Analysis of Terroristic Networks

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Abstract

Terroristic networks form a great danger to our society. Identifying the key players within a terroristic network is useful, since counterterrorism efforts can be focused on these essential players. As a result the chances of destabilizing a network rise and an attack might be prevented. In this thesis we analyze two existing game theoretic approaches to identify key players and we also introduce a third one. The advantage of using game theory for analyzing terroristic networks is that information about both the network structure and individual parameters can be taken into account. For the different games we derive several theoretical results and we apply the methods on the network of hijackers that executed the September 11 attacks. The methods all result in a ranking of the terrorists in the network. We find that the rankings are robust regarding changes in the communication structure and the individual parameters. Therefore we recommend to use these methods in the analysis of terroristic networks.

Contents

1

Introduction

Terroristic networks form a big threat to our society, as is illustrated by the attacks of 9/11 and the bombings in Madrid and London. Since then, there has been an increasing development in the research of counterterrorism. This research consists of two parts; first, there is collected as much data as possible about a terroristic network. This includes a wide variety of data, such as the communication structure, frequency and content of the communications, and personal details such as age and nationality. The second part involves using this information to determine the key individuals of the network and the most important communication links. In this thesis we focus on different methods to determine the key individuals in a network, and we also check how robust each method is. Robustness is important since terroristic networks try to stay as hidden as possible, and therefore it is difficult to collect complete and accurate data. The methods we use are based both on graph theory and game theory. In the graph that represents a terroristic network, the vertices correspond to individuals and the edges indicate that there is communication between two individuals. For every method we construct a game based on the available data. Then we compute the Shapley value for this game, which we use as a measure for the importance of an individual. In the first method we construct a game solely based on the graph structure. These games are called *connectivity* *games* and are defined by Amer and Giminez(2004). In the second method we also incorporate person specific information. On basis of this information a weight is assigned to every individual. Thus, we translate the person specific information to a weight vector and together with the information about the graph structure a game is constructed. Lindelauf(2011) was the first to do this and he defined the so-called *weighted connectivity games*. We also develop a third method, in which we construct a game that is also based on both the graph structure and person specific information, but we define these games in such a way that the marginal contribution of a person to a coalition cannot be negative. We call these games *monotonic weighted connectivity games*.

For the different connectivity games described above we derive several theoretical results about the core. Among others we show that the core of a connectivity game is non-empty if and only if the corresponding graph is a star graph, and that the core of a weighted connectivity game equals the weight vector if and only if the degree of each vertex of the corresponding graph is at least two.

For the star graph and the complete graph we derive closed formulas for the Shapley value of the different connectivity games. We find that for these graphs the Shapley value of the monotonic weighted connectivity game is equal to the Shapley value of the weighted connectivity game, but that there are also graphs for which this doesn't hold. Also we show how a coordinate of the Shapley value of the weighted connectivity game of a cycle graph depends on the weight of the corresponding individual, and that for all graphs it holds that all coordinates of the Shapley value depend more on the weight of the corresponding individual than on any other weight.

We apply the methods on the network of the hijackers that executed the 9/11 attacks. We use data that is collected by Krebs(2002) from open sources. We use the different methods to make rankings of the hijackers and we check how robust these rankings are by making adjustments to the data. We find that the rankings derived from the weighted connectivity game of the 9/11 network are more robust concerning weights than the rankings derived from the monotonic weighted connectivity game.

1.1 Outline

We start with providing some mathematical background in chapter 2. In chapter 3 we introduce the different connectivity games and obtain theoretical results about the core. In chapter 4 we look at the Shapley value of several standard graphs. In chapter 5 we will use the network of the terrorists that executed the 9/11 attacks as an example to apply the different methods and also we check how robust the results are.

2

Preliminaries

In this chapter we present some basic notions and definitions that will be used throughout this thesis.

A *graph* g is an ordered pair (N, E), where N with $|N| \geq 2$ represents the finite set of vertices and the set of edges E is a subset of the set of all unordered pairs of vertices. An *edge* $\{i, j\}$ connects the vertices i and j and is also denoted by ij. We say that vertex i is a *neighbour* of vertex j when there is an edge that connects i and j. The set of neighbours of i is denoted by $N_i = \{j \in N | ij \in E\}$. The degree of a vertex i of a graph is the number of neighbours $|N_i|$ and is denoted by $d_i(g)$. The *order* of a graph is the number of vertices $|N|$ and the size equals its number of edges $|E|$. The set of graphs of order n is denoted by $\mathbb{G}^n$.
There is a *path* between the vertices i and j when there is a sequence of vertices that starts with i and ends with j such that every 2 subsequent vertices in the sequence are connected by an edge and such that no vertex is repeated. A graph g is *connected* when there is a path between any two vertices of g. For $S \subset N$, the *subgraph* S_g is the graph (S, E'), whose edge set E' consists of all the edges $ij \in E$ of the original graph g that connect players $i, j \in S$. We say that $S \subset N$ *is connected by* g when subgraph S_g is connected.

S_g is a *component* (Chartrand, 1977) of g when S_g is connected and subgraph $\{S \cup \{i\}\}_g$ is not connected for all $i \in N \backslash S$. A set P is a *partition* of S when the union of the elements in P equals S and the intersection of 2 different elements in P is empty.
We denote by S/g the set that is a partition of S such that for all $T \in S_g$ the subgraph T_g is a component of g .

We call a graph of order n a *complete graph* when $d_i(g) = n - 1$ for all $i \in N$ and we denote this graph by g^n_{comp}. A *star graph* of order n is a connected graph that consists of a vertex $i \in N$ such that $d_i(g) = n-1$, and for all $j \in N \backslash \{i\}$ it holds that $d_j(g) = 1$. This graph is denoted by g^n_{star}. A *ring graph* of order n is a connected graph where $d_i(g) = 2$ for all vertices and is denoted by g^n_{ring}. See Figure 2.1 for examples of these graphs. For a complete overview of graph theory, see Bollobas(1998).

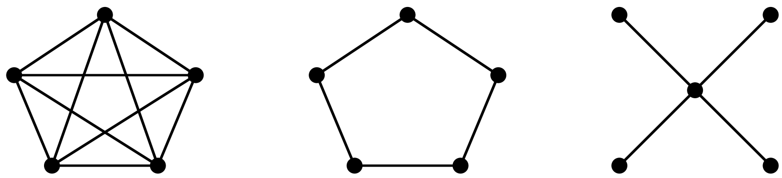

Figure 2.1: Complete graph of order 5(left), ring graph of order 5(middle) and star graph of order 5(right).

A *cooperative transferable utility game* is an ordered pair (N, v) where $N = \{1, ..., n\}$ is a finite set of players and $v : 2^N \mapsto \mathbb{R}$ is a map assigning to each coalition $S \in 2^N$ a real number $v(S)$, which is the worth coalition S can obtain by cooperation, with $v(\{\emptyset\}) = 0$.
A game (N, v) satisfies *superadditivity* when

$$v(S \cup T) \geq v(S) + v(T) \textit{ for all } S, T \in 2^N \textit{ with } S \cap T = \emptyset.$$

A game is called monotonic when

$$v(S) \leq v(T) \textit{ for all } S, T \in 2^N \textit{ with } S \subset T.$$

The *core* (cf. Gillies (1953)) $C(v)$ of a game $v \in TU^N$ is defined by

$$C(v) = \{x \in \mathbb{R}^N | x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\}$$

where $x(S) = \sum_{i \in S} x_i$. Core elements can be seen as stable divisions of the value of the grand coalition. This is done in such a way that there is no coalition which can be better off by deviating from the grand coalition. A cooperative game (N, v) is called *balanced* if

$$\sum_{S \in N} \lambda(S) v(S) \leq v(N)$$

for all functions $\lambda : 2^N \mapsto \mathbb{R}_+$ satisfying $\sum_{S \subset N : i \in S} \lambda(S) = 1$ for all $i \in N$. Such a function λ is called a *balanced map*. The following result about the core and balancedness is derived by Bondareva (1963) and Shapley (1967):

Theorem 2.0.1. *(Bondareva, 1963; Shapley, 1967) Let (N, v) be a cooperative game. Then $C(v) \neq \emptyset$ if and only if (N, v) is balanced.*

Let $\Pi(N)$ be the set of all permutations of $N = \{1, ..., n\}$. Then the i-th coordinate of the *marginal vector* $m^\sigma(v), \sigma \in \Pi(N)$, is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), ..., \sigma(k)\}) - v(\{\sigma(1), ..., \sigma(k-1)\}),$$

where $\sigma(k)$ is the player that is on position k in permutation σ.
The Shapley value $\Phi(v)$ (Shapley, 1953) of (N, v) is defined as the average of all marginal vectors, i.e.,

$$\Phi(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m^\pi(v)$$

The Shapley value is *efficient*, which means that $\sum_{i \in N} \Phi_i(v) = v(N)$ for all cooperative transferable utility games v.
Also the Shapley value is *symmetric*, which means that

$$\Phi_i(v) = \Phi_j(v) \quad \text{ when } v(S \cup \{i\}) = v(S \cup \{j\}) \text{ for all } S \subset N \backslash \{i, j\}$$

for all cooperative transferable utility games v.

3

Connectivity games

In this chapter we discuss different kind of connectivity games which correspond to a graph. We give the definitions of unweighted connectivity games and weighted connectivity games, and introduce a new kind of connectivity games, namely monotonic weighted connectivity games. Unweighted connectivity games need as input only information about the communication structure. For weighted connectivity games and monotonic weighted connectivity games also person specific information is used. On basis of this information a weight is assigned to every individual, which results in a weight vector. Monotonic weighted connectivity games are defined in such a way that they are monotonic. For each of these games we check whether monotonicity is satisfied and whether the core is non-empty. We show that the core of a connectivity game is non-empty if and only if the corresponding graph is a star graph, and we show that the core of a weighted connectivity game equals the weight vector if and only if the degree of each vertex of the corresponding graph is at least two.

