The impact of collateralization on swap curves and their users

Master Thesis Investment Analysis

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Abstract

Derivative contracts are increasingly being collateralized in order to mitigate counterparty credit risk. The fact that collateralization makes the contract (close to) riskless has consequences for the discount rate used. This discount rate should not be based on the swap curve, but on the growth rate of the collateral specified in the Credit Support Annex (CSA). If the collateral is cash based, which it usually is, the overnight index rate (OIS) is closest to matching the single day credit risk of collateralized derivatives and should therefore be used as the discount rate. The result is a different valuation which can have big impact if the spread between the swap curve and the overnight curve increases. After years of a stable near-zero spread, the Euribor-OIS basis exploded in 2007, resulting in a market-wide shift to OIS discounting of collateralized derivatives. The effects of this reach beyond the derivative markets, because collateralized interest rate swaps are the basis for building zero curves used to discount all future cash flows. With the swaps being the instruments to build zero curves, the changes affect the valuation of all cash flows to be discounted.

The theoretical evidence leaving no doubt, an attempt is made to empirically determine the impact on different hedging strategies in the stress scenario that has occurred during the credit crunch. This stress test leads us to conclude that there is a significant difference in hedge effectiveness when switching to OIS discounting, confirming the theoretical need for proper accounting. It is determined that more sophisticated two-curve hedging methods are promising but require more testing and support in the form of liquid OIS markets. It can be concluded that there is little doubt that collateralized derivatives should be discounted based on the underlying CSA and that, when setting up a hedge or building a zero curve, multiple curves have to be accounted for.

Contents

1. Introduction

The traditional approach to interest rate swap (IRS) valuation treats a swap as a portfolio of forward contracts on the underlying floating interest rate. Under specific assumptions regarding the nature of default and the credit risk of the counterparties, Duffie and Singleton (1997) prove that swap rates are par bond rates of an issuer who remains at LIBOR quality (AA credit rating, Hull, 2010a) throughout the life of the contract. Due to increasing popularity, more diverse counterparties entered the market which led to measures to mitigate credit risk on swap contracts. (Johannes et al. 2007)

In the case of a swap, what matters to a counterparty in the event of a default is the market value and the replacement cost of the swap: what it costs to put on a new swap with equal characteristics. Even though market risk is covered by the swap, counterparty credit risk arises as a result of the contract. A way to lower counterparty risk is collateralization combined with (daily) Marking to Market (MtM).

ISDA (2001) finds that "more than 65% of plain vanilla derivatives, especially interest rate swaps" are collateralized according to the CSA (Credit Support Annex, a standardized legal agreement on collateral). Most collateral is posted in the form of either cash or government bonds. Other, riskier assets are possible but these are likely subject to a bigger haircut than the small one on government bonds (ISDA, 2003). ISDA (2001) also finds through a survey that 74% of market participants MtM at least daily. A collateralized interest rate swap that is marked to market daily has (almost) no credit risk. He (2000) states about this: 'The current industry practice has essentially removed (in a significant way if not completely) the risk of default by either counterparty so that, for all practical purposes, swaps shall be valued without the consideration of counterparty risk.' ISDA(2011) finds that in 2010, 70% of all OTC derivatives transactions were subject to a collateral agreement, of which 81% was collateralized with cash, and most of the remainder with government securities. It is possible that only one of the parties in an OTC contract is obligated to post collateral, especially when there is a big difference in credit rating. However, ISDA (2003) concludes that bilateral collateral agreements are market practice on swaps, meaning that both parties post collateral if their position has a negative market value.

Before the credit crunch, the default-risky Libor curve was used to discount the future cash flows of a swap contract, even though most contracts were already collateralized at the time. (Johannes et al. 2007) This was not a problem because the risky Euribor curve and the risk free curve were almost equal so that the valuation differences from using another curve for discounting were negligible. However, the spread between the risk free Eonia[1] overnight rate and the swap rate has increased since then, coming from nearly zero. As a result, collateralized swaps are increasingly discounted using the Eonia index while uncollateralized swaps are still discounted using the swap curve. (Koers, 2010) Also Fujii et al. (2009b) conclude that 'Libor discounting is inappropriate for the proper pricing and hedging of collateralized contracts.'

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[1] Euro Over Night Index Average

The 6 month Eonia-Euribor spread is depicted in graph 1 below, to illustrate the magnitude of the spread increase after years of near perfect correlation and a near-zero spread.

Graph 1: 6month EONIA versus 6month EURIBOR
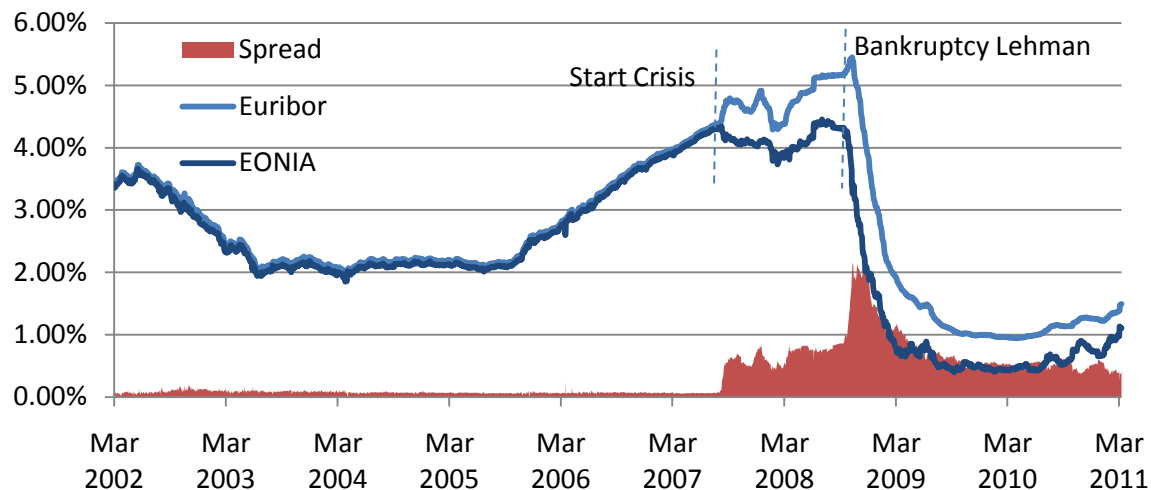
The spread starts to widen at the beginning of the crisis, and reaches its peak at just over 200 basis points around the Lehman Brothers bankruptcy event. After that, the curves converges to a reasonably stable spread of about 40 basis points. This pattern is also observed outside the euro area; in the US, the 3 month Libor – OIS[2] basis reached a peak of 366 basis points in October 2008 (Whittal, 2010a), after which the curves converged to a lower spread.

Another indicator of credit risk, or as Paul Krugman (2008) calls it, an indicator of lack of trust in the economy, is the TED spread. This is the spread between 3 month US Libor and the interest rate on 3 month treasury bills.

Graph 2: 3m US Libor versus 3m T-Bill: The TED spread

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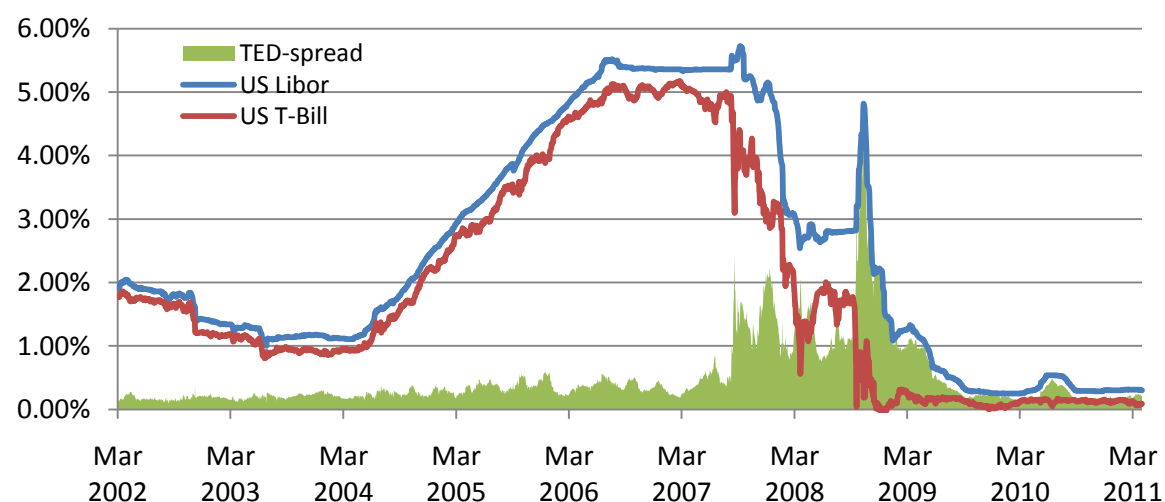
[2] Overnight index swap, the overnight rate which is Eonia in the Euro area. In the US this is the Federal funds rate.

The TED spread has different absolute values but follows the same pattern as the Euribor-Eonia spread, with the same two peaks indicated in graph 1. This illustrates the global reach of the liquidity crisis around those two events and makes clear that a spread that has been constant for years can suddenly explode.

Given that collateralization leads to risk-free discounting, the impact of credit spreads can be very large. The shift is supported by Fujii et al. (2009b) who show that when a contract is collateralized, the collateral rate should be used for discounting. For cash collateral, this is the Overnight Indexed Swap rate (OIS) which is Eonia in the euro zone, the federal funds rate being its counterpart in the US. Also, one of the biggest clearinghouses in the world, LCH.Clearnet, recently shifted to OIS discounting, after consulting its members and concluding that 'the market has come to the consensus that this is the correct way to value swap trades.'(Whittal, 2010b)

Accounting rules nowadays are aimed at transparency through fair value accounting, for which IAS39 states for over the counter derivatives: 'the valuation technique is consistent with accepted economic methodologies for pricing financial instruments and incorporates all factors that market participants would consider in setting a price.' Given the market consensus described above, OIS discounting should be accounted for by all OTC derivative users. Besides the transparency argument, proper discounting should also reduce to likelihood of closing collateralized derivative positions with positive mark-to-market value against too low of a price. If one party in the transaction uses a non-risk-free curve, the other party could benefit by using its knowledge advantage. This should not be possible in the sophisticated financial world today.

The consequences of the shift to risk free discounting in combination with a spread increase are not limited to the swap market. All derivatives and future cash flows are discounted with a discount curve which is primarily built from swaps subject to a change in valuation. This means that the swap curves change which has an impact on the zero curve used for discounting, thus affecting the value of all financial objects that are discounted.

On the contract level, collateral other than cash can be used, which would make the discount rate different. An appropriate rate when bonds are used as collateral is the repo rate for that bond. However, this imposes some problems, as repo rates are not as widely and publicly available, 'always leaving room for discussion on the exact amount' (Verheijen, 2009). This is contrary to OIS rates, which are published daily. The haircut imposed on the repo transaction varies and this has an effect on the repo rate, making the effective rate hard to observe.

Regardless of the collateral used, a collateralized swap has exposure to more than just the underlying swap curve. Since the discounting is done with a risk-free curve with or without a repo spread, there is also exposure to the discount curve. Besides this, the exposure to the Euribor curve changes as it is no longer used for discounting. This is discussed in section 4.

The above can have major impact on financial institutions with large interest rate exposure, such as pension funds and insurance companies. The liabilities of defined benefit (DB) pension funds have to be discounted with a market interest rate according to IFRS. Due to the long duration of the liabilities, the

funding ratio of a pension fund is very sensitive to the interest rate used for discounting. The funding ratio is defined as:

$$FR = \frac{Market\ value\ Assets}{Market\ value\ Liabilities}$$

Here, in the case of a DB scheme, the liabilities are the built up pension rights. In the Netherlands, which has a large pension sector, the discount curve is the euro swap curve under the FTK regulation for pension funds. The regulator also prescribes a minimum funding ratio of 105%. As a result, all Dutch pension funds that hedge their interest rate risk, do so with respect to the euro swap curve to protect their funding ratio from falling below 105%. In this process, the euro swap curve is not only used to determine the swap quotes, but also used for discounting of the contract. As the valuation section will show, this is perfectly fine for uncollateralized swaps. But given the fact that the great majority of the swap contracts is collateralized, this is no longer appropriate. The same can be said about the interest rate sensitivity and resulting hedge strategy of the fund. New hedging policies might be required.
In particular, sections 6 and 7 will focus on a stylized Dutch pension fund, which will be used in an attempt to quantify the hedge effectiveness of different hedging strategies in a crisis scenario. The testing environment will be the situation where the hedge is most needed: a crisis scenario where Euribor and Eonia go down and diverge in a short time span. The recent credit crisis in which the interest rates moved as depicted in graph 1 will be used as the stress scenario.

The goal of this thesis is to analyze the impact of collateralization on swap curves and their users. The first part will cover the effects of collateralization on the valuation and interest rate sensitivity of interest rate swaps. When the technicalities are treated, the different hedging strategies will be researched to reach a conclusion about the optimal hedging policy for pension funds and other financial institutions, given the new financial order after the crisis.

In the next section, the pricing of swaps with and without collateral is discussed, after which the construction of curves will be elaborated on in section 3. Section 4 investigates the interest rate sensitivity for both collateralized and uncollateralized swaps to one or more curves. The technicalities end in section 5 where the testing methodology is described. Section 6 presents the results from the tests described in section 5. Finally, section 7 concludes.

2. The pricing of interest rate swaps

In this section, the valuation of interest rate swaps (IRS) is discussed for contracts with and without collateral and different underlying curves.

2.1. Uncollateralized IRS

The traditional approach to interest rate swap valuation treats a swap as a portfolio of forward contracts on the underlying floating interest rate. It is then valued by separately assessing both the legs of the swap. The fixed leg is equal to a fixed rate bond without the notional being exchanged. The floating leg is equal to a floating rate bond without exchanged notional. At initiation, the value of both the legs is equal, making an interest rate swap a product which does not require an initial investment. The fixed rate which makes the value of the contract zero; the par swap rate; is quoted in the market.

Both legs are valued by discounting all the future cash flows. Since the future payments of the floating leg are unknown, forward rates are used for valuation. If this would not be accurate, arbitrage opportunities would occur.

For the sake of illustration, we value a simple 1-year swap with a notional of 1, with the floating leg paying the 6 month Euribor rate every 6 months and the fixed leg paying the agreed fixed rate annually. This is the common market practice in the Euro zone.[3] First, some definitions:

$N = Notional\ of\ the\ swap$

$\alpha_{t,k} = Daycount\ fraction\ from\ time\ t\ to\ time\ k.$

$$The\ day\ count\ convention\ (DCC)\ of\ the\ fixed\ leg\ is\ \frac{30}{360}, while\ the\ floating\ leg\ has\ \frac{ACT}{360}$$

$$Alpha\ is\ the\ year\ fraction\ between\ 2\ dates\ counted\ according\ to\ the\ appropriate\ DCC.$$

$Z_k = Euribor\ zero\ rate\ with\ maturity\ k$

$$D_k = Euribor\ discount\ factor\ with\ maturity\ k\ ^4 = \frac{1}{1 + Z_k * \alpha_{0,k}} \tag{1}$$

$$F_{t,k} = Euribor\ forward\ rate\ from\ time\ t\ to\ time\ k^5\ = \frac{1}{\alpha_{t,k}}\left(\frac{D_t}{D_k} - 1\right) \tag{2}$$

When a symbol of the above is used to indicate an overnight index swap rate, and is thus related to EONIA, the superscript OIS will be added.

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[3] For USD and GBP swaps, the leg tenors are different.
[4] Simple compounding is used here, since the underlying Euribor curve is simple compounded. Ametrano and Bianchetti (2009) also use simple compounding in their work.
[5] Derivation on the next page.

We will now value a one year swap, which has two legs, each with a stream of cash flows that have to be discounted. The fixed leg pays annually and thus pays one fixed cash flow, which is valued as:

$$PV(fixed) = D_{1y} * \alpha_{0,1y} * K_{1y}$$

To value the floating leg, we first need to define the forward rates as function of discount factors.
In the simple compounded world, there would be arbitrage opportunities if the following relation would not hold for the two year case:

$$\left(1 + Z_1 * \alpha_{0,1y}\right) * \left(1 + F_{1y,2y} * \alpha_{1y,2y}\right) = 1 + Z_{2y} * \alpha_{0,2y}$$

The one to two year forward rate is then easily found as:

$$F_{1y,2y} = \left(\frac{1 + Z_{2y} * \alpha_{0,2y}}{1 + Z_{1y} * \alpha_{0,1y}} - 1\right) * \frac{1}{\alpha_{1y,2y}}$$

Then, we reverse (1) to express discount factors as a function of zero rates:

$$D_k = \frac{1}{1 + Z_k * \alpha_{0,k}}$$

$$Z_k = \left(\frac{1}{D_k} - 1\right) * \frac{1}{\alpha_{0,k}}$$

We can use this result to rewrite the one to two year forward rate:

$$F_{1y,2y} = \left(\frac{1 + \left(\frac{1}{D_{2y}} - 1\right) * \frac{1}{\alpha_{0,2y}} * \alpha_{0,2y}}{1 + \left(\frac{1}{D_{1y}} - 1\right) * \frac{1}{\alpha_{0,1y}} * \alpha_{0,1y}} - 1\right) * \frac{1}{\alpha_{1y,2y}}$$

$$F_{1y,2y} = \frac{\frac{1}{D_{2y}}}{\frac{1}{D_{1y}}} * \frac{1}{\alpha_{1y,2y}} = \left(\frac{D_{1y}}{D_{2y}} - 1\right) * \frac{1}{\alpha_{1y,2y}}$$

This result is generalized in (2), and is required for valuing the floating leg of the one year interest rate swap. The floating leg is valued by discounting all cash flows; the payment in 6 months and the final payment in one year:

$$PV(flt) = \alpha_{0,6m} * F_{0,6m} * D_{6m} + \alpha_{6m,1y} * F_{6m,1y} * D_{1y}$$

$$By \ rewriting \ the \ forward \ rates \ using \ (2) \ we \ get:$$

$$PV(flt) = \alpha_{0,6m} * \frac{1}{\alpha_{0,6m}} \left(\frac{D_0}{D_{6m}} - 1\right) * D_{0,6m} + \alpha_{6m,1y} * \frac{1}{\alpha_{6m,1y}} \left(\frac{D_{6m}}{D_{1y}} - 1\right) * D_{0,1y}$$

$$PV(flt) = \left(\frac{D_0}{D_{6m}} - 1\right) * D_{6m} + \left(\frac{D_{6m}}{D_{1y}} - 1\right) * D_{1y}$$

$$PV(flt) = D_0 - D_{6m} + D_{6m} - D_{1y}$$

$$PV(flt) = D_0 - D_{1y}$$

This result also holds for multi-year swaps; which makes it an easy pricing method.

The results can be generalized to the following[6]. Note that the alphas are differentiated here, because the floating leg has semi-annual periods while the fixed leg has annual periods.[7]

$$PV(fixed) = N * K_n * \sum_{k=1}^{n} D_k * \alpha_{k,k-1}^{fxd} \tag{3}$$

$$PV(floating) = N * \sum_{k=1}^{n} D_k * \alpha_{k,k-1}^{flt} * F_{k-1,k} \tag{4}$$

$$Which\ reduces\ to:$$

$$PV(floating) = N * (D_0 - D_n) \tag{5}$$

The value of a receiver swap is then the fixed leg minus the floating leg, while a payer swap is the opposite.

2.2. Overnight index swap

An overnight index swap (OIS) is a collateralized product which swaps a fixed rate for the overnight interest rate. The fixed leg is valued similar to an uncollateralized single curve swap, but now the underlying curve is the OIS curve. The floating leg is different, as it pays the daily rate instead of the six month rate. The actual payment, however, only takes place yearly. This means that the reset dates are not simultaneous with the payout dates, and thus interest is accrued over these reset payments.

Equation (4) can be rewritten with OIS discount factors and a fixed rate from the OIS curve, which leads to:

$$PV(fixed) = N * K_n^{OIS} * \sum_{k=1}^{n} D_k^{OIS} * \alpha_{k,k-1}^{fxd} \tag{6}$$

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[6] The fixed and floating leg are valued separately and not summed to obtain the value of a swap. The 'value of a swap' is trivial because it is not clear whether a payer or receiver swap is meant. If there is reference to the value of a swap in this thesis, the value to the counterparty being discussed is meant.
[7] From now on, every time it could be unclear what period is meant, the superscripts will be added.

The floating leg has, as stated, a different approach. The daily realizations of the floating OIS rate accrue the Eonia interest rate, which leads to the following:

$$PV(floating) = D_n^{OIS} * \left(\left[\prod_{k=1}^{n} (1 + F_{k-1,k}^{OIS} * \alpha_{k-1,k}) \right] - 1 \right)$$

Note that the day count fractions concern business day periods here.[8] By rewriting the forward rates similar to (2) we get:

$$PV(floating) = D_n^{OIS} * \left(\left[\prod_{k=1}^{n} \left(1 + \frac{1}{\alpha_{k-1,k}} \left(\frac{D_{k-1}^{OIS}}{D_k^{OIS}} - 1 \right) * \alpha_{k,k-1}^{OISflt} \right) \right] - 1 \right)$$

$$PV(floating) = D_n^{OIS} * \left(\left[\prod_{k=1}^{n} \left(\frac{D_{k-1}^{OIS}}{D_k^{OIS}} \right) \right] - 1 \right)$$

For a one-year OI swap, this results in:

$$PV(floating) = D_{365d}^{OIS} * \left(\left[\frac{D_0^{OIS}}{D_{1d}^{OIS}} * \frac{D_{1d}^{OIS}}{D_{2d}^{OIS}} * \frac{D_{2d}^{OIS}}{D_{3d}^{OIS}} \cdots \cdots * \frac{D_{364d}^{OIS}}{D_{365d}^{OIS}} \right] - 1 \right)$$

This reduces to

$$PV(floating) = D_{365d}^{OIS} * \left(\frac{D_0^{OIS}}{D_{365d}^{OIS}} - 1 \right)$$

$$PV(floating) = D_0^{OIS} - D_{365d}^{OIS}$$

Which can be generalized to:

$$PV(floating) = N * \left(D_0^{OIS} - D_n^{OIS} \right) \tag{7}$$

This is equal to (6). We can thus conclude that the OI par swap valuation method is equal to that of an uncollateralized Euribor swap, but that a different curve is used.

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[8] It looks here as if there are 365 resets in a year. This is not the case, resets only occur on business days. In the weekends and on holidays, the compounding takes place using the daily rate of the last business day.

