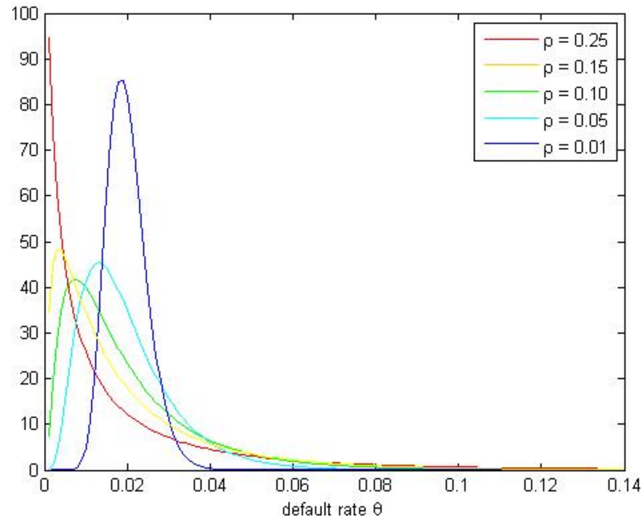


Fig. 3.1: Density of the Vasicek distribution for $\rho = 0.25$, $\rho = 0.15$, $\rho = 0.10$, $\rho = 0.05$ and $\rho = 0.01$ and $PD = 0.02$ for an infinitely granular portfolio.

We have arrived at the cdf of the default fraction of an infinitely granulated portfolio. The pdf can easily be acquired by differentiating the cdf, and is defined as follows

$$\frac{\partial}{\partial x} P[\theta \leq x] = \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \frac{\phi\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}(x) - \frac{1}{\sqrt{\rho}}\Phi^{-1}(PD)\right)}{\phi\left(\Phi^{-1}(x)\right)}, \tag{3.12}$$

where ϕ denotes the pdf of the standard normal distribution. Figure 3.1 shows the Vasicek default fraction distribution for different values of ρ. The density function is skewed due to positive correlation between defaults. Furthermore, the probability mass moves to the expected loss as $\rho \to 0$. Note that the cdf of the default fraction equals the cdf of the losses in the portfolio if we assume that the principal on every loan consists of 1 euro, and the loss given default equals 1. Hence, the Vasicek default fraction distribution is sometimes also referred to as the Vasicek loss distribution.

3.2 Capital Requirements

In Chapter 2 we explained that Basel II gives an explicit formula for the calculation of capital requirements under the IRB approach. This formula is based on the concept of the Value-at-Risk (VaR), where the underlying distribution is the Vasicek distribution as described in Section 3.1. In order to arrive at the capital requirement formula as found in the Basel capital accords, we multiply the VaR implied by the Vasicek distribution by the loss given default (LGD).

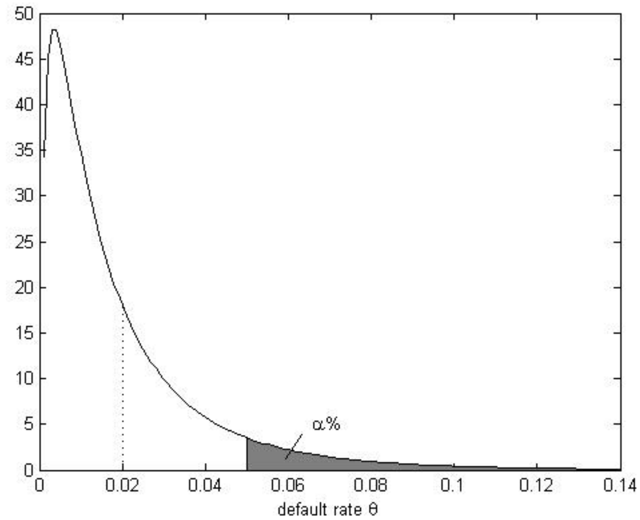


Fig. 3.2: Density of the Vasicek distribution for $\rho = 0.15$ and $PD = 0.02$ for an infinitely granular portfolio. The line adjacent to the left side of the grey area equals the $(1-\alpha)\%$ VaR.

Indeed, in case of default, banks may only lose a percentage of the exposure of a loan due to, for example, collateral. Furthermore, it is believed that banks already account for the expected loss of a portfolio. Hence, we subtract the expected loss from the product of the VaR and the LGD to arrive at the capital requirements, also referred to as the unexpected loss.

The $(1-\alpha)\%$ VaR can be calculated by determining the $(1-\alpha)$-quantile of the distribution of θ. Hence, we obtain the $(1-\alpha)\%$ VaR by setting the right hand side of equation (3.11) equal to $(1-\alpha)\%$ and solving for x, i.e.

$$VaR_{(1-\alpha)} = \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(PD) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(1-\alpha)\right). \tag{3.13}$$

Figure 3.2 shows the density of the Vasicek distribution for an infinitely granular portfolio. The grey area in the figure equals $\alpha\%$ and hence represents the probability that losses greater than the VaR occur. The line adjacent to the left side of the grey area equals the $(1-\alpha)\%$ VaR. The dotted line equals the expected loss (EL). Note that the Basel Committee on Banking Supervision (2006) prescribes $\alpha = 0.1\%$ for the calculation of VaR. Furthermore, we set $\rho = 0.15$ as determined by the Basel Committee on Banking Supervision (2006) for residential mortgage exposures. Thus, we will restrict our attention to residential mortgage exposures. Finally, as the focus of this thesis will be the influence of PD on capital requirements, the scope of this thesis is not the value of LGD and, hence, we set the value of LGD equal to 100%.

We arrive at the capital requirement formula as described in the Basel capital accords. The formula is given by the following Equation:

$$Cap = LGD \cdot \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(PD) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(\alpha)\right) - LGD \cdot PD. \quad (3.14)$$

Note that the risk-weighted-assets (RWA) can be determined by multiplying the obtained capital requirement by 12.5 as well as by the exposure at default (EAD).

Figure 3.3 shows the capital requirement, the VaR and the expected loss as a function of PD. Note that VaR equals the expected loss plus the capital requirement. The VaR equals zero for PD equal to zero and one. Note that a loan will certainly not go into default in case PD equals zero, while a loan will certainly default in case PD equals one. Hence, in these cases, there is no risk and, resultantly, the VaR equals zero. Furthermore, from Figure 3.3, the

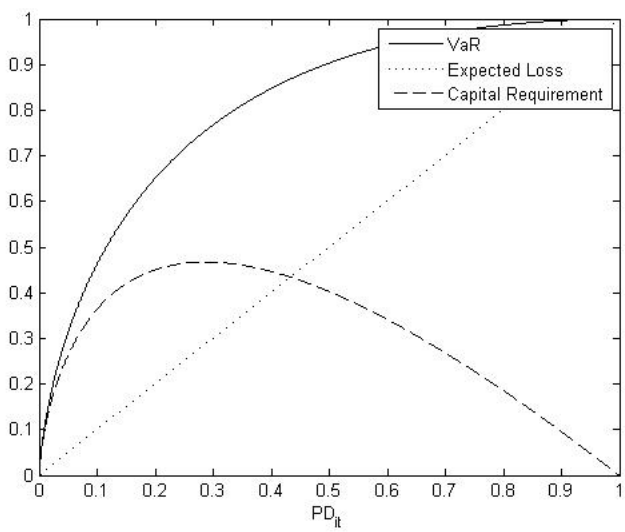


Fig. 3.3: VaR, Expected Loss and Capital Requirement against PD for $\rho = 0.15$ and $\alpha = 0.01\%$. Note that VaR equals Expected Loss plus Capital Requirement.

capital requirement function looks concave in PD. However, as will be proven in appendix A, it is not concave for all possible values of PD. In particular, the capital requirement function is concave for $PD \in \left[\Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right), 1\right]$. For retail exposures we have that $\Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right) = 7.38 \cdot 10^{-16}$, the value of which falls below the minimum value of PD for retail exposures (0.03%) as defined by De Nederlandsche Bank (2010). Hence, the capital requirement function is concave in the range of values of PD allowed for by De Nederlandsche Bank (2010).

Proposition 1. *The capital requirement formula for residential mortgage exposures, i.e. for $\rho = 0.15$ and $\alpha = 0.999$, as described by the Basel capital accords is concave in PD for values of $PD \in \left[\Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right), 1\right]$.*

Hence, by Jensen's inequality, the function applied to the (weighted) average of different values of PD is greater than the (weighted) average of the function applied to different values of PD, as long as $PD \in \Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right)$. In general,

$$\frac{\sum_{i=1}^{N} a_i \cdot Cap(PD_i)}{\sum_{i=1}^{N} a_i} \leq Cap\left(\frac{\sum_{i=1}^{N} a_i \cdot PD_i}{\sum_{i=1}^{N} a_i}\right). \tag{3.15}$$

As explained in the previous section, capital requirements are usually calculated per bucket. Total capital requirements are then obtained by taking the sum over the capital requirements per bucket. Hence, if we look at PD_i as the representative PD for bucket i and PD_j as the representative PD for bucket j, we have that the average capital requirement over bucket i and j is smaller than the capital requirement over the average of PD_i and PD_j. Note that, from Equation 3.15, the average can also be a weighted average. Because the sizes of the buckets will, in general, not be equal we are particularly interested in the weighted average, where the weights are based on the values of EAD_i and EAD_j. EAD_i and EAD_j are defined as the representative EAD for bucket i and bucket j, respectively. In this case, Equation 3.15 reduces to[2]

$$\frac{EAD_i \cdot Cap(PD_i) + EAD_j \cdot Cap(PD_j)}{EAD_i + EAD_j} \leq Cap\left(\frac{EAD_i \cdot PD_i + EAD_j \cdot PD_j}{EAD_i + EAD_j}\right). \tag{3.16}$$

Hence, following Proposition 1, it is beneficial for banks to increase the number of buckets in the portfolio, as this leads to lower total capital requirements. Furthermore, note that the capital requirement function peaks around 30%. Although the capital requirements are lower for values of PD greater than 30%, this does not mean that a bank has to hold fewer total reserves for loans with PD greater than 30%. However, the extra capital is not reflected by an increase in the capital requirements but by an increase in the expected loss.