3.1 Unweighted connectivity games

In this paragraph we give the definition of connectivity games corresponding to a graph. Next we investigate these games on monotonicity and non-emptiness of the core.

Definition (Amer and Giminez, 2004). Let $g \in \mathbb{G}^n$ be a graph. The corresponding *connectivity game* is defined by

$$v_g^c(S) = \begin{cases} 1 \text{ if } S \subset N \text{ is connected by } g \text{ and } |S| > 1 \\ 0 \text{ otherwise} \end{cases} \tag{3.1}$$

We will refer to these type of games both as *connectivity games* and *unweighted connectivity games.*

Example Let $g = (N, E)$ be a graph with $N = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 23,$
$34, 35, 45\}$. We will refer to this graph as the *bow* graph. In Figure 3.1 we present the visualization of the bow graph and in Table 3.1 we present the corresponding connectivity game v_g^c.

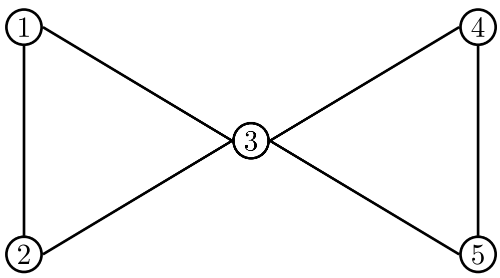

Figure 3.1 The bow graph.

S	{1}	{2}	{3}	{4}	{5}	{1,2}	{1,3}
$v_g^c(S)$	0	0	0	0	0	1	1
S	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}
$v_g^c(S)$	0	0	1	0	0	1	1
S	{4,5}	{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}
$v_g^c(S)$	1	1	0	0	1	1	0
S	{2,3,4}	{2,3,5}	{2,4,5}	{3,4,5}	{1,2,3,4}	{1,2,3,5}	{1,2,4,5}
$v_g^c(S)$	1	1	0	1	1	1	0
S	{1,3,4,5}	{2,3,4,5}	{1,2,3,4,5}				
$v_g^c(S)$	1	1	1				

Table 3.1: The connectivity game corresponding to the bow graph.

Observe that v_g^c is not monotonic, since $v_g^c(\{1,2\}) = 1 > v_g^c(\{1,2,4,5\}) = 0$. Recall that the core of a game is non-empty if and only if the game is balanced. Take the balanced map

$$\lambda(S) = \begin{cases} 1 \text{ if } S \in \{\{1,2\},\{3,4,5\}\} \\ 0 \text{ otherwise} \end{cases}$$

Now $\sum_{S \in N} \lambda(S) v_g^c(S) = 2$ and $v_g^c(N) = 1$, so this game is not balanced. Hence, $C(v_g^c) = \emptyset$ △

It turns out that only connectivity games that correspond to a star graph are balanced.

Theorem 3.1.1. *Let $g \in \mathbb{G}^n$ be a connected graph. Let v_g^c be the corresponding connectivity game.*
Then $C(v_g^c) \neq \emptyset$ if and only if the graph is a star graph.

Proof. $' \Rightarrow'$ It suffices to show that when a connected graph is not a star graph, the core of the corresponding connectivity game is empty. If g is not a star graph then there exist $i, j \in N$ such that $d_i(g) \geq 2$ and $d_j(g) \geq 2$. Denote by $H = \{i\} \cup \{j\} \cup N_i \cup N_j$ the set of vertices i, j and their neighbours. Clearly, $|H| \geq 3$. We now make a distinction between 2 cases: $|H| = 3$ and $|H| > 3$.
When $|H| = 3$, the subgraph H_g is a complete graph. Denote by n_{ij} the

player that is connected with both i and j. Take as balanced map on the corresponding connectivity game:

$$\lambda(S) = \begin{cases} 1 \text{ if } S \in N\backslash\{\{i\},\{j\},\{n_{ij}\}\} \text{ and } |S| = 1 \\ \frac{1}{2} \text{ if } S \in \{\{i,j\},\{i,n_{ij}\},\{j,n_{ij}\}\} \\ 0 \text{ otherwise} \end{cases}$$

Now $\sum_{S \in N} \lambda(S) v_g^c(S) = \frac{3}{2} > 1 = v_g^c(N)$, so the game is not balanced and hence the core is empty.
In the other case, when $|H| > 3$, there exist vertices $k, l \in N\backslash\{i,j\}$ such that $ik, jl \in E$. To show $C(v_g^c) = \emptyset$, take as balanced map on v_g^c:

$$\lambda(S) = \begin{cases} 1 \text{ if } S \in N\backslash\{\{i\},\{j\},\{k\},\{l\}\} \text{ and } |S| = 1 \\ 1 \text{ if } S \in \{\{i,k\},\{j,l\}\} \\ 0 \text{ otherwise} \end{cases}$$

From $\sum_{S \in N} \lambda(S) v_g^c(S) = 2 > 1 = v_g^c(N)$ it follows that also in this case the core is empty.

$' \Leftarrow '$ Let g be a star graph. There exists $s \in N$ such that $d_s(g) = n - 1$. Define $r^s \in \mathbb{R}^N$ such that

$$r_i^s = \begin{cases} 1 \text{ if } i = s \\ 0 \text{ otherwise} \end{cases}$$

Then $r^s \in C(v_g^c)$, hence the core is non-empty. □

3.2 Weighted connectivity games

In this paragraph we give the definition of weighted connectivity games corresponding to a graph. The difference with connectivity games is that to determine the weighted connectivity game corresponding to a graph also a weight vector is required, which assigns to every vertex a number. We will investigate these games on monotonicity and non-emptiness of the core.

Definition Let $g \in \mathbb{G}^n$. The *weight vector* w is a function $N \to \mathbb{R}_+$ that assigns to every vertex $i \in N$ a positive number. We denote by w_i the number that is assigned to vertex i for all $i \in N$. The set of weight vectors for graphs of order n is denoted by $\mathbb{W}^n$

Definition (Lindelauf, 2011). Let $g \in \mathbb{G}^n$ and let $w \in \mathbb{W}^N$. The corresponding *weighted connectivity game* is defined by

$$v_g^w(S) = \begin{cases} \sum_{i \in S} w_i \text{ if } S \subset N \text{ is connected by } g \text{ and } |S| > 1 \\ 0 \qquad \text{otherwise} \end{cases} \tag{3.2}$$

Note that in general there is not a weight vector available such that the weighted connectivity game of a graph is equal to the unweighted connectivity game.

Example Let g be the bow graph. Let the weight vector be $w = (1, 2, 5, 3, 4)$. In table 3.2 we present the corresponding weighted connectivity game v_g^w.

S	{1}	{2}	{3}	{4}	{5}	{1,2}	{1,3}
$v_g^w(S)$	0	0	0	0	0	3	6
S	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}
$v_g^w(S)$	0	0	7	0	0	8	9
S	{4,5}	{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}
$v_g^w(S)$	7	8	0	0	9	10	0
S	{2,3,4}	{2,3,5}	{2,4,5}	{3,4,5}	{1,2,3,4}	{1,2,3,5}	{1,2,4,5}
$v_g^w(S)$	10	11	0	12	11	12	0
S	{1,3,4,5}	{2,3,4,5}	{1,2,3,4,5}				
$v_g^w(S)$	13	14	15				

Table 3.2: The weighted connectivity game corresponding to the bow graph

Observe that v_g^w is not monotonic, since $v_g^w(\{1, 2\}) = 3 > v_g^w(\{1, 2, 4, 5\}) = 0$. Furthermore v_g^w is not superadditive, because $v_g^w(\{1, 2\}) + v_g^w(\{4, 5\}) = 10 > v_g^w(\{1, 2\} \cup \{4, 5\}) = 0$. Moreover $C(v_g^w) \neq \emptyset$, since $w \in C(v_g^w)$. $\triangle$

Now we show that $C(v_g^w) \neq \emptyset$ for all $g \in \mathbb{G}^n$.

Theorem 3.2.1. *Let $g = (N, E)$ be a connected graph and let $w \in \mathbb{W}^N$. Let v_g^w be the corresponding weighted connectivity game. Then $w \in C(v_g^w)$.*

Proof. When g is connected, $v_g^w(N) = \sum_{i \in N} w_i$. By definition $v_g^w(S) \leq \sum_{i \in S} w_i$. From this it directly follows that $w \in C(v_g^w)$. $\square$

Moreover, w is the only core element if and only if $d_i(g) \geq 2$ for all $i \in N$.

Theorem 3.2.2. *Let $g = (N, E)$ be a connected graph and let $w \in \mathbb{W}^N$. Let v_g^w be the corresponding weighted connectivity game.*
$C(v_g^w) = \{w\}$ if and only if $d_i(g) \geq 2$ for all $i \in N$.

Proof. $' \Leftarrow'$ From Proposition 3.2.1 we already know that $w \in C(v_g^w)$. To show that this is the only core element, assume that there is another core element $x \neq w$. Since $\sum_{t \in N} x_t = \sum_{t \in N} w_t$, there exists $i \in N$ such that $x_i < w_i$. Let $j \in N_i$. Let $R_i \in (N \backslash \{j\})/g$ such that $R_i \ni i$. Since $d_i \geq 2, |R_i| \geq 2$, and for x to be a core element it is therefore required that

$$v_g^w(R_i) = \sum_{t \in R_i} w_t \leq \sum_{t \in R_i} x_t \tag{3.3}$$

Now let $N \backslash R_i$ be the complement of R_i. Since $\sum_{t \in N} w_t = \sum_{t \in N} x_t$, it follows from (3.3) that

$$\sum_{t \in N \backslash R_i} w_t \geq \sum_{t \in N \backslash R_i} x_t$$

Since $x_i < w_i$,

$$\sum_{t \in (N \backslash R_i) \cup \{i\}} w_t > \sum_{t \in (N \backslash R_i) \cup \{i\}} x_t \tag{3.4}$$

Now $(N \backslash R_i)$ is connected by g, because j is connected with all $R \in (N \backslash \{j\})/g$. Since $j \in N_i$, also $(N \backslash R_i) \cup \{i\}$ is connected by g, and therefore for x to be a core element it is required that

$$v_g^w((N \backslash R_i) \cup \{i\}) = \sum_{t \in (N \backslash R_i) \cup \{i\}} w_t \leq \sum_{t \in (N \backslash R_i) \cup \{i\}} x_t \tag{3.5}$$

Equations (3.4) and (3.5) form a contradiction.