2.3. Cash Collateralized IRS[9]

If the swap contract is subject to a CSA which specifies Euro cash to be used as collateral, there is exchange of collateral when the market value of the contract changes. Assuming that the marked-to-market value (MtM) of a bilateral swap is positive for the bank, the counterparty, in this case a pension fund, has to post collateral. When the CSA specifies that collateral can only be posted in cash, and also specifies a single currency (here: EUR), the cash flows look like the following:

Figure 1: Cash collateral process
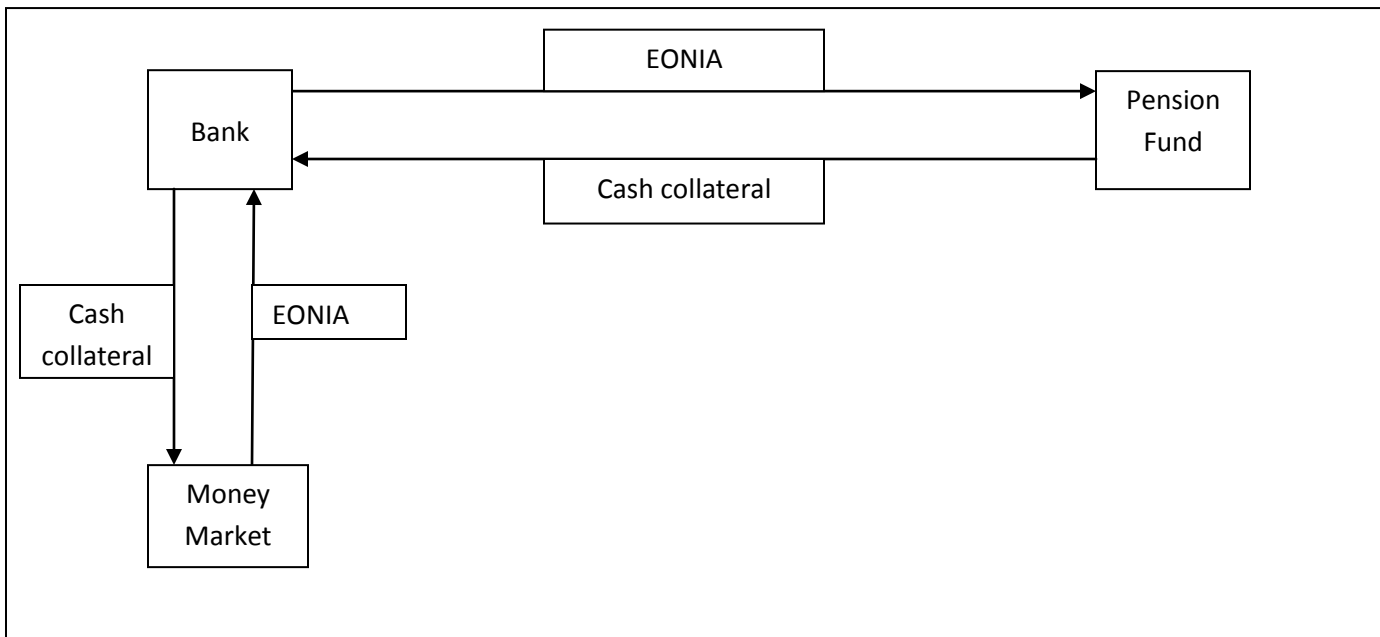
The pension fund has to post collateral to the bank and does so with EUR cash. The bank pays daily Eonia[10] as a fee over the collateral to the pension fund. This is common practice in the swap market (Johannes and Sundaresan, 2007) The bank finances this by posting the collateral in a money market account[11] on which it receives EONIA. The collateral account is balanced daily and the reverse position is also possible. In this case, the MtM of the swap is positive to the pension fund and thus negative to the bank. The process is then reversed; the bank borrows money in the money market to post as collateral to the pension fund, for which it receives EONIA. The latter is used to pay the fee to the money market counterparty.

The amount of collateral posted equals the present value of the future cash flow. The interest rate of the collateral thus determines how much collateral should be posted, making the growth rate of the collateral the discount rate for the value of the derivative. (Lansink & Potters, 2011) This can be

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[9] See Fujii et al. (2009b) and Piterbarg (2010) for the proof.

[10] For British pounds, SONIA would be the EONIA equivalent. In the US this would be the Federal funds rate.

[11] The money market is exemplary to illustrate the revenue from the collateral for the bank. In practice, the bank could loan out the collateral cash to another bank for one day, for which it would receive EONIA over the amount. If the bank does not lend out the money because it needs it to finance other activities, it has less need to borrow. As the bank would have to pay EONIA over this loan, it 'saves' the interest burden due to the collateral posting. Regardless of what it does with the money, the bank earns EONIA over the amount posted by the pension fund.

illustrated by assuming a positive MtM for one party. Suppose the bank has a positive MtM and thus receives, ceteris paribus, a cash flow at the next settlement date, which is the date the contract matures. The collateral the pension fund has to post is the present value of this cash flow. This present value is calculated by multiplying the future cash flow with a discount factor. What is known, is that the collateral amount posted today will grow to the cash flow at the settlement date, one year from now. We thus have 2 equations:

$$t = 0 \quad Collateral(0) = Market\ value\ derivative = [CF(1) * D_1]$$

$$t = 1 \quad Collateral(0) * (1 + R_{collateral}) = CF(1)$$

$$From\ these\ 2\ equations, we\ can\ derive\ that$$

$$D_1 = 1/(1 + R_{collateral}) \tag{8}$$

This illustrates that when collateralization is applied, the return on the collateral is the basis for the discount factor. If this were not the case, there would be arbitrage opportunities.

For cash collateral, the overnight rate on the cash is the discount rate for the derivative, regardless of the credit quality of the counterparties (the collateralization makes it riskless). For bond collateral, the discount rate is then the repo rate. This causes a problem, since this rate is generally unobservable, as it depends on the credit quality of the party borrowing the bonds and the applied haircut, which differs per transaction. This all depends on the type of bonds used, which is specified in the CSA. Hence the term CSA discounting.

The most important result is that the funding cost of the collateral determines the discount curve used to value the derivative. If we translate this to a cash collateralized normal (Euribor) swap, the Euribor discount rate does no longer apply. The risk free nature due to the collateral is to be captured in the risk free discount rate, which is the OIS rate. A second curve is thus introduced for valuation purposes. The pricing formulas (3) and (5) above are then no longer accurate because those formulas only work for one underlying curve.

When using the OIS curve for discounting on a normal Euribor swap, we have to combine the Euribor swap valuation with the risk free nature from the OI swaps. For the fixed leg, this results in combining (3) and (6):

$$PV(fixed) = N * K_n * \sum_{k=1}^{n} D_k^{OIS} * \alpha_{k,k-1}^{fxd} \tag{9}$$

The fixed cash flows determined by the Euribor curve are discounted with the OIS curve since the collateral made the swap risk-free.

The same operation for the floating cash flows is performed by combining (4) with Eonia discounting:

$$PV(floating) = N * \sum_{k=1}^{n} D_k^{OIS} * \alpha_{k,k-1}^{flt} * F_{k-1,k} \tag{10}$$

The forward rates used to value the floating leg are still Euribor rates while the discounting of this leg is done using the OIS curve. Note that the simplification to (5) is no longer possible, as the discount factors used to calculate the forward rates originate from the Euribor curve while the discounting itself is done with OIS discount factors. This is a direct consequence of equation (8), which states that the growth rate of the collateral (here: Eonia) is the basis for the discount factors used to value the derivative.
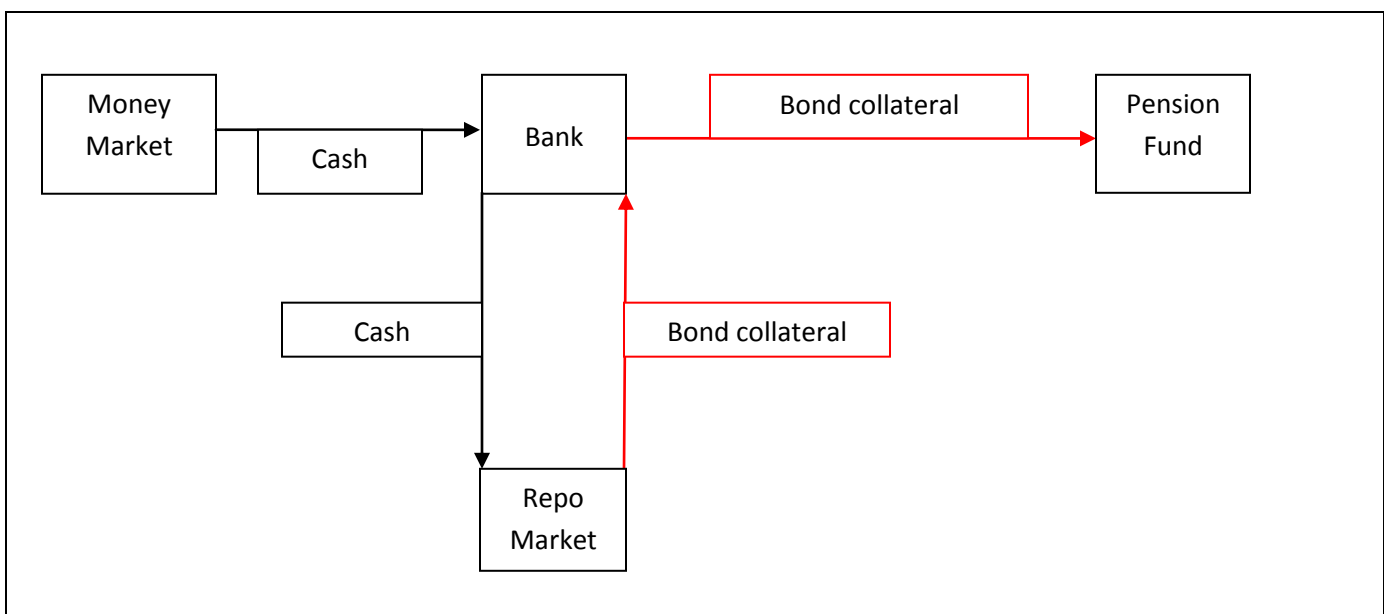
2.4. Bond collateral

Contrary to banks, pension funds do not usually have a lot of cash at hand. They do typically have a large asset allocation to government bonds, which can also be used as collateral. (Verheijen, 2009) Therefore, pension funds specify their CSAs such that collateral on both sides is posted in the form of government bonds. Sometimes, a further specification is added; such as triple A rated only, or only government bonds from a specific set of countries. The latter has become more important since the euro debt crisis has resulted in differences in quality as collateral between countries' government bonds.

Let's assume that the bank has a negative MtM , and thus has to post collateral, which it does in the form of bonds. The flow chart then looks like:

Figure 2: Bonds collateral process, after shift in MtM

As stated earlier, banks do not usually have a large portfolio of bonds on their balance sheet. To obtain the bonds needed for the collateral, the bank lends out cash in the repo market. The repo transaction is bond collateralized, so the bank receives bonds as collateral for the loan. The financing of this transaction is still done through the money market, similar to cash collateral. However, the cash is lent out to the repo counterparty in exchange for the bonds as collateral. (Lansink & Potters, 2011)

During the repo, there is transfer of legal and beneficial title, allowing "re-use". (Wood, 2011) The bank is thus allowed to post the bonds as collateral to the pension fund. Even though the legal rights have been transferred to the bank, the repo counterparty still receives the coupons, just as the pension fund transfers these coupons to the bank when paid out. Because the value of the collateral drops with the payout of the coupon, the bank then will post more bonds to take away the credit risk for the pension fund. The coupons can thus be left out of the valuation process, and will be ignored in the remainder of the thesis.

The amount of collateral is settled on a daily basis, and thus varies during the life of the contract. The settlement is performed such that at the end of each day, the collateral amount equals the present value of the contract.[12] This will be discussed further below.

Now suppose that one day later, the pension fund wants to unwind the position, thus receiving the present value of the contract by selling it back to the bank. The process is then reversed:

Figure 3: Bonds collateral process, at unwinding of the contract
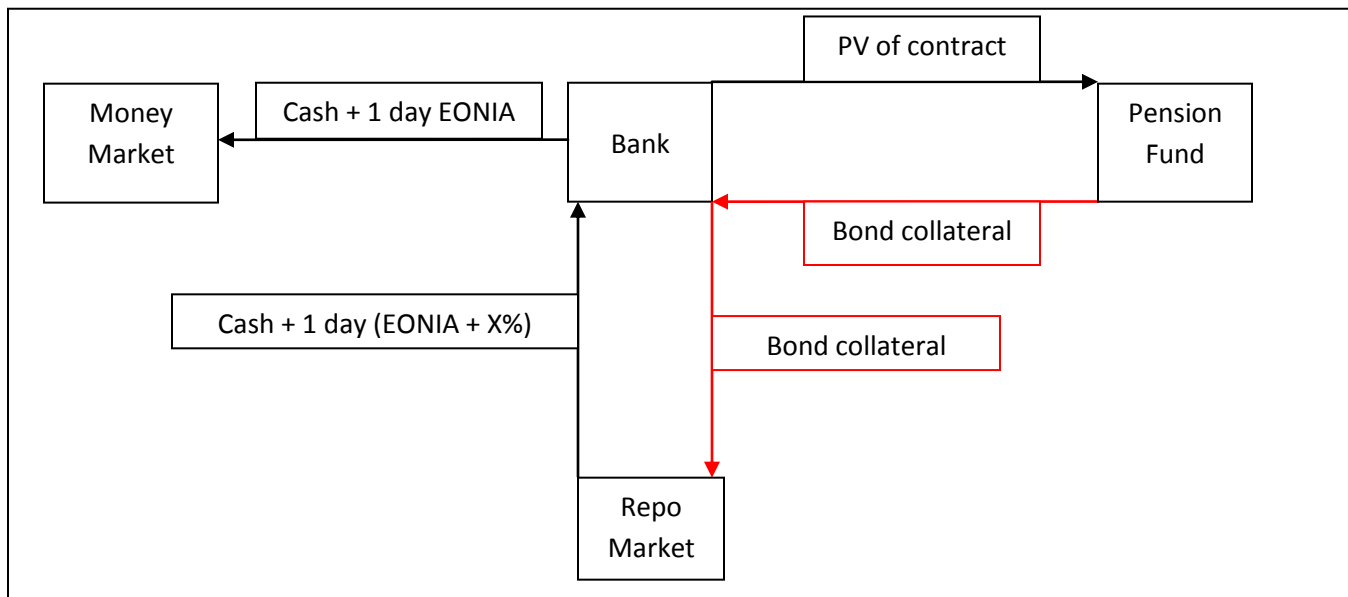
The bank pays the present value of the contract to the pension fund and receives back the bonds posted as collateral, and then reverses the repo transaction. The bank receives back the cash plus interest, being the repo rate. The cash plus Eonia is then used to pay back the financer of the collateral. When the repo rate equals EONIA, X=0 and the process is basically similar to the one above with cash collateral. The only difference is the repo market as intermediary to swap cash collateral to bond collateral. However, X% is a profit for the bank when the repo rate is higher than EONIA[13]. The total loss of the contract to the bank is then the present value of the contract it has to pay to the pension fund plus the gain or loss on the collateral posting, which is dependent on X.

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[12] In practice, not all MtM is settled daily. When the change is very small, the cost of posting the extra collateral can become relatively high. Therefore, a minimum transfer amount (MTA) is specified in the CSA. This means that the position is only settled when the amount that has to be exchanged exceeds the MTA. Obviously, the higher the MTA, the higher the risk. In this thesis, the MTA is assumed to be low enough not to cause significant credit risk.
[13] In theory and in practice, repo spreads can be negative. This has occurred for German government bonds. This is not market failure, it just reflects the strength of the demand for a security. (Wood, 2011) In this case, the bank would make a loss on posting bonds as collateral.

When bonds are used as collateral, equations (9) and (10) are no longer valid because the collateral cannot be discounted using the OIS rate (assuming X≠ 0). Following (8), the repo rate is then the relevant rate to use. The forward rates in the floating leg are still based on the curve used, but the discount factors are now based on the repo rate, which is Eonia + a spread which is unknown. Regardless of the spread, the pricing formulas would then be transformed to:

$$PV(fixed) = N * K_n * \sum_{k=1}^{n} D_k^{Repo} * \alpha_{k,k-1}^{fxd} \tag{11}$$

$$PV(floating) = N * \sum_{k=1}^{n} D_k^{Repo} * \alpha_{k,k-1}^{flt} * F_{k-1,k} \tag{12}$$

Since the allowed collateral is specified in the CSA, the practice of discounting with the collateral rate is referred to as CSA discounting. This is discussed next.

2.4.1. <u>CSA optionality</u>

The above examples assume that the CSA is specified such that one particular way of collateral posting is allowed. However, frequently there is optionality in the CSA when it specifies that both cash and bonds are accepted as collateral. The party that has to post will then choose the cheapest way of doing so. Hence the term cheapest to deliver (CTD) option.

For the example above, if the CSA would allow both bonds and cash as collateral, the bank would choose bonds when X>0. The bank will pick the allowed bonds with the highest repo rate, which usually is the lowest rated bond. If the pension fund fails to restrict the collateral to a certain quality, the bank will profit by posting low-grade collateral in which the pension fund possibly would not normally invest.

Another form of optionality occurs when different currencies are allowed. For example, the CSA can specify that only Euros, Dollars or British Pounds can be used, or bonds from all of the countries. In this case, the party that has to post collateral will choose the currency that is cheapest to borrow. The reason that currency optionality is present is due to a lack of liquidity in some currencies. For example, there are not enough British government bonds (Gilts) being traded to support all the collateral calls in Britain's large financial and pension industries. Specifying the CSA in a very strict way could lead to higher collateral costs for both parties, which is obviously not desirable. A widely specified CSA does, however, lead to extra complexity in the valuation of derivative contracts.

This effect is enlarged when collateral substitution is not prohibited. In this case, the CSA allows for daily substitution of the collateral, meaning that the counterparty in a collateralized contract with a negative MtM can switch collateral every day. Thus, every day the party will reevaluate what is cheapest to deliver, and act accordingly. This creates a series of options, one or more for every day until the maturity of the contract. Sawyer (2011) reports occurrences where dealers have refused requests for collateral substitution, and concludes that multicurrency CSAs change the valuation of plain vanilla swaps from simple to very complex. The instruments needed to hedge these exposures or derive pricing from are not currently traded, so he states that a new standardized CSA may be the only option.

The optionality that can be in a CSA will not be further discussed, as it is not relevant for the research question. The assumption throughout the thesis is a clearly specified Euro-cash-only CSA, unless stated otherwise. The reason that bond collateral is only briefly mentioned is that is does not make a significant difference when the CSA is conservatively specified. If the CSA only specifies AAA rated government bonds or other very high quality securities to qualify as collateral, 'the OIS index is generally a good proxy for the repo rate.' (Shepley, 2001) This would make X equal zero, and would reduce the methodology for valuation and hedging to the cash collateral approach. For this reason and the lack of consistent data on repo spreads, in the remainder of this thesis all mentioned collateral concerns cash collateral, so that the OIS curve can be used for discounting.

3. Curve construction

Now that the valuation of interest rate swaps is sorted out, we can use the results from the previous section to build a zero rate curve, which are essential for any valuation problem. Since these curves are primarily built from swaps, we can use par swap quotes to construct a zero curve, based on the valuation section. How this is done and some related issues are discussed in this section.

3.1. Curve instruments

As visible in the appendix, the shortest two instruments used to build the Euribor curve in this thesis are the six month and one year cash instruments, while liquid instruments of for example one week and three months are available. However, since the short end lacks importance in this thesis due to the long horizons, only these two zero coupon instruments are chosen to build the short end of the curve.

The swap instruments used all have a floating tenor of six months. The reason for this is that many authors, including Bianchetti (2009), explicitly stress the importance of using curve instruments that are homogeneous in the underlying rate tenor. This means that the floating side of the swap should be equal for all instruments. Market characteristics such as liquidity and credit risk premiums are very different for different floating rate tenors. The fixed-for-3month floating swap market has different dynamics than the fixed-for-6month floating swap market. This has not always been the case, but the crisis has 'segmented the market in sub-areas corresponding to different underlying rate tenors', according to Bianchetti (2010).

For the Eonia curve, the floating tenor of all instruments always is a single day, so that no such problems could arise. All instruments used to build the OIS curve are therefore swaps, being most liquid. Other instruments could be added to the short end but the effect on the high duration portfolio would be negligible. To retain oversight, the same instruments are used as for the Euribor curve, except for the three highest maturity instruments which are not available for Eonia. This is discussed in subsection 3.2.2.

3.2. Curve building[14]

With the knowledge from the valuation section, market swaps can be analyzed. In particular, the fact that the value is zero at inception offers possibilities in combination with a market quote. For valuation of a large set of cash flows at varying maturities, a zero curve is required so that the discount factor for each cash flow can be derived. The short end of the zero curve, up to one year, is constructed from cash instruments and is therefore known. However, for longer maturities, the swap market is more liquid making that a better choice. To extract the zero rate from a swap, we can rewrite equations (3) and (4), given that the value of the floating leg should equal the value of the fixed leg:

$$N * K_n * \sum_{k=1}^{n} D_k * \alpha_{k,k-1}^{fxd} - N * \sum_{k=1}^{n} D_k * \alpha_{k,k-1}^{flt} * F_{k-1,k} = 0$$

---

[14] Subsection based on Hull (2007)

$$K_n * \sum_{k=1}^{n} D_k * \alpha_{k,k-1}^{fxd} - (D_0 - D_n) = 0$$

Since the one-year instrument is a cash instrument, we can derive D_{1y} using (1) for Euribor. For Eonia, only swap instruments are used so that we get:

$$K_{1y}^{OIS} * \sum_{k=1}^{n} D_{1y}^{OIS} * \alpha_{k,k-1}^{fxd} - (D_0 - D_{1y}) = 0$$

Since the fixed rate is quoted in the market, the one year discount factor is the only unknown which can be solved for. From this point on the procedure is the same for both curves, starting from the two year swap value:

$$K_{2y} * (D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y}) - (1 - D_{2y}) = 0$$

Since we have calculated D_{1y} already, and K_{2y} is the swap rate quoted in the market for the 2-year instrument, we are left with only one unknown parameter; D_{2y}. We can solve for D_{2y}:

$$K_{2y} * (D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y}) = 1 - D_{2y}$$

$$K_{2y} * D_{1y} * \alpha_{0,1y} + K_{2y} * D_{2y} * \alpha_{1y,2y} + D_{2y} = 1$$

$$D_{2y} * (1 + K_{2y} * \alpha_{1y,2y}) = 1 - K_{2y} * D_{1y} * \alpha_{0,1y}$$

$$D_{2y} = \frac{1 - K_{2y} * D_{1y} * \alpha_{0,1y}}{(1 + K_{2y} * \alpha_{1y,2y})}$$

This can be generalized for k year swaps up to maturity n to:

$$D_n = \frac{1 - K_n * (\sum_{k=1}^{n-1} D_k * \alpha_{k,k-1}^{fxd})}{(1 + K_n * \alpha_{k,k-1}^{fxd})} \tag{13}$$

This way, we can step-by-step solve for all the discount factors of the curve's instruments. Using (1) we can then convert the discount factors to zero rates to obtain the required curve. The process described above is generally known as bootstrapping.