[2] Note that this implies that PD_{i+j} (the PD representative for the sum of bucket i and j) is calculated as the weighted average of PD_i and PD_j, where the weights are based on the values of EAD_i and EAD_j

4. A REGIME SWITCHING VERSION OF THE VASICEK MODEL

In Chapter 3 we derived the distribution of the default rate as implied by the Vasicek model. This distribution is based on the value of the long term probability of default. The short term PD is calculated from the long term probability of default and a standard normally distributed, independent random variable M_t (see Equation (3.7)), which can be seen as a variable representing economy. In Chapter 3 this variable generates cross-sectional dependence. However, due to a lack of autocorrelation in M_t, the short term PD lacks serial dependence. Empirical observations of default rates indicate that the short term PD does exhibit serial dependence, and, in particular, that the PD depends on the state of the economy. For example, Nickell et al. (2000) state that "business cycle effects make an important difference especially for lowly graded issuers. Default probabilities in particular depend strongly on the stage of the business cycle." Furthermore, Crook and Bellotti (2010) claim that "there is considerable evidence that the state of a country's macroeconomy affects, on average, the chance that an applicant will default in the future and the ranking in terms of risk of individuals who apply for a loan." Hence, in this chapter we will incorporate a regime switching model, such that the probability of default behaves differently during economic downturns and economic upturns than during times with normal economic conditions.

Similar to the framework of the Vasicek single factor model, we define the latent variables D^*_{it}.

$$D^*_{it} = \sqrt{\rho}M_t + \sqrt{1-\rho}\epsilon_{it}, \tag{4.1}$$

where ϵ_{it} are iid $N(0,1)$ over i and t, independent of M_t iid $N(0,1)$. We remark that the variable M_t lacks a clear interpretation, as opposed to the variable M_t in Chapter 3. Although both variables are meant to introduce cross-sectional dependence, M_t should not be seen as a variable representing economy. Hence, in this chapter we define a reduced form model as opposed to the structural model from Chapter 3[1]. Next, we define the default indicators D_{it} as follows

$$D_{it} = \begin{cases} 1 & \text{if} \quad D^*_{it} \leq c_i(S_{t-1}), \\ 0 & \text{if} \quad D^*_{it} > c_i(S_{t-1}), \end{cases} \tag{4.2}$$

where the function $c_i(S_{t-1})$ will be defined below. M_t and ϵ_{it} are independent of S_{t-1}, where S_{t-1} can be thought of as a variable representing the state of the

[1] Reduced form models begin by identifying relations between variables, while structural models are based on theories about the economy.

economy. S_{t-1} is based on the variable Z_{t-1}, which represents the economy. In order to mimic autocorrelation present in variables representing the economy, we will use a first order autoregressive model, i.e.

$$Z_t = \tau \cdot Z_{t-1} + \sqrt{1-\tau^2} \cdot \upsilon_t, \tag{4.3}$$

where τ determines the strength of the serial correlation and both Z_0 and υ_t are *iid* $N(0,1)$. Next, we define three different states of the economy, namely downturn, upturn and normal. The economy is in the downturn state if the value of Z_t is smaller than -1, while the economy is in the upturn state if the value of Z_t is greater than 1. Values between -1 and 1 are considered as normal. Hence, define

$$S_{t-1} = \begin{cases} DT & \text{if} \quad Z_{t-1} < -1, \\ N & \text{if} \quad -1 \leq Z_{t-1} < 1, \\ UT & \text{if} \quad Z_{t-1} \geq 1, \end{cases} \tag{4.4}$$

where S_{t-1} denotes the state of the economy at the beginning of $(t-1, t]$. Hence, the unconditional probability that the economy is in either the upturn or the downturn state equals $\Phi(-1) \approx 0.16$, while the probability that the economy is in the normal state equals $1 - 2 \cdot \Phi(-1) \approx 0.68$. Furthermore, as Z_t exhibits autocorrelation, the conditional probability of S_t being in one of the three possible states depends on the value of Z_{t-1}. This can be seen as follows:

$$\begin{aligned} P[S_t = DT \mid Z_{t-1}] &= P\left[Z_t < -1 \mid Z_{t-1}\right] \\ &= P\left[\tau \cdot Z_{t-1} + \sqrt{1-\tau^2} \cdot \upsilon_t < -1 \,\middle|\, Z_{t-1}\right] \\ &= P\left[\upsilon_t < \frac{-1}{\sqrt{1-\tau^2}} - \frac{\tau}{\sqrt{1-\tau^2}} \cdot Z_{t-1} \,\middle|\, Z_{t-1}\right] \\ &= \Phi\left(\frac{-1}{\sqrt{1-\tau^2}} - \frac{\tau}{\sqrt{1-\tau^2}} \cdot Z_{t-1}\right). \end{aligned} \tag{4.5}$$

Similarly, the conditional probabilities that S_t equals either N or UT also depend on the value of Z_{t-1}. Finally, $c_i(S_{t-1})$ is defined as follows:

$$c_i(S_{t-1}) = \begin{cases} c_i^{DT} & \text{if} \quad S_{t-1} = DT, \\ c_i^{N} & \text{if} \quad S_{t-1} = N, \\ c_i^{UT} & \text{if} \quad S_{t-1} = UT, \end{cases} \tag{4.6}$$

where c_i^{DT}, c_i^N and c_i^{UT} are constants. A default of obligor i thus occurs when the latent variable drops below the threshold $c_i(S_{t-1})$. From Equation (4.1) we can see that, at one point in time, the default events of loans are correlated only through M_t. Thus, M_t represents systematic risk, or cross-sectional dependence. Furthermore, ρ is a measure of the strength of cross-sectional dependence. In contrast to the framework described in Chapter 3, the occurence of default is also dependent on the state of the economy. Due to the autocorrelation in Z_t,

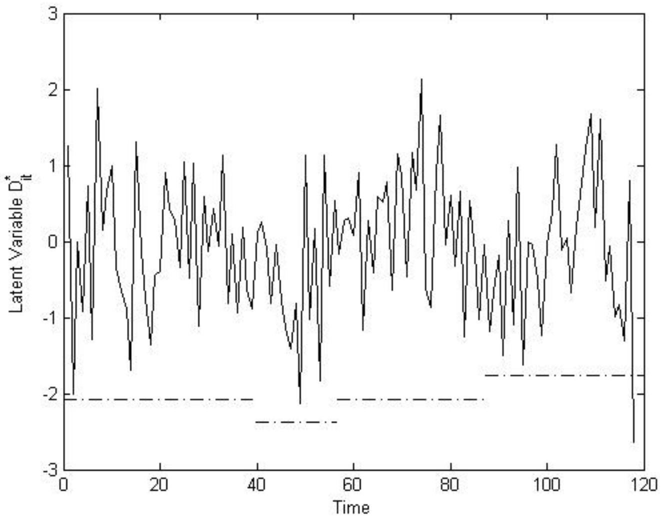


Fig. 4.1: Visualisation of the Vasicek model with regime switching for one obligor i.

the model exhibits serial dependence. Thus, the default events of loans exhibit dependence over time.

The framework described thus far can be visualised by means of Figure 4.1. The dotted line equals the threshold $c_i(S_{t-1})$. Default occurs when the latent variable D^*_{it}, represented by the black line, drops below the threshold. In this picture, default occurs at $t = 118$. Note that the threshold is not a constant function of time, but rather moves depending on the value of S_{t-1}. Hence, the probability that D^*_{it} drops below the threshold, and thus the probability that default occurs, also depends on the value of S_{t-1}.

4.1 Vasicek distribution with regime switching

As described in Section 3.1, loans with similar characteristics are grouped into buckets. Hence, we are not only interested in the default probability of one single loan, but rather in the distribution of the default fraction of a homogeneous credit portfolio. Similar to our derivation in Section 3.1, the framework described above can be elaborated to arrive at the distribution of the default fraction of a credit portfolio. In order to do so, we will assume again that the credit portfolio is infinitely granular.

Because both M_t and ϵ_{it} are standard normally distributed and independent of each other, the variable D^*_{it} is standard normally distributed as well. As S_{t-1} is independent of D^*_{it}, we can easily calculate the probability of default conditional on the state of the economy[2], i.e.

$$\begin{aligned} PD_{it}^{DT} &:= P[D_{it} = 1 \mid S_{t-1} = DT] = P[D_{it}^* \leq c_i^{DT}] = \Phi(c_i^{DT}), \\ PD_{it}^{N} &:= P[D_{it} = 1 \mid S_{t-1} = N] = P[D_{it}^* \leq c_i^{N}] = \Phi(c_i^{N}), \\ PD_{it}^{UT} &:= P[D_{it} = 1 \mid S_{t-1} = UT] = P[D_{it}^* \leq c_i^{UT}] = \Phi(c_i^{UT}). \end{aligned} \tag{4.8}$$

Hence, $c_i^{DT} = \Phi^{-1}(PD_{it}^{DT})$, $c_i^N = \Phi^{-1}(PD_{it}^N)$ and $c_i^{UT} = \Phi^{-1}(PD_{it}^{UT})$. Note that, in this framework, the value of the long term probability of default behaves according to a regime switching model, and, hence, also the value of the short term PD behaves differently during economic downturns and upturns, compared to normal states.

Analogous to our derivation in Section 3.1 we have, conditional on M_t and S_{t-1}, that

$$\begin{aligned} P[D_{it} = 1 \mid M_t, S_{t-1}] &= P[D_{it}^* \leq c_i(S_{t-1}) \mid M_t, S_{t-1}] \\ &= P[\sqrt{\rho} M_t + \sqrt{1-\rho}\epsilon_{it} \leq c_i(S_{t-1}) \mid M_t, S_{t-1}] \\ &= P[\epsilon_{it} \leq \frac{c_i(S_{t-1})}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t \mid M_t, S_{t-1}] \\ &= \Phi\left(\frac{c_i(S_{t-1})}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right). \end{aligned} \tag{4.9}$$

Loans are dependent only through the common risk factor M_t and the state of the economy S_{t-1}. Hence, conditional on M_t and S_{t-1} loans are independent and identically distributed. Thus, the default fraction θ, defined as in Equation 3.8, in an infinitely granular portfolio, conditional on M_t and S_{t-1}, equals

$$\theta = \Phi\left(\frac{c_i(S_{t-1})}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}} M_t\right). \tag{4.10}$$

The cdf of the default fraction in the portfolio is defined by the probability that θ takes on a value less than a possible given value. Hence, by using the law of iterated expectations we have that the cdf is defined as follows (for $x \in [0, 1]$):

$$P[\theta \leq x] = E[P[\theta \leq x \mid S_{t-1}]]. \tag{4.11}$$

[2] Note that the probability of default conditional on Z_{t-1} equals the probability of default conditional on S_{t-1}, given by Equation (4.8). This can be seen as follows.

$$\begin{aligned} P[D_{it} = 1 \mid S_{t-1}] &= E\left[P[D_{it} = 1 \mid Z_{t-1}, S_{t-1}] \mid S_{t-1}\right], \\ &= E\left[P[D_{it} = 1 \mid Z_{t-1}] \mid S_{t-1}\right], \\ &= P[D_{it} = 1 \mid Z_{t-1}]. \end{aligned} \tag{4.7}$$