$' \Rightarrow '$ It suffices to show that if there exists a vertex i with $d_i(g) = 1$, $C(v_g^w)\backslash\{w\} \neq \emptyset$. Let $n_i \in N_i$. From Proposition 3.2.1 we know that w is a core element. Now let $w' \in \mathbb{W}^N$ such that $w'_j = w_j$ for all $j \in N\backslash\{\{i\},\{n_i\}\}$, $w'_i = \frac{1}{2}w_i$ and $w'_{n_i} = w_{n_i} + \frac{3}{2}w_i$. Now $\sum_{t\in N} w'_t = \sum_{t\in N} w_t$. Moreover, if $i \in S$ and $v_g^w(S) > 0$, then $n_i \in S$ and then $\sum_{i\in S} w'_i = \sum_{i\in S} w_i = v_g^w(S)$. When $i \notin S$, $\sum_{i\in S} w'_i \geq \sum_{i\in S} w_i \geq v_g^w(S)$. Hence $\sum_{i\in S} w'_i \geq v_g^w(S)$ for all $S \in 2^N$ and hence $w' \in C(v_g^w)$. □

3.3 Monotonic weighted connectivity games

In this paragraph we introduce a new kind of connectivity games, namely monotonic weighted connectivity games. Just like weighted connectivity games, they require a weight vector. Moreover, monotonic weighted connectivity games are defined in such a way that they are monotonic. Furthermore we will investigate these games on superadditivity and emptiness of the core.

Definition Let $g \in \mathbb{G}^n$ and let $w \in \mathbb{W}^N$. We define the corresponding *monotonic weighted connectivity game* by

$$v_g^m(S) = \begin{cases} \max\limits_{T\in S/g:|T|\geq 2} \sum\limits_{i\in T} w_i & \text{if } |S/g| < |S| \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

Theorem 3.3.1. *Let $g \in \mathbb{G}^n$ and let $w \in \mathbb{W}^N$. Let v_g^m and v_g^w respectively be the corresponding monotonic weighted connectivity game and the corresponding weighted connectivity game. Then $v_g^m \geq v_g^w$ for all $S \in 2^N$.*

Proof. For every $S \in 2^N$ with $v_g^w(S) > 0, v_g^w(S) = v_g^m(S)$, and $v_g^m(S) \geq 0$ for all $S \in 2^N$. Therefore $v_g^m \geq v_g^w$ for all $S \in 2^N$. □

By defining these games in this way, they are monotonic:

Theorem 3.3.2. *Let $g \in \mathbb{G}^n$ be a graph and let $w \in \mathbb{W}^N$. Then the corresponding monotonic weighted connectivity game is monotonic.*

Proof. Let $T \in 2^N$ such that $S \subset T \subset N$. For every $U \in S/g$, there exists a $U' \in T/g$ such that $U \subset U'$, and therefore

$$\max_{U \in S/g} \sum_{i \in U} w_i \leq \max_{U' \in T/g} \sum_{i \in U'} w_i$$

Also, when $|S/g| < |S|$ holds, $|T/g| < |T|$ holds. Therefore $v_g^m(S) \leq v_g^m(T)$. □

Example Let g be the bow graph. Again we use $w = (1, 2, 5, 3, 4)$. In table 3.3 we present the corresponding monotonic weighted connectivity game v_g^m.

S	{1}	{2}	{3}	{4}	{5}	{1,2}	{1,3}
$v_g^m(S)$	0	0	0	0	0	3	6
S	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}
$v_g^m(S)$	0	0	7	0	0	8	9
S	{4,5}	{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}
$v_g^m(S)$	7	8	3	3	9	10	7
S	{2,3,4}	{2,3,5}	{2,4,5}	{3,4,5}	{1,2,3,4}	{1,2,3,5}	{1,2,4,5}
$v_g^m(S)$	10	11	7	12	11	12	7
S	{1,3,4,5}	{2,3,4,5}	{1,2,3,4,5}				
$v_g^m(S)$	13	14	15				

Table 3.3: The monotonic weighted connectivity game corresponding to the bow graph

We already know from Theorem 3.3.2 that all monotonic weighted games are monotonic, and therefore v_g^m as well.
Observe that v_g^m is not superadditive, because $v_g^m(\{1,2\}) + v_g^m(\{4,5\}) = 10 > v_g^m(\{1,2\} \cup \{4,5\}) = 7$. Again $C(v_g^m) \neq \emptyset$, since $w \in C(v_g^m)$. △

Similar to weighted connectivity games, the weight vector is in the core of the monotonic weighted connectivity game.

Theorem 3.3.3. *Let $g = (N, E)$ be a connected graph and let $w \in \mathbb{W}^N$. Let v_g^m be the corresponding monotonic weighted connectivity game. Then $w \in C(v_g^m)$.*

Proof. When g is connected, $v_g^m(N) = \sum_{i \in N} w_i$. By definition $v_g^m(S) \leq \sum_{i \in S} w_i$. From this it directly follows that $w \in C(v_g^m)$. $\square$

Also similar is that there exist graphs in which the weight vector is not the only core element.

Theorem 3.3.4. *Let $g = (N, E)$ be a connected graph and let $w \in \mathbb{W}^N$. Let v_g^m be the corresponding monotonic weighted connectivity game. There exist graphs g such that $\{w\} \subsetneq C(v_g^m)$.*

Proof. Let $g = g_{star}^n$. For all $S \in 2^N$, $v_g^m(S) = v_g^w(S)$ and therefore $v_g^m = v_g^w$. Since in a star graph there exist vertices i such that $d_i < 2$, it follows from Theorem 3.2.1 and Theorem 3.2.2 that $\{w\} \subsetneq C(v_g^m)$. $\square$

Now we show that for a given graph the core of the corresponding monotonic weighted connectivity game is a subset of the core of the corresponding weighted connectivity game.

Theorem 3.3.5. *Let $g \in \mathbb{G}^n$ be a connected graph and let $w \in \mathbb{W}^N$.*
Then $C(v_g^m) \subset C(v_g^w)$.

Proof. It follows directly from Theorem 3.3.1 that $v_g^w(S) \leq v_g^m(S)$ for all $S \in 2^N$.
Moreover, $v_g^w(N) = \sum_{i \in N} w_i = v_g^m(N)$ and therefore $C(v_g^m) \subset C(v_g^w)$. $\square$

4

Shapley value of connectivity games

In this chapter we derive closed formulas for the Shapley value of the different kind of connectivity games corresponding to several standard graphs.

4.1 Shapley value of unweighted connectivity games

In this section we derive closed formulas for the Shapley value of the unweighted connectivity game of complete graphs and star graphs.

4.1.1 Complete graphs

Theorem 4.1.1. *Let* $g = g^n_{comp}$ *and let* v^c_g *be the corresponding connectivity game. Then* $\Phi_i(v^c_g) = \frac{1}{n}$ *for all* $i \in N$

Proof. For all $i \in N$ it holds that $v(S \cup \{i\}) = 1$ when $|S| \geq 1$, and $v(S \cup \{i\}) = 0$ when $S = \emptyset$. From the symmetry property of the Shapley value it follows that $\Phi_i(v^c_g) = \Phi_j(v^c_g)$ for all $i, j \in N$. Together with the efficiency property it follows that $\Phi_i(v^c_g) = \frac{1}{n}$. □

4.1.2 Star graphs

Theorem 4.1.2. *Let* $g = g^n_{star}$ *and let* v^c_g *be the corresponding connectivity game. Let* $s \in N$ *be the vertex with* $d_s(g^n_{star}) = n - 1$. *Then* $\Phi_s(v^c_g) = \frac{n-1}{n}$

and $\Phi_i(v_g^c) = \frac{1}{n(n-1)}$ *for all* $i \in N\backslash\{s\}$.

Proof. For all $i \in N\backslash\{s\}$ it holds that $v(S \cup \{i\}) = 1$ when $s \in S$ and $v(S \cup \{i\}) = 0$ when $s \notin S$. It follows from the symmetry property from the Shapley value therefore that $\Phi_i(v_g^c) = \Phi_j(v_g^c)$ for all $i, j \in N\backslash\{s\}$. We start with the expression for $\Phi_s(v_g^c)$:

$$\begin{aligned}\Phi_s(v_g^c) &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m_s^\sigma(v_g^c)\\ &= \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma^{-1}(s)\geq 2} m_s^\sigma(v_g^c)\\ &= \frac{(n-1)!(n-1)}{n!}\\ &= \frac{n-1}{n}\end{aligned}$$

The first equality sign follows directly from the definition of the Shapley value.

When $\sigma^{-1}(s) = 1$, it holds that $m_s^\sigma(v_g^c) = 0$, because $v_g^c(S) = 0$ for all $S \subset N$ with $|S| = 1$.

When $\sigma^{-1}(s) \geq 2$, it holds for all $(n-1)!(n-1)$ permutations that $m_s^\sigma(v_g^c) = 1$, because $v_g^c(S) = 0$ and $S \cup \{s\}$ is connected by g_{star}^n for all $S \subset N\backslash\{s\}$.