3.2.1. Curve interpolation

As is visible in the appendix, there are no (liquid) instruments for all maturities. For example, the 13 and 14 year Euribor swap quotes are missing. We then jump to the 15 year quote, but this means that there are three unknowns and only one equation. However, by assuming an interpolation method, the 13 and 14 year discount factors can be written as functions of the 12 and 15 year discount factors, so that one unknown is left and the unknown discount factor can be solved for using (13). The method used is smooth cubic splines interpolation as described in Hagan and West (2006) and will also be discussed in

the interest rate sensitivity section. The curve is thus sensitive to two input types; the instrument rates and the interpolation method. The grid points of the curve where the instruments are located are not subject to interpolation while the rest of the curve depends upon the interpolation method which connects the grid points into one continuous curve.

The reason for choosing a smooth interpolation technique over a computationally much less complex linear or log-linear method lies purely in quality. Even though Hagan and West (2006) state that log-linear interpolation is quite popular, they conclude that there are multiple problems. The method allows for negative forward rates, which are not always differentiable and the forward curve suffers from 'zig-zag instability'. Also Bianchetti (2009) notes that 'this still diffused market practice produces zero curves without apparent problems, but sag saw forward curves with unnatural oscillations in the forward basis.' For pure discounting purposes the method can thus hold, but since we are treating almost solely swaps which are valued by their forward cash flows, a smooth bootstrapping technique is required.

3.2.2. <u>Curve use</u>

Once a curve is built, any cash flow at any future time can be properly discounted. For cash flows with a maturity of more than 50 years, the 50 year zero rate is assumed, implying flat extrapolation of the zero rates. This is done due to lack of better alternatives, and the observation that very long term rates correlate strongly.

An Eonia zero curve can also be built using the procedure above, as OI swaps are only dependent on one curve even though the contracts are collateralized. The swap and forward quotes are indeed retrieved from the Eonia curve which is also used for discounting. Recall that (3) and (5) are equal to (6) and (7) except for the curve used. Inversely, the two different curves can be built using the same method, but with different instruments.

3.3. <u>**Forward adjustment**</u>

Since swap quotes for Euribor swaps are set under the assumption of collateralization, a completely realistic Euribor curve cannot be built in the way described in section 4.1. This is due to the OIS discount factors in the valuation formula of the swap instruments. However, these OIS discount factors are not unknown, they can be derived from the Eonia curve built first. Given the Euribor swap quotes, we then have from (9) and (10) :

$$N * K_n * \sum_{k=1}^{n} D_k^{OIS} * \alpha_{k,k-1}^{fxd} - N * \sum_{k=1}^{n} D_k^{OIS} * \alpha_{k,k-1}^{flt} * F_{k-1,k} = 0$$

For a two year swap this becomes:

$$
\begin{aligned}
K_{2y} * \big(D_{1y}^{OIS} * \alpha_{0,1y} &+ D_{2y}^{OIS} * \alpha_{1y,2y} \big) \\
&- \big(F_{0,6m} * D_{6m}^{OIS} * \alpha_{0,6m} + F_{6m,1y} * D_{1y}^{OIS} * \alpha_{6m,1y} + F_{1y,18m} * D_{18m}^{OIS} * \alpha_{1y,18m} \\
&+ F_{18m,2y} * D_{2y}^{OIS} * \alpha_{18m,2y} \big) = 0
\end{aligned}
$$

Since all the OIS discount factors are known from the Eonia curve, only the forward rates are unknown. The short end of the Euribor curve is derived from cash instruments, so the first 2 forward rates are given. By then interpolating between the one and two year forward rate to obtain $F_{1y,18m}$ as a function of $F_{6m,1y}$ and $F_{18m,2y}$, only the last forward rate is unknown. This way, we can again use a step-by-step procedure to obtain all the forward rates. We then have all the parameters required in (9) and (10), and thus we can value a collateralized Euribor swap.

The implied forward curve obtained this way is slightly different than the forward rates calculated using (2). This is due to the fact that even though both legs are discounted with the same, but now lower curve, the distribution of cash flows over time is not exactly equal. The present value is the same at inception, but the payouts occur yearly for the fixed leg and semi-annually for the floating leg. Also, the forward rate could be below the fixed rate during the first years of the swap and above the fixed rate in later years, or vice versa. As a result, the forward curve obtained using (2) is recalibrated using implied rates from the swap quotes. Mercurio (2009) accounts for this difference by using forward rate agreements to determine the forward rates and then discount these using the OIS curve.

In the next chapter, the interest rate sensitivity of a swap is determined. The forward adjustment described here is not taken into account, as the computational difficulties outweigh the benefit from a more exact approach. The expectation is that the minor adjustment resulting from this more refined approach will only show up as unidentifiable noise on the funding ratio level. It is therefore ignored, so that (2) can be used in both the valuation and sensitivity calculations.

4. Interest rate sensitivity

Present values of future cash flows are discounted using a curve built as described in the previous section. This present values are thus sensitive to the discount curve. If the future cash flows itself are also dependent on a curve, which is the case with a swap, there is a second base of exposure to that curve. The curve itself has sensitivity to the instruments with which it is built, as described in the previous section. The goal of this section is to identify the exposure of any claim on future cash flows to the curve instruments. This information is a prerequisite for being able to hedge the exposure.

For small parallel shifts the modified duration, which is the change in present value of the claim as a result of a one percent parallel shift in the interest rate curve, is appropriate for determining the interest rate sensitivity of an IRS. However, since interest rates also shift separately from each other, a more refined approach is advisable. The sensitivity can be determined to each grid point on the interest rate curve, where the grids represent the instruments used to build the curve. The method the curve was built with determines the sensitivity of a cash flow to that curve. Thus, the calculation of the sensitivity is done according to the method used to build the curve. This results in reporting zero deltas to the cash instruments, while reporting par swap rate deltas to the swap instruments. The whole procedure is illustrated using a two year swap example, combined with a curve which has only two instruments, a one and a two year swap. Where relevant, the effect of using a cash instrument instead of a swap instrument is discussed. The two year swap will be illustrated numerically as well as in formulas. For every important result, the generalized formula is presented and numbered.

First, the sensitivity of a single curve swap is derived, for Euribor and OI swaps. The results will then be used as the basis for deriving the sensitivity of a collateralized Euribor swap to both relevant curves.

4.1. Single curve interest rate sensitivity[15]

The sensitivity of a product is calculated with respect to an interest rate curve, which in turn is sensitive to the rates of the underlying instruments. We thus want to calculate the sensitivity of a product to the curve's rates; this is done by defining the sensitivity of the value of a contingent claim C to every rate r_i as (using the chain rule):

$$\frac{\partial C}{\partial r_i} = \frac{\partial C}{\partial D_i} \cdot \frac{\partial D_i}{\partial r_i}$$

Note that r_i depends on the instrument used. This could be a zero rate or a swap rate.

---

[15] Derivation based on Lord (2009)

Since we want to know the sensitivity with respect to the whole curve, and thus all the instruments, we switch to matrix notation, where the under bar depicts the vector property of the variable:

$$\frac{\partial \underline{C}}{\partial \underline{r}} = \frac{\partial \underline{C}}{\partial \underline{D}} \cdot \frac{\partial \underline{D}}{\partial \underline{r}} \tag{14}$$

The sensitivity of the claim to the *n* rates equals the product of the *m* replicating flows[16] of the claim and a Jacobian[17] matrix with the sensitivities of the discount factors to the underlying instrument rates. The left-hand side is a row vector of sensitivities to all the instruments' rates, which is a result of the row vector of repflows multiplied with the Jacobian. The claim that we are going to investigate is a two year uncollateralized interest rate swap with a notional of 100. Recall that the value of a two year receiver swap is determined by the fixed leg minus the floating leg. As it concerns a quoted swap, this means that the fixed rate is exactly set to make the total value zero. This results in the two year claim[18]:

$$C_{2y} = N * K_{2y} * \left(D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y} \right) - N * \left(1 - D_{2y} \right) = 0$$

For our illustrative example, the following applies:

- The curve is built from two swap instruments:
    - 1y   4%
    - 2y   5%
- As a result, $K_{2y} = 5\%$ as this is the two year swap rate quoted in the market
- N=100
- For simplicity, the alphas are assumed one.
- The discount factors can be solved for using the curve build section(3.2):
    - $D_{1y} = 0.9615$
    - $D_{2y} = 0.9066$

This means we have all the required parameters. We can then take the derivative of the value of this claim with respect to its discount factors to obtain the replicating future cash flows, referred to as *repflows* in this thesis.

$$\frac{\partial C_{2y}}{\partial D_{1y}} = N * K_{2y} * \alpha_{0,1y}$$

$$\frac{\partial C_{2y}}{\partial D_{2y}} = N * (K_{2y} * \alpha_{1y,2y} + 1)$$

---

[16] The present value of a cash flow is the product of that cash flow and its discount factor, + the initial investment. The derivative to the discount factor thus only leaves the future cash flow, defined here as replicating flow or repflow. (except for the derivative to D_0, which will leave the initial investment. The investment is not present in the matrix, as it is not relevant, it has no interest rate sensitivity.) This holds true for linear products, which is suitable here as only plain vanilla swaps are discussed in this thesis.

[17] A Jacobian matrix is the matrix of all first order partial derivatives of a vector with respect to another vector.

[18] Because all day count fractions relating to the floating side drop out for all sensitivity calculations, all the alphas shown are fixed leg alphas with annual periods.

This can be put together in a row vector. The notional will be ignored in the generalized formulas in an attempt to maintain oversight:

$$\frac{\partial C_{2y}}{\partial \underline{D}} = [K_{2y} * \alpha_{0,1y} \qquad 1 + K_{2y} * \alpha_{1y,2y}]$$

For the example swap, this comes down to:

$$\frac{\partial C_{2y}}{\partial \underline{D}} = [5 \quad 105]$$

To make this intuitive; recall that a swap is basically a fixed rate bond minus a floating rate bond. A floating rate bond has no or minimal interest rate sensitivity, all intermediate payment terms drop out already in the valuation section. The discount exposure of this swap can be described as the nominal coupon payment in one year and the coupon plus notional in two years. The numbers should be interpreted as the following: Any increase in the one year discount factor directly results in a five times as big an increase in the present value of the swap, any increase in the two year discount factor is immediately followed by a 105 times as big an increase in the PV of the claim.

The above can be generalized for n discount factors with subscript i:

$$\frac{\partial C}{\partial \underline{D}} = [K_n * \alpha_{0,i} \quad K_n * \alpha_{i,i+1} \quad \dots \quad K_n * \alpha_{n-2,n-1} \quad 1 + K_n * \alpha_{n-1,n}] \qquad (15)$$

Note that only the last cash flow has the added '1', representing the notional of the embedded fixed rate bond.

What is described here is the sensitivity of the claim to the discount factors, while the latter only changes as a result of a change in the rate of the underlying instrument. What is required is thus the sensitivity of the discount factors with respect to the underlying rates. This will be discussed below.

4.1.1. Intermezzo: Cash instrument

Now that when $\frac{\partial C}{\partial \underline{D}}$ (the replicating flows) are known, the sensitivities of the discount factors to the interest rates represent the only unknown in (14). If the one year instrument would be a zero coupon bond instead of a swap, the derivatives can simply be taken from (1):

$$D_{1y} = \frac{1}{1 + Z_{1y} * \alpha_{0,1y}}$$

We then take the derivative with respect to all the zero rates:

$$\frac{\partial D_{1y}}{\partial Z_{6m}} = 0$$

$$\frac{\partial D_{1y}}{\partial Z_{1y}} = \frac{-\alpha_{0,1y}}{(1 + Z_{1y} * \alpha_{0,1y})^2}$$

$$\frac{\partial D_{1y}}{\partial Z_{2y}} = 0$$

For zero coupon instruments, which are the cash instruments used here to build the curve, there is no cross-sensitivity with other rates. Therefore, derivatives to other rates are zero, just like the derivative to the two year rate here.

4.1.2. <u>Swap instruments cont'd</u>

Unfortunately, the derivative of the discount factor to the respective instrument rate is not directly derivable for swap instruments. The relation described in (1) does not hold for the swap instruments, as zero rates are not the same as swap rates. There is a way around this, however. Relation (14) also holds for the instruments used to build the curve. We can thus write the swap instrument's present value sensitivity to the underlying swap rate similarly to the claim's sensitivity:

$$\frac{\partial PV}{\partial \underline{K}} = -\frac{\partial PV}{\partial \underline{D}} \cdot \frac{\partial \underline{D}}{\partial \underline{K}} \tag{16}$$

Note that the PV here refers to the present value the swap instrument, which is similar to the claim C of which we are trying to calculate the deltas. The deltas calculated here are par swap deltas, contrary to the zero deltas which were calculated in de the cash instrument section. $\underline{K}$ then refers to a vector of swap rates. Equation (16) can be rearranged by multiplying both sides with the inverse of the repflow matrix:

$$\frac{\partial \underline{D}}{\partial \underline{K}} = -\frac{\partial \underline{PV}}{\partial \underline{D}}^{-1} \cdot \frac{\partial \underline{PV}}{\partial \underline{K}} \tag{17}$$

The matrices on the right hand side can both be derived for both the instruments. For the one year swap we have the instruments' repflows similar to the repflows of the claim C:

$$PV_{1y} = K_{1y} * \left(D_{1y} * \alpha_{0,1y}\right) - \left(1 - D_{1y}\right)$$

$$\frac{\partial PV_{1y}}{\partial D_{1y}} = 1 + K_{1y} * \alpha_{0,1y}$$

$$\frac{\partial PV_{1y}}{\partial D_{2y}} = 0$$

For the two year swap instrument we then have:

$$PV_{2y} = K_{2y} * \left(D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y}\right) - \left(1 - D_{2y}\right)$$

$$\frac{\partial PV_{2y}}{\partial D_{1y}} = K_{2y} * \alpha_{0,1y}$$

$$\frac{\partial PV_{2y}}{\partial D_{2y}} = 1 + K_{2y} * \alpha_{1y,2y}$$

This can be put together in a square matrix:

$$\frac{\partial PV}{\partial \underline{D}} = \begin{bmatrix} 1 + K_{1y} * \alpha_{0,1y} & 0 \\ K_{2y} * \alpha_{0,1y} & 1 + K_{2y} * \alpha_{1y,2y} \end{bmatrix}$$

For simplicity we take a notional of 100 for these instruments too regarding the example, which results in:

$$\frac{\partial PV}{\partial \underline{D}} = \begin{bmatrix} 104 & 0 \\ 5 & 105 \end{bmatrix}$$

This matrix represents the replicating flows of both the instruments with respect to both the one year and two year discount factor. The middle term of (17) is now known. The third term from (17) can also be derived from the swap valuation formulas. We then take the derivative of the PV of both the one and two year instrument with respect to the fixed rates instead of the discount factors.

For the one year swap:

$$\frac{\partial PV_{1y}}{\partial K_{1y}} = D_{1y} * \alpha_{0,1y}$$

$$\frac{\partial PV_{1y}}{\partial K_{2y}} = 0$$

For the two year swap:

$$\frac{\partial PV_{2y}}{\partial K_{1y}} = 0$$

$$\frac{\partial PV_{2y}}{\partial K_{2y}} = D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y}$$

This results in a diagonal matrix, as there is no cross-sensitivity here:

$$\frac{\partial \underline{PV}}{\partial \underline{K}} = \begin{bmatrix} D_{1y} * \alpha_{0,1y} & 0 \\ 0 & D_{1y} * \alpha_{0,1y} + D_{2y} * \alpha_{1y,2y} \end{bmatrix}$$

For our example, this results in:

$$\frac{\partial \underline{PV}}{\partial \underline{K}} = \begin{bmatrix} 96.1538 & 0 \\ 0 & 186.8132 \end{bmatrix}$$

Note that the numbers are multiplied by the notional of 100 here. It then becomes clear that the derivative to the fixed rate of a swap results in the sum of the discount factors needed to discount all future payments. (Multiplied by the notional) The floating side drops out which basically leaves a stream of fixed cash flows that only have discount exposure. With these results, the Jacobian of the discount factors to the swap instrument rates can be generally defined with i denoting the n instruments as:

$$
\begin{bmatrix} \dfrac{\partial D_i}{\partial K_i} & \cdots & \dfrac{\partial D_i}{\partial K_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial D_n}{\partial K_i} & \cdots & \dfrac{\partial D_n}{\partial K_n} \end{bmatrix} = \begin{bmatrix} 1 + K_i * \alpha_{0,i} & 0 & 0 \\ \vdots & \ddots & 0 \\ K_n * \alpha_{0,i} & .. & 1 + K_n * \alpha_{n-1,n} \end{bmatrix}^{-1} * \begin{bmatrix} D_i * \alpha_{0,i} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum\limits_{i=1}^{n} D_i * \alpha_{i-1,i} \end{bmatrix}
$$

For both matrices, the diagonal contains the information regarding the sensitivity of the instrument with its own grid point. Note that there is no cross sensitivity in the present values to the rates, but that the middle matrix makes clear that swaps with longer maturities have exposure to the shorter maturity discount factors. This pictures the discount exposure the fixed payments contain.

Note that the result above only holds for curves built completely from swaps. Given that the curve used in this thesis is built from cash instruments up to the one year point, the top rows can be replaced with the cash instrument derivatives described in the cash instrument section.

This result can be plugged in equation (14) :

$$
\frac{\partial C}{\partial \underline{K}} = \frac{\partial C}{\partial \underline{D}} \cdot - \frac{\partial \underline{PV}^{-1}}{\partial \underline{D}} \cdot \frac{\partial \underline{PV}}{\partial \underline{K}} \tag{18}
$$

Equation (18) shows that all the terms can be derived now. For our example, we can first calculate the sensitivity of the discount factors with respect to the underlying par swap rates:

$$
\frac{\partial \underline{D}}{\partial \underline{K}} = - \begin{bmatrix} 104 & 0 \\ 5 & 105 \end{bmatrix}^{-1} * \begin{bmatrix} 96.1538 & 0 \\ 0 & 186.8132 \end{bmatrix} = \begin{bmatrix} -0.9246 & 0 \\ 0.0440 & -1.7792 \end{bmatrix}
$$

we can now calculate the deltas by multiplying the repflows with this result:

$$
\frac{\partial C_{2y}}{\partial \underline{K}} = \begin{bmatrix} 5 & 105 \end{bmatrix} * \begin{bmatrix} -0.9246 & 0 \\ 0.0440 & -1.7792 \end{bmatrix} = \begin{bmatrix} 0 & -0.0186 \end{bmatrix}
$$

This result is the answer to the original question: what is the sensitivity of the present value of the claim to the yield curve? As it turns out, the one year rate has no influence at all, while the present value of our two year par swap with notional 100 will decrease by 1.86 cents if ceteris paribus the two year par swap rate goes up with one basis point. (So from 5% to 5.01% in this case)

From the start, we have been analyzing a receiver swap, meaning that the valuation and sensitivity have been viewed from the party that pays the floating rate and receives the fixed rate. The value of the contract is then the fixed side minus the floating side. If the perspective of the counterparty is to be analyzed, the valuation formula changes to floating minus fixed. The only relevant change is the minus sign, which then appears for the fixed leg instead of the float leg. All signs above are then reversed and the final answer would be that the swap increases 1.86 due to a one basis point increase in the swap rate. This is true by definition, as the value of the contract to one party should always be the opposite of the value of the same contract for the counterparty.[19]

4.1.3. <u>Interpolation</u>

The approach considered makes one unrealistic assumption, namely that all cash flows of the contingent claim occur exactly on the grid points of the curve where the instruments are located. Cash flows should be able to occur at any point of the curve. If this is the case, the rate is interpolated between the grid points using smooth interpolation, identical to the curve build.

We then get an additional derivative, that of the discount factor of the cash flow at time τ to the discount factors of the nearby grid points. We account for this by redefining the Jacobian:

$$(19)$$

$$\frac{\partial \underline{D}(\tau)}{\partial \underline{K}} = \frac{\partial \underline{D}(\tau)}{\partial \underline{D}} \cdot \frac{\partial \underline{D}}{\partial \underline{K}}$$

Here τ represents the exact time of the future cash flow, so that $\underline{D}(\tau)$ Is the vector of discount factors for all payment dates. The $\underline{D}$ term then only contains the discount factors on the grid points of the curve, where the instruments are located, so that we get an interpolation matrix containing the sensitivities of all relevant discount factors to all discount factors on the grid points. This is required to calculate the sensitivity of the present value of the claim with respect to the instruments' rates.

Recall from (5) that the intermediate floating payments dropped out in the valuation formula using one curve. Therefore, no interpolation is required when valuing a swap on its start date. However, if the swap started in the past, the fixed payments will often not occur on the grid points, and thus interpolation is required if for example the hedge has to be rebalanced.

We introduce an interpolation term which defines the sensitivities of the pay date discount factors to the grid point discount factors of the instruments. Suppose the 2 year swap started 6 months ago but we want to know its deltas today (with the value of the contract still zero). We then have two replicating flows, in 6 months from now and in 18 months. The first flow only has exposure to the one year instrument, albeit partially, and the second flow is influenced by both the one and two year instrument.

---

[19] In practice, there can be disagreement between the two parties about the value, for example due to different curve building techniques used, which can have an impact on the valuation. The value is then renegotiated. This possible inequality is ignored in this thesis.

The function of the interpolation matrix is thus to divide the cash flow sensitivities to the corresponding discount factor to all the instrument discount factors:

$$\frac{\partial \underline{D}(\tau)}{\partial \underline{D}} = \begin{bmatrix} \dfrac{\partial D_{6m}}{\partial D_{1y}} & \dfrac{\partial D_{6m}}{\partial D_{2y}} \\[2ex] \dfrac{\partial D_{18m}}{\partial D_{1y}} & \dfrac{\partial D_{18m}}{\partial D_{2y}} \end{bmatrix}$$

The full derivation is not provided, as it is beyond the scope of the thesis here. The method used is cubic-splines interpolation as described in Hagan and West (2006).

Since the example curve used in this section only has two instruments, and the replicating flows of the two year example swap occur exactly at the instrument maturities, the interpolation matrix will have no impact.(It will be an identity matrix) An example explaining this will be given later on.