Since M_t independent of S_{t-1} and $M_t \sim N(0,1)$ we have

$$\begin{aligned}
P[\theta \leq x \mid S_{t-1}] &= P\left[\Phi\left(\frac{c_i(S_{t-1})}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}}M_t\right) \leq x \;\middle|\; S_{t-1}\right] \\
&= P\left[M_t \geq -\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - c_i(S_{t-1})\right) \;\middle|\; S_{t-1}\right] \\
&= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - c_i(S_{t-1})\right)\right). \qquad (4.12)
\end{aligned}$$

Hence, combining Equation (4.11) with Equation (4.12) gives

$$\begin{aligned}
P[\theta \leq x] &= E\left[\Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - c_i(S_{t-1})\right)\right)\right] \\
&= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD^{DT})\right)\right) \cdot P[S_{t-1} = DT] \\
&\quad + \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD^{N})\right)\right) \cdot P[S_{t-1} = N] \\
&\quad + \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(PD^{UT})\right)\right) \cdot P[S_{t-1} = UT]. \qquad (4.13)
\end{aligned}$$

We have arrived at the cdf of the default fraction of an infinitely granular portfolio, where the long term value of the probability of default follows a regime switching model. Note that the cdf of the Vasicek model with regime switching equals the weighted average of the three cdf's (corresponding to the three different values of the probability of default) of the Vasicek model without regime switching. The weights equal the probabilities that the economy is in one of the three different states. Furthermore, the pdf of the default fraction of an infinitely granular portfolio can be acquired by differentiating the cdf. Figure 4.2 shows the pdf of the Vasicek model both with and without regime switching. Furthermore, Figure 4.3 shows details of the left and right tails of the distribution. Note that the pdf with regime switching is higher than the pdf without regime switching on both the left and right tails of the distribution, while it is lower in the middle.

4.2 Capital Requirements

The capital requirement function under the Basel capital accords is based on the concept of Value-at-Risk (VaR), where the underlying distribution is the Vasicek distribution as described in Section 3.1. However, in Section 4.1 we argued that a regime switching version of the Vasicek model leads to a different distribution. Hence, the corresponding VaR differs from the VaR based on the Vasicek distribution. The $(1-\alpha)\%$ VaR for the Vasicek model with regime switching can be calculated by determining the value of θ for which the cumulative distribution function equals $(1-\alpha)\%$. The cumulative distribution function

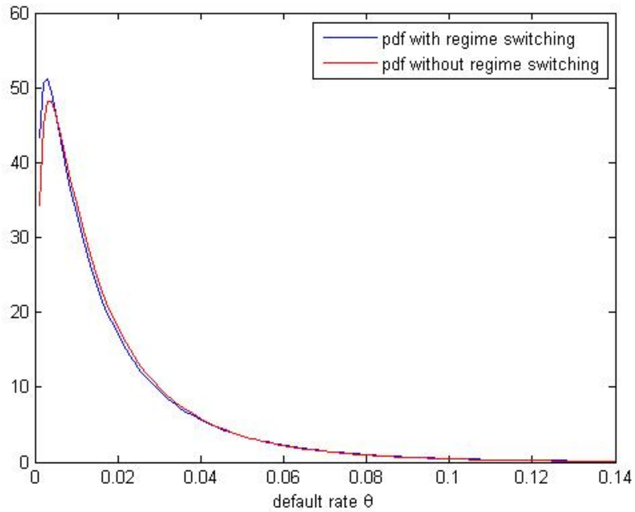
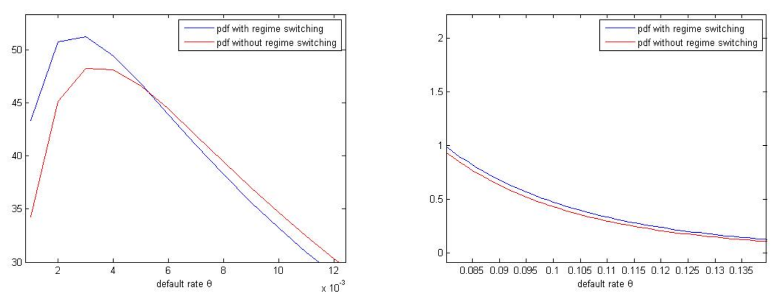


Fig. 4.2: Density of the Vasicek distribution with and without regime switching for $\rho = 0.15$ and $PD = 0.02$. Furthermore, $PD^{DT} = 0.03$, $PD^{N} = 0.02$ and $PD^{UT} = 0.01$. Note that, under the regime switching version of the Vasicek distribution, we have that $E[\theta] = 0.02$, which equals $E[\theta]$ under the Vasicek distribution.



(a) A detailed picture of the left tail of the distribution.

(b) A detailed picture of the right tail of the distribution.

Fig. 4.3: Detailed pictures of the density of the Vasicek distribution with and without regime switching for $\rho = 0.15$ and $PD = 0.02$.

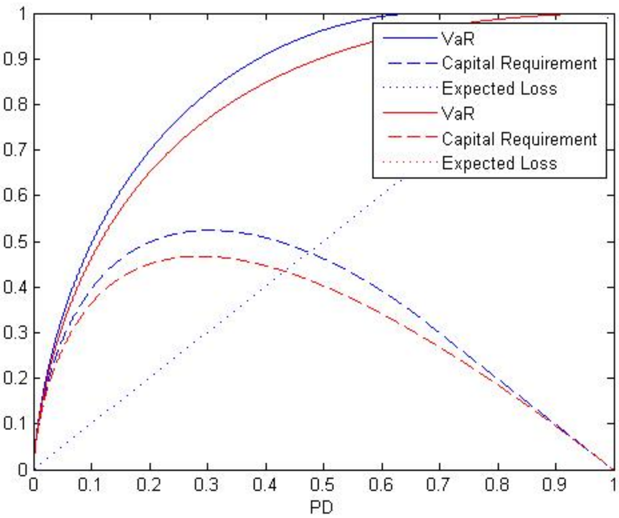


Fig. 4.4: VaR, Expected Loss and Capital Requirement against PD for $\rho = 0.15$ and $\alpha = 0.01\%$. Note that VaR equals Expected Loss plus Capital Requirement. The blue lines are based on the Vasicek model with regime switching, while the red lines are based on the Vasicek model. Furthermore, $PD^N = PD$, $PD^{DT} = PD \cdot 150\%$, $PD^{UT} = PD \cdot 50\%$. Note that, under the regime switching version of the Vasicek model, we have that $E[\theta] = PD^N$, which equals $E[\theta]$ under the Vasicek distribution.

is given by Equation (4.13). However, as this function is rather complex, we are unable to find an explicit formula for the VaR. Hence, we will determine the VaR numerically. Figure 4.4 shows the capital requirement, the VaR and the expected loss as a function of PD, for both the Vasicek distribution and the Vasicek distribution with regime switching. The capital requirement based on the Vasicek distribution with regime switching is always higher than the capital requirement based on the Vasicek model. Hence, if the Vasicek model with regime switching is the appropriate model to describe reality, the capital requirement is consistently being underestimated when using the Vasicek model as underlying distribution. Underestimation is highest for PD approximately equal to 41% and, in this case, amounts more than 7%.

5. THE VASICEK MODEL AND TRANSITION MATRICES

In Chapter 3 we examined the Vasicek model underlying the capital requirements as prescribed by the Basel Committee on Banking Supervision. In Chapter 4 we introduced serial dependence as well as regime switching, in order to make the Vasicek model more consistent with observed default rates.

Capital requirements are determined for the different buckets in a bank's portfolio. Over time, the amount of loans per bucket does not necessarily need to stay the same. As mentioned in Section 3.1, banks determine the risk that is associated to a loan by means of, among others, individual characteristics and macroeconomic variables such as houseprices. As the characteristics underlying the credit scores change, credit scores are revised over time. Hence, over time, loans may transition between buckets. In this chapter we will model the fluctuations in the distribution of loans over the buckets by introducing transition matrices.

5.1 The transition matrix

First, we will introduce transition matrices in the case where there is no cross-sectional dependence. Note that assuming that there is no cross-sectional dependence is the same as assuming that $\rho = 0$ in the Vasicek model. A transition matrix describes the probability that a loan migrates from one bucket to another. As scorecards are very complex, and so are the dynamics of the characteristics underlying the scorecards, we will use the transition matrix as a simplified way to mimic the fluctuations observed in credit scores.

Suppose there are five buckets. Then we define a 6x6 transition matrix P as follows:

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} & p_{1,def} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} & p_{2,def} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} & p_{3,5} & p_{3,def} \\ p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} & p_{4,5} & p_{4,def} \\ p_{5,1} & p_{5,2} & p_{5,3} & p_{5,4} & p_{5,5} & p_{5,def} \\ p_{def,1} & p_{def,2} & p_{def,3} & p_{def,4} & p_{def,5} & p_{def,def} \end{bmatrix}, \tag{5.1}$$

where $p_{k,j}$ represents the probability that a loan transitions from bucket k to bucket j for $k, j \in \{1, 2, 3, 4, 5\}$. Furthermore, $p_{k,def}$ and $p_{def,j}$ represent the probability that a loan in bucket k will go into default and the probability that a defaulted loan will be replaced by a loan in bucket j, respectively. Note that, for simplicity, we discard the loans that leave the portfolio due to other reasons than default, for example because of reaching maturity of the loan. Furthermore, we assume that new loans only enter the portfolio to replace defaulted loans, and that all defaulted loans will be replaced by a new loan.

Next, we define a vector b_t that represents the probability that a loan sits in a bucket at time t as follows:

$$b_t = \begin{bmatrix} b_{1,t} & b_{2,t} & b_{3,t} & b_{4,t} & b_{5,t} & b_{def,t} \end{bmatrix}, \tag{5.2}$$

where $b_{j,t}$ represents the probability that a loan is situated in bucket j at time t. Hence, at time $t+1$ the vector that represents the probability that a loan is situated in a bucket equals

$$b_{t+1} = b_t \cdot P. \tag{5.3}$$

Thus, the framework can be described as a Markov chain, as the next state depends solely on the current state through b_t, and not on other states.