From the symmetry property and the efficiency property from the Shapley value it follows that

$$\Phi_i(v_g^c) = \frac{1 - \frac{n-1}{n}}{n-1} = \frac{1}{n(n-1)} \text{ for all } i \in N\backslash\{s\}$$

□

4.2 Shapley value of weighted connectivity games

In this section we derive closed formulas for the Shapley value of weighted connectivity games corresponding to complete graphs and star graphs, while for cycle graphs we only find an closed formula for how a coordinate of the Shapley value depends on its corresponding weight.

We start this section with introducing a notation for the different parts of a coordinate of a marginal vector and for the different parts of a coordinate of the Shapley value. This notation we will keep using throughout the thesis.

Let $g \in \mathbb{G}^n$ and let $w \in \mathbb{W}^N$. Let v_g^w be the corresponding weighted connectivity game. From the definition of weighted connectivity games it follows that the value of each coalition is linearly dependent of the weights. Therefore all coordinates of all marginal vectors are also linear dependent on the weights, i.e. for all $i \in N$ and for all $\sigma \in \Pi(N)$ there is a $b^{\sigma i} \in \mathbb{R}^N$ such that $m_i^\sigma(v_g^w) = \sum_{t \in N} b_t^{\sigma i} w_t$. Since the Shapley value is an average of the marginal vectors, it follows directly that for all $i \in N$ there is a $c^i \in \mathbb{R}^N$ such that $\Phi_i(v_g^w) = \sum_{t \in N} c_t^i w_t$.

Example Let $g = g^3_{comp}$ be the complete graph of order 3. Let $w \in \mathbb{W}^N$. The corresponding weighted connectivity game is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_g^w(S)$	0	0	0	$w_1 + w_2$	$w_1 + w_3$	$w_2 + w_3$	$w_1 + w_2 + w_3$

The marginal vectors of v_g^w are given by

σ	$m_1^\sigma(v_g^w)$	$m_2^\sigma(v_g^w)$	$m_3^\sigma(v_g^w)$
(123)	0	$w_1 + w_2$	w_3
(132)	0	w_2	$w_1 + w_3$
(213)	$w_1 + w_2$	0	w_3
(231)	w_1	0	$w_2 + w_3$
(312)	$w_1 + w_3$	w_2	0
(321)	w_1	$w_2 + w_3$	0

Now for example $m_2^{(321)}(v_g^w) = \sum_{t \in N} b_t^{(321)2} w_t = b_1^{(321)2} w_1 + b_2^{(321)2} w_2 + b_3^{(321)2} w_3 = w_2 + w_3$. So $b^{(321)2} = (0, 1, 1)$.

The Shapley value is the average of the six marginal vectors:

$$\Phi(v_g^w) = (\frac{4}{6}w_1 + \frac{1}{6}w_2 + \frac{1}{6}w_3, \frac{1}{6}w_1 + \frac{4}{6}w_2 + \frac{1}{6}w_3, \frac{1}{6}w_1 + \frac{1}{6}w_2 + \frac{4}{6}w_3)$$

Now for example $\Phi_3(v_g^w) = \sum_{t\in N} c_t^3 w_t = c_1^3 w_1 + c_2^3 w_2 + c_3^3 w_3 = \frac{1}{6}w_1 + \frac{1}{6}w_2 + \frac{4}{6}w_3$. So $c^3 = (\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$. $\triangle$

4.2.1 Complete graphs

Theorem 4.2.1. *Let $g = g^n_{comp}$ be a complete graph, let $w \in \mathbb{W}^N$ and let v_g^w be the corresponding weighted connectivity game. Then $\Phi_i(v_g^w) = \frac{(n-1)w_i}{n} + \sum_{j\in N\setminus\{i\}} \frac{w_j}{n(n-1)}$ for all $i \in N$.*

Proof.

$$\begin{aligned}
\Phi_i(v_g^w) &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m_i^\sigma(v_g^w) \\
&= \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma(2)=i} m_i^\sigma(v_g^w) + \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma^{-1}(i)\geq 3} m_i^\sigma(v_g^w) \\
&= \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma(2)=i} m_i^\sigma(v_g^w) + \frac{(n-1)!(n-2)w_i}{n!} \\
&= \frac{1}{n!}\sum_{j\in N\setminus\{i\}} (w_i + w_j)(n-2)! + \frac{(n-2)w_i}{n} \\
&= \sum_{j\in N\setminus\{i\}} \frac{w_j}{n(n-1)} + \frac{(n-1)w_i}{n} \qquad \text{for all } i \in N
\end{aligned}$$

The first equality sign follows directly from the definition of the Shapley value. The second equality sign divides the permutations into different cases. When $\sigma^{-1}(i) = 1, m_i^\sigma(v_g^w) = 0$, because $v_g^w(S) = 0$ for all $S \subset N$ with $|S| = 1$.

When $\sigma^{-1}(i) \geq 3$, it holds for all $(n-1)!(n-2)$ permutations that $m_i^\sigma(v_g^w) = w_i$, because S and $S \cup \{i\}$ are both connected by g^n_{comp} for all $S \subset N$ with $|S| \geq 2$ and for all $i \in N\setminus S$.

When $\sigma^{-1}(i) = 2$, it holds for all $(n-2)!$ permutations that $m_i^\sigma(v_g^w) = w_i + w_{\sigma(1)}$, because $v_g^w(S) = 0$ and $S \cup \{i\}$ is connected by g^n_{comp} for all

$S \subset N$ with $|S| = 1$ and for all $i \in N \backslash S$. Rewriting concludes the proof. $\square$

4.2.2 Star graphs

Theorem 4.2.2. *Let g^n_{star}, let $w \in \mathbb{W}^N$ and let v^w_g be the corresponding weighted connectivity game. Let $s \in N$ be the vertex with $d_s(g^n_{star}) = n-1$. Then $\Phi_s(v^w_g) = \frac{(n-1)w_s}{n} + \sum_{j \in N\backslash\{s\}} \frac{w_j}{2}$ and $\Phi_i(v^w_g) = \frac{w_i}{2} + \frac{w_s}{n(n-1)}$ for all $i \in N\backslash\{s\}$.*

Proof. The first part of the proof consists of showing that $\Phi_s(v^w_g) = \frac{(n-1)w_s}{n} + \sum_{j \in N\backslash\{s\}} \frac{1}{2} w_j$ and in the second part we show that $\Phi_i(v^w_g) = \frac{1}{2}w_i + \frac{1}{n(n-1)} w_s$ for all $i \in N\backslash\{s\}$.

$$\begin{aligned}
\Phi_s(v^w_g) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma_s(v^w_g) \\
&= \sum_{j \in N} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} b^{\sigma s}_j w_j \\
&= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} b^{\sigma s}_s w_s + \sum_{j \in N\backslash\{s\}} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} b^{\sigma s}_j w_j \\
&= \frac{(n-1)w_s}{n} + \sum_{j \in N\backslash\{s\}} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} b^{\sigma s}_j w_j \\
&= \frac{(n-1)w_s}{n} + \sum_{j \in N\backslash\{s\}} \frac{1}{n!} \sum_{\sigma \in \Pi(N); \sigma^{-1}(s) > \sigma^{-1}(j)} b^{\sigma s}_j w_j \\
&= \frac{(n-1)w_s}{n} + \sum_{j \in N\backslash\{s\}} \frac{w_j}{2}
\end{aligned}$$

The first equality sign follows directly from the definition of the Shapley value. The second equality follows directly from Definition 4.2.1. The third equality divides the sum into 2 parts. The fourth equality sign holds because for all $(n-1)!(n-1)$ permutations with $\sigma^{-1}(s) \geq 2$, $b^{\sigma s}_s w_s = w_s$, since S is not connected by g and $S \cup \{s\}$ is connected by g for all $S \subset N\backslash\{s\}$. The fifth equality sign holds because $b^{\sigma s}_j w_j = 0$ for all $j \in N\backslash\{s\}$ when $\sigma^{-1}(s) < \sigma^{-1}(j)$. The last equality sign holds because $b^{\sigma s}_j w_j = w_j$ for all $j \in N\backslash\{s\}$, when $\sigma^{-1}(s) > \sigma^{-1}(j)$. There are $\frac{n!}{2}$ such permutations.

Now we come to the second part of the proof.

$$\begin{aligned}
\Phi_i(v_g^w) &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m_i^\sigma(v_g^w) \\
&= \sum_{j\in N}\frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_j^{\sigma s} w_j \\
&= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_i^{\sigma i} w_i + \frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_s^{\sigma i} w_s \\
&= \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma^{-1}(i)>\sigma^{-1}(s)} b_i^{\sigma i} w_i + \frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_s^{\sigma i} w_s \\
&= \frac{w_i}{2} + \frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_s^{\sigma i} w_s \\
&= \frac{w_i}{2} + \frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma(1)=s;\sigma(2)=i} b_s^{\sigma i} w_s \\
&= \frac{w_i}{2} + \frac{w_s}{n(n-1)} \qquad \text{for all } i \in N\{s\}
\end{aligned}$$

At the third equality sign it is acknowledged that $b_j^{\sigma i} w_j = 0$ for all $j \in N\backslash\{\{i\},\{s\}\}$ and for all $i \in N\backslash\{s\}$. This holds because for all $S \subset N\backslash\{\{i\},\{s\}\}$ $v_g^w(S) = 0$ and also $v_g^w(S \cup \{i\}) = 0$. The fourth and fifth equality sign holds because $b_i^{\sigma i} w_i \neq 0$ only if $\sigma^{-1}(i) > \sigma^{-1}(s)$. For these $\frac{n!}{2}$ permutations $b_i^{\sigma i} w_i = w_i$. The last two equality signs hold because $b_s^{\sigma i} w_s \neq 0$ only if $\sigma(1) = s$ and $\sigma(2) = i$. For these $(n-2)!$ permutations $b_s^{\sigma i} w_s = w_s$, which concludes the second part of the proof. □

4.2.3 Cycle graphs

For cycle graphs we give an expression for how a coordinate of the Shapley value of the corresponding weighted connectivity game depends on its corresponding weight.