The interpolation term can be integrated in (18) by applying the chain rule:

$$\frac{\partial C}{\partial \underline{K}} = \frac{\partial C}{\partial \underline{D}(\tau)} \cdot \frac{\partial \underline{D}(\tau)}{\partial \underline{D}} \cdot -\frac{\partial PV^{-1}}{\partial \underline{D}} \cdot \frac{\partial PV}{\partial \underline{K}} \tag{20}$$

The result is an analytic solution for the sensitivity of any cash flow to any rate on the curve, defined as the change in value of the contingent claim as a consequence of a one basis point change in the respective rate. This is the single curve solution which is used to set up hedges to Euribor without taking a separate discount curve into account. We will now turn to the two-curve solution where the OIS discount curve is explicitly accounted for.

4.2. Double curve interest rate sensitivity

Recall from the valuation section that collateralized swaps are discounted with the OIS curve. Therefore, the valuation formula to derive the interest rate sensitivity from is for a collateralized two year swap given below. The claim C is the same as in the previous subsection, but now it is collateralized. We use the double curve valuation from (9) and (10):

$$
\begin{aligned}
C_{2y} = K_{2y} * &\left(D_{1y}^{OIS} * \alpha_{0,1y}^{EUR} + D_{2y}^{OIS} * \alpha_{1y,2y}^{EUR} \right) \\
&- \left(F_{0,6m} * D_{6m}^{OIS} * \alpha_{0,6m}^{EUR} + F_{6m,1y} * D_{1y}^{OIS} * \alpha_{6m,1y}^{EUR} + F_{1y,18m} * D_{18m}^{OIS} * \alpha_{1y,18m}^{EUR} \right. \\
&\left. + F_{18m,2y} * D_{2y}^{OIS} * \alpha_{18m,2y}^{EUR} \right) = 0
\end{aligned}
$$

We can rewrite the forward rates using (2):

$$
\begin{aligned}
K_{2y} * &\left(D_{1y}^{OIS} * \alpha_{0,1y}^{EUR} + D_{2y}^{OIS} * \alpha_{1y,2y}^{EUR} \right) \\
&- \left(\frac{1}{\alpha_{0,6m}^{EUR}} \left(\frac{D_0^{EUR}}{D_{6m}^{EUR}} - 1 \right) * D_{6m}^{OIS} * \alpha_{0,6m}^{EUR} + \frac{1}{\alpha_{6m,1y}^{EUR}} \left(\frac{D_{6m}^{EUR}}{D_{1y}^{EUR}} - 1 \right) * D_{1y}^{OIS} * \alpha_{6m,1y}^{EUR} \right. \\
&\left. + \frac{1}{\alpha_{1y,18m}^{EUR}} \left(\frac{D_{1y}^{EUR}}{D_{18m}^{EUR}} - 1 \right) * D_{18m}^{OIS} * \alpha_{1y,18m}^{EUR} + \frac{1}{\alpha_{18m,2y}^{EUR}} \left(\frac{D_{18m}^{EUR}}{D_{2y}^{EUR}} - 1 \right) * D_{2y}^{OIS} * \alpha_{18m,2y}^{EUR} \right)
\end{aligned}
$$

As in the valuation, the alphas on the floating side drop out[20]:

$$
\begin{aligned}
K_{2y} * &\left(D_{1y}^{OIS} * \alpha_{0,1y}^{EUR} + D_{2y}^{OIS} * \alpha_{1y,2y}^{EUR} \right) \\
&- \left(\left(\frac{D_0^{EUR}}{D_{6m}^{EUR}} - 1 \right) * D_{6m}^{OIS} + \left(\frac{D_{6m}^{EUR}}{D_{1y}^{EUR}} - 1 \right) * D_{1y}^{OIS} + \left(\frac{D_{1y}^{EUR}}{D_{18m}^{EUR}} - 1 \right) * D_{18m}^{OIS} + \left(\frac{D_{18m}^{EUR}}{D_{2y}^{EUR}} - 1 \right) * D_{2y}^{OIS} \right)
\end{aligned}
$$

Removing the brackets we get:

$$
\begin{aligned}
K_{2y} * &\left(D_{1y}^{OIS} * \alpha_{0,1y}^{EUR} + D_{2y}^{OIS} * \alpha_{1y,2y}^{EUR} \right) \\
&- \left(\frac{D_0^{EUR} * D_{6m}^{OIS}}{D_{6m}^{EUR}} - D_{6m}^{OIS} + \frac{D_{6m}^{EUR} * D_{1y}^{OIS}}{D_{1y}^{EUR}} - D_{1y}^{OIS} + \frac{D_{1y}^{EUR} * D_{18m}^{OIS}}{D_{18m}^{EUR}} - D_{18m}^{OIS} + \frac{D_{18m}^{EUR} * D_{2y}^{OIS}}{D_{2y}^{EUR}} - D_{2y}^{OIS} \right)
\end{aligned}
$$

For the sensitivity to the Euribor discount factors we then see that the fixed leg drops out completely, as that leg only has discount exposure. This exposure is to be captured in the derivation to the OIS discount factors. The floating leg is of course dependent on Euribor, while the cash flows emerging from this leg are also discounted with the OIS curve.

Bianchetti (2009) states in his presentation at the Quant congress that 'delta risk with respect to both curves [i.e. the forward and the discount curve] should be calculated in order to determine the delta of

---

[20] The Day count convention (DCC) of EONIA is ACT/360, for both the fixed and floating leg. Since the floating leg of Euribor swaps also has this DCC, this turns out to be convenient when rewriting the forward rates; the alpha term denoting the DCC then drops out similar to the uncollateralized case.

any portfolio of interest rate derivatives.' So what is required is a split of the sensitivities of the claims to both curves. We thus rewrite (14) in two ways[21]:

$$\frac{\partial C}{\partial \underline{K}^{EUR}} = \frac{\partial C}{\partial \underline{D}^{EUR}} \cdot \frac{\partial \underline{D}^{EUR}}{\partial \underline{K}^{EUR}} \tag{21}$$

$$\frac{\partial C}{\partial \underline{K}^{OIS}} = \frac{\partial C}{\partial \underline{D}^{OIS}} \cdot \frac{\partial \underline{D}^{OIS}}{\partial \underline{K}^{OIS}} \tag{22}$$

The terms of interest are the repflow terms in the middle as the instrument sensitivities on the right hand side are already explained in the previous subsections. We already know the sensitivities of the discount factors to the instrument rates, as these can be derived for both curves using (16).[22]

However, the example was only performed for the Euribor curve. Note however that the valuation of an OI swap is performed equally to that of a normal IRS except for the underlying curve. (Compare (3) and (5) with (6) and (7)) This means that the method to determine the derivatives is also equal, but that the input parameters are different. For illustration, we use the following instruments:

Table 1: Curve instruments

	EUR swap curve instruments	OI swap curve instruments
1y	4%	4%
2y	5%	5%

Indeed, we use two identical curves, but we will now separate the forward and the discount exposure, to determine the order of magnitude of the two deltas.

Given that both curves are built from the same instruments, the instrument discount factor sensitivities are equal to the single curve case:

$$\frac{\partial \underline{D}^{EUR}}{\partial \underline{K}^{EUR}} = \frac{\partial \underline{D}^{OIS}}{\partial \underline{K}^{OIS}} = \begin{bmatrix} -0.9246 & 0 \\ 0.0440 & -1.7792 \end{bmatrix}$$

The difference is going to be in the repflows, which we will now determine below separately, starting with the sensitivity of the collateralized claim to the forward curve.

---

[21] We switch notation; from now on, the superscript 'EUR' is added to all Euribor (forward) terms, while the Eonia (discount) terms retain the superscript 'OIS' in order to avoid confusion.
[22] This is true when the Euribor swap instrument is quoted as uncollateralized.

As the fixed leg has no exposure to the forward curve and thus drops out in this derivation, the sensitivity is determined for the floating leg which has the value[23]:

$$-\left(\frac{D_0^{EUR} * D_{6m}^{OIS}}{D_{6m}^{EUR}} - D_{6m}^{OIS} + \frac{D_{6m}^{EUR} * D_{1y}^{OIS}}{D_{1y}^{EUR}} - D_{1y}^{OIS} + \frac{D_{1y}^{EUR} * D_{18m}^{OIS}}{D_{18m}^{EUR}} - D_{18m}^{OIS} + \frac{D_{18m}^{EUR} * D_{2y}^{OIS}}{D_{2y}^{EUR}} - D_{2y}^{OIS}\right)$$

The floating leg then has the following sensitivity:

$$\frac{\partial C_{2y}}{\partial D_{6m}^{EUR}} = -\left(\frac{-D_0^{EUR} * D_{6m}^{OIS}}{(D_{6m}^{EUR})^2} + \frac{D_{1y}^{OIS} * D_{1y}^{EUR}}{(D_{1y}^{EUR})^2}\right) = \frac{D_{6m}^{OIS}}{(D_{6m}^{EUR})^2} - \frac{D_{1y}^{OIS}}{D_{1y}^{EUR}}$$

$$\frac{\partial C_{2y}}{\partial D_{1y}^{EUR}} = \frac{D_{6m}^{EUR} * D_{1y}^{OIS}}{(D_{1y}^{EUR})^2} - \frac{D_{18m}^{OIS}}{D_{18m}^{EUR}}$$

$$\frac{\partial C_{2y}}{\partial D_{18m}^{EUR}} = \frac{D_{1y}^{EUR} * D_{18m}^{OIS}}{(D_{18m}^{EUR})^2} - \frac{D_{2y}^{OIS}}{D_{2y}^{EUR}}$$

$$\frac{\partial C_{2y}}{\partial D_{2y}^{EUR}} = \frac{D_{18m}^{EUR} * D_{2y}^{OIS}}{(D_{2y}^{EUR})^2}$$

These results can be put together in a vector. Note that the derivative is to the cash flow discount factors, not yet to the instrument discount factors, since there are no semi-annual instruments.

$$\frac{\partial C_{2y}}{\partial D(\tau)^{EUR}} = \left[\frac{D_{6m}^{OIS}}{(D_{6m}^{EUR})^2} - \frac{D_{1y}^{OIS}}{D_{1y}^{EUR}} \quad \frac{D_{6m}^{EUR} * D_{1y}^{OIS}}{(D_{1y}^{EUR})^2} - \frac{D_{18m}^{OIS}}{D_{18m}^{EUR}} \quad \frac{D_{1y}^{EUR} * D_{18m}^{OIS}}{(D_{18m}^{EUR})^2} - \frac{D_{2y}^{OIS}}{D_{2y}^{EUR}} \quad \frac{D_{18m}^{EUR} * D_{2y}^{OIS}}{(D_{2y}^{EUR})^2}\right]$$

This vector represents the Euribor forward replicating flows. The same vector should be determined for the (Eonia) discount curve. Both legs have OIS discount exposure, resulting in the following:

$$\frac{\partial C_{2y}}{\partial D_{6m}^{OIS}} = -\left(\frac{D_0^{EUR} * D_{6m}^{EUR}}{(D_{6m}^{EUR})^2} - 1\right) = \frac{-1}{D_{6m}^{EUR}} + 1$$

$$\frac{\partial C_{2y}}{\partial D_{1y}^{OIS}} = K_{2y} * \alpha_{0,1y}^{EUR} - \left(\frac{D_{6m}^{EUR} * D_{1y}^{EUR}}{(D_{1y}^{EUR})^2} - 1\right) = K_{2y} * \alpha_{0,1y}^{EUR} - \frac{D_{6m}^{EUR}}{D_{1y}^{EUR}} + 1$$

$$\frac{\partial C_{2y}}{\partial D_{18m}^{OIS}} = -\left(\frac{D_{1y}^{EUR} * D_{18m}^{EUR}}{(D_{18m}^{EUR})^2} - 1\right) = \frac{-D_{1y}^{EUR}}{D_{18m}^{EUR}} + 1$$

$$\frac{\partial C_{2y}}{\partial D_{2y}^{OIS}} = K_{2y} * \alpha_{1y,2y}^{EUR} - \left(\frac{D_{18m}^{EUR} * D_{2y}^{EUR}}{(D_{2y}^{EUR})^2} - 1\right) = K_{2y} * \alpha_{1y,2y}^{EUR} - \frac{D_{18m}^{EUR}}{D_{2y}^{EUR}} + 1$$

---

[23] The minus sign is to illustrate the point that we are still working on a receiver swap, which value is determined by subtracting the float leg from the fixed leg. As only par swaps are used, the two legs have equal value because the total value of a par swap is zero by definition.

These results can also be put together in a vector:

$$\frac{\partial C_{2y}}{\partial D(\tau)^{OIS}} = \left[\frac{-1}{D_{6m}^{EUR}} + 1 \quad K_{2y} * \alpha_{0,1y}^{EUR} - \frac{D_{6m}^{EUR}}{D_{1y}^{EUR}} + 1 \quad \frac{-D_{1y}^{EUR}}{D_{18m}^{EUR}} + 1 \quad K_{2y} * \alpha_{1y,2y}^{EUR} - \frac{D_{18m}^{EUR}}{D_{2y}^{EUR}} + 1 \right]$$

The two repflow vectors can be generalized for n cash flow tenors of six months with subscript i to:

$$\frac{\partial C}{\partial D(\tau)^{EUR}} = \left[\frac{D_i^{OIS}}{(D_i^{EUR})^2} - \frac{D_{i+1}^{OIS}}{D_{i+1}^{EUR}} \quad \frac{D_i^{EUR} * D_{i+1}^{OIS}}{(D_{i+1}^{EUR})^2} - \frac{D_{i+1}^{OIS}}{D_{i+1}^{EUR}} \quad \cdots \quad \frac{D_{n-2}^{EUR} * D_{n-1}^{OIS}}{(D_{n-1}^{EUR})^2} - \frac{D_{n-1}^{OIS}}{D_{n-1}^{EUR}} \quad \frac{D_{n-1}^{EUR} * D_n^{OIS}}{(D_n^{EUR})^2} \right] \quad (23)$$

$$\frac{\partial C}{\partial D(\tau)^{OIS}} = \left[\frac{-1}{D_i^{EUR}} + 1 \quad K_n * \alpha_{0,i+1}^{EUR} - \frac{D_i^{EUR}}{D_{i+1}^{EUR}} + 1 \quad \cdots \quad \frac{D_{n-2}^{EUR}}{D_{n-1}^{EUR}} + 1 \quad K_n * \alpha_{n-2,n}^{EUR} - \frac{D_{n-1}^{EUR}}{D_n^{EUR}} + 1 \right] \quad (24)$$

These same vectors can be calculated for our collateralized two year swap example, as all the terms are known. The results are simply presented:

$$\frac{\partial C_{2y}}{\partial D(\tau)^{EUR}} = [2.0082 \quad 1.9534 \quad 2.8086 \quad 103.1618]$$

$$\frac{\partial C_{2y}}{\partial D(\tau)^{OIS}} = [-2.0078 \quad 3.0470 \quad -2.8082 \quad 1.8364]$$

The most obvious difference is the large exposure to the highest maturity discount factor of the forward curve, which is much smaller when regarding the discount curve. The two negative value in the OIS derivation concerns the two (negative) floating payments, which have opposite discount exposure than the positive (netted) cash flows on the full-year points.

Now that we have the repflow vectors for both curves, we can use this in (21) and (22). However, the repflow vectors contain 4 elements for a period that only has two instruments on the curve. We thus have to add the same interpolation term as in (20). Even though it now concerns a floating payment instead of the fixed repflow in (20), the method does not change as the goal is still the same: determining the sensitivities of the repflow discount factors with respect to the grid point discount factors. Therefore we can simply add (19) to (21) and (22). The result from (17) is also added so that we end up with the full two curve interest rate sensitivities:

$$\frac{\partial C}{\partial K^{EUR}} = \frac{\partial C}{\partial D(\tau)^{EUR}} \cdot \frac{\partial D(\tau)^{EUR}}{\partial D^{EUR}} \cdot - \frac{\partial PV^{EUR-1}}{\partial D^{EUR}} \cdot \frac{\partial PV^{EUR}}{\partial K^{EUR}} \quad (25)$$

$$\frac{\partial C}{\partial K^{OIS}} = \frac{\partial C}{\partial D(\tau)^{OIS}} \cdot \frac{\partial D(\tau)^{OIS}}{\partial D^{OIS}} \cdot - \frac{\partial PV^{OIS-1}}{\partial D^{OIS}} \cdot \frac{\partial PV^{OIS}}{\partial K^{OIS}} \quad (26)$$

These two results will be the main tools in determining the optimal two-curve hedge in the empirical part, as the deltas calculated with the above are the exposure that has to be hedged.

If we run this procedure for the two year example swap, we have to add the interpolation term first. Here the interpolation matrix for both repflow vectors is presented, as deriving it is beyond scope here.

$$\frac{\partial D(\underline{\tau})^{EUR}}{\partial \underline{D}^{EUR}} = \frac{\partial D(\underline{\tau})^{OIS}}{\partial \underline{D}^{OIS}} = \begin{bmatrix} 0.5140 & 0 \\ 1 & 0 \\ 0.6998 & 0.3711 \\ 0 & 1 \end{bmatrix}$$

Recall that there are only two instruments, a one and two year swap. There are four cash flows with sensitivities, every six months, which are all discounted using a discount factor. The second and fourth row thus represent the full year discount factors, which occur on the grid points on the curve where the instruments are located. Therefore, the second row has a one to depict the perfect sensitivity to the one year instrument discount factor and a zero because the two year instrument has no influence in that point. The fourth row represents the two year cash flow which only has (perfect) sensitivity to the two year instrument discount factor. The first row concerns the six month payment, which has only sensitivity to the one year discount factor. (The zero-year interest rate is zero by definition, and is left out). The third row depicts the sensitivity of the 1,5 year discount factor which is determined by the two instruments' discount factors, more so by the one year swap instrument.

Now we have all the input needed for equation (24) and (25). The interpolation matrix converts the repflow matrix to the right dimension, so we get:

$$\frac{\partial C_{2y}}{\partial \underline{D}^{EUR}} = [4.9511 \quad -104.2041]$$

$$\frac{\partial C_{2y}}{\partial \underline{D}^{OIS}} = [-0.0499 \quad -0.7959]$$

This shows some interesting differences with the single curve situation, which is again depicted below.

$$\frac{\partial C_{2y}}{\partial \underline{D}} = [5 \quad 105]$$

Note that the single curve repflows are the sum of the separate curve repflows. We can thus conclude, that if

$$\underline{K}^{EUR} = \underline{K}^{OIS} \qquad then \qquad \frac{\partial C}{\partial \underline{D}} = \frac{\partial C}{\partial \underline{D}^{EUR}} + \frac{\partial C}{\partial \underline{D}^{OIS}}$$

In other words, if the discount and forward curve are built from the same instruments, the curves are identical and their separate repflows sum up to the single curve repflows.

The same holds for the deltas; we keep the repflows separate and multiply them with the sensitivity of the discount factors to the rates to obtain the deltas[24]:

$$\frac{\partial C_{2y}}{\partial \underline{K^{EUR}}} = [0.000001 \quad -0.018539]$$

$$\frac{\partial C_{2y}}{\partial \underline{K^{OIS}}} = [-0.000001 \quad -0.000141]$$

Recall the single curve deltas, which match the sum of the separate deltas:

$$\frac{\partial C}{\partial \underline{K}} = [0 \quad -0.01868]$$

The delta to the lower maturity instruments is non-zero but small if taken separately for discount and forward exposure.[25] There is also a clear separate exposure to the OIS curve to the highest maturity instrument, albeit much smaller in magnitude compared to the forward curve delta. Note however that this is a short swap, and that the discount exposure becomes larger relative to the forward exposure for longer maturity swaps.

The next subsection will show the additive property symbolically.

4.2.1. <u>Validation</u>

The results in (23) and (24) can easily be validated by taking the same instruments for both curves. So suppose that the discount curve and the forward curve are exactly equal in each point, because the curves are built from the same instruments. The superscripts then drop out, as the OIS curve = EUR curve so the discount factors are also equal.

$$\frac{\partial C}{\partial \underline{D(\tau)^{EUR}}} = \left[\frac{D_i}{(D_i)^2} - \frac{D_{i+1}}{D_{i+1}} \quad \frac{D_i * D_{i+1}}{(D_{i+1})^2} - \frac{D_{i+1}}{D_{i+1}} \quad \ldots \quad \frac{D_{n-2} * D_{n-1}}{(D_{n-1})^2} - \frac{D_{n-1}}{D_{n-1}} \quad \frac{D_{n-1} * D_n}{(D_n)^2}\right]$$

This can be reduced to :

$$\frac{\partial C}{\partial \underline{D(\tau)^{EUR}}} = \left[\frac{1}{D_i} - 1 \quad \frac{D_i}{D_{i+1}} - 1 \quad \ldots \quad \frac{D_{n-2}}{D_{n-1}} - 1 \quad \frac{D_{n-1}}{D_n}\right] \tag{27}$$

---

[24] Somewhat more decimals are used her to show the non-zero nature of the lower maturity sensitivities. Of course, a higher notional could have been used, but that would create the same problem elsewhere in the calculation.

[25] The forward exposure is positive contrary to the negative exposure on the last instrument, but that is not the general case. When 20 instruments are used, the shorter maturity instrument deltas can have both positive and negative signs, depending on the curve.

We do the same to the OIS repflows:

$$\frac{\partial C}{\partial \underline{D(\tau)^{OIS}}} = \begin{bmatrix} \frac{-1}{D_i} + 1 & K_n * \alpha_{0,i+1}^{EUR} - \frac{D_i}{D_{i+1}} + 1 & \dots & \frac{-D_{n-2}}{D_{n-1}} + 1 & K_n * \alpha_{n-2,n}^{EUR} - \frac{D_{n-1}}{D_n} + 1 \end{bmatrix} \quad (28)$$

$$Since \ \underline{D(\tau)^{EUR}} = \underline{D(\tau)^{OIS}} \quad \rightarrow \quad \frac{\partial C}{\partial \underline{D(\tau)^{EUR}}} + \frac{\partial C}{\partial \underline{D(\tau)^{OIS}}} = \frac{\partial C}{\partial \underline{D(\tau)}}$$

We can thus add the (27) and (28) to obtain:

$$\frac{\partial C}{\partial \underline{D(\tau)}} = \begin{bmatrix} 0 & 1 + K_n * \alpha_{0,i+1}^{EUR} & \dots & 0 & 1 + K_n * \alpha_{n-2,n}^{EUR} \end{bmatrix}$$

The zeros in the above vector represent sensitivity to semi-annual points, which played no role in the single curve case. For the two year swap case, the interpolation matrix would be:

$$\frac{\partial \underline{D}}{\partial \underline{D(\tau)}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

All the rows representing a semi-annual cash flow only have zeros. If we then multiply the summed up repflows with the matrix with the appropriate dimension, the zeros in the repflow vector drop out and we get:

$$\frac{\partial C}{\partial \underline{D}} = \begin{bmatrix} K_n * \alpha_{0,i} & K_n * \alpha_{i,i+1} & \dots & K_n * \alpha_{n-2,n-1} & 1 + K_n * \alpha_{n-1,n} \end{bmatrix}$$

This is exactly equal to (15), the repflow vector in the single curve case.