Note that under certain conditions[1], we have that

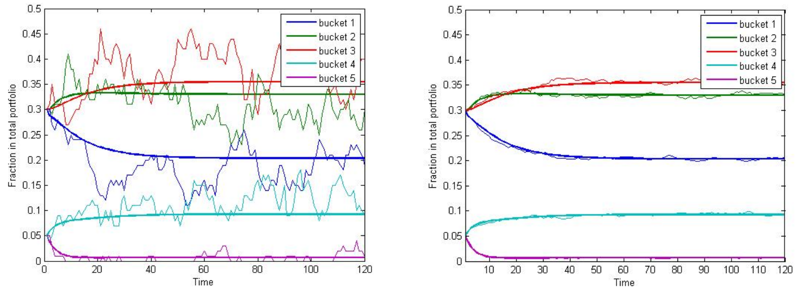
$$\lim_{n \to \infty} P^n_{k,j} = \pi_j, \tag{5.4}$$

for every $k,\ j \in \{1, 2, 3, 4, 5, def\}$, where $\pi = [\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \pi_5 \quad \pi_{def}]$ is the left eigenvector of P, corresponding to eigenvalue 1. Hence, in the long run, the system evolves to a stationary state meaning that the probability of being in state j does not depend on the initial state k. Thus, in the long run, the probability of being in a particular state does not change over time.

To illustrate the dynamics of the distribution of loans over the buckets within the framework described above, we will simulate the sizes of the buckets using the transition matrix P described in Appendix B. This matrix is based upon the transition matrix as described by Nickell et al. (2000)[2]. Figure 5.1 shows the dynamics of the buckets of loans over time. The thick lines represent the probability that a loan sits in one particular bucket, while the thin lines represent a simulation of the number of loans in one particular bucket. As implied by Equation (5.4), we see that the probabilities that a loan sits in one particular bucket converge to certain values, and remain at these values after some period of time. In particular, we have that the probabilities that a loan sits in a bucket converge to the left eigenvector of P, corresponding to eigenvalue 1. Hence, the left eigenvector of P equals $\pi = [0.2014 \quad 0.3228 \quad 0.3624 \quad 0.0947 \quad 0.0073 \quad 0.0114]$. Furthermore, in Figure 5.1 (a), we see that the thin line fluctuates around the thick line. In contrast, in Figure 5.1 (b), we see that the thin line follows the thick line very closely. This observation can be explained by the fact that our

[1] Equation (5.4) holds if the Markov chain P is irreducible, aperiodic and positive recurrent. A Markov chain is said to be irreducible if it is possible to access any state from any state, aperiodic if a loan can return to its current bucket in an irregular number of steps and positive recurrent if a loan always returns to its current bucket in finitely many steps.

[2] Note that it is not the purpose of this thesis to find the best estimate of the transition matrix. Therefore, we will adopt the transition matrix as described by Nickell et al. (2000). Because the matrix is based on corporate and sovereign bond ratings, we will slightly adapt it to fit our framework.



(a) N=100 *(b)* N=10,000

Fig. 5.1: The distribution of loans over buckets for one transition matrix and no cross-sectional dependence, for portfolios with 100 and 10,000 obligors, respectively. The thick line represents the probability that a loan sits in a particular bucket, while the thin line represents a realization of the fraction of loans per bucket.

framework lacks systematic risk. The remaining risk is idiosyncratic risk, which will be diversified away when the number of loans in the portfolio gets larger. Thus, there is still a lot of idiosyncratic risk left in a portfolio with 100 obligors, while the idiosyncratic risk is diversified to a great extend in a portfolio with 10,000 obligors.

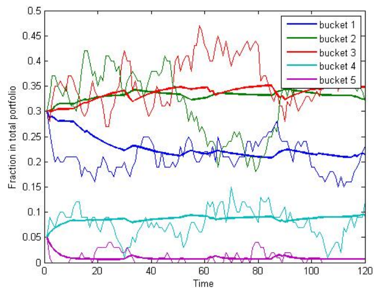
5.2 The transition matrix and regime switching

As mentioned in Chapter 4, empirical observations of default rates indicate that the short term PD depends on the state of the economy. This observation does not only apply to default rates, but can be extended to the entire transition matrix. For example, Crook and Bellotti (2010) state that the chance that an applicant will default in the future depends on the state of a country's macroeconomy, but also that the ranking in terms of risk of individuals who apply for a loan depends on the state of the macroeconomy. Furthermore, Nickell et al. (2000) find that transition matrices differ during business cycle peaks and throughs.

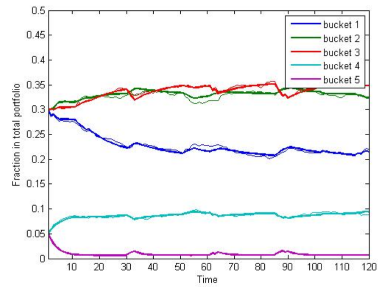
Note that we still assume that there is no cross-sectional dependence. In this section we will incorporate regime switching in the transition matrix. Hence, we define the variable S_{t-1}, which can be thought of as a variable that denotes the state of the economy at the beginning of period $(t-1,t]$. Note that the variable equals the variable S_{t-1} as defined by Equation 4.4, i.e.

$$S_{t-1} = \begin{cases} DT & \text{if} \quad Z_{t-1} < -1, \\ N & \text{if} \quad -1 \leq Z_{t-1} < 1, \\ UT & \text{if} \quad Z_{t-1} \geq 1. \end{cases} \tag{5.5}$$

The variable Z_t is as defined in Equation (4.3). We will make the transition



(a) N=100



(b) N=10,000

Fig. 5.2: The distribution of loans over buckets for multiple transition matrices exhibiting regime switching and no cross-sectional dependence, for portfolios with 100 and 10,000 obligors, respectively. The thick line represents the probability that a loan sits in a particular bucket, while the thin line represents a realization of the fraction of loans per bucket.

matrix dependent on the state of the economy as follows:

$$P_t = \begin{cases} P^{DT} & \text{if} \quad S_{t-1} = DT, \\ P^N & \text{if} \quad S_{t-1} = N, \\ P^{UT} & \text{if} \quad S_{t-1} = UT, \end{cases} \tag{5.6}$$

where P^{DT}, P^N and P^{UT} are transition matrices as defined by Equation (5.1). Note that, by Equation (5.4), each of the transition matrices has its own stationary state. If S_t stays in, for example, the downturn state long enough, b_t evolves to π^{DT}, where π^{DT} is the left eigenvector of P^{DT} corresponding to eigenvalue 1. The same reasoning can be applied to the normal and upturn states.

To illustrate the dynamics of the distribution of loans over the buckets, we will simulate the sizes of the buckets using the transition matrices described in Appendix B. Again, we adopt the transition matrices as described by Nickell et al. (2000). Figure 5.2 shows the dynamics of the buckets of loans over time. In contrast to Figure 5.1, the probability that a loan sits in a particular bucket does not converge to a certain value. Rather, the thick lines fluctuate over time. This is due to the fact that different transition matrices correspond to different moments in time. Similar to our observation in Figure 5.1, we again see that idiosyncratic risk is diversified to a great extend in the portfolio with 10,000 obligors, while there is still a lot of idiosyncratic risk in the portfolio with 100 obligors.

5.3 The transition matrix and the Vasicek model

In the previous sections we assumed that transitions do not exhibit cross-sectional dependence. Hence, within the framework of the Vasicek model, we have that $\rho = 0$. However, in Chapter 3 and 4, we assumed that the default events of loans in a portfolio are subject to a single common risk factor. Therefore, part of the credit risk in a portfolio of loans is systematic. In this section, we will incorporate cross-sectional dependence in the transitions between different buckets, thereby relaxing the assumption that $\rho = 0$. Similar to Chapters 3 and 4, we introduce a latent variable B^*_{it}, which consists of a variable representing systematic risk and a variable representing idiosyncratic risk, i.e.

$$B^*_{it} = \sqrt{\rho} M_t + \sqrt{1-\rho}\epsilon_{it}, \tag{5.7}$$

where ϵ_{it} are *iid* $N(0,1)$ over i and t, independent of M_t *iid* $N(0,1)$. Next, we define the function B_{it} that indicates which bucket is assigned to loan i at time t as follows:

$$B_{it} = \begin{cases} 1 & \text{if} \quad c_{k,1} \leq B^*_{it} < c_{k,0}, \\ 2 & \text{if} \quad c_{k,2} \leq B^*_{it} < c_{k,1}, \\ 3 & \text{if} \quad c_{k,3} \leq B^*_{it} < c_{k,2}, \\ 4 & \text{if} \quad c_{k,4} \leq B^*_{it} < c_{k,3}, \\ 5 & \text{if} \quad c_{k,5} \leq B^*_{it} < c_{k,4}, \\ default & \text{if} \quad c_{k,6} \leq B^*_{it} < c_{k,5}, \end{cases} \tag{5.8}$$

where k represents the bucket assigned at time $t-1$ and $c_{k,j}$ is a constant for all $k, j \in \{1,2,3,4,5,default\}$. Furthermore, $c_{k,0} = \infty$ and $c_{k,6} = -\infty$ for all $k \in \{1,2,3,4,5,default\}$[3].

The transition of a loan from bucket k to bucket j thus occurs when the value of the latent variable falls in between the thresholds related to the transition from bucket k to bucket j. Each of the transitions has its own thresholds. From Equation (5.7) we can see that, at one point in time, the transitions of loans are correlated only through M_t. Thus, M_t represents systematic risk, or cross-sectional dependence. ρ is a measure of the strength of cross-sectional dependence.

Because both M_t and ϵ_{it} are standard normally distributed and independent of each other, the variable B^*_{it} is standard normally distributed as well. Due to this observation, we can easily calculcate the probability that a loan transitions from bucket k to bucket j as follows:

$$\begin{aligned} P[B_{it} = j \mid B_{i,t-1} = k] &= P[c_{k,j} \leq B^*_{it} < c_{k,j-1}] \\ &= \Phi(c_{k,j-1}) - \Phi(c_{k,j}). \end{aligned} \tag{5.9}$$

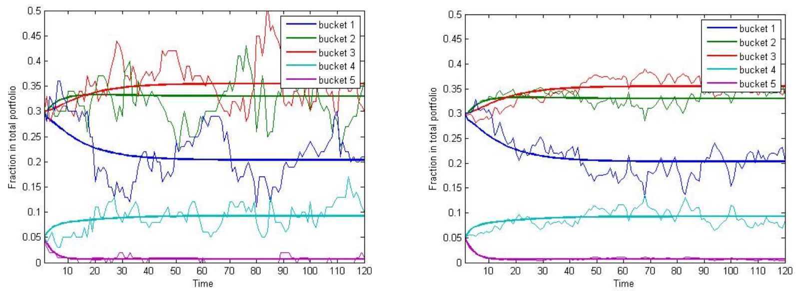
Note that

$$P[B_{it} = default \mid B_{i,t-1} = k] = \Phi(c_{k,5}). \tag{5.10}$$

Resultantly, by iteratively applying Equation (5.10) for all buckets, we have that

$$\Phi(c_{k,j}) = \sum_{l=j+1}^{6} P[B_{it} = l \mid B_{i,t-1} = k]. \tag{5.11}$$

[3] The values for $c_0(B_{i,t-1}, S_{t-1})$ and $c_6(B_{i,t-1}, S_{t-1})$ follow from the fact that B^*_{it} can take any real value and each i must be classified in one of the categories.