Theorem 4.2.3. *Let $g = g^n_{cycle}$, let $w \in \mathbb{W}^N$ and let v_g^w be the corresponding weighted connectivity game. Let $\Phi_i(v_g^w) = \sum_{t\in N} c_t^i w_t$.*
Then $c_i^i = \sum_{j=1}^{n-1}\binom{n}{j}^{-1}$ for all $i \in N$.

Proof.

$$
\begin{aligned}
c_i^i &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} b_i^{\sigma i} \\
&= \sum_{j=1}^{n}\frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma(j)=i} b_i^{\sigma i} \\
&= \sum_{j=2}^{n-1}\frac{1}{n!}\sum_{\sigma\in\Pi(N);\sigma(j)=i} b_i^{\sigma i} + \sum_{\sigma\in\Pi(N);\sigma(n)=i} b_i^{\sigma i} \\
&= \sum_{j=2}^{n-1}\frac{j(j-1)!(n-j)!}{n!} + \frac{(n-1)!}{n!} \\
&= \sum_{j=2}^{n-1}\binom{n}{j}^{-1} + \frac{1}{n} \\
&= \sum_{j=1}^{n-1}\binom{n}{j}^{-1}
\end{aligned}
$$

The first equality sign follows from the definitions of the Shapley value and Definition 4.2.1. At the third equality sign this expression is divided into 2 parts. At the fourth equality sign, $b_i^{\sigma i} w_i$ is determined for all permutations. $b_i^{\sigma i} w_i \neq 0$ only if the set vertices $C \subset N$, with $c \in C$ when $\sigma^{-1}(c) < \sigma^{-1}(i)$ and $C \cap N_i \neq \emptyset$, is connected by g. Since for every $j = \{2, ..., n-1\}$ with $\sigma(j) = i$ there are j different sets C, since $|C| = j - 1$ and since $n - |C| - |i| = n - j$ there are $j(j-1)!(n-j)!$ different permutations for every $j = \{2, ..., n-1\}$ for which $b_i^{\sigma i} \neq 0$. For all these permutations $b_i^{\sigma i} = 1$. When $\sigma^{-1}(i) = n$, $b_i^{\sigma i} = 1$. Adding the parts together concludes the proof. □

It is more difficult to determine how a coordinate of the Shapley value depends on the other weights. For a cycle graph of order 4 it is not complicated yet and we give an expression for this dependency in the following example.

Example Let $g = g^4_{cycle}$. See Figure 4.1.

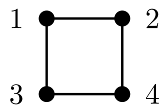

Figure 4.1: A cycle graph of order 4.

For this graph we determine how the coordinate of the Shapley value of the corresponding weighted connectivity game of individual 1 depends on the weights of all the individuals. From Theorem 4.2.3 it follows that

$$c^1_1 = \binom{4}{1}^{-1} + \binom{4}{2}^{-1} + \binom{4}{3}^{-1} = \frac{2}{3}.$$

$b^{\sigma 1}_2 \neq 0$ either when $\sigma(1) = 2$ and $\sigma(2) = 1$ or when $\sigma(3) = 1$, $\sigma^{-1}(2) \leq 2$ and $\sigma^{-1}(3) \leq 2$. The are 2 permutations for which $\sigma(1) = 2$ and $\sigma(2) = 1$ and also 2 permutations for which $\sigma(3) = 1$,$\sigma^{-1}(2) \leq 2$ and $\sigma^{-1}(3) \leq 2$. In all these cases $b^{\sigma 1}_2 = 1$. Therefore $c^1_2 = \frac{2+2}{24} = c^1_3$. $b^{\sigma 1}_4 = 0$ for all permutations and therefore $c^1 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0)$. △

4.2.4 Dependency on weights

In this paragraph we show that for every graph it holds that for every vertex the corresponding coordinate of the Shapley value of the corresponding weighted connectivity game depends more on his own weight than on any other weight.

Theorem 4.2.4. *Let $g \in \mathbb{G}^n$ be a connected graph and let $w \in \mathbb{W}^N$. Let $i \in N$. Let v^w_g be the corresponding weighted connectivity game. Let $\Phi_i(v^w_g) = \sum_{t \in N} c^i_t w_t$. Then $c^i_i > c^i_j$ for all $j \in N \backslash \{i\}$.*

Proof. For all σ with $m^\sigma_i(v^w_g) \neq 0$ it holds that either $m^\sigma_i(v^w_g) = w_i$, $m^\sigma_i(v^w_g) = \sum_{j|\sigma(j) \leq \sigma(i)} w_j$ or $m^\sigma_i(v^w_g) = \sum_{j|\sigma(j) < \sigma(i)} w_j$. In all cases, $b^{\sigma i}_i \geq b^{\sigma i}_j$

for all $j \in N \backslash \{i\}$. Since $d_i(g) \geq 1$ for all $i \in N$ there exists a σ such that $m_i^\sigma(v_g^w) = w_i$ and hence $b_i^{\sigma i} > b_j^{\sigma i}$. Now since the Shapley value is the average of the marginal vectors, $c_i^i > c_j^i$ for all $j \in N \backslash \{i\}$. □

4.3 Shapley value of monotonic weighted connectivity games

In this section we show that for complete graphs and star graphs the Shapley value of the corresponding monotonic weighted connectivity game is the same as the Shapley value of the corresponding weighted connectivity game. We also show that this does not hold for all graphs.

4.3.1 Complete graphs and star graphs

Theorem 4.3.1. *Let $g = g^n_{comp}$, let $w \in \mathbb{W}^N$ and let v_g^m be the corresponding monotonic weighted connectivity game. Then $\Phi(v_g^m) = \Phi(v_g^w)$.*

Proof. Since $v_g^m(S) = v_g^w(S)$ for all $S \in 2^N$, $\Phi(v_g^m) = \Phi(v_g^w)$. □

Theorem 4.3.2. *Let g^n_{star}, let $w \in \mathbb{W}^N$ and let v_g^m be the corresponding monotonic weighted connectivity game. Then $\Phi(v_g^m) = \Phi(v_g^w)$.*

Proof. Since $v_g^m(S) = v_g^w(S)$ for all $S \in 2^N$, $\Phi(v_g^m) = \Phi(v_g^w)$. □

4.3.2 Other graphs

It does not hold for all graphs that $\Phi_i(v_g^m) = \Phi_i(v_g^w)$.

Example In the previous chapter we have already seen in the example of the bow graph that there exist $S \in 2^N$ for which $v_g^{mwconn} \neq v_g^{wconn}$. Computing the Shapley value gives $\Phi_i(v_g^{wconn}) = (\frac{80}{120}, \frac{140}{120}, \frac{1040}{120}, \frac{240}{120}, \frac{300}{120})$ and $\Phi_i(v_g^{mwconn}) = (\frac{132}{120}, \frac{192}{120}, \frac{752}{120}, \frac{332}{120}, \frac{392}{120})$. △

4.3.3 Dependency on weights

Note that the value of a coalition in a monotonic connectivity game is not always linearly dependent of the weights, see the following example.

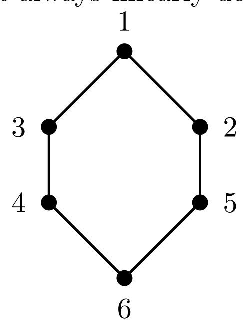

Figure 4.2: A cycle graph of order 6.

Example Let v be the monotonic weighted connectivity game corresponding to Figure 4.1. Now $v(1356) = max(w_1 + w_3, w_5 + w_6)$. Since this expression contains a maximum operator, it is not linear. $\triangle$

5

Case: The September 11 attacks

In this chapter we discuss a real-life example, namely the network of hijackers that executed the terroristic attack on 9/11. For the different connectivity games we will use the Shapley value as a measure for the importance of the individuals within this network. We will look at the most important individuals according to this measure and we also look how robust these results are.

In the morning of September 11, 2001, a series of suicide attacks were committed in the United States by a group of hijackers from the militant Islamitic organization Al Qaida. They hijacked four airplanes, and intentionally directed two of them into the World Trade Center, and one of them into the Pentagon. The fourth air plane they could not fully control and ended in a field in Pennsylvania. Around 3000 people got killed by these attacks and more than 6000 people got injured, which makes it the biggest terroristic attack in history. Soon after it the FBI already published a list of the 19 hijackers. Later on all kind of information about these hijackers became available. We use information that was collected by Krebs(2002) from open sources. The vertices of our theoretical framework are now replaced by the hijackers and the edges indicate whether there was communication between the hijackers. See Figure 5.1 for a schematic representation of Al Qaida's network of hijackers.