This can then be multiplied with the instrument sensitivities to the rates, which are equal for both curves since the curves were set equal;

$$\frac{\partial \underline{D^{EUR}}}{\partial \underline{K^{EUR}}} = \frac{\partial \underline{D^{OIS}}}{\partial \underline{K^{OIS}}} = \frac{\partial \underline{D}}{\partial \underline{K}}$$

We can thus conclude that when the Euribor-Eonia spread is zero at every point in the curve, the two-curve approach delivers the exact same results as the single curve approach.

This result justifies the methods used in the past, as the spread was negligible before the crisis (check graph 1) and both methods would have delivered the same deltas, resulting in the same hedge strategy. This is also pointed out by Mercurio (2009) who refers to the OIS curve before the crisis as chasing the swap curve making the spread negligible.

The results above, however, do outline the need for a two-curve approach when the Euribor-Eonia spread is significant. (25) and (26) cannot be added up in this case so the distinctive approach is methodologically advisable. One could argue that $\frac{\partial D^{EUR}}{\partial \underline{K}^{OIS}} \neq 0$ which is indeed the case, and this is exactly what is captured in the forward adjustment described in the curve build section. The influence, however, is so small that it is ignored. Another statement that can be made is that the correlation between the two curves is so high that a completely separate treatment misses an important parameter. This is discussed below.

4.2.2. **Intermezzo: Correlation**

As quoted before, Mercurio (2009) stated that swap rates of the same maturity but different underlying rate tenor would chase each other on a negligible spread before the crisis, so that a separate treatment would make no sense. However, this correlation is no longer equal to one, not even if calm times return. As Bianchetti (2010) noted, the fact that the OIS market has a different underlying rate tenor than the Euro swap market, makes them segmented. This does not mean that the correlation cannot be close to one, but there are different dynamics at work.

Another reason for not taking into account the correlation is that institutions that want to hedge interest rate risk do so to protect themselves against extreme events. Exactly in these extreme events, the correlation will be far from one so that any hedging strategy based on static correlation will not perform very well. Graphs 3 and 4 on the next page show that this is the case.

Graph 3: 6month Euribor vs 6m Eonia
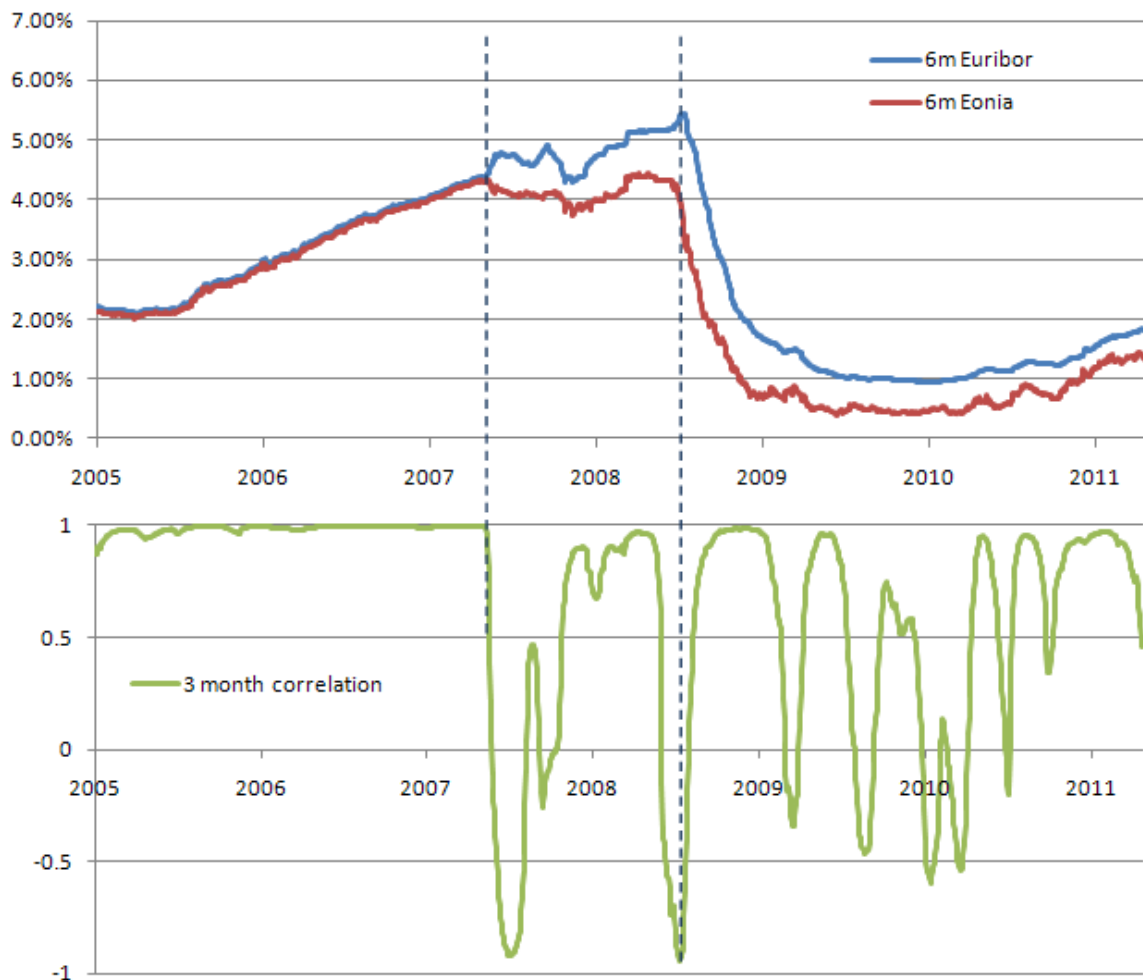
Graph 4: Correlation 6m Euribor swap rate with 6m Eonia swap rate (3 month backwards rolling level average, so t-3m to t)

A hedging strategy for stress scenarios should not be dependent on variables that behave differently during stress events. It is clear that the correlation between the two curves is very unpredictable and varies across the full range from minus one to one. There is also no indication that the old times with a stable correlation close to one is likely to return in the near future, just as it looks unlikely that the spread will completely close again. Any assumption made about future correlation to build a hedge with is therefore futile, which is the reason that it is completely left out of this thesis.

There are two risks that should be hedged; the future nominal floating payments vary with the Euribor curve, while both the fixed and floating payments are exposed to the OIS discount curve. We will thus proceed in a setting where the sensitivities to both curves are calculated separately in order to optimally determine the exposure. This will give the best starting point for hedging.

4.2.3. <u>Double curve sensitivities cont'd</u>

If we lower the discount curve while keeping the forward curve equal, we get the situation where the double curve approach is more appropriate. We again follow equations (25) and (26).

Table 2: Curve instruments with non-zero spread

	EUR swap curve instruments	OI swap curve instruments
1y	4%	3%
2y	5%	4%

We repeat all calculations done in the sensitivity section, but now with a lower discount curve. We start with the instrument sensitivities:

$$\frac{\partial \underline{D}^{EUR}}{\partial \underline{K}^{EUR}} = \begin{bmatrix} -0.9246 & 0 \\ 0.0440 & -1.7792 \end{bmatrix}$$

$$\frac{\partial \underline{D}^{OIS}}{\partial \underline{K}^{OIS}} = \begin{bmatrix} -0.9426 & 0 \\ 0.0363 & -1.82217 \end{bmatrix}$$

Note that the OIS instruments now have higher sensitivity. This is caused by the lower curve, which gives the present value of all cash flows of the OIS instrument a higher value.

The repflows change for the derivative to the forward curve discount factors, while the OIS repflows remain constant. This is due to the OIS discount factors in the forward curve repflows, as shown in (23). At the same time, (24) shows that the OIS repflows do not contain any OIS terms.

$$\frac{\partial C_{2y}}{\partial \underline{D(\tau)}^{EUR}} = \begin{bmatrix} 1.5379 & 1.4748 & 2.3763 & 105.1656 \end{bmatrix}$$

$$\frac{\partial C_{2y}}{\partial \underline{D(\tau)}^{OIS}} = \begin{bmatrix} -2.0078 & 3.0470 & -2.8082 & 1.8364 \end{bmatrix}$$

The interpolation matrix for the forward curve remains constant contrary to the OIS one. This is caused by different discount factors that partly determine the interpolation matrix:

$$\frac{\partial D(\underline{\tau})^{EUR}}{\partial \underline{D^{EUR}}} = \begin{bmatrix} 0.5140 & 0 \\ 1 & 0 \\ 0.6998 & 0.3711 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial D(\underline{\tau})^{OIS}}{\partial \underline{D^{OIS}}} = \begin{bmatrix} 0.5116 & 0 \\ 1 & 0 \\ 0.7082 & 0.3720 \\ 0 & 1 \end{bmatrix}$$

We can then multiply the repflows with the interpolation matrices to obtain the sensitivity of the present value of the claim to the instrument discount factors. We then get the instrument repflows:

$$\frac{\partial C_{2y}}{\partial \underline{D^{EUR}}} = [3.9282 \quad -106.0474]$$

$$\frac{\partial C_{2y}}{\partial \underline{D^{OIS}}} = [-0.0311 \quad -0.7918]$$

Note that these vectors do no longer add up to the single curve equivalent, which had 5 and 105 as values. This also translates into the deltas, which are obtained by multiplying the above with the instrument sensitivities.

$$\frac{\partial C_{2y}}{\partial \underline{K^{EUR}}} = [0.000104 \quad -0.018867]$$

$$\frac{\partial C_{2y}}{\partial \underline{K^{OIS}}} = [-0.000000 \quad -0.000144]$$

The sum of these deltas is quite different from the case with two equal (and thus one) curves, which is repeated here for comparison:

$$\frac{\partial C}{\partial \underline{K}} = [0 \quad -0.01868]$$

Regardless of the fact that the outcomes do not match, these two delta vectors are no longer theoretically additive. They represent sensitivities to different curves.

The results from this section are the main input for setting up hedges in the next section. Whether the double curve methodology can be used to improve the existing single curve hedging methods, is to be discussed in section 6.

5. Application in a Pension fund

Now that the required valuation and sensitivity tools are defined, we can point our attention to the second part of the research question: the effect of collateralization on the users of swap curves, where the focus lies of institutions which operate in an asset- liability management context such as insurance companies and pension funds. The latter is picked out due to its importance in the Dutch financial sector.

The users will need the curves primarily for discounting and hedging purposes, so that will be the settings in which we will operate. In the remainder of the thesis, all practical issues will be illustrated by using the hypothetical Dutch defined benefit pension fund xyz. The asset and liability data are from a real pension fund but have been anonymised and aggregated in broad asset classes.

5.1. Pension fund characteristics
The pension fund has the following balance sheet(in millions):

Table 3: Balance sheet pension fund xyz

Assets		Duration	Liabilities		Duration
Equity	€ 1,500	0	Pension rights	€ 3,100	-17.50
Government bonds	€ 2,100	-11.68	Surplus	€ 500	0
Total	€ 3,600	-11.68	Total	€ 3,600	-17.50

 The 42% equity portfolio is not further specified as it is not of interest. We assume the duration of the equity to be zero, even though this might not be true in practice. However, since all hedging strategies will be tested under this same assumption, and a non-zero duration would lead to the same adjustment to all hedges, it is considered negligible. The equity is also kept constant in order to minimize noise in the results. The effects of this will be discussed in section 5.2.5.

The presence of a corporate sponsor is possible, but there are many real world cases where a corporate sponsor does either not exist or has gone bankrupt. (Kocken, 2008) Therefore we will proceed with a standalone pension fund without a corporate sponsor.

The rest of the assets will be referred to as fixed income portfolio, or matching portfolio, which forms 58% of assets. The current funding ratio is 116%, which is above the regulatory level, but lower than convenient given the volatility of the investments. The surplus acting as buffer could be wiped out in adverse events, but this risk can be lowered by taking out the interest rate risk with a hedge.

The board of the pension fund has decided to hedge all interest rate risk, and thus needs to know which positions should be taken in the market to offset the current exposure. For simplicity, it is assumed that

the pension fund is closed, so that no new participants can enter and no new rights or contributions are added. This actually happens regularly, and does not cause any loss in generalizability. (Kocken, 2008)

The liabilities consist of the built up pension rights, which have to be paid out in the future. The distribution of nominal liabilities is illustrated in graph 5:

Graph 5: Nominal liability distribution of pension fund xyz over time
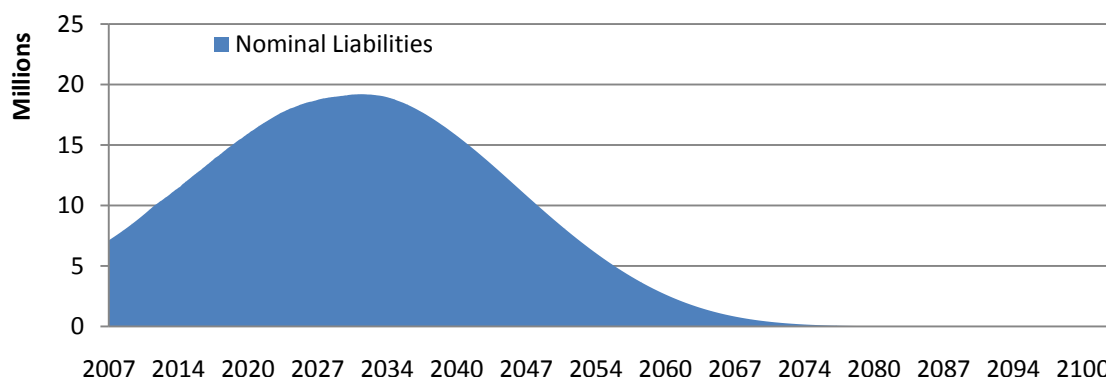
The graph makes clear that the maximum yearly payout will be reached in 24 years from the startdate, with an afterwards steadily declining payout until 90 years from now. However the payouts in the near future also have a large magnitude, and in present value terms have more impact:

**Graph 6: Distribution of present value of liabilities of pension fund xyz over time**
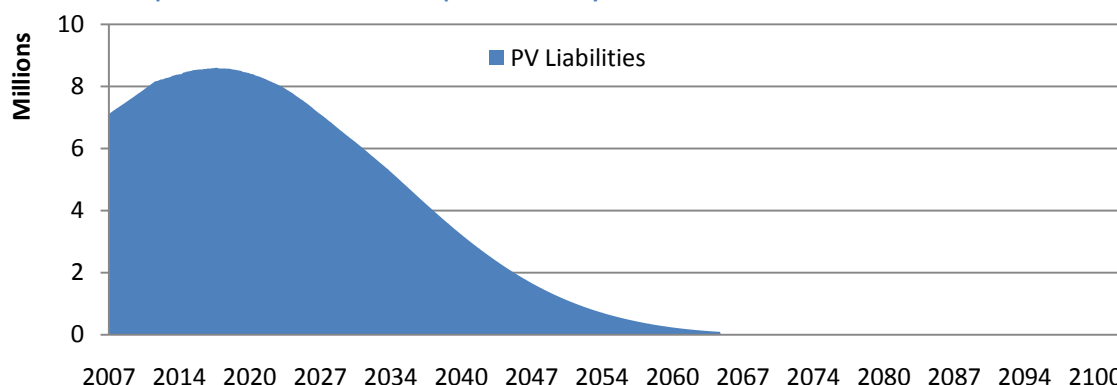
The present value of the current pension rights is largely concentrated in the coming 30 years. To obtain the present value, the liabilities are discounted using the Euribor swap curve[26], which is mandatory by FTK regulation in the Netherlands.

Because of the high duration, the fund has a large exposure to the discount rate it uses, and is thus sensitive to the euro swap curve. However, as shown on the balance sheet, fund xyz has a substantial fixed income portfolio which partly offsets the sensitivity of the liabilities.

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[26] The swap curve of 02-07-2007 is used here, as this is the date the hedge is set up later in the thesis. The asset and liability values are of the same date.

This results in the following bucket basis point values (BPVs)[27] :

Graph 7: Interest rate exposure fund xyz
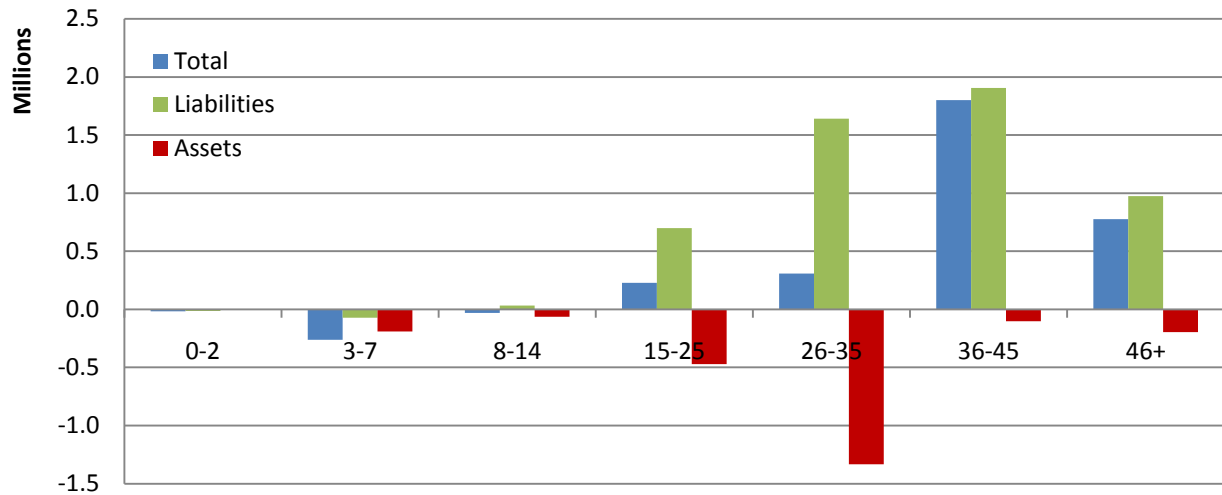
Graph 7 shows that the exposure is concentrated in the longer maturities, and that the asset cash flows are concentrated in the middle maturities. The buckets are already defined here, as shown on the x-axis. The buckets are centered around the most liquid instruments[28], being the 2 and 5 year and the 10, 20, 30,40 and 50 year swaps. The exposures are thus summed per bucket, which makes clear that the matching portfolio offsets quite a big part of the middle maturity liabilities.[29] The green bars represent the exposure that the board wants to hedge, given the assumption that shifts in the rates occur parallel within the buckets. For example, if the rates in the 36 to 45 year bucket shift downwards by one basis point, the surplus of the fund would fall by about 1.8 million. A hedge should compensate this effect by gaining the same value as the loss in surplus.

 The buckets are wider for longer maturities, as long term instruments tend to be more correlated than short term instruments. (Potters, 2011) This statement is supported by the correlation matrix in the appendix; it clearly shows that longer term rates are more correlated. The 30, 50 and 40 year rates are nearly perfectly correlated. This could imply that all these rates can be put in one bucket. However, graph 8 shows that there is quite some variation in the spreads between these rates, implying that correlation is not a good measure in this case. Therefore the bucket selection is largely the result of availability and liquidity of the underlying curve instruments.

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[27] Where the duration is the PV change resulting from a one percent parallel change in the interest rate curve , the BPV is the PV change resulting from a one basis point parallel change in the curve. A bucket BPV is then the PV change following a change of one basis point of all rates within the bucket.

[28] List of instruments is available in the appendix

[29] The sensitivity of the liabilities is shown positively here, while the duration on the balance sheet is negative. This is because when it concerns the other side of the balance sheet, the fund is basically short the liability cash flows, and thus the sign flips.
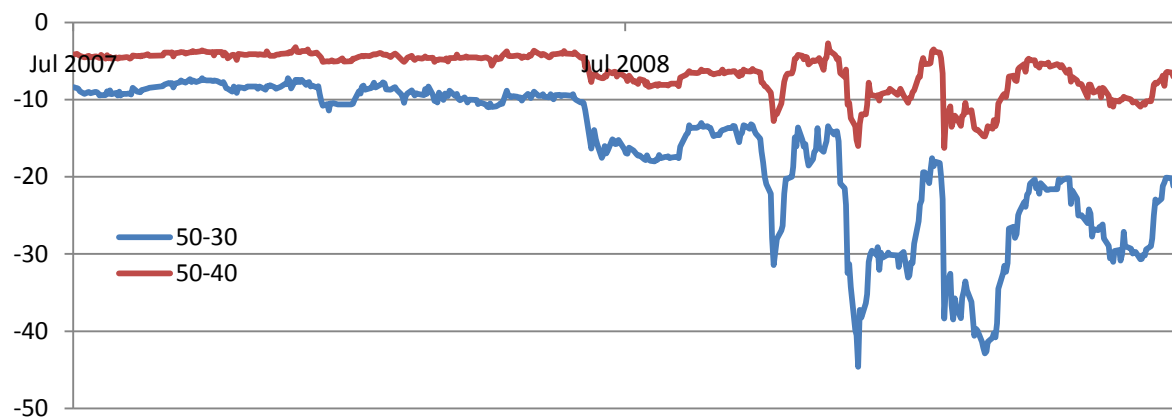
The co-movement is obvious from graph 8 but the volatility of the spread, combined with a large concentration of liability cash flows in the high end of the curve, make that these instruments will be in separate buckets.

 Liquidity also plays a role in the selection, but the two year instrument is not more liquid than for example the one year. However, graph 7 shows that the interest rate exposure in the short end is so small that it would not make sense to take out the one year swap separately. The two year instrument is chosen for methodological reasons; the sample period lasts just under two years and a static hedge is used. A one year swap would expire halfway the sample so the hedge would have to be rebalanced, which is undesirable for the testing procedure. Therefore all hedges will be static over the whole sample period, meaning that no rebalancing will take place.

5.2. Testing methodology

To be able to compare the hedging strategies empirically, they are all tested for effectiveness in a crisis scenario. Specifically, the test will evaluate the hedge quality during the recent credit crisis. The choice of a historic event for a stress test has the advantage that 'these events generally resonate with higher levels of management, as everybody is aware that these kinds of events actually occurred and that there is a chance that it might take place again in the future.' (Kocken, 1997) A randomly generated stress scenario could be less effective in acting according to the recommendations from the stress test, but has the advantage of easily increasing the number of scenarios that can be analyzed.

Because of the large potential impact of an interest rate shock to pension funds, which occurred during the recent crisis, a hedge is set up at the second of July 2007[30], a day where the crisis was not yet present in the newspapers or in the Euribor-Eonia spread. The period ends two years later, at June 30th 2009. Goal is to see how different hedging strategies hold up during this period.