(a) N=100 *(b)* N=10,000

Fig. 5.3: The distribution of loans over buckets for multiple transition matrices exhibiting cross-sectional dependence and no regime switching, for portfolios with 100 and 10,000 obligors, respectively. The thick line represents the probability that a loan sits in a particular bucket, while the thin line represents a realization of the fraction of loans per bucket.

From this it follows that

$$P[B_{it} = j \mid B_{i,t-1} = k] = \Phi(c_{k,j-1}) - \sum_{l=j+1}^{6} P[B_{it} = l \mid B_{i,t-1} = k]. \tag{5.12}$$

Hence, we have that

$$c_{k,j-1} = \Phi^{-1}\left(\sum_{l=j}^{6} P_{k,j}\right), \tag{5.13}$$

where $P_{k,j} := P[B_{it} = j \mid B_{i,t-1} = k]$, which is given by the $(k, j)th$ entry in P.

Again, we will illustrate the dynamics of the distribution of loans over the buckets. Under the framework described in this section, we simulate the sizes of the buckets using the transition matrix described in Appendix B. Note that we are able to determine the thresholds $c_{k,j}$ by means of Equation (5.13) and the transition probabilities given by the transition matrix. Figure 5.3 shows the dynamics of the buckets of loans over time. Note that the thick lines, representing the probability that a loans sits in a particular bucket, are equal to the thick lines in Figure 5.1. Note that, again, from Figure 5.3 (a), we can see that there is a lot of idiosyncratic risk present. However, in contrast to our observations in previous figures, the thin lines in Figure 5.3 (b) also fluctuate, albeit less than in Figure 5.3 (a). This is due to the fact that we have introduced systematic risk. Hence, since systematic risk cannot be diversified away when the number of loans in the portfolio goes to infinity, the realized number of loans per bucket can be quite different from the expected number of loans.

5.4 The transition matrix and a regime switching version of the Vasicek model

In the previous section we assumed that transitions exhibit cross-sectional dependence, but no regime switching. However, in Section 5.2 we argued that transition matrices depend on the state of the economy. Hence, we will introduce regime switching within the methodoloy described in the previous section. We will do so by making the transition thresholds dependent on the state of the economy, in a similar fashion as the default thresholds depend on the state of the economy in Chapter 4. Resultantly, we define

$$B_{it} = \begin{cases} 1 & \text{if} \quad c_{k,1}(S_{t-1}) \leq B^*_{it} < c_{k,0}(S_{t-1}), \\ 2 & \text{if} \quad c_{k,2}(S_{t-1}) \leq B^*_{it} < c_{k,1}(S_{t-1}), \\ 3 & \text{if} \quad c_{k,3}(S_{t-1}) \leq B^*_{it} < c_{k,2}(S_{t-1}), \\ 4 & \text{if} \quad c_{k,4}(S_{t-1}) \leq B^*_{it} < c_{k,3}(S_{t-1}), \\ 5 & \text{if} \quad c_{k,5}(S_{t-1}) \leq B^*_{it} < c_{k,4}(S_{t-1}), \\ default & \text{if} \quad c_{k,6}(S_{t-1}) \leq B^*_{it} < c_{k,5}(S_{t-1}), \end{cases} \tag{5.14}$$

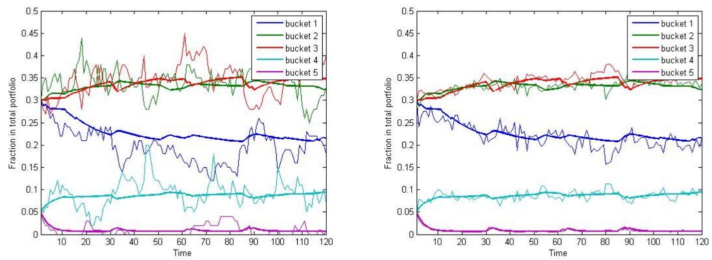
where k represents the bucket assigned at time $t-1$, $c_{k,0} = \infty$ and $c_{k,6} = -\infty$ for all $k \in \{1, 2, 3, 4, 5, default\}$. Furthermore, M_t and ϵ_{it} are independent of S_{t-1}, where S_{t-1} is a variable representing the state of the economy, defined in Equation 4.4. For all $j \in \{1, 2, 3, 4, 5\}$, the function $c_{k,j}(S_{t-1})$ is defined as

$$c_{k,j}(S_{t-1}) = \begin{cases} c^{DT}_{k,j} & \text{if} \quad S_{t-1} = DT, \\ c^{N}_{k,j} & \text{if} \quad S_{t-1} = N, \\ c^{UT}_{k,j} & \text{if} \quad S_{t-1} = UT, \end{cases} \tag{5.15}$$

where $c^{DT}_{k,j}$, $c^{DT}_{k,j}$ and $c^{DT}_{k,j}$ are constants for all $k, j \in \{1, 2, 3, 4, 5, default\}$. The transition of a loan from bucket k to bucket j thus occurs when the value of the latent variable falls in between the thresholds related to the transition from bucket k to bucket j and the prevalent state of the economy. Each of the transitions has its own thresholds. From Equation (5.7) we can see that, at one point in time, the transitions of loans are correlated only through M_t. Thus, M_t represents systematic risk, or cross-sectional dependence. ρ is a measure of the strength of cross-sectional dependence. Furthermore, transitions are also dependent on the state of the economy and over time. Due to the autocorrelation in Z_t, the model exhibits serial dependence.

Because both M_t and ϵ_{it} are standard normally distributed and independent of each other, the variable B^*_{it} is standard normally distributed as well. As S_{t-1} and $B_{i,t-1}$ are independent of B^*_{it}, we can easily calculate the probability of default conditional on the state of the economy and conditional on the bucket assigned at time $t-1$. The conditional probability of default equals

$$\begin{aligned} P[B_{it} = j \mid B_{i,t-1} = k, S_{t-1} = DT] &= P[c^{DT}_{k,j} \leq B^*_{it} < c^{DT}_{k,j-1}] \\ &= \Phi(c^{DT}_{k,j-1}) - \Phi(c^{DT}_{k,j}). \end{aligned} \tag{5.16}$$



(*a*) N=100 (*b*) N=10,000

Fig. 5.4: The distribution of loans over buckets for multiple transition matrices exhibiting regime switching as well as cross-sectional dependence, for portfolios with 100 and 10,000 obligors, respectively. The thick line represents the probability that a loan sits in a particular bucket, while the thin line represents a realization of the fraction of loans per bucket.

Hence, similar to our derivation in Section 5.3, we see that

$$\begin{aligned} c_{k,j-1}^{DT} &= \Phi^{-1}\left(\sum_{l=j}^{6} P_{k,j}^{DT}\right), \\ c_{k,j-1}^{N} &= \Phi^{-1}\left(\sum_{l=j}^{6} P_{k,j}^{N}\right), \\ c_{k,j-1}^{UT} &= \Phi^{-1}\left(\sum_{l=j}^{6} P_{k,j}^{UT}\right), \end{aligned} \tag{5.17}$$

where $P_{k,j}^{DT} := P[B_{it} = j \mid B_{i,t-1} = k, S_{t-1} = DT]$, $P_{k,j}^{N} := P[B_{it} = j \mid B_{i,t-1} = k, S_{t-1} = N]$ and $P_{k,j}^{UT} := P[B_{it} = j \mid B_{i,t-1} = k, S_{t-1} = UT]$. These probabilities are given by the $(k,j)th$ entries in P^{DT}, P^{N} and P^{UT}, respectively.

Again, we will illustrate the dynamics of the distribution of loans over the buckets. Under the framework described in this section, we simulate the sizes of the buckets using the transition matrices described in Appendix B. Note that we are able to determine the thresholds $c_{k,j}$ by means of Equation (5.17) and the transition probabilities given by the transition matrices. Figure 5.4 shows the dynamics of the buckets of loans over time. Note that the thick lines, representing the probability that a loans sits in a particular bucket, are equal to the thick lines in Figure 5.2. Note that, again, from Figure 5.4 (a), we can see that there is a lot of idiosyncratic risk present. Furthermore, the thin lines in Figure 5.4 (b) fluctuate, albeit less than in Figure 5.4 (a). This is due to systematic risk. Since systematic risk cannot be diversified away when the number of loans in the portfolio goes to infinity, the realized number of loans per bucket can be quite different from the expected number of loans.

6. RATING PHILOSOPHY

In Chapters 3 to 5 we have developed a dynamic version of the Vasicek model by introducing regime switching, serial dependence and transition matrices. Particularly, we have been interested in the implications of our model for the capital requirements as described in the Basel capital accords. In this chapter we will investigate the implications of banks' rating philosophy on the capital requirements, within the framework described in preceding chapters.

In general, we can distinguish two different rating philosophies: through-the-cycle (TTC) and point-in-time (PIT). We will adopt the definition of TTC and PIT as given by Tasche (2006). Tasche (2006) explains the TTC rating philosophy as the philosophy "where rating grades are assumed to express the same degree of creditworthiness at any time and economic downturns are only reflected by a shift of the score distribution towards the worse scores." In contrast, the PIT rating philosophy is defined as the philosophy "according to which one and the same rating grade can reflect different degrees of creditworthiness, depending on the state of the economy." The Basel Committee on Banking Supervision (BCBS) is ambiguous about the philosophy banks should use in the estimation of the PD.

By Tasche (2006), "PD estimates can be based (or, technically speaking, conditioned) on the current state of the economy, for instance by inclusion of macro-economic co-variates in a regression process. [...] The resulting PD estimates are then called point-in-time (PIT)." Hence, within the framework described in Chapter 4, the PIT estimate of PD equals

$$PD_t^{PIT} = P[D_{it} = 1|S_{t-1}], \tag{6.1}$$

for all i in the concerning bucket. In contrast, "unconditional PD estimates are not based on a current state of the economy. Unconditional PDs that are estimated based on data from a complete economic cycle are called through-the-cycle (TTC)." Thus, the TTC estimate of PD can be defined as the expected value of PD_t^{PIT}, i.e.

$$PD^{TTC} = E[PD_t^{PIT}]. \tag{6.2}$$

Note that, if PD_t^{PIT} is stationary over time, PD^{TTC} does not depend on time.