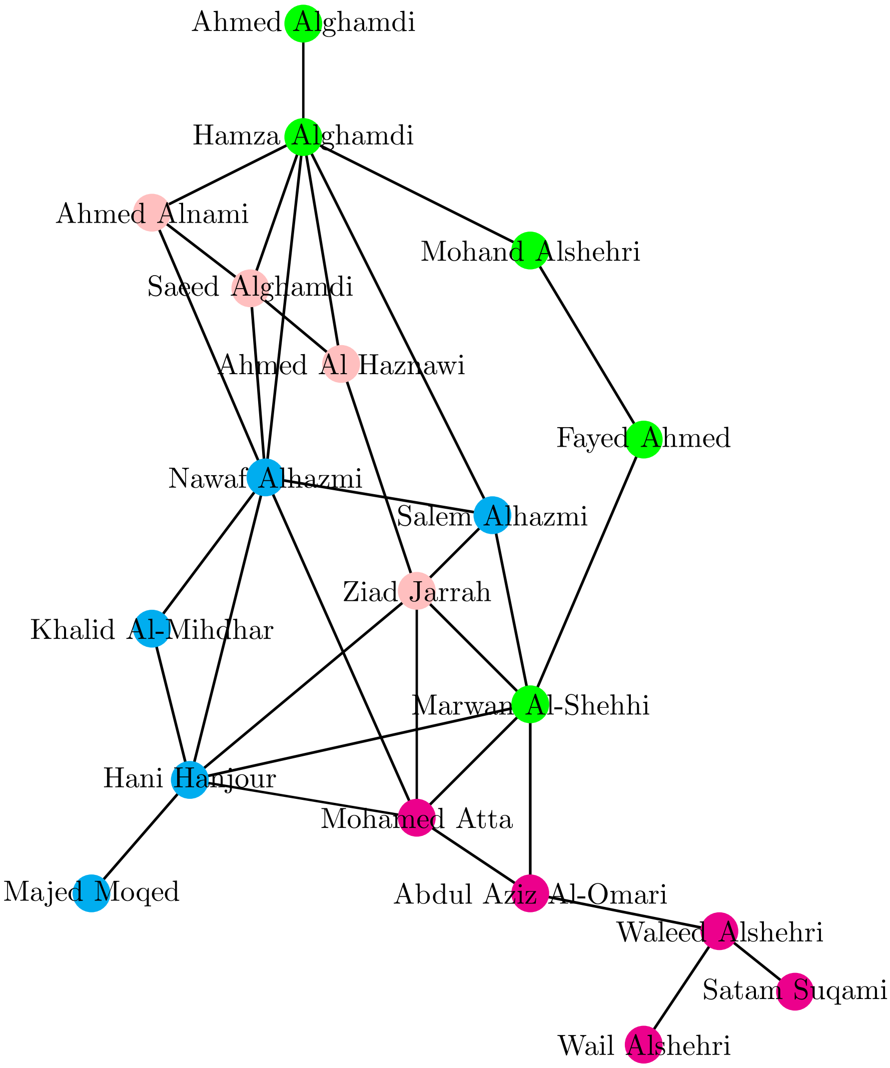

Figure 5.1: Al Qaida's 9/11 hijackers network, UA-175 (green), UA-93 (pink), AA-77(cyan) and AA-11(magenta).

For the different connectivity games we calculate the Shapley value, which provides an indication for the importance of every individual. Based on the Shapley value we make a ranking, in which hijackers with a higher Shapley value are higher in the ranking. Because of the amount of individuals it was necessary to develop computer programs to determine the different connectivity games and to compute the Shapley value.

Robustness Analysis

In analyzing covert networks it is difficult to collect precise data. When the methods are robust, there can be made quite good conclusions even when the data is not complete. We will check for robustness in 2 different ways. First we will add one or more links in the network and calculate the Shapley value for the new situation. Since computing the Shapley value for this network takes between 15 and 30 minutes, we do only 10 simulations for each method. For the connectivity games that also require a weight vector we will also check the robustness by varying the weights. For weighted connectivity games we can do much more simulations in the same time, since the Shapley value is linearly dependent on the weights.

5.1 Results using an unweighted connectivity game

To be able to calculate the corresponding connectivity game for the network of Figure 5.1 we needed a program that checks for every graph whether it is connected. See Appendix A1 for this program. Also we developed a program to calculate the Shapley value. We did this with help of the program in Appendix A2. See Table 5.1 for the Shapley value of each hijacker.

Hijacker	$\Phi^i(v^c_{911})$	Hijacker	$\Phi^i(v^c_{911})$
Abdul Aziz Al-Omari	0.2366	Salem Alhazmi	0.0060
Hamza Alghamdi	0.2344	Saeed Alghamdi	0.0053
Hani Hanjour	0.2234	Fayez Ahmed	-0.0024
Waleed Alshehri	0.2034	Ahmed Alnami	-0.0038
Marwan Al-Shehhi	0.0934	Khalid Al-Mihdhar	-0.0088
Nawaf Alhazmi	0.0864	Satam Suqami	-0.0424
Mohamed Atta	0.0476	Wail Alshehri	-0.0424
Ziad Jarrah	0.0323	Ahmed Alghamdi	-0.0449
Ahmed Al Haznawi	0.0150	Majed Moqed	-0.0530
Mohand Alshehri	0.0138		

Table 5.1 Connectivity of the 19 hijackers.

Observe that this table is identical to Table 7.13 of Lindelauf(2011). This method suggests that Abdul Aziz Al-Omari, Hamza Alghamdi, Hani Hanjour and Waleed Alshehri were the most important individuals within the network. Al-Omari and Waleed Alshehri were hijackers in American Airlines Flight 11, which was the first plane that hit the World Trade Center. Looking at Figure 5.1 it is easy to see why they were so important. Without one of them, at least 3 hijackers of the AA-11 plane would be not connected anymore with the rest of the network, which would have made the operation much less likely to succeed. Hani Hanjour was the pilot hijacker of American Airlines Flight 77, which hit the Pentagon. Hanjour was the only one to know Majed Moqed, so without Hanjour also Moqed would not have been in the network anymore. Hamza Alghamdi was one of the hijackers of United Airlines Flight 175, which was the second plane that hit the World Trade Center. He was the only one who knew Ahmed Alghamdi.

Robustness Analysis

In this paragraph we look at how the results change when one or more links between different hijackers are added. First we look what happens when we add an edge between 2 random hijackers who haven't communicated according to the data. We do 10 different simulations and check how often a hijacker is among the 5 hijackers with the highest Shapley value, which we refer to as the *top 5.* After this, we also look what happens if we add 3 links instead of one.

Adding a link

We did 10 simulations, so 10 times we added randomly an edge to the network of Figure 5.1 and we computed the Shapley value for the new corresponding connectivity game. See Table 5.3 for the percentage that a hijacker is in the top 5 of hijackers with the highest Shapley value.

Hijacker	% in top 5	Hijacker	% in top 5
Abdul Aziz Al-Omari	70	Salem Alhazmi	10
Hamza Alghamdi	100	Saeed Alghamdi	0
Hani Hanjour	90	Fayez Ahmed	0
Waleed Alshehri	60	Ahmed Alnami	10
Marwan Al-Shehhi	100	Khalid Al-Mihdhar	0
Nawaf Alhazmi	50	Satam Suqami	0
Mohamed Atta	0	Wail Alshehri	0
Ziad Jarrah	10	Ahmed Alghamdi	0
Ahmed Al Haznawi	0	Majed Moqed	0
Mohand Alshehri	0		

Table 5.2: Percentage of simulations that a hijacker is in the top 5 after randomly adding a link.

In Table 5.3 we see that the same 5 hijackers that were in the top 5 of the original network, are for these 10 simulations most often in the top 5 as well. Noteworthy is that in the simulations in which Al-Omari or Waleed Alshehri are not in the top 5, there is a link added between another hijacker and either Wail Alshehri or Satam Suqami. It makes sense that in this case Al-Omari and Waleed Alshehri become less important according to this measure, because in these cases Wail Alshehri and Satam Suqami don't need them to stay connected with the rest of the network.

Adding 3 links

Again we did 10 simulations, but this time we added randomly 3 edge to the network of Figure 5.1 for each simulation and we computed the Shapley value for the new corresponding connectivity game. See Table 5.4 for the percentage that a hijacker is in the top 5 of hijackers with the highest Shapley value.

Hijacker	% in top 5	Hijacker	% in top 5
Abdul Aziz Al-Omari	80	Salem Alhazmi	0
Hamza Alghamdi	100	Saeed Alghamdi	0
Hani Hanjour	90	Fayez Ahmed	20
Waleed Alshehri	70	Ahmed Alnami	0
Marwan Al-Shehhi	70	Khalid Al-Mihdhar	0
Nawaf Alhazmi	60	Satam Suqami	10
Mohamed Atta	0	Wail Alshehri	0
Ziad Jarrah	10	Ahmed Alghamdi	0
Ahmed Al Haznawi	0	Majed Moqed	0
Mohand Alshehri	0		

Table 5.3: Percentage of simulations that a hijacker is in the top 5 after randomly adding 3 links.

Again we see that the hijackers that were in the top 5 of the original network, are in the top 5 in most of the simulations. Also a sixth hijacker, Nawaf Alhazmi, is in the top 5 in 6 of the simulations. Nawaf Alhazmi was one of the hijackers of the plane that crashed in to the Pentagon. The results are quite similar to adding only 1 link, which is an indication for robustness.

Conclusion
The rankings based on the Shapley value of the connectivity game corresponding to the hijackers network seem to be pretty robust regarding adding links to the network. There are 6 hijackers which fill the top 5 for almost all the simulations. The most important important individual seems to be Hamza Alghamdi, which is in the top 5 for all the simulations.

5.2 Results using a weighted connectivity game

The idea behind weighted connectivity games is that it incorporates person specific variation. To construct the weighted connectivity game corresponding to the hijackers networks we use the same weight vector as Lindelauf(2011). This weight vector has been determined by starting with giving all the hijackers weight 1, and then add 1 when they were involved in an activity which could indicate that they were key individuals within the network. Such activities include attending terror training camps and attending meetings on terror attack planning. Some of the hijackers participated in several such activities. See Table 5.4 for the assigned weights.