Graph 9: Sample period of hedge effectiveness tests, 6m Euribor vs. 6m EONIA
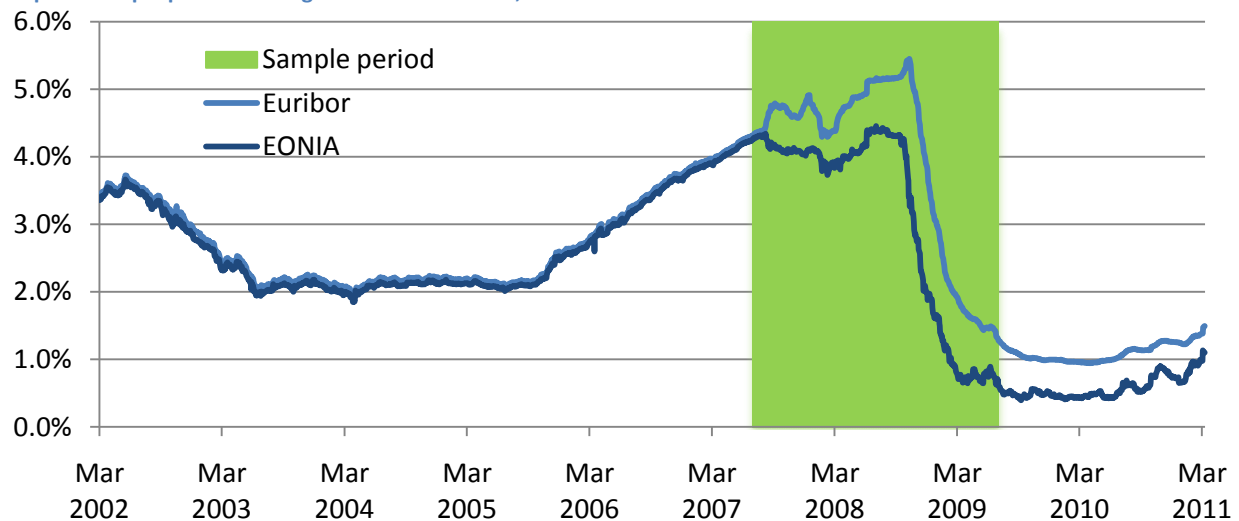
As can be seen in graph 9, the sample period stretches from one of the last dull moments to the recovery of a somewhat stable spread, albeit a much larger one than before. This is the scenario for which the hedge is set up primarily.

The sample period contains 501 business days, for which the interest rates of the instruments of both the Euribor and the Eonia curve are taken from Bloomberg. The instrument codes can be found in the appendix. Over the time series, the following parameters will be monitored:

- Volatility/Variance of the funding ratio
- Minimum of the FR during the period

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[30] The first of July 2007 was a Sunday, so trading was only possible the day after.

The board has asked to take away all interest rate risk. This can be translated in a FR volatility due to interest rate movements of zero. The prime measure for hedge quality will therefore be the variance of the funding ratio, which should be as low as possible. This is also the only measure that will be statistically tested, the other one providing secondary information over the hedge. The minimum funding ratio is particularly interesting for regulatory reasons, as FTK requires the FR to be above 105% at all times, and a recovery plan with corrective measures when the FR falls below this threshold.

The statistical procedure will be a two-sided F-test for the ratio of two variances of two strategies. The hypotheses will be that there is a significant difference in variances, where the method which is assumed to be better(usually the more complicated one), has a lower variance. Whether the latter is true, is to be found in the results, as the opposite can of course also be true. As we are dealing with interest rates here, our sample may suffer from autocorrelation. The statistical consequences of this will be discussed in the limitation section.

5.2.1. <u>Hedge ratio</u>

The hedge described above is a surplus hedge; it is designed to exactly mirror the value change in the balance sheet of the pension fund in order to keep the surplus constant. This method focuses on the absolute amount of money in the portfolio, while pension funds are regulated and operated by funding ratio, a relative measure. A 500 million surplus is very different for a 2-billion Euro fund than for a fund with 20-billion Euro in liabilities, as the funding ratios will be 125% and 102,5%, respectively. The hedge ratio, which is 100% for a surplus hedge, has to be chosen with the goal of keeping the funding ratio intact instead of the surplus. This can be expressed as:

$$FR * V_L * D_L = V_A * D_A + N_S * D_S$$

$With\ V\ being\ the\ value\ \ of\ the\ \underline{L}iabilities\ and\ \underline{A}ssets, D\ being\ the\ duration$[31]$\ and$

$N\ the\ Notional\ of\ the\ \underline{S}wap.$

In order to keep the FR constant at 116%, the asset side of the balance sheet has to move with 116% of the percentage change in liabilities. In the above equation, the only unknown is the value of the swaps, the other variables can be picked from the balance sheet except for the duration of the swaps, which can of course be picked as required by picking the right instrument. (Or, in the bucket approach, by properly combining the different instruments) For the notional of the swaps we then get:

$$N_S = \frac{FR * V_L * D_L - V_A * D_A}{D_S} \tag{29}$$

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[31] We deviate from the bucket approach here by using the duration, for the sake of simplicity. The same approach can be followed using the bucket method, but in that case by solving a system of equations.

This way, a funding ratio hedge can be set up by first inflating the deltas of the liabilities with the funding ratio, calculate total deltas of the balance sheet and then building the hedge based on these deltas.

5.2.2. <u>Setting up a hedge: Dividing delta in buckets</u>[32]

To determine a bucket hedge, a number of steps are performed. Suppose a pension fund has the following liabilities:

Table 4: Simplified liabilities

1y	1000
2y	1000
3.5y	1250
4y	1500
5y	1500
6y	2000

Given a curve built from cash and swap instruments as described in the curve build section, we then calculate the sensitivities of these claims using (20)[33], which results in table 3[34]:

Table 5: Delta's of the claims to all instruments

6m	0.00
1y	-0.06
2y	-0.13
3y	-0.13
4y	-0.65
5y	-0.59
6y	-0.98
7y	0.00

The right column shows the delta to each instrument; a one basis point rise in the one year swap rate would lower the present value of the claim portfolio by about six cents. The sensitivity of the cash flow in 3.5 years is basically divided over the three and four year instrument. What is noticeable is that the six year delta is much higher than the one year delta, even though the underlying amount is only twice as big. This shows that longer maturities translate into higher deltas.

We could, of course, now inflate (or deflate) these deltas as described in the previous subsection, but we assume here that the asset side has exactly the same value as the liabilities and is fully invested in equity. The hedge ratio is then 100% and there is no natural hedge in place in the form of a fixed income

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[32] Based on Hull (2010)

[33] The approach is similar for one or more curves. For simplicity, only the single curve case is shown.

[34] The calculation is performed for all instruments described in the appendix. The eight year and higher instruments are not shown here as the delta is zero, like the seven year delta.

portfolio. In that case, we now have the delta of the balance sheet to all instruments, which should be aggregated on the subset of most liquid market instruments according to Bianchetti(2009). The chosen instruments are then referred to as the hedge instruments.

The sum of the deltas tells how much money is won or lost when the curve moves in a parallel way by one basis point. In that sense, it is highly related to the definition of duration. A single hedge would be performed using a four year swap, as that one is most in the middle. But if we use the buckets specified above in graph 7, only the two and five year instrument qualify and the short and middle rates are separated. Table four contains the bucket sensitivities. The two year bucket contains all delta up to and including the two year sensitivity, the five year bucket contains the rest.

Table 6: Required hedge notional calculation

Tenor	Claim delta	Swap unit delta	$\dfrac{Claim\ delta}{Swap\ delta} = required\ notional$
2y	-0.196356	-0.000186848	1050.88
5y	-2.339233	-0.000436473	5359.39

The swap unit delta column shows the delta of a swap of the bucket maturity with a notional of 1. By dividing the portfolio delta by the swap delta, the required notional to delta-hedge the bucket is obtained. The portfolio from table 2 can thus be bucket hedged by taking a position of 1051 in a two year par receiver swap and a position of 5359 in a five year par receiver swap.

If the pension fund wants a single hedge with a four year swap, basically one bucket remains. In this case, the sum of all deltas would be the exposure to hedge with the single swap, and this number would then be divided by the per-unit delta of a four year swap. The notional required for this single hedge would be the result.

The exact same procedure can be followed with OI swaps, but there are some caveats there, which will be discussed in section 5.2.4.

5.2.3. Single hedge vs bucket hedge

Under the assumption of only parallel shifts of the whole curve, a full hedge can be implemented with just one instrument. In this case, the exposure of the claim to all instruments would be summed up and the notional of the hedging swap is based on this number. The result for fund xyz looks like the following:

Graph 10: Single swap hedge deltas
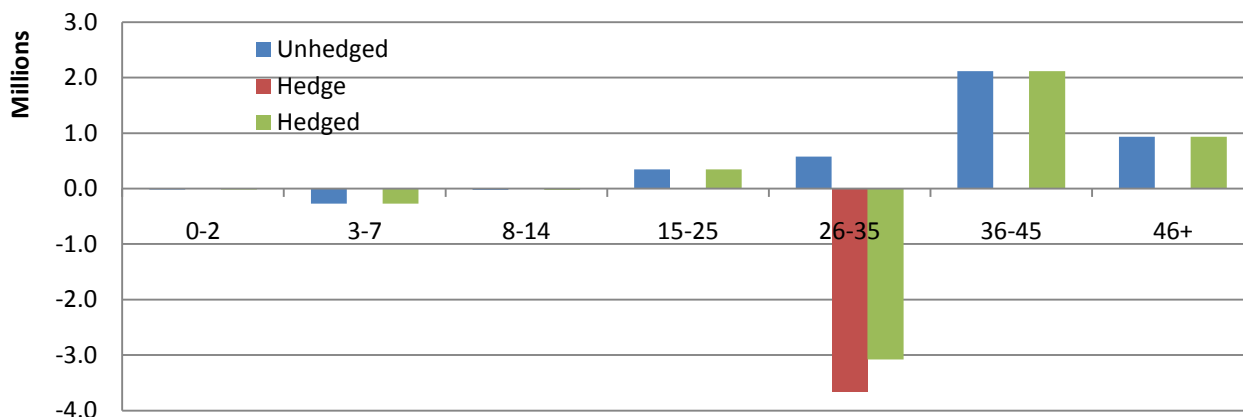
The 30 year receiver swap used for hedging has a notional sufficient to cover all the sensitivities, the red bar equals the sum of the blue bars. It is, however, only effective for parallel shifts. The graph makes clear that the hedge is ineffective for non-parallel curve changes or shifts in separate maturity buckets. Also, if only the 30 year rate changes, the fund is massively over hedged, which leads to extra volatility, opposite to the goal of the hedge. The problems described here can be greatly mitigated using the more refined approach of bucket hedging. The assumption of parallel shifts is weakened by only assuming parallel shifts within the bucket, as explained earlier. Of course, the best possible hedge using this method is achieved by bucketing per instrument. However, this is often not possible due to lack of liquidity of some of the instruments. Also the cost-effectiveness can be an issue, but that is beyond the scope of this thesis. If the bucket hedge approach is followed, the result is a more intuitive graph:

Graph 11: Bucket hedge deltas
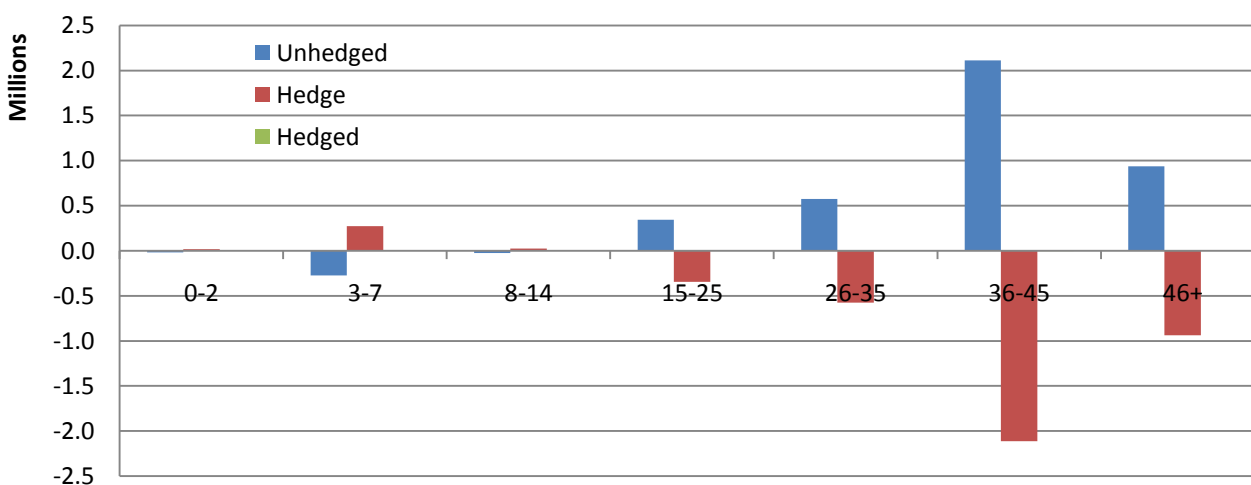
When the seven buckets are treated separately, the hedged positions have zero delta exposure in the model. In the short term buckets, the exposure is opposite to the long buckets, because the assets dominate the liabilities with respect to interest rate sensitivity. Since the board has elected to take away all exposure to the interest rate, this opposite position also has to be hedged. This is done using three (small) payer swaps. The four highest maturity buckets are hedged with four receiver swaps. This way, the delta interest rate exposure is reduced to zero on the bucket level. Note that in both graphs 10 and 11, the adjusted liability deltas have been accounted for. Total delta is therefore somewhat higher than in graph 7, where the intrinsic balance sheet exposure is shown.

5.2.4. Eonia liquidity

Trading of short Eonia swaps has been around for 12 years, while the middle maturities up to ten years were added in 2005. The curve was only extended to 30 years on 28th of May 2008, while the 50 year instrument was added in April 2011. This means that for about half of the sample period, Eonia data is missing. In an attempt to reconstruct the long end of the OIS curve for the first half of the dataset, the Eonia-Euribor spread for all instruments of 10 years maturity or higher is determined on every day both rates were available. All these spreads are then divided by the 10 year Euribor-Eonia spread of the same day, which serves as a reference as the longest maturity instrument being traded over the whole sample period. The outcome is the instruments' spread as a percentage of the 10 year spread. This percentage is then averaged over all days, and is shown in table 7:

Table 7: Average Euribor-Eonia spread as percentage of 10 year spread (28-05-2008 to 30-06-2009, 279 business days)

11y	12y	15y	20y	25y	30y
98.02%	95.33%	89.65%	83.97%	80.16%	77.72%

The table shows that the absolute spread is smaller for longer rates. The Eonia rates for the missing data points are then generated as:

$$r_t^{OIS} = r_t^{EUR} - S_t^{average} * (r_{10}^{EUR} - r_{10}^{OIS})$$

So the OIS rate on a particular day is the Euribor rate minus the 10 year Euribor-Eonia spread multiplied by the predetermined spread average for that maturity. This way, the dataset is made complete up to the 30 year point. We cannot calculate the spread average for the rates higher than 30 years, as these were not available within the sample period. Therefore, the 30 year Eonia zero rate is extrapolated flat for the whole sample period, similar to the flat extrapolation performed on the Euribor curve from the 50 year point onwards.

The fact that the Eonia swaps have not been trading as long as Euribor swaps extends to a lower liquidity in the overnight market. Goldman Sachs (2008) describes the OIS market as liquid up to two years. They do add the expectation that the long dated OI swaps market will develop in the future. Also, 'the development of an OIS option market is inevitable.' This means that at this point the middle and high maturity Eonia swap market is not liquid enough for a somewhat large pension fund such as fund xyz. There is definitely by far not enough liquidity for all pension funds, so the tradability will be low. Regardless, an attempt to hedge Eonia exposure with OI swaps is made, but the feasibility of this in the market is not present today. It is, therefore, a theoretical experiment. It will take regulation and some

large parties to make the OIS market sufficiently liquid, and there is a chance that will happen. This is the reason why the Eonia swap hedge is considered in section 6.

5.2.5. <u>Cash account & equity</u>

During the sample period, the fund is in normal operative status, and thus has to pay out pensions every month. Also, fixed income investments pay out coupons and mature. Any hedge strategy is built from swaps, which make reset payments every six months. All these cash flows have to be accounted for. This is done using a cash account, which can be both positive and negative. If it is negative, it means money is borrowed, if it is positive, the money earns interest. Borrowing and lending is done at the Eonia rate of that day. This approach is taken in order to prevent reinvestment issues with maturing bonds and asset allocation distortions. The cash account is added to the assets every day, so that the surplus and funding ratio are calculated taking the operating cash into account. This way, the effect on the results should be minimized.

The same can be said of the assumption that the equity remains constant. This assumption causes the FR to drop over time because the liabilities are larger than the fixed income portfolio, while the present value of both amounts grows over time. This is even the case then interest rates stay constant over the sample. Of course the equity is normally expected to grow also, but is we were to add a drift with a random component to the equity portfolio the (FR volatility) results would be distorted. Another option would be adding the equity to the cash account on day one and letting it grow with daily Eonia, but given that the Eonia curve changes during the sample differently from the Euribor curve used to discount the remainder of the balance sheet, this would not solve the problem. Also, given the crisis during the sample, holding equity constant is already optimistic, let alone growing equity. The whole asset class could of course be dropped and assume fixed-income assets only, but nowadays it is hard to find such a pension fund.

Given that the FR volatility should solely reflect the effects of interest rate changes, the equity is assumed constant. Adding a constant to every component of a time series does not change the variance of that time series, which makes this method the best way of comparing different hedging strategies. Since all strategies suffer from the same downward trend of the FR, they are comparable without bias.

5.2.6. Sample period events

The situation the hedges have to cope with is illustrated in graph 12. It pictures the interest rates of the high maturity instruments[35] over the sample period. The interest rates drop 200 to 250 basis points before bouncing back to a level around 4%. Note that the basis spreads rise sharply during the period and do not recover to the low levels as before.

Graph 12: Euribor long interest rates
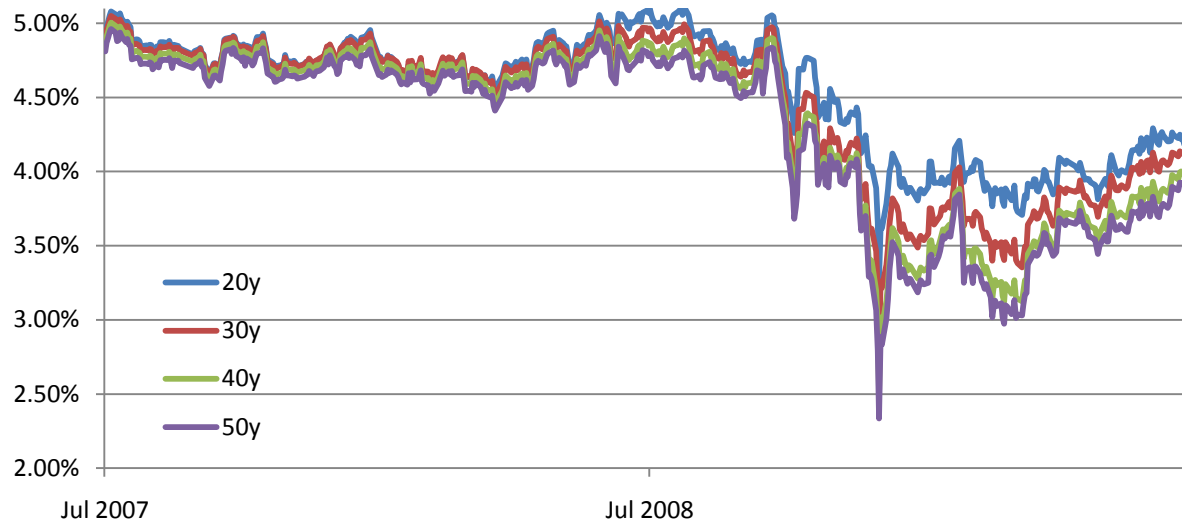
This event causes the liabilities to greatly increase in value; remember from the balance sheet that the duration of the fund is about 17.5 years. This should thus result in an around 40% increase in the value of the liabilities. Graph 13 shows that the liabilities increase by more than 50% before falling again, which pictures the convexity effect. This is the effect that the duration is not constant over different levels of interest rates, immediately showing a disadvantage of delta hedging.

Graph 13: Assets and liabilities of fund xyz during the sample period

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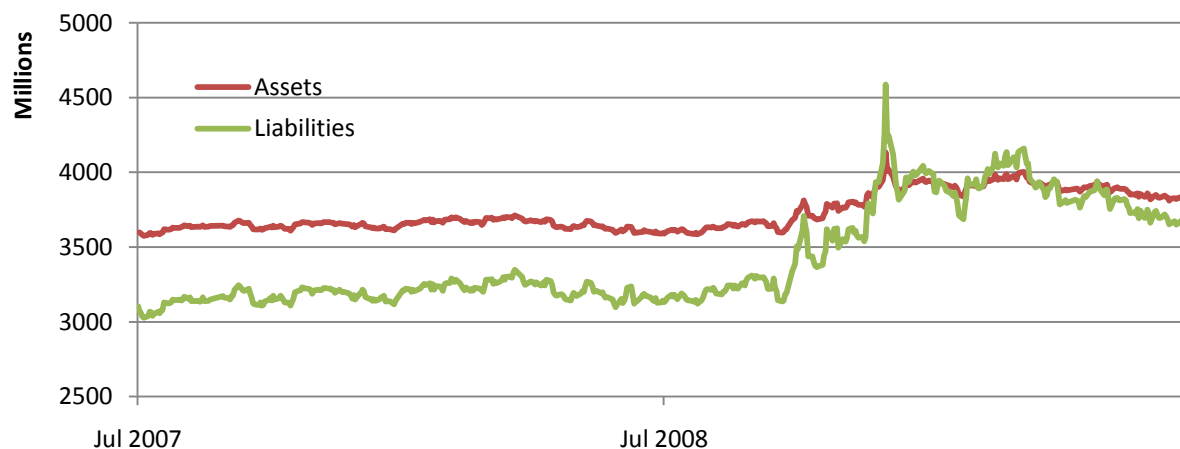
[35] The shortest three bucket instruments are not shown here, since their delta exposure is very small compared to the other four, and has the opposite sign due to the natural (over) hedge from the fixed income portfolio.

What a surplus hedge should do, is exactly mirror the movement of the surplus of the fund, being the difference between the assets and the liabilities. Graph 14 shows how the simplest hedge, a single swap in a single curve setting with a hedge ratio of 100%, performs in this task.