6.1 Capital Requirements per bucket

In this section we will investigate the implications of the rating philosophy for the capital requirements per bucket. The VaR per bucket is calculated by means of Equation (4.13), i.e.

$$\begin{aligned} P[\theta \leq x] &= E\left[\Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - c_i(S_{t-1})\right)\right)\right] \\ &= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD^{DT})\right)\right) \cdot P[S_{t-1} = DT] \\ &\quad + \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD^{N})\right)\right) \cdot P[S_{t-1} = N] \\ &\quad + \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD^{UT})\right)\right) \cdot P[S_{t-1} = UT]. \end{aligned} \tag{6.3}$$

Note that, under the TTC rating philosophy, we do not condition on the state of the economy. Hence, as explained in Chapter 4, we arrive at the capital requirements by setting Equation (6.3) equal to 99.9% and solving for x, and subtracting the expected loss. The obtained capital requirements are higher than the capital requirements under the Vasicek single factor model, even if the expected value of the default fraction is the same.

In contrast, under the PIT rating philosophy we do condition on the state of the economy. If we extend this philosophy to Equation (6.3), we see that

$$\begin{aligned} P[\theta \leq x \mid S_{t-1}] &= E\left[\Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - c_i(S_{t-1})\right)\right) \,\middle|\, S_{t-1}\right] \\ &= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - c_i(S_{t-1})\right)\right), \end{aligned} \tag{6.4}$$

where $c_i(S_{t-1})$ as defined in Equation (4.6). Again, we arrive at the capital requirements by setting Equation 6.4 equal to 99.9% and solving for x, and subtracting the expected loss. Note that Equation 6.4 equals the cdf of the Vasicek single factor model, the only difference being the value of PD. In the Vasicek single factor model, the value of PD remains constant over time, while the value of PD in Equation (6.4) depends on the state of the economy. Resultantly, the capital requirements based on the Vasicek single factor model equal the capital requirements based on Equation (6.4) when the value of PD equals the value of PD conditional on the state of the economy. Furthermore, due to concavity of the capital requirement formula, the expected value of the capital requirement based on Equation (6.4) is lower than the capital requirement based on the Vasicek single factor model, even if the expected value of the default fraction is the same.

6.1.1 Procyclicality

Figure 6.1 shows the (conditional and unconditional) capital requirements under the Vasicek model with regime switching. Note that the PIT capital requirements are correlated to the economy. In particular, the estimated correlation for

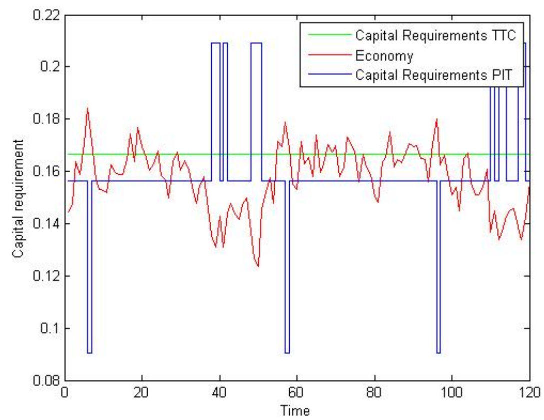


Fig. 6.1: A simulation of capital requirements over time for $PD^{DT} = 0.03$, $PD^{N} = 0.02$ and $PD^{UT} = 0.01$. Note that the capital requirements under the PIT rating philosophy represent the conditional capital requirements as given by Equation (6.4), while the capital requirements under the TTC rating philosophy represent the unconditional capital requirements as given by Equation (6.3). Furthermore, Z is scaled such that we can easily compare its dynamics to the capital requirements.

the simulation shown in Figure 6.1 equals -0.653. Negative correlation between capital requirements and the economy is an important subject, as it leads to procyclicality. According to Borio et al. (2001) “the movement in a financial indicator is said to be procyclical if it tends to amplify business cycle fluctuations. According to this definition, for instance, provisions behave procyclically if they fall in economic upswings and rise in downswings.” This can be seen as follows. When economy deteriorates, risk goes up and thus capital requirements go up. In order to satisfy capital requirements banks will have to attract money. However, since it is expensive to attract money in periods of economic distress, banks may reduce lending. This causes household spending to drop, thereby amplifying economic distress. From this point of view, procyclicality is an undesirable property. Hence, in order to reduce procyclicality in capital requirements, BCBS introduced a countercyclical buffer in their Basel III accord. This countercyclical buffer increases in economic upswings, thereby reducing the tendency of capital requirements to fall during upswings and rise in downswings.

Thus, a disadvantage of the PIT rating philiosophy is its tendency to increase procyclicality while, when examining one bucket, this is not apparent for the TTC rating philosophy.

6.2 Total Capital Requirements

In this section, we will investigate the implications of the rating philosophy for the capital requirements for the total portfolio. As pointed out in the previous sections, the PD assigned to a bucket under the TTC rating philosophy will not depend on the state of the economy. Hence, the PD of a bucket stays the same over time. However, in order to reflect the increase or decrease in risk caused by a change in the economy, loans will transition from one bucket to another. Resultantly, the probability that a loan transitions from bucket i to bucket j will depend on the state of the economy. In contrast, under the PIT rating philosophy, an increase or decrease in risk caused by a change in the economy is reflected by a change in the PD assigned to a bucket. If, under the PIT rating philosophy, the probability that a loan transitions form bucket i to bucket j depends on the state of the economy, then differences in risk are reflected by different PDs assigned to the buckets as well as by a difference in the distribution of loans over the buckets. If, on the other hand, the probability that a loan transitions from bucket i to bucket j under the PIT rating philosophy does not depend on the state of the economy, then differences in risk are entirely reflected by different PDs assigned to the buckets.

Within the framework described in Chapters 4 and 5, we are able to model total capital requirements under both the TTC and PIT rating philosophy. Similar to Section 6.1, the capital requirements per bucket under the TTC and PIT rating philosophies are given by the unconditional and conditional capital requirements per bucket, respectively. By definition, we arrive at the total capital requirements by taking the sum of capital requirements per bucket multiplied by the total exposure per bucket. If we assume that all loans consist of one euro, the total exposure per bucket is equal to the number of loans per bucket. The number of loans per bucket can be calculated by means of Chapter 5. In particular, as transitions between buckets under the TTC rating philosophy depend on the state of the economy, we will use the methodology described in Section 5.4 for modelling the transitions under the TTC philosophy. On the other hand, as transitions between buckets under the PIT rating philosophy do or do not depend on the state of the economy, we will use the methodology described in Section 5.4 as well as the methodology described in Section 5.3 for modelling the transitions under the PIT philosophy.

In the remainder of this section, we will perform simulations in order to analyse the performance of the total capital requirements under both the PIT and TTC rating philosophy. For the simulation of the capital requirements per bucket, we will base the default probabilities on the default probabilities given by the last column of the transition matrices in Appendix B. However, due to rounding to two decimal places, the default probabilities in the appendix for the first three buckets are not very precise. Hence, we will use the following default probabilities for the downturn state, the normal state and the upturn state, respectively:

$$PD^{DT} = \begin{bmatrix} 0.00045 \\ 0.00375 \\ 0.01125 \\ 0.10000 \\ 0.23000 \end{bmatrix}, \; PD^{N} = \begin{bmatrix} 0.00030 \\ 0.00250 \\ 0.00750 \\ 0.07000 \\ 0.16000 \end{bmatrix}, \; PD^{UT} = \begin{bmatrix} 0.00015 \\ 0.00125 \\ 0.00375 \\ 0.05000 \\ 0.18000 \end{bmatrix}. \tag{6.5}$$

Furthermore, for the simulation of the distribution of the loans over the buckets, we will use the transition matrices as given in Appendix B. We assume that the portfolio consists of 10,000 loans, meaning that idiosyncratic risk is diversified to a great extend. Furthermore, simulations are based on 10,000 scenarios.

6.2.1 Performance

Capital requirements are designed to protect against unexpected losses. Hence, in order to evaluate capital requirements, it is important to take into account the performance of capital requirements with respect to the protection against these losses. In general, high capital requirements are more likely to give a good protection against losses. However, high capital requirements might lead to a low return on capital, as capital cannot be used to invest in risky opportunities generating a high return. Hence, in theory, there is a tradeoff between protection against losses and the level of capital requirements. Based on 10,000 simulations of capital requirements over 120 time periods, we calculated the mean and standard deviation of capital requirements under the TTC rating philosophy, PIT rating philosophy with regime switching and PIT rating philosophy without regime switching. Table 6.1 shows the results. Note that the average

	Mean	Std. dev.
TTC	0.0797	0.0079
PIT, w/ regime switching	0.0751	0.0114
PIT, w/o regime switching	0.0761	0.0116

Tab. 6.1: The mean and standard deviation of total capital requirements. Simulations are performed under the TTC rating philosophy, PIT rating philosophy with regime switching and PIT rating philosophy without regime switching, for the transition matrices given in Appendix B and values of PD as given by Equation (6.5).

capital requirement under the TTC philosophy is higher than the average capital requirement under both PIT philosophies. This can be explained by our conclusions from Section 6.1, as we found that the value of capital requirements per bucket under the TTC rating philosophy is higher than the expected value of capital requirements per bucket under the PIT rating philosophy. Furthermore, we find that the standard deviation of capital requirements under the TTC rating philosophy is lower.

Figure 6.2 shows the distribution of losses at t=50 for 10,000 simulations over 120 time periods of the losses in a portfolio of 10,000 loans, where transitions

	Mean	Std. dev.
TTC	0.0917%	0.0301%
PIT, w/ regime switching	0.1387%	0.0368%
PIT, w/o regime switching	0.1314%	0.0356%

Tab. 6.2: The average number of times the unexpected losses exceed the capital requirements under the TTC rating philosophy, the PIT rating philosophy with regime switching and the PIT rating philosophy without regime switching.

between buckets are modelled as described in Section 5.4. The losses shown in

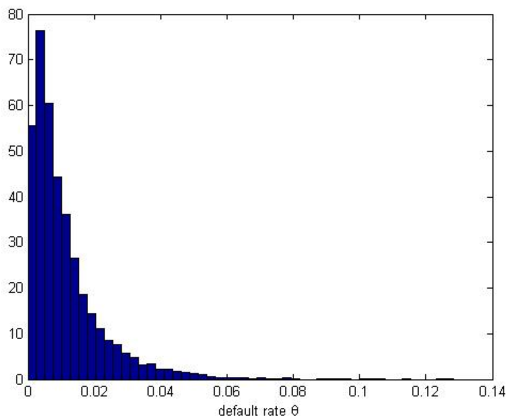


Fig. 6.2: The loss distribution at t=50 for 10,000 simulations of the losses in a portfolio of 10,000 loans over 120 time periods, for the methodology described in Section 5.4.