Hijacker	Weight	Hijacker	Weight
Ahmed Alghamdi	1	Khalid Al-Mihdhar	3
Hamza Alghamdi	1	Marwan Al-Shehhi	3
Ahmed Alnami	1	Hani Hanjour	1
Mohand Alshehri	1	Mohamed Atta	4
Saeed Alghamdi	1	Majed Moqed	1
Ahmed Al Haznawi	1	Abdul Aziz Al-Omari	1
Fayez Ahmed	1	Waleed Alshehri	1
Nawaf Alhazmi	2	Satam Suqami	1
Salem Alhazmi	1	Wail Alshehri	1
Ziad Jarrah	4		

Table 5.4 Weights of the 19 hijackers.

We computed the Shapley value of the weighted connectivity game corresponding to the network of the 19 hijackers. See Table 5.3 for the Shapley value of each hijacker.

Hijacker	$\Phi^i(v_{911}^w)$	Hijacker	$\Phi^i(v_{911}^w)$
Abdul Aziz Al-Omari	6.0957	Ahmed Al Haznawi	0.4966
Hamza Alghamdi	5.5770	Fayez Ahmed	0.2920
Waleed Alshehri	5.5622	Salem Alhazmi	0.2804
Hani Hanjour	5.4026	Saeed Alghamdi	0.2336
Marwan Al-Shehhi	2.2026	Ahmed Alnami	0.1496
Mohamed Atta	1.6003	Satam Suqami	-0.3690
Nawaf Alhazmi	1.5696	Wail Alshehri	-0.3690
Ziad Jarrah	1.3108	Ahmed Alghamdi	-0.5351
Mohand Alshehri	0.6300	Majed Moqed	-0.6911
Khalid Al-Mihdhar	0.5612		

Table 5.5 Weighted connectivity of the 19 hijackers.

Observe that this table coincides with Lindelauf(2011). The rankings for the weighted connectivity game are similar to those of the unweighted connectivity games. The same 5 hijackers are in the top 5, though Hanjour and Waleed Alshehri switched places. Mohamed Atta, which was the pilot hijacker of AA-11, is now on the sixth place instead of the seventh place.

Robustness Analysis

In this paragraph we will not only look how the result change when one or more links are added, but also how sensitive the results are to changes in the weight vector.

Adding a link

We did 10 simulations, so 10 times we added randomly an edge to the network of Figure 5.1 and we computed the Shapley value for the new corresponding weighted connectivity game. See Table 5.5 for the percentage that a hijacker is in the top 5 of hijackers with the highest Shapley value.

Hijacker	% in top 5	Hijacker	% in top 5
Abdul Aziz Al-Omari	90	Ahmed Al Haznawi	0
Hamza Alghamdi	100	Fayez Ahmed	10
Waleed Alshehri	90	Salem Alhazmi	0
Hani Hanjour	80	Saeed Alghamdi	0
Marwan Al-Shehhi	100	Ahmed Alnami	0
Mohamed Atta	10	Satam Suqami	0
Nawaf Alhazmi	20	Wail Alshehri	0
Ziad Jarrah	0	Ahmed Alghamdi	0
Mohand Alshehri	0	Majed Moqed	0
Khalid Al-Mihdhar	0		

Table 5.6: Percentage of simulations that a hijacker is in the top 5 after randomly adding a link.

We see that the hijackers that were in the top 5 of the original network are still in the top 5 for almost all the simulations. So the weighted connectivity games seem to be robust regarding adding links.

Adding 3 links

Now we take a look at what might happen when there are 3 links added to the original network.

Hijacker	% in top 5	Hijacker	% in top 5
Abdul Aziz Al-Omari	80	Ahmed Al Haznawi	10
Hamza Alghamdi	100	Fayez Ahmed	10
Waleed Alshehri	40	Salem Alhazmi	10
Hani Hanjour	70	Saeed Alghamdi	20
Marwan Al-Shehhi	90	Ahmed Alnami	10
Mohamed Atta	30	Satam Suqami	0
Nawaf Alhazmi	20	Wail Alshehri	10
Ziad Jarrah	0	Ahmed Alghamdi	0
Mohand Alshehri	0	Majed Moqed	0
Khalid Al-Mihdhar	0		

Table 5.7: Percentage of simulations that a hijacker is in the top 5 after randomly adding 3 links.

In this case we see that Waleed Alshehri, which was in the top 5 in the original network, is not in the top 5 anymore for most of the observations. Like we saw in the previous paragraph, this happens when there is a link added between another hijacker and either Wail Alshehri or Satam Suqami. In these cases Waleed Alshehri becomes suddenly much less important in keeping the network together. This shows that the robustness of the methods is not only dependent on the method itself, but also on the data that is used.

Changing the weights

Now we look how robust the results are regarding changes in the weight vector. First see what happens to the Shapley value when every hijacker would have weight 1 instead of the weights of Table 5.4.

Hijacker	In top 5	Hijacker	In top 5
Abdul Aziz Al-Omari	yes	Ahmed Al Haznawi	no
Hamza Alghamdi	yes	Fayez Ahmed	no
Waleed Alshehri	yes	Salem Alhazmi	no
Hani Hanjour	yes	Saeed Alghamdi	no
Marwan Al-Shehhi	yes	Ahmed Alnami	no
Mohamed Atta	no	Satam Suqami	no
Nawaf Alhazmi	no	Wail Alshehri	no
Ziad Jarrah	no	Ahmed Alghamdi	no
Mohand Alshehri	no	Majed Moqed	no
Khalid Al-Mihdhar	no		

Table 5.8: Weighted connectivity of the 19 hijackers when all weights would be 1.

The top 5 is still the same, which indicates that the results are robust regarding the weights. Just to be sure, we now run 10000 simulations in which we let the weights for every hijacker vary between 0 and 10 and check for every hijacker how often he is in the top 5:

Hijacker	% in top 5	Hijacker	% in top 5
Abdul Aziz Al-Omari	100	Ahmed Al Haznawi	0
Hamza Alghamdi	100	Fayez Ahmed	0
Waleed Alshehri	100	Salem Alhazmi	0
Hani Hanjour	100	Saeed Alghamdi	0
Marwan Al-Shehhi	95.2	Ahmed Alnami	0
Mohamed Atta	0	Satam Suqami	0
Nawaf Alhazmi	4.8	Wail Alshehri	0
Ziad Jarrah	0	Ahmed Alghamdi	0
Mohand Alshehri	0	Majed Moqed	0
Khalid Al-Mihdhar	0		

Table 5.9: Percentage of simulations that a hijacker is in the top 5 with random weights.

It turns out that the results are extremely robust regarding changing weights. In around 5% of the simulations Nawaf Alhazmi is in the top 5 instead of Marwan Al-Shehhi, but in all the other simulations the same 5 hijackers are in the top 5. This means that for this example weighted connectivity games are in fact an ineffective way to incorporate person specific information.

Conclusion

The rankings for the weighted connectivity game are similar to those of the unweighted connectivity games. Although the idea of weighted connectivity games is to include person specific information by means of a weight vector, it turns out that the weights barely influence the Shapley value. Therefore weighted connectivity games seem to be ineffective for this purpose.

5.3 Results using a monotonic weighted connectivity game

To calculate the monotonic weighted connectivity game corresponding to the network of Figure 5.1, we needed a program that determines for every graph for which component the sum of the weights of the hijackers is the biggest. This is done by a program which is similar to the program that checks for connectivity. The program checks for a given hijacker i and a given graph to which other hijackers there is a path. These hijackers form together with i a component. From this component the sum of the weights is computed. This is done for all hijackers. The value of the coalition is equal to the value of the component with the biggest sum of weights.
Using above methods we computed the Shapley value of the weighted connectivity game corresponding to the network of the 19 hijackers. We obtained the same results as Lindelauf(2011). See Table 5.3 for the Shapley value of each hijacker.

Hijacker	$\Phi^i(v^m_{911})$	Hijacker	$\Phi^i(v^m_{911})$
Mohamed Atta	3.8850	Salem Alhazmi	1.0182
Ziad Jarrah	3.8671	Fayez Ahmed	0.8552
Marwan Al-Shehhi	3.4761	Saeed Alghamdi	0.8155
Nawaf Alhazmi	3.0681	Mohand Alshehri	0.7530
Hani Hanjour	2.1477	Ahmed Alnami	0.6944
Khalid Al-Mihdhar	1.9925	Majed Moqed	0.5025
Abdul Aziz Al-Omari	1.8487	Ahmed Alghamdi	0.4509
Hamza Alghamdi	1.6066	Satam Suqami	0.3952
Waleed Alshehri	1.1522	Wail Alshehri	0.3952
Ahmed Al Haznawi	1.0757		

Table 5.10 Monotonic weighted connectivity of the 19 hijackers.

Compared with the previous sections we now have another top 5. Mohamed Atta en Ziad Jarrah have entered the top 5, while Al-Omari and Waleed Alshehri are not in the top 5 anymore. Ziad Jarrah was the pilot hijacker of the plane that crashed into the pentagon. Atta and Jarrah are the hijackers that got a weight of 4, which is an indication that monotonic weighted connectivity games is sensitive to weights. We will see more about this in the robustness analysis.

Robustness Analysis

In this paragraph we look how sensitive the rankings are to adding links and changing weights.

Adding a link

Again we do 10 simulations, and in each simulation we add randomly a link between 2 hijackers who are not connected yet.

Hijacker	% in top 5	Hijacker	% in top 5
Mohamed Atta	100	Salem Alhazmi	0
Ziad Jarrah	100	Fayez Ahmed	0
Marwan Al-Shehhi	100	Saeed Alghamdi	0
Nawaf Alhazmi	100	Mohand Alshehri	0
Hani Hanjour	70	Ahmed Alnami	0
Khalid Al-Mihdhar	10	Majed Moqed	0
Abdul Aziz Al-Omari	20	Ahmed Alghamdi	0
Hamza Alghamdi	0	Satam Suqami	0
Waleed Alshehri	0	Wail Alshehri	0
Ahmed Al Haznawi	0		

Table 5.11: Percentage of simulations that a hijacker is in the top 5 after adding a link.