Graph 14: Hedged surplus of fund xyz
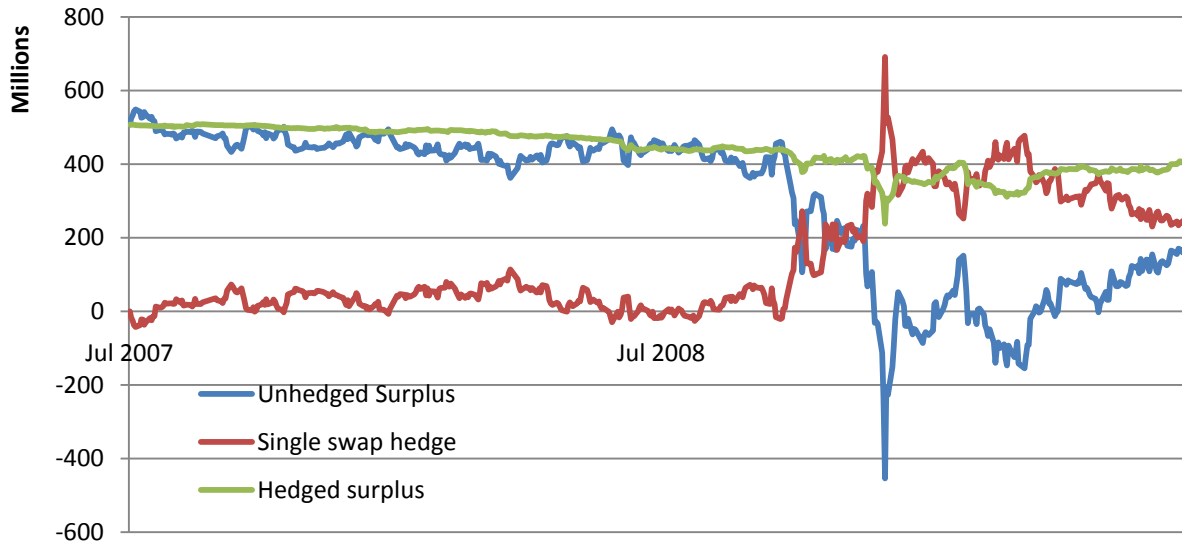
If the hedge were perfect, the green line would be a straight line without any bumps.[36] It is clearly not, which shows that improvement is still possible. Whether this can be achieved with a static hedge is to be analyzed in the next section. A dynamic hedge might do better but all hedges are static to make them comparable. If the board of a pension fund or corporation wants to take away its interest rate exposure, it wants an effective hedge. The result section therefore tests the performance of more complicated hedge strategies.

Recall from subsection 5.2.1 that the real testing is performed over funding ratio hedges, not over the surplus hedge used in graph 14. The surplus hedge is used only for illustrative value in showing how to counter the effect of decreasing interest rates on the value of the liabilities. All hedges shown in the results will be funding ratio hedges, as hedging the FR is the goal here.

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[36] Remember that it would not be horizontal, due to the downward trend explained in the cash account section.

6. Results

By using the methodology discussed in the sections three through six, the results are generated and presented below. The results will cover a number of hedge strategies, which are all analyzed over the sample period from graph 9. The strategies to be discussed are:

1. No hedging
2. Single curve hedging (Only swap deltas to Euribor curve)
   a. Euribor discounting of derivatives
      i. Single swap hedging
      ii. Bucket hedging
   b. OIS discounting of derivatives
      i. Single swap hedging
      ii. Bucket hedging
3. Double curve hedging (Swap deltas to both the forward curve and the discount curve)
   a. Euribor bucket hedge with swap notionals adjusted for two curves
   b. Bucket hedge to two curves

The first strategy speaks for itself; the others will be judged based on the performance compared to not hedging and compared to each other. The most interesting pairs will be picked out, where three questions pop up as most important:

- What is the difference caused by faulty Euribor discounting where OIS discounting is required? In other words, to what degree can a pension fund be harmed by using the wrong discount curve for its derivatives?
- Does bucket hedging outperform hedging with a single swap?
- Does the split of sensitivities improve the quality of the hedge? This can be divided over two cases; the case where only the notionals of the Euribor swaps are adjusted for OIS discounting and the case where OI swaps are added to the hedge in order to hedge the exposure to the discount curve.

The different hedge strategies should allow us to answer these questions. As a little spoiler, the reason that the single swap hedges are not implemented for two curves is the fact that the results turn out to confirm the hypotheses that the more refined bucket hedging works better.

Note that the deltas of the assets and liabilities are all calculated to the Euribor curve only, as that is the regulatory restriction. All statements about deltas to multiple curves only relate only to the swaps, as these are the collateralized instruments to which the new approach applies. This means that the second column of table 8, which depicts the total Euribor deltas from the balance sheet, is the basis for all the hedges. Exposure to the OIS curve can only originate from collateralized swaps designed to hedge the original exposure of the balance sheet to the Euribor curve.

6.1. Not hedging

It is widely believed that a risk premium is rewarded for taking risk. This leads some entities to the decision not to hedge, because in the long run the risk premium is supposed to pay out. What the consequences of such a policy can be for pension fund xyz, is depicted in graph 15:

Graph 15: Unhedged FR fund xyz over the sample period
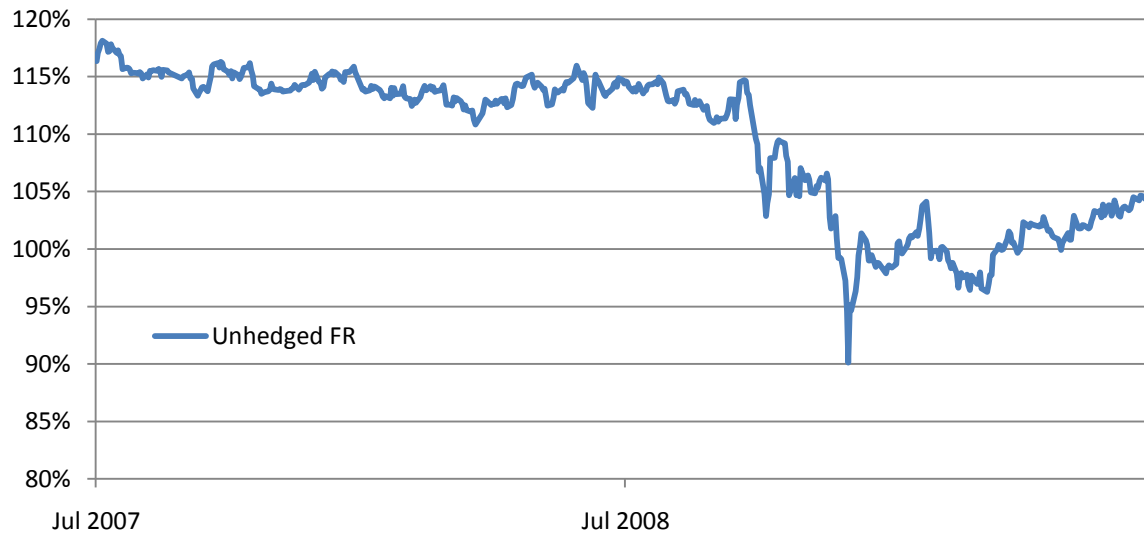
The funding ratio drops to 90% and does not reach above the regulatory level of 105% afterwards. This means that the fund is in serious trouble, with the regulator likely to intervene at the bottom point. Recall that this is the funding ratio with the equity at a constant level. Given that during the crisis, worldwide equity dropped over 30%, the fund is likely to have become insolvent.

6.2. Hedging to Euribor only

In this section, the notional of the hedging instruments is determined by deriving the swap deltas to one curve, so by using equation (20) from the sensitivity section.

Following the approach from section 5.2.2 on bucket hedging, we collect the deltas from the assets and the inflated deltas from the liabilities and sum these to obtain the balance sheet deltas to all instruments. These are then divided in buckets and listed in column two of table eight. We then divide these bucket deltas by their respective instrument per-unit delta. We then obtain the notionals required to delta hedge all the interest rate risk on the balance sheet.

Table 8: Hedge notional calculation fund xyz

| Bucket	Delta A&L	Single curve Euribor instrument unit delta	Single curve Euribor swap notional
0-2	18,362	-0.000186848	-85,415,526
3-7	272,777	-0.000436473	-598,567,490
8-14	14,316	-0.000781453	-20,794,982
15-25	-345,479	-0.001261082	182,082,291
26-35	-532,402	-0.001556815	167,052,154
36-45	-2,158,585	-0.001744951	1,059,592,126
46+	-922,872	-0.001868351	408,981,098
Total			1,112,929,670
Single 30y swap	-3,653,883	-0.001556815	2,347,024,335

The single swap hedge is obtained here by dividing the summed bucket deltas by the unit delta of the 30 year swap instrument. Because the 30 year swap unit delta is lower than the 40 and 50 year unit deltas, the resulting notional in the single swap is much higher than the sum of the bucket swap notionals. The 5 year payer swap in the bucket hedge has such a high notional because its unit delta is much lower than that of the high maturity swaps, which is makes sense given the difference in duration between the short and long term swaps.

How these hedges will perform in a stress scenario will be analyzed next.

### 6.2.1. Hedging with Euribor discounted swaps

Before the crisis, it was largely unknown or simply ignored that OIS has to be used for the discounting of collateralized derivatives. This was not without reason, the Euribor-Eonia spread was nearly zero anyway so an Euribor-only policy can be argued for at the time. Back then, a hedge was set up to protect the fund against its exposure to Euribor, and the hedge derivatives were to be discounted with Euribor, regardless of collateralization. To illustrate how such a hedge would come up in the accounting of the fund, graph 16 shows how three hedge strategies perform over time. All discounting is done using Euribor, Eonia does not play any role here.

Graph 16: Euribor discounted hedged FRs
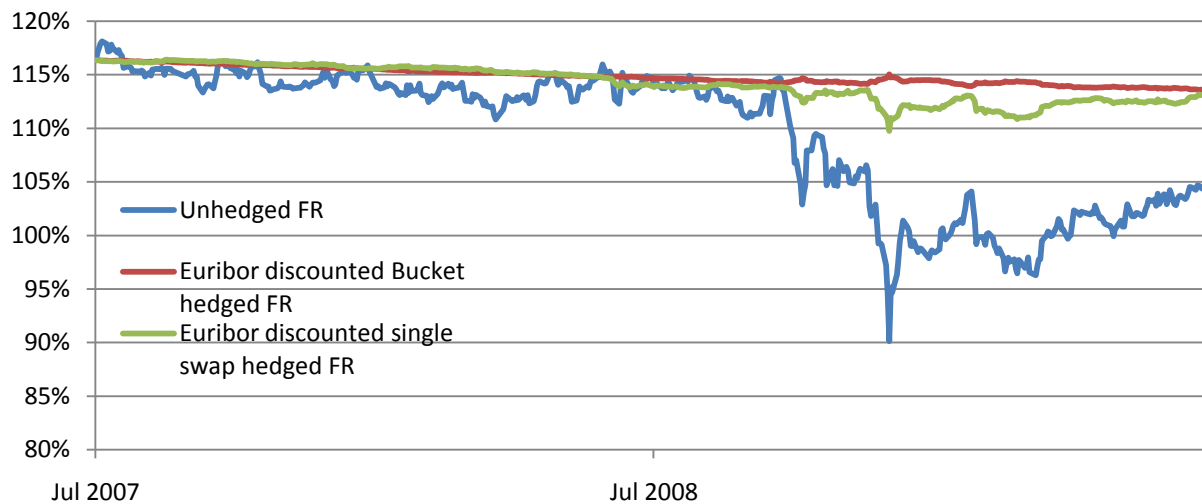
Given the adverse scenario hedging is obviously preferred here, as even the simplest hedge keeps the FR above the regulatory 105% level, while the more refined bucket hedge makes sure that the FR barely drops at all, separate from the trend described in the cash account section. There is even a small upward movement at the time that the unhedged and single hedged FRs are at their minimum. This looks like the effect of an over hedge, but it has a sound explanation. Recall from graph 8 that the 50-40 and 50-30 year spreads are not constant through time. Graph 17 shows some of the instrument swap rates during the end of 2008:

Graph 17: Swap rates of four Euribor instruments during November-December 2008
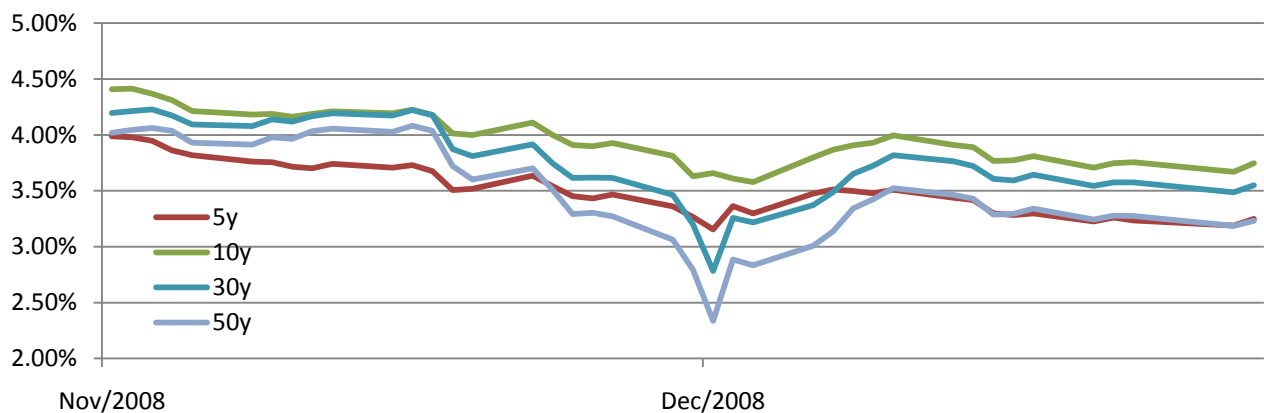
The biggest downward movement of the funding ratio is caused by a crash of especially the 40 and 50 year rate,[37] the 50-30 year spread in December is much higher than the spread in November .On top of that, the short term rates do not make the same movement, but stay rather calm compared to the long term rates, as graph 17 shows. This has the effect that the two receiver swaps with the highest notional, the 40 and 50 year swap, gain a lot of value, while the 30 year swap gains relatively less. This also explains the big decrease in the single hedged funding ratio, because the present value of the liabilities rises sharply due to the lower long term rates while the single swap fails to compensate this effect by being linked solely to the 30 year rate. This is a prime example of the failed assumption of parallel movements of the yield curve. Of course, the payer swaps in the bucket hedge have a negative value, but graph 17 shows that the short rates do not exhibit the huge downward pike that the long rates do. So the pension fund wins a lot on its receiver swaps while losing much less on its payer swaps.

The falling short rates of course increase the value of the fixed income portfolio, but this is neutralized by the payer swaps. The total result is that the movements of both the assets and the liabilities are effectively mirrored by the hedge, greatly reducing the volatility of the funding ratio.

Table 9: FR volatilities with Euribor discounted hedges

	Unhedged FR	Euribor discounted Single swap hedged FR	Euribor discounted Bucket hedged FR
FR volatility	6.43%	1.63%	0.78%

Table 9 shows the difference in funding ratio volatility between the strategies. We can now draw a conclusion about the decision whether or not to hedge interest rate risk. If the graph is not clear enough, table 9 certainly is. For statistical confirmation of these numbers, table 10 depicts the F-values of the difference in variability of the funding ratio during the sample period between the unhedged FR and the hedged FRs. The rejection region of the hypothesis that hedging makes no difference is the area above the F-bound of 1.19.[38]

Table 10: F-values Hedging vs not hedging

Hedge type	Euribor discounted Single swap	Euribor discounted Bucket hedge
Unhedged FR	15.46*	103.14*

The conclusion is unambiguous, hedging makes a huge difference, which is not surprising in a stress test.[39]

Remember however, that in this graph the present value of the swaps is calculated using Euribor discounting, which is deemed inappropriate given the collateralization of all hedge instruments. Therefore, the next subsection shows the difference between Euribor and Eonia discounting. The latter is also referred to as CSA discounting, as the reason that this discount rate should be used is the fact

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[37] The 40 year rate is not shown here to retain overview of the lines in the graph. It behaves like the 50 year rate.
[38] It concerns a two sided F-test with 95% confidence level. All samples have 501 data points, being the business days on which the FR is calculated. We then get $F_{0.025,501,501} = 1.19$
[39] Significant F-values will be indicated by an asterisk (*)

that the Credit Support Annex specifies that collateral should be used, and of which type. (Here: EUR cash)

6.2.2. Hedging with OIS discounted swaps

When the exact same derivatives are used with the notionals from table 8, but now discounted with Eonia, the picture changes. The unhedged case is left out as it is clear that hedging is preferable. This allows for a change in the y-axis to a 108% minimum, as all hedges keep the funding ratio above this level. What is shown in graph 18 is are the same hedges as in graph 16 but then zoomed in. The colors are kept the same; the red line the Euribor discounted bucket hedged FR and the green line being its single swap hedged counterpart. The two added lines are the exact same hedged funding ratios, but the swaps are discounted with Eonia instead of Euribor. Recall that the assets and liabilities are still discounted using the Euribor curve.

Graph 18: Funding ratios with a single hedge or a bucket hedge, discounted with two different curves
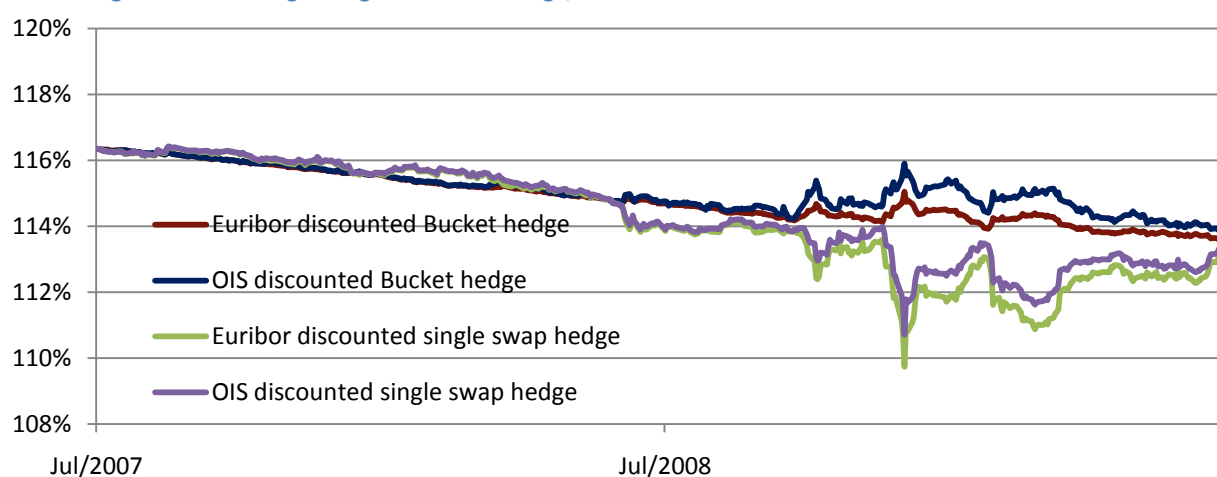
As the hedges have a positive market value, discounting with the lower OIS curve results in higher value for the swaps and thus a higher funding ratio. This does not say anything about the quality of the hedge, only about the impact of using another discount curve. This is confirmed by the statistics in table 11, indicating that the difference is not only theoretically wrong but also statistically significant.

Table 11: FR volatilities with Euribor and OIS discounted hedges, and the F-values of the differences

Hedge type	Euribor discounted Single swap	Euribor discounted Bucket hedge	OIS discounted Single swap	OIS discounted Bucket hedge
FR volatility	1.63%	0.78%	1.42%	0.63%
Euribor discounting vs OIS discounting F-values				
Single swap			**Bucket**	
1.32*			1.50*	

For both the single swap and bucket hedge case, there is a significant difference in FR volatility due to different swap valuation. Analysis shows that at the maximum, there is an over 38 million euro

difference in valuation of the bucket swap portfolio, about six percent of the total value of the swaps at that time. This corresponds to almost one percent of assets resulting in a difference in funding ratio of about one percent. This one percent is clearly visible in graph 18. Note however that this one percent understates the potential impact of the discounting for separate swaps. The swap portfolio includes multiple instruments including payer swaps with a negative value, compensating part of the impact. If the 50 year swap is singled out at the maximum 50 year Euribor zero – 50 year Eonia zero spread, which is shortly after the Lehmann Brothers bankruptcy event, there is a valuation difference of almost 18% in using the two different curves for discounting. This illustrates the need for using the appropriate discount curve.

The next subsection will examine the double curve approach as described in the valuation and sensitivity sections. This means that the deltas of the swaps are determined to both curves instead of one. The results will be compared to the best performing single curve hedge; the OIS discounted bucket hedge.

6.3. Using two curves to determine the hedge

For the hedges used in this section, the deltas of the swaps are derived to both the forward curve and the discount curve, following equations (24) and (25) from the sensitivity section. The OIS discounted bucket hedge from the previous subsection will be compared with two extensions in an attempt to improve performance. The first extension considers Euribor par swap deltas that are determined while taking the separate discount curve into account. The second extension will be treated in the next subsection, the first extension treated here will be referred to as the two-curve adjusted bucket hedge.

Recall from the sensitivity section that when the delta is derived to the two underlying curves separately, the instruments have sensitivity to all rates with an equal or lower maturity. For example, in the single curve case, a 20 year par swap only has sensitivity to the 20 year swap rate. In the two-curve case, the 20 year swap has exposure mainly to the 20 year rate, but to all the lower maturity rates also. On top of that, due to the smooth interpolation technique used to build the curve, the 20 year swap also has sensitivity to the 25 year swap rate, albeit very small. The solution, then, is found by solving a system of linear equations. This is not shown here, but to illustrate the difference with the single curve case, the unit deltas are provided in the column next to the single curve unit deltas in table 7. The next two columns show what the effect is on the notionals of the swaps that are used for hedging. The last column shows the relative difference in notionals.

Table 12: Hedge notional calculation using two different methods, but both with the goal of hedging the balance sheet deltas in the second column. The single curve method uses the swap deltas calculated with Euribor as discount curve, the double curve method takes the separate OIS discount curve into account when calculating the unit deltas of the swaps. The differences in unit deltas translate into different notionals.

Bucket	Delta A&L	Single curve Euribor instrument deltas	Two-curve Euribor instrument deltas	Single curve Euribor swap notionals	Two-curve adjusted Euribor swap notionals	Double notionals / Single notionals
0-2	18,362	-0.000186848	-0.000184645	-85,415,526	-82,463,751	96.54%
3-7	272,777	-0.000436473	-0.000433454	-598,567,490	-585,696,797	97.85%
8-14	14,316	-0.000781453	-0.000782364	-20,794,982	-26,702,983	128.41%
15-25	-345,479	-0.001261082	-0.001271362	182,082,291	150,816,258	82.83%
26-35	-532,402	-0.001556815	-0.001567551	167,052,154	161,037,837	96.40%
36-45	-2,158,585	-0.001744951	-0.001709394	1,059,592,126	1,065,815,297	100.59%
46+	-922,872	-0.001868351	-0.001709394	408,981,098	440,012,485	107.59%
Total				**1,112,929,670**	**1,122,818,346**	**100.89%**
Single swap	-3,653,883	**(30y)** -0.001556815	**(30y)** -0.001567551	2,347,024,335	2,330,950,740	99.32%

The last column is actually the most important one, it tells how big the difference in notional of the hedging instruments is. What will be done now is check whether these differences in hedge notionals deliver a better hedge over the sample period.