Figure 6.2 exceeded the capital requirements under the TTC rating philosophy at t=50 in 0.11% of the simulations. In contrast, at t=50, losses exceeded capital requirements under the PIT rating philosophy in 0.17% and 0.16% of the simulations, in the case with regime switching and in the case without regime switching, respectively. The average percentage of exceedences, along with the standard deviation, over 120 time periods are shown in Table 6.2. The capital requirements calculated under the TTC rating philosophy give, on average, the lowest number of exceedences. Hence, the capital requirements calculated under the TTC rating philosophy provide the best protection against unexpected losses. Note that we used a confidence level of 99.9% in the calculation of VaR. Hence, following the definition of VaR, we expect that losses exceed the VaR in less than 0.1% of the simulations. However, this limit is not satisfied under the PIT philosophies. This observation might be explained by the fact that total capital requirements are calculated as the sum of capital requirements per bucket, while the true VaR does not equal the sum of the VaRs per bucket.

6.2.2 Procyclicality

Although we have not observed procyclicality in the capital requirements under the TTC rating philosophy when examining one bucket, procyclicality might be introduced by a shift in the distribution of loans over time. For example, when the economy is in a downturn houseprices fall and unemployment rates rise, causing loans to transition to buckets with a higher probability of default. Figure 6.3 shows the total capital requirements under the TTC rating philosophy and both versions of the PIT rating philosophy. Indeed, the capital require-

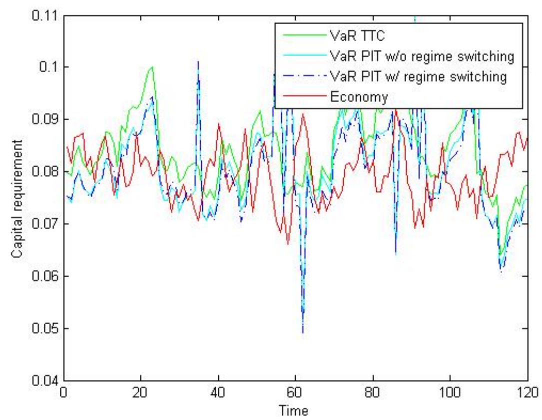


Fig. 6.3: A simulation of total capital requirements over time for the transitions matrices as given in Appendix B and values of PD as given by Equation (6.5). Furthermore, Z is scaled such that we can easily compare its dynamics to the capital requirements.

ments under both the TTC and PIT rating philosophy shown in Figure 6.3 are correlated with the economy. The correlation under the TTC rating philosophy equals 0.0489, the correlation under the PIT rating philosophy with regime switching equals -0.4022, while the correlation under the PIT rating philosophy without regime switching equals -0.4195.

However, it could be the case that correlation observed in Figure 6.3 is due to a coincidental correlation between Z_t and M_t. In order to rule out coincidental correlations, we performed 10,000 simulations and calculated the correlation between the capital requirements and the economy for each simulation. We calculated the average, standard deviation and confidence intervals of the obtained correlations. Table 6.3 contains the results for the TTC rating philosophy, the PIT philosophy with regime switching and the PIT philosophy without regime switching.

Note that we can only draw conclusions based on the average correlation if the system is stationary over time. Although it is beyond the scope of this thesis to prove stationarity, we do note that the correlation between the capital requirements and the economy looks stationary. This can be seen from Figure

	Mean	Std. dev.	95% confidence interval mean
TTC	0.0184	0.1945	(0.0146, 0.0223)
PIT, w/ regime switching	-0.5057	0.1493	(-0.5086, -0.5028)
PIT, w/o regime switching	-0.5188	0.1492	(-0.5217, -0.5159)

Tab. 6.3: The mean, standard deviation and confidence interval of the mean of correlations between total capital requirements and the economy. Simulations are performed under the TTC rating philosophy, PIT rating philosophy with regime switching and PIT rating philosophy without regime switching, for the transition matrices given in Appendix B and values of PD as given by Equation (6.5).

6.4, which shows the correlation between the capital requirements and the economy for each moment in time. The correlation seems to converge over time for all three rating philosophies, indicating stationarity.

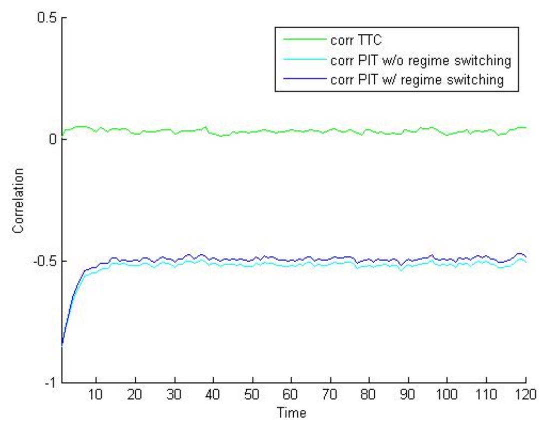


Fig. 6.4: The correlation between the economy and total capital requirements over 10,000 simulations, at different moments in time. Capital requirements are based on the TTC rating philosophy, the PIT rating philosophy with regime switching and the PIT rating philosophy without regime switching.

The average correlation under the TTC rating philosophy is positive and significant on a 5% level, while the average correlation under the PIT rating philosophy without regime switching is negative and significant. The average correlation under the PIT rating philosophy with regime switching is somewhat less negative than the correlation under the PIT rating philosophy without regime switching. Hence, from Table 6.3 we can conclude that there is no procyclicality in capital requirements under the TTC rating philosophy, while there is procyclicality under both versions of the PIT rating philosophy.

The positive correlation between the economy and the capital requirements under the TTC rating philosophy is rather counter intuitive. Intuitively, capital

	Mean	Std. dev.	95% confidence interval mean
TTC	-0.0987	0.2101	(-0.1029, -0.0946)
PIT, w/ regime switching	-0.6737	0.0834	(-0.6753, -0.6721)
PIT, w/o regime switching	-0.6762	0.0860	(-0.6778, -0.6745)

Tab. 6.4: The mean, standard deviation and confidence interval of the mean of correlations between average PD and the economy. Simulations are performed under the TTC rating philosophy, PIT rating philosophy with regime switching and PIT rating philosophy without regime switching, for the transition matrices given in Appendix B and values of PD as given by Equation (6.5).

requirements increase due to an increase in risk during economic distress, and capital requirements decrease due to a decrease in risk during economic booms. Indeed, we do find that the average PD (which can be seen as a measure of risk) of all three rating philosophies is negatively correlated to the economy, as can be seen in Table 6.4. Also, negative correlation can be observed in Figure 6.5. The observation that total capital requirements under the TTC rating philosophy are positively correlated to economy, while the average PD under the TTC rating philosophy is negatively correlated to economy can be explained as follows. Although the PD per bucket under the TTC rating philosophy stays the same over time, capital requirements per bucket increase or decrease due to an increase or decrease in the number of loans in the bucket. In particular, under the transition matrices used in this thesis, the number of loans in bucket 1 and bucket 4 are positively correlated to the economy. Hence, capital requirements for buckets 1 and 4 correlate positively with the economy. As the capital requirement as a function of PD is concave, bucket 4 gets relatively more weight in the calculation of total capital requirements, compared to its weight in the calculation of the average PD. Hence, bucket 4 causes the total capital requirements to correlate positively with economy, while it fails to do so in the calculation of the average PD.

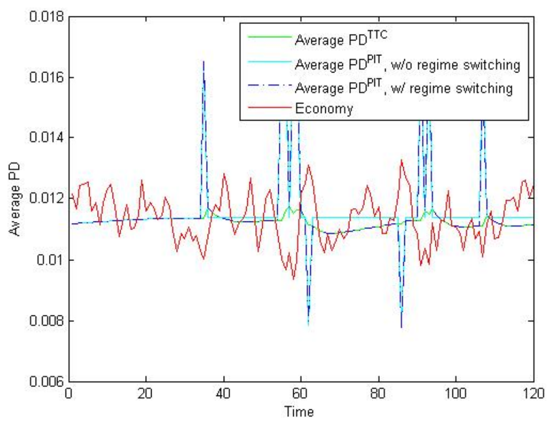


Fig. 6.5: A simulation of the average PD over time for the transitions matrices as given in Appendix B and values of PD as given by Equation (6.5). Furthermore, Z is scaled such that we can easily compare its dynamics to the average PDs.

7. CONCLUSION

The Basel Committee on Banking Supervision provides an explicit formula for the calculation of capital requirements under the IRB approach. This formula is based on the VaR of the Vasicek distribution and is to be determined for different buckets within the total portfolio. Total capital requirements are then obtained by summing the capital requirements per bucket. In order to calculate capital requirements per bucket, banks have to estimate the EAD, LGD and PD, representative for the bucket under consideration, as input parameters. However, different philosophies underlying the determination of those input parameters, and the PD input parameter in particular, lead to differences in the obtained capital requirements. Furthermore, the Basel capital accords do not explicitly prescribe the use of one particular philosophy.

In order to examine the differences in capital requirements per bucket over time, we developed a dynamic version of the Vasicek single factor model exhibiting regime switching and autocorrelation. The resulting cdf of the Vasicek model with regime switching has a fatter tail, and hence the capital requirements under this model exceed the capital requirements under the Vasicek single factor model. Hence, when the true PD is believed to exhibit regime switching, capital requirements are being underestimated when based on the Vasicek distribution.

Furthermore, capital requirements are influenced by the exposure per bucket. Even if we assume that all loans consist of one euro, the exposure per bucket fluctuates due to loans transitioning between buckets. The underlying cause of these transitions are the variables determining credit scores, such as individual characteristics as well as e.g. houseprices and unemployment rates. Hence, transitions between buckets are believed to exhibit regime switching, in line with the regime switching found in the value of PD. Hence, in a similar fashion as the regime switching version of the Vasicek model, we developed a regime switching version of the transition matrix, exhibiting cross-sectional dependence. This model forms the basis of our analysis of total capital requirements over time.

In the final chapter, we examined the influence of different philosophies on capital requirements. First, underlying the calculation of capital requirements per bucket, we distinguish two different philosophies: through-the-cycle (TTC) and point-in-time (PIT). The capital requirements per bucket under the TTC philosophy equal the capital requirements as obtained via the regime switching version of the Vasicek model. In contrast, the capital requirements per bucket under the PIT philosophy equal the capital requirements as obtained via the regime switching version of the Vasicek model, conditioned on the state of the economy. If the expected value of PD under both philosophies is equal, we find that the average capital requirement under the TTC philosophy exceed the average capital requirement under the PIT philosophy. Furthermore, capital requirements per bucket calculated under the PIT philosophy are correlated to the economy, while capital requirements per bucket under the TTC philosophy are not.