The results seem to be robust regarding adding a link. 4 hijackers stay in the top 5 in all the simulations, only Hani Hanjour is not in the top 5 anymore for a few simulations.

Adding 3 links

We also check what happens if we add 3 links instead of one.

Hijacker	% in top 5	Hijacker	% in top 5
Mohamed Atta	100	Salem Alhazmi	0
Ziad Jarrah	100	Fayez Ahmed	0
Marwan Al-Shehhi	100	Saeed Alghamdi	0
Nawaf Alhazmi	100	Mohand Alshehri	0
Hani Hanjour	50	Ahmed Alnami	0
Khalid Al-Mihdhar	50	Majed Moqed	0
Abdul Aziz Al-Omari	0	Ahmed Alghamdi	0
Hamza Alghamdi	0	Satam Suqami	0
Waleed Alshehri	0	Wail Alshehri	0
Ahmed Al Haznawi	0		

Table 5.12: Percentage of simulations that a hijacker is in the top 5 after adding 3 links.

The results are quite robust regarding adding links. From the original top 5 only Hani Hanjour is not in the top 5 anymore for half of the simulations. Looking at Table 5.10 this is not very surprising, since the difference in the Shapley value between Hanjour and Al-Mihdhar is small.

Changing the weights

First we check what happens to the Shapley value when every hijacker would have weight 1 instead of the weights of Table 5.4.

Hijacker	In top 5	Hijacker	In top 5
Mohamed Atta	no	Salem Alhazmi	no
Ziad Jarrah	no	Fayez Ahmed	no
Marwan Al-Shehhi	yes	Saeed Alghamdi	no
Nawaf Alhazmi	yes	Mohand Alshehri	no
Hani Hanjour	yes	Ahmed Alnami	no
Khalid Al-Mihdhar	no	Majed Moqed	no
Abdul Aziz Al-Omari	yes	Ahmed Alghamdi	no
Hamza Alghamdi	yes	Satam Suqami	no
Waleed Alshehri	no	Wail Alshehri	no
Ahmed Al Haznawi	no		

Table 5.13: Monotonic weighted connectivity of the 19 hijackers when all weights would be 1.

We see that Ziad Jarrah en Mohamed Atta, who had a high weight in the original network, are not in the top 5 anymore. This indicates that the Shapley value of monotonic weighted connectivity games are sensitive to changes in the weight structure.

Now we take a weight structure which is more similar to the weights of Table 5.4. We take for every hijacker a weight randomly between 0 and twice its weight.

Hijacker	% in top 5	Hijacker	% in top 5
Mohamed Atta	90	Salem Alhazmi	0
Ziad Jarrah	90	Fayez Ahmed	0
Marwan Al-Shehhi	100	Saeed Alghamdi	0
Nawaf Alhazmi	90	Mohand Alshehri	0
Hani Hanjour	20	Ahmed Alnami	0
Khalid Al-Mihdhar	50	Majed Moqed	0
Abdul Aziz Al-Omari	30	Ahmed Alghamdi	0
Hamza Alghamdi	20	Satam Suqami	0
Waleed Alshehri	0	Wail Alshehri	0
Ahmed Al Haznawi	10		

Table 5.14: Monotonic weighted connectivity of the 19 hijackers when all weights would be between 0 and twice its original weight.

The hijackers which were in the top 4 using the original weights are in the top 5 for almost all the simulations. Noteworthy is that Hani Hanjour falls out of the simulation quite often. A possible explanation is that his Shapley value in the original situation was not much higher than some of the other hijackers, so if one of them gets a relatively high weight, Hani Hanjour is passed by this hijacker. The results seem to be quite robust to changing the weights, though the stability could also be explained by the big difference between the Shapley value of the top 4 with the rest.

Now we change the weight of one of the hijackers with one, and look what happens with the Shapley value. We take the same weight vector as Table 5.1 but now we give Mohamed Atta a weight of 3 instead of 4. Compare

Table 5.15 with Table 5.10.

Hijacker	$\Phi^i(v_{911}^m)$	Hijacker	$\Phi^i(v_{911}^m)$
Mohamed Atta	3.0531	Salem Alhazmi	1.0182
Ziad Jarrah	3.8320	Fayez Ahmed	0.8558
Marwan Al-Shehhi	3.4410	Saeed Alghamdi	0.8162
Nawaf Alhazmi	3.0329	Mohand Alshehri	0.7542
Hani Hanjour	2.1139	Ahmed Alnami	0.6952
Khalid Al-Mihdhar	1.9918	Majed Moqed	0.5022
Abdul Aziz Al-Omari	1.8157	Ahmed Alghamdi	0.4517
Hamza Alghamdi	1.6081	Satam Suqami	0.3951
Waleed Alshehri	1.1513	Wail Alshehri	0.3951
Ahmed Al Haznawi	1.0764		

Table 5.15 Monotonic weighted connectivity of the 19 hijackers with the weight of Mohamed Atta equal to 3 instead of 4.

The Shapley value of Atta dropped with 0.73, and the Shapley value of the other hijackers with a high Shapley value also dropped a little bit. The Shapley value of the remaining hijackers increased, but only slightly. The rankings stay the same except for the top 3, where Atta drops 2 places.

Conclusion

Monotonic weighted connectivity games seem to be very robust regarding adding links. Also regarding weights they are robust, but less robust than weighted connectivity games. Therefore monotonic weighted connectivity games seem to be a good way to incorporate person specific information.

5.4 Overview

In this chapter we computed the different connectivity games, computed the Shapley value and looked how robust these rankings were. In table 5.16 we give a quick overview of the robustness of the methods for the case of the September 11 attacks.

Method	Adding a link	Adding 3 links	Changing weights
Connectivity	+	+	
Weighted connectivity	+	+	+++
Monotonic weighted connectivity	++	++	+

Table 5.16 Overview robustness of the methods for the case of the September 11 attacks.

All the methods have turned out to be robust for the case of the September 11 attacks. Robustness can be good, since when the rankings are robust there can be made quite good conclusions about who the key individuals are, even when the data is not complete. However, a method should not be too robust. For example the weighted connectivity games are extremely robust to changing weights for this particular example. This means that it does not matter which weight vector there is used, the rankings are always more or less the same. Since for this example monotonic weighted connectivity games are less robust regarding weights than weighted connectivity games, but still robust, monotonic weighted connectivity games seem to be a more effective way to incorporate individual specific information.

Conclusion and Recommendations

In this thesis we analyzed different methods to identify the key players within a terroristic network. These methods combine graph theory and game theory and are suitable to use information about both the network structure and individual parameters. Relevant information about individuals in a terroristic network include for example whether they were part of earlier attacks and whether they showed signs of radicalization. We showed that the core of a connectivity game is non-empty if and only if the corresponding graph is a star graph, and that the core of a weighted connectivity game equals the weight vector if and only if the degree of each vertex of the corresponding graph is at least two. For several standard graphs we derived closed formulas for the Shapley value of the different connectivity games. We applied the methods to the network of hijackers that executed the September 11 attacks. The methods all resulted in rankings of the terrorists in the network. We found that for this example all rankings are robust regarding adding links and changing weights. Weighted connectivity games are maybe even too robust regarding changing weights, and therefore monotonic weighted connectivity games are in our opinion the best way to incorporate both the network structure and individual specific information.

Since these methods make it possible to identify the most important individuals within a network even with limited data, we recommend to keep developing these methods and to use them in counterterrorism research.

Future research could include analyzing a strictly monotonic weighted connectivity game in which the value of a coalition always increases when an individual is added to this coalition. This could be done either by taking the sum of the values of the components or in such a way that the component with the biggest sum of weights is still the most significant, but that smaller components are also taken into account.
Another way to determine the value of a component could be not to look only at whether it is connected, but also at the structure of a graph. Lindelauf(2011) analyzed optimal structures of covert networks and it could be argued that the better the structure of a component, the higher the value it should get.
Another idea is to use the Shapley value of the connectivity game as a weight vector for the weighted connectivity game. For this it is necessary to analyze the effects of allowing for negative weights. Also interesting is to determine for a given graph the stable weight vector, i.e. the weight vector for which the Shapley value equals the weight vector. Note that in these cases the weight vector should not be interpreted as information about individual details, but rather as a measure for the centrality of an individual in the network. We finish with an example to show how this works.

Example Let $g = g^n_{star}$. Let individual i be the individual with $d_i = n - 1$. Then if $w_i = \frac{\sum_{k \in N} w_k}{n+2}$ and $w_k = \frac{2 * \sum_{k \in N} w_k}{(n-1)(n+2)}$ for all $k \neq i$, it holds that the Shapley value of the corresponding weighted connectivity game is equal to the weight vector, i.e. $\Phi(v^w_g) = w$ △

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Appendix A

Matlabfiles

A.1 Connectivity check

The following program determines for a given graph g whether it is connected or not.

```
function connected=checkc(g)
n=length(g);
I=zeros(1,n);
I(1)=1;
count=1;
while sum(I)<n && count<n
for i=1:n
if I(i)==1
for j=1:n if g(i,j)==1 I(j)=1;
end end end end count=count+1; end
if sum(I)==n connected=1; else connected=0; end
```

A.2 Shapley value

The following program computes for a given lexicographically ordered game v the Shapley value for individual 1.

```
function Shapley=Shappie(v)
n=log2(length(v)+1);
M=zeros(1,2^n-2);
M(1,1)=v(1)*factorial(n-1);
c=2; i=1;
while i<n
for j=c:c+nchoosek(n-1,i)-1
M(1,j)=(v(j+nchoosek(n-1,i))-v(j))*((factorial(n-1))/(nchoosek(n-1,i))); end
c=c+2*(nchoosek(n-1,i)); i=i+1;
end
Shapley=sum(M)/factorial(n); end
```