The effect of this adjustment is shown in graph 18. Note that we tightened the y-axis of the graph again, because no OIS-discounted bucket hedge falls below 110% during the sample period. We have taken the best strategy from graph 18, being the single-curve OIS discounted bucket hedge, and compared this with the two-curve adjusted bucket hedge.
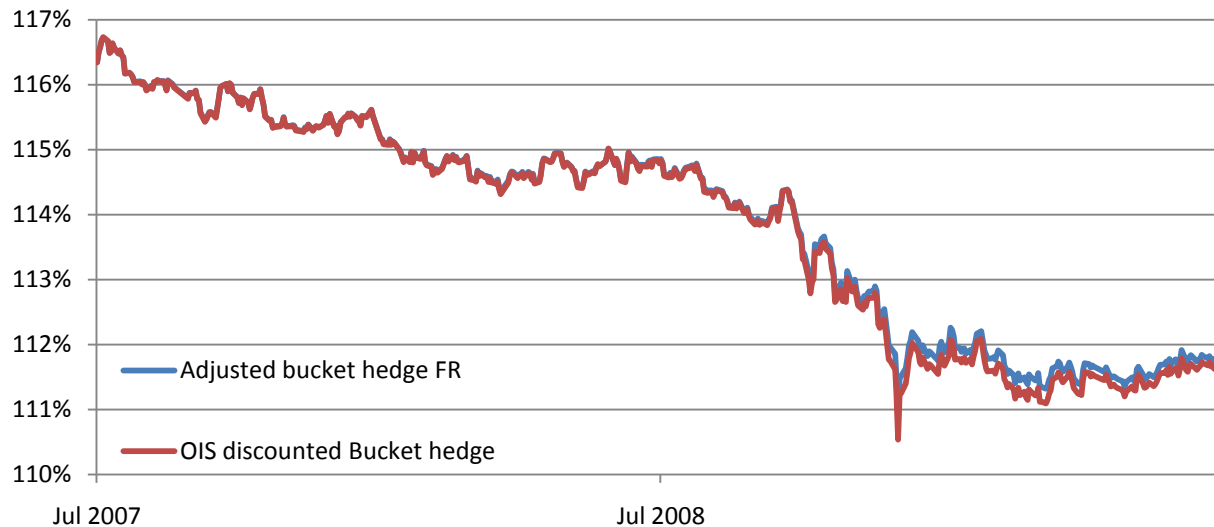
The adjusted hedge seems to do better when the interest rate shocks get large at the end of 2008, mostly due to the higher notional in the 50 year swap. Note however, that even though the separate, discount curve is taken into account when calculating the swap deltas, the OIS deltas have been left open. The next section hedges this residual exposure with Eonia swaps and analyzes the performance.

6.3.1. **Double curve hedging**

Now, the first extension is extended even further by adding OI swaps in order to take away the Eonia exposure. This means that we take the two-curve adjusted bucket hedge from the previous subsection and attempt to take away the OIS exposure resulting from the collateralized Euribor swaps. This extension will be referred to as the double curve hedge; which it is as it hedges the deltas to both relevant curves.

The assets and liabilities are still discounted using the Euribor curve and do not have any Eonia exposure. Table 13 then shows the Eonia deltas resulting from the initial Euribor hedge and how this translates into OI swap notionals.

Table 13: OIS hedge notional calculation. The Euribor swaps have exposure to the OIS curve which has to hedged with OI swaps. The highest two buckets are empty because the highest OI swap being traded is the 30 year instrument, so the OI deltas from the 40 and 50 year Euribor swaps are added to the 26-35 year bucket.

Bucket	Two-curve Euribor swap notionals	Two-curve Eonia deltas	OI swap unit deltas	Two-curve OI swap notionals
0-2	-94,187,156	-81	-0.000187302	432,161
3-7	-606,616,125	-188	-0.000438118	430,027
8-14	-25,123,427	19,505	-0.000785742	-24,823,549
15-25	235,099,683	65,804	-0.001273287	-51,680,504
26-35	334,740,268	-153,436	-0.001575864	97,366,380
36-45	1,243,822,280			
45+	531,423,471			

Note that all the OI exposure of the highest three buckets is gathered in the 26 to 35 year bucket, because the 30 year OI swap was the highest traded during the sample period. Also recall that the addition of Eonia swaps is difficult to achieve for high notionals due to lack of liquidity. The notional of almost a 100 million in the 30 year OI swap is unlikely to be easily tradable. For the sake of curiosity whether this extension is useful, we continue as if the trade can be executed without problems. When the Eonia bucket hedge from table 13 is added to the existing hedge, graph 20 results:

Graph 20: Double curve hedge vs two-curve adjusted hedge
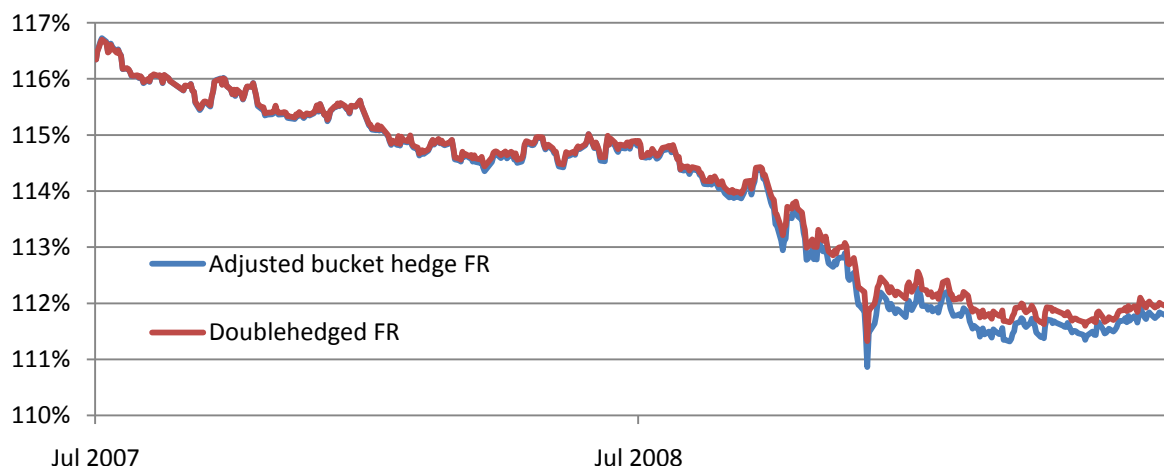
Again, the more complicated hedge performs slightly better in terms of keeping the FR from falling.

Even though this hedge may not be executable in practice at this time, possibly in the future it will be. Table 14 shows that it may be worthwhile, as the double curve hedge performs significantly better than the normal OIS discounted bucket hedge.

Table 14: FR volatilities under different hedge strategies, and the resulting F-statistics of the differences in variance.

Hedge type	OIS Discounted Bucket hedge	OIS Discounted two-curve adjusted hedge	Double curve hedge
FR volatility	0.63%	0.60%	0.57%
	OIS discounted bucket hedge	**OIS Discounted two-curve adjusted hedge**	**double curve**
OIS discounted bucket hedge	1		
OIS Discounted two-curve adjusted hedge	1.11	1	
double curve	1.24*	1.12	1

The adjusted notionals, however, do not by itself make a significant difference. Also, the addition of OI swaps over the adjusted hedge does not have a significant effect. Taking into account that liquidity constraints will make it more expensive (if even possible) to obtain the high maturity OI swaps discards this method as fruitful at this time. Also keep in mind that the F-bound should possibly be somewhat higher than 1.19 to correct for autocorrelation, further reducing the feasibility of this strategy.

6.4. Upswing scenario

As a back test for the hedge quality, an opposite scenario is also analyzed. A large increase in the interest rates is used to test robustness of the results. The scenario chosen is reversed to the original one, while the initial curves are equal to that of the downward scenario. This means that the hedges, which are based on the curves at the start date, are the same as before. Graph 21 shows the six month rates for both curves, to indicate the scenario that the hedges have to cope with.

Graph 21: Simulated 6m Euribor and 6m Eonia over the sample period
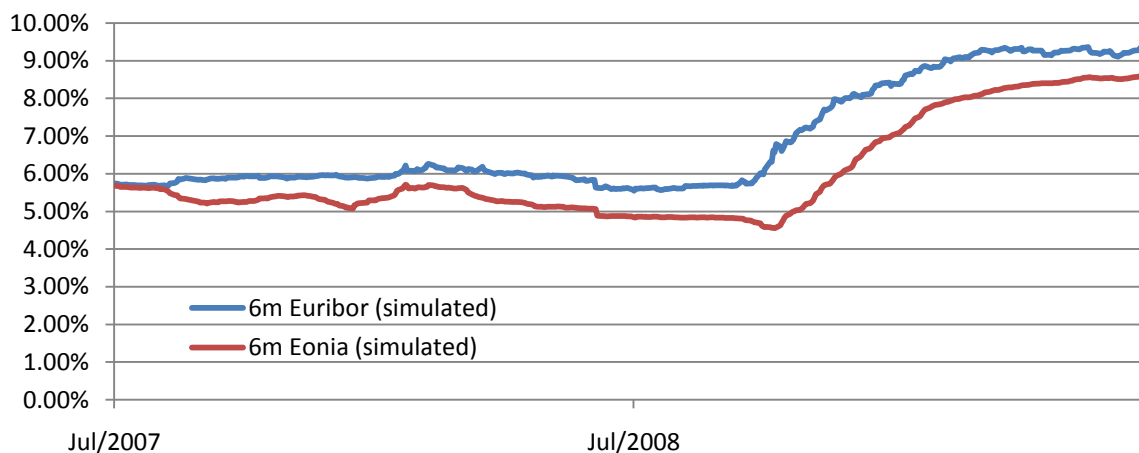

This scenario is usually not bad for a pension fund, since the value of the liabilities will go down in this scenario. As we are using linear products to hedge, the effect of all hedge strategies should be the opposite compared with the original scenario. Funding ratio volatility is still the most important determinant, but it is also analyzed how much 'damage' the hedge does in this circumstance. We will show a summarized report below. Since we have already established the right way to discount, only the OIS discounted hedges will be shown.

6.4.1. Upswing scenario analysis

Graph 22 shows the obvious conclusion that not hedging is best here, given the interest rate upswing. It is clear that the volatility of the FR averaged over the two scenarios is very high if no hedge is put in place. The hedged FRs barely deviate from their path in the other scenario in graph 17. In other words, volatility over the two extreme scenarios is low, while the unhedged FR again shows large peeks and bumps. Note that the downward trend of the funding ratios[40] is greater than in the downward interest rate scenario. This makes sense; the equity is still constant while the discounting effect has a larger impact on the present values of the liabilities and the fixed income portfolio when interest rates are higher. This is the problem discussed in section 5.2.5, but has no consequences for the conclusion that the bucket hedge performs best.

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[40] The unhedged scenario inherits the same trend, but this is not visible due to its high volatility.
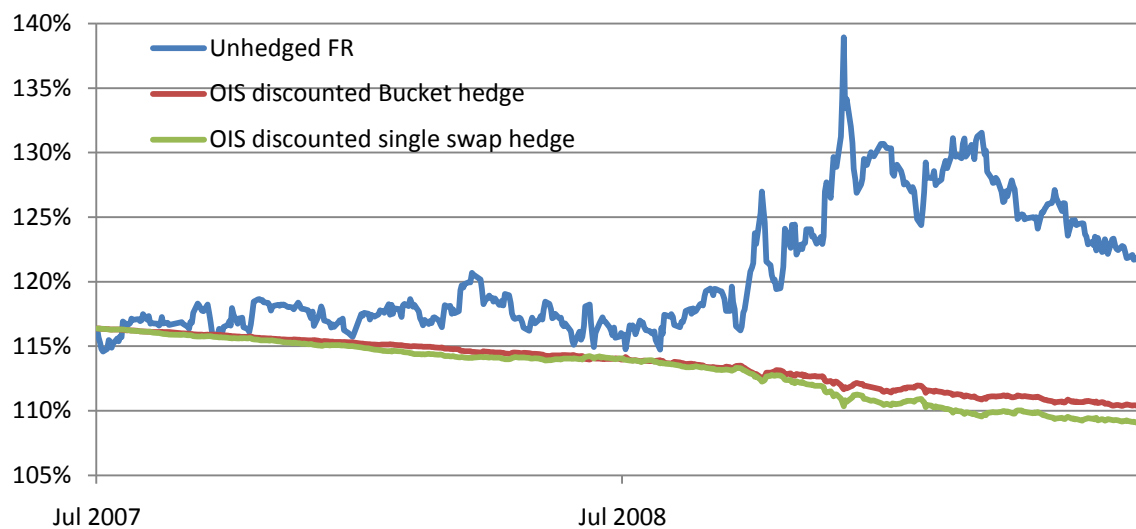
Again, the bucket hedge outperforms the single hedge, confirming that the curve does make non-parallel movements and that this can be accounted for with a bucket hedge. Of course, there are scenarios where the 30 year rate would move much more than the other rates so that the single swap hedged FR would do better, but that does not disregard the conclusion about the volatility in table 15. The bucket hedge still outperforms its simpler opponent with a significant difference.

Table 15: Upswing FR volatilities

Hedge type	OIS discounted Single swap hedge	OIS discounted Bucket hedge
FR volatility	1.42%	0.63%
Single swap hedge vs Bucket hedge, both OIS discounted		
F-value	1.49*	

The more complicated strategies are not shown here, because the variances barely deviate from the OIS discounted bucket hedge strategy. The adjusted notionals and the addition of OI swaps slightly harms in this case in the sense that the market value of the swap portfolio is even more negative than for the OIS discounted bucket hedge. This results in a somewhat higher FR volatility. Another test with inverted initial curves yielded differences so small that the variances ratios approach one and are thus far from significant. This does not discard the performance in the downward scenario, but does question the use of a much more complicated strategy which adds very little performance, if any. We can conclude that the bucket hedge holds up as outperforming the single swap hedges while the more complicated strategies do not conclusively stand with performance different from a single curve OIS discounted bucket hedge.

7. Conclusion

7.1. What is the impact of collateralization on swap curves and their users?

Derivative contracts are increasingly being collateralized to prevent shifting market risk to credit risk when hedging financial risks using derivatives. This has impact on the valuation of these now riskless contracts because the discount rate of the embedded future cash flows is no longer extracted from the swap curve but is determined by the growth rate of the underlying collateral specified in the CSA. For cash and high-quality bond collateral the discount rate can be extracted from the OIS curve which is more appropriate for the risk profile of collateralized cash flows. This has the consequence that swap instruments are no longer viewed as the single determinant of the swap market, because market wide collateralization has forced the Euribor swap markets' discounting to be dependent on another class of instruments which no longer behaves on a predictable spread. The swap curve that is left now has different dynamics than before and should therefore be interpreted accordingly by its users, being effectively the whole financial sector and treasury departments. This thesis attempts to hand them the tools required for the current reality in this field.

The change in swap markets extends to the discounting of all future cash flows because swaps are the main instruments for building zero curves used for discounting. The impact of collateralization of swap contracts combined with an increased Euribor-OIS spread is therefore a market wide shift to a multiple curve framework. Secured cash flows should be discounted with the OIS curve while the swap curve is still the base for unsecured discounting but with different dynamics than before.

Not only is it theoretically the right method to discount collateralized derivatives with the overnight interest rate curve, it has now also empirically been established that wrong discounting can have negative consequences on the hedge effectiveness. After deriving the deltas to two separate curves, the impact is quantified for the case of a pension fund. It turns out that the valuation difference due to using OIS discounting of the hedge instruments can be as much as one full percent of the reported funding ratio. Given that these funds as well as insurance companies are regulated on this measure of solvency, the need for proper accounting cannot be understated.

As for the different hedging strategies, bucket hedging clearly outperforms a single swap hedge by weakening the unrealistic assumption of parallel yield curve movements and successfully acting according to this new situation. The more complicated approaches using two curve deltas look promising from a theoretic viewpoint, but require more research and feasibility in the sense of liquid OIS markets. If these are to develop, the hedge efficiency could be improved with the help of new instruments such as Euribor-Eonia basis swaps. This is left to the future.

7.2. Limitations & extensions

Some assumptions made tighten the generalizability of the methodology but not of the results. Assuming a tight CSA which only allows for Euro cash collateral seems very strict, but in the case of bond collateral the repo rate is often not far from the OIS rate, as noted in section 2.4.1. Also, specifying the Euro as only currency was necessary to prevent optionality from entering the swap valuation, but as discussed in the section on CSA optionality, this is a market wide problem. The results are expected to hold under other currencies, but the methodology will have to be adjusted to account for other leg maturities which are the standard in the USD and GBP market. These changes are mere technicalities, however.

A real limitation is the assumption that the forward curves can simply be derived from the Euribor curve, which is not as simple in the real, collateralized world. In this world, formula (2) will not hold, and the forward curve should be derived as in section 3.3, or from other products such as FRAs. Applying this would greatly increase complexity while it is not expected to distort the results. For research on forward curves, the reader should follow Mercurcio (2009), whereas this thesis has focused on the discount curves.

One could argue that assuming a closed pension fund is a limitation, but consider then that there are many such funds in the world. Of course, extending this work to an open fund is possible, but then an assumption like constant in and outflow of participants and cash flows is probably required. Fact is that a one-size-fits-all approach is not optimal; every fund should be analyzed first before hedging. There are, however, general principles that should be applied and that is exactly what this thesis is trying to show.

Making the equity a fixed number over the whole sample period is of course not realistic, but the goal was to minimize the impact unrelated factors on the FR volatility. Since adding a constant to every value in a time series does not change the variance of the time series, this seemed the best way to prevent noise from equity movements. Of course, the absolute funding ratios at the midst of the crisis would be lower if the equity was booked against market value. However, this would completely distort the interest rate effects we are trying to extract. The results should therefore not be looked at in the sense of absolute FR declines, but purely comparing different strategies under the same circumstances. I believe that has been achieved.

Even though outside distortions have been minimized, the sample is not entirely free of bias. Given that interest rates are the basis for the results, the fact that they are not free of serial correlation puts pressure on the reported F-values. Of course, some reported variance ratios leave little doubt, but the F-bounds are probably wider than is reported. This should be taken into account when judging feasibility. For the future, these problems could be taken away in an ALM analysis, where the interest rates are the independent variable. Averaging over 1000 scenarios should solve the problem and provide more insight in hedge behavior over different curves. This has not been performed here because the focus has been on stress testing, not on ALM.

7.3. <u>Recognitions</u>

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9. Appendix

Table 16: Bloomberg codes of data used. Data used from 1-7-2007 to 30-6-2009

Maturity	Euribor	EONIA
1d[41]		EONIA Index
6m	EUR006M Index	EUSWEF Curncy
1y	EUSA1 Index	EUSWE1 Curncy
2y	EUSA2 Curncy	EUSWE2 Curncy
3y	EUSA3 Curncy	EUSWE3 Curncy
4y	EUSA4 Curncy	EUSWE4 Curncy
5y	EUSA5 Curncy	EUSWE5 Curncy
6y	EUSA6 Curncy	EUSWE6 Curncy
7y	EUSA7 Curncy	EUSWE7 Curncy
8y	EUSA8 Curncy	EUSWE8 Curncy
9y	EUSA9 Curncy	EUSWE9 Curncy
10y	EUSA10 Curncy	EUSWE10 Curncy
11y	EUSA11 Curncy	EUSWE11 Curncy
12y	EUSA12 Curncy	EUSWE12 Curncy
15y	EUSA15 Curncy	EUSWE15 Curncy
20y	EUSA20 Curncy	EUSWE20 Curncy
25y	EUSA25 Curncy	EUSWE25 Curncy
30y	EUSA30 Curncy	EUSWE30 Curncy
35y	EUSA35 Curncy	
40y	EUSA40 Curncy	
50y	EUSA50 Curncy	

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[41] Not used to build a curve, only for calculating daily cash account accrual.

Table 17: Correlation matrix Euribor instruments over sample period

	50y	40y	35y	30y	25y	20y	15y	12y	11y	10y	9y	8y	7y	6y	5y	4y	3y	2y	1y	6m
6m	0.798	0.801	0.799	0.800	0.814	0.846	0.890	0.910	0.915	0.918	0.920	0.921	0.923	0.928	0.934	0.941	0.948	0.958	0.982	1.000
1y	0.864	0.869	0.868	0.871	0.883	0.909	0.939	0.954	0.957	0.960	0.962	0.964	0.966	0.970	0.975	0.981	0.986	0.993	1.000	
2y	0.881	0.886	0.887	0.891	0.904	0.930	0.958	0.971	0.975	0.978	0.980	0.982	0.985	0.988	0.992	0.995	0.998	1.000		
3y	0.884	0.890	0.891	0.895	0.909	0.937	0.966	0.979	0.982	0.985	0.988	0.990	0.992	0.995	0.997	0.999	1.000			
4y	0.887	0.893	0.894	0.899	0.914	0.942	0.971	0.983	0.987	0.990	0.992	0.994	0.996	0.998	0.999	1.000				
5y	0.892	0.899	0.901	0.906	0.920	0.948	0.976	0.988	0.991	0.993	0.995	0.997	0.998	0.999	1.000					
6y	0.899	0.905	0.907	0.913	0.927	0.954	0.981	0.991	0.994	0.996	0.997	0.999	0.999	1.000						
7y	0.905	0.912	0.914	0.919	0.933	0.960	0.985	0.994	0.996	0.998	0.999	1.000	1.000							
8y	0.909	0.915	0.918	0.923	0.937	0.963	0.987	0.996	0.997	0.999	1.000	1.000								
9y	0.913	0.920	0.922	0.928	0.941	0.967	0.990	0.997	0.999	1.000	1.000									
10y	0.918	0.924	0.926	0.932	0.945	0.970	0.992	0.998	0.999	1.000										
11y	0.923	0.929	0.931	0.937	0.950	0.974	0.995	1.000	1.000											
12y	0.928	0.934	0.936	0.942	0.955	0.979	0.997	1.000												
15y	0.947	0.953	0.955	0.960	0.972	0.990	1.000													
20y	0.978	0.982	0.984	0.988	0.994	1.000														
25y	0.993	0.996	0.997	0.999	1.000															
30y	**0.997**	0.999	1.000	1.000																
35y	0.999	1.000	1.000																	
40y	**0.999**	1.000																		
50y	1.000																			