Underlying the calculation of total capital requirements, we distinguish three different philosophies: TTC with regime switching in the transitions between buckets, PIT with regime switching in the transitions between buckets and PIT without regime switching in the transitions between buckets. Under the first philosophy, differences in risk are reflected by a difference in the distribution of loans over the buckets. Under the second philosophy, differences in risk are reflected by different PDs assigned to the buckets as well as by a difference in the distribution of loans over the buckets. Under the third philosophy, differences in risk are entirely reflected by different PDs assigned to the buckets. We find that the average value of capital requirements under the TTC philosophy exceeds the average value of capital requirements under the PIT philosophy. Also, the standard deviation under the TTC philosophy is lower. Furthermore, we find that the TTC philosophy does not lead to procyclicality in total capital requirements, while both PIT philosophies do lead to procyclicality. Note that the observation that capital requirements under the TTC rating philosophy do not exhibit procyclicality is not in line with the common idea that capital requirements do exhibit procyclicality. The conclusions drawn in this thesis are subject to chosen parameters for e.g. the transition matrix and PD. Hence, the choice of different parameters might lead to different conclusions. Finally, we find that the TTC philosophy provides a better protection against future losses.

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APPENDIX

A. PROOF OF CONCAVITY OF THE BASEL CAPITAL REQUIREMENT FORMULA

In Chapter 3 we noticed that the capital requirement formula as a function of PD_{it} looks concave. In this appendix we will proof that the capital requirement formula indeed is concave on a particular interval. Thus, we are looking for an interval $[a, b]$ such that the capital requirement formula is concave in PD_{it} if $PD_{it} \in [a, b]$ The Basel capital requirement formula, given by Equation (3.14), equals

$$Cap = LGD \cdot \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(PD_{it}) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(\alpha)\right) - LGD \cdot PD_{it}. \quad \text{(A.1)}$$

In order to prove the concavity of A.1 as a function of PD_{it} on the interval $[a, b]$, we will use the fact that a twice-differentiable function is concave on the interval $[a, b]$ if and only if the second derivative on that interval is nonpositive. Note that Equation (A.1) is concave if the following function is concave.

$$h(x) = \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(x) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(\alpha)\right). \quad \text{(A.2)}$$

For convenience, we will define $f(x) = \frac{1}{\sqrt{1-\rho}}\Phi^{-1}(x) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(\alpha)$. Thus,

$$h(x) = \Phi\left(f(x)\right). \quad \text{(A.3)}$$

To compute the first and second derivative of Equation (A.3), and hence prove concavity, we will need the following results.

$$\frac{\partial}{\partial x}\Phi^{-1}(x) = \frac{1}{\phi\left(\Phi^{-1}(x)\right)}, \quad \text{(A.4)}$$

$$\frac{\partial}{\partial x}\phi(f(x)) = -f'(x) \cdot f(x) \cdot \phi(f(x)). \quad \text{(A.5)}$$

By the chain rule, the first derivative of Equation (A.3) equals

$$\frac{\partial}{\partial x}h(x) = \phi\left(f(x)\right) \cdot f'(x). \quad \text{(A.6)}$$

We use (A.4) to find that the derivative of $f(x)$ equals

$$f'(x) = \frac{1}{\sqrt{1-\rho}} \cdot \frac{1}{\phi\left(\Phi^{-1}(x)\right)}. \quad \text{(A.7)}$$

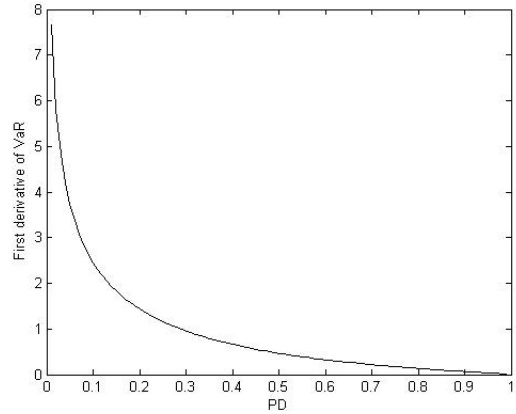

Fig. A.1: The first derivative of h(x) for values of PD between zero and one. ρ equals 0.15.

Figure A.1 displays $\frac{\partial}{\partial x}h(x)$ for $x \in [0,1]$. By Equation (A.5) and the product rule, the second derivative of Equation (A.3) equals

$$\frac{\partial^2}{\partial x^2}h(x) = -f'(x) \cdot f(x) \cdot \phi(f(x)) \cdot f'(x) + \phi(f(x)) \cdot f''(x). \tag{A.8}$$

Again, we use Equation (A.4) to find that the second derivative of $f(x)$ equals

$$f''(x) = \frac{1}{\sqrt{1-\rho}} \cdot \frac{\Phi^{-1}(x)}{(\phi(\Phi^{-1}(x)))^2}. \tag{A.9}$$

Figure A.2 displays $\frac{\partial^2}{\partial x^2}h(x)$ for $x \in [0,1]$.

Now, to prove concavity, we need that Equation (A.8) is nonpositive for all $x \in [a,b]$. Hence,

$$f''(x) \leq (f'(x))^2 \cdot f(x).$$

Which is equivalent to

$$\frac{1}{\sqrt{1-\rho}} \cdot \frac{\Phi^{-1}(x)}{(\phi(\Phi^{-1}(x)))^2} \leq \frac{1}{\sqrt{1-\rho}} \cdot \frac{1}{(\phi\,(\Phi^{-1}(x))} \cdot \frac{1}{\sqrt{1-\rho}} \cdot \frac{1}{\phi\,(\Phi^{-1}(x))} \cdot \left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(x) + \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\Phi^{-1}(\alpha)\right). \tag{A.10}$$

The inequality of Equation (A.10) is satisfied if and only if

$$\Phi^{-1}(x) \leq \frac{1}{1-\rho}\Phi^{-1}(x) + \frac{\sqrt{\rho}}{1-\rho}\Phi^{-1}(\alpha) \iff \tag{A.11}$$

$$-\rho\ \Phi^{-1}(x) \leq \sqrt{\rho}\ \Phi^{-1}(\alpha) \iff \tag{A.12}$$

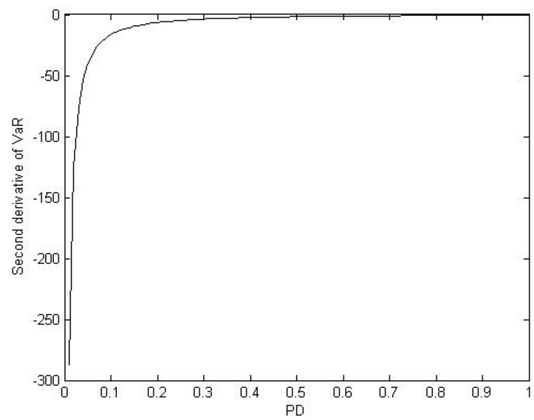

Fig. A.2: The second derivative of h(x) for values of PD between zero and one. ρ equals 0.15.

$$x \geq \Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right). \tag{A.13}$$

When considering the case of a retail portfolio, the Basel accord prescribes that $\rho = 0.15$ and $\alpha = 0.999$. Hence, $\Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right) \approx 7.38 \cdot 10^{-16}$. By Figure A.2 $f''(x)$ appears to be nonpositive for every $x \in [0, 1]$. However, this is not the case as $f''(x)$ is positive for very small values of x by Equation (A.13). This can be seen from Figure A.3. Thus, the capital requirement formula as described in

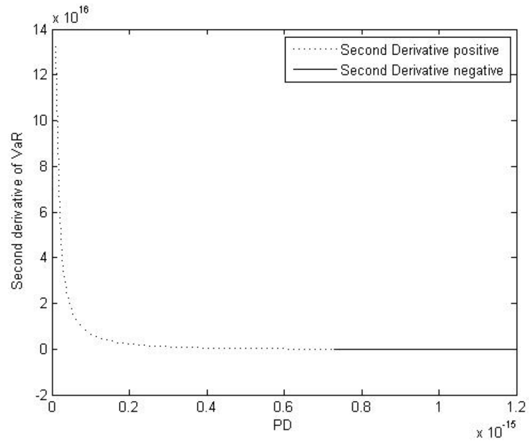

Fig. A.3: The second derivative of h(x) for very small values of PD. ρ equals 0.15.

the latest Basel capital accord is concave on the interval $\left[\Phi\left(-\frac{1}{\sqrt{\rho}}\Phi^{-1}(\alpha)\right), 1\right]$.

B. TRANSITION MATRICES

The transition matrix in the general case is defined by P. It is given by

$$P = \begin{bmatrix} 0.94 & 0.05 & 0.01 & 0 & 0 & 0.00 \\ 0.02 & 0.92 & 0.06 & 0 & 0 & 0.00 \\ 0 & 0.03 & 0.92 & 0.04 & 0 & 0.01 \\ 0 & 0 & 0.07 & 0.84 & 0.02 & 0.07 \\ 0 & 0 & 0.01 & 0.09 & 0.74 & 0.16 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{B.1}$$

The transition matrices for the downturn, normal and upturn states are defined by P^{DT}, P^N and P^{UT}, respectively. They are given by

$$P^{DT} = \begin{bmatrix} 0.94 & 0.06 & 0 & 0 & 0 & 0.00 \\ 0.03 & 0.91 & 0.06 & 0 & 0 & 0.00 \\ 0 & 0.04 & 0.90 & 0.04 & 0.01 & 0.01 \\ 0 & 0 & 0.07 & 0.80 & 0.03 & 0.10 \\ 0 & 0 & 0.01 & 0.08 & 0.68 & 0.23 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.2}$$

$$P^{N} = \begin{bmatrix} 0.94 & 0.05 & 0.01 & 0 & 0 & 0.00 \\ 0.02 & 0.92 & 0.06 & 0 & 0 & 0.00 \\ 0 & 0.03 & 0.92 & 0.04 & 0 & 0.01 \\ 0 & 0 & 0.07 & 0.84 & 0.02 & 0.07 \\ 0 & 0 & 0.01 & 0.09 & 0.74 & 0.16 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.3}$$

$$P^{UT} = \begin{bmatrix} 0.96 & 0.04 & 0 & 0 & 0 & 0.00 \\ 0.02 & 0.93 & 0.04 & 0.01 & 0 & 0.00 \\ 0 & 0.02 & 0.94 & 0.03 & 0 & 0.00 \\ 0 & 0 & 0.07 & 0.86 & 0.02 & 0.05 \\ 0 & 0 & 0.03 & 0.05 & 0.74 & 0.18 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{B.4}$$

Note that defaulted loans are always replaced by loans in the first two buckets. Also, note that the transition matrix P is equal to the transition matrix P^N.