Pricing Guarantee Option Contracts in a Monte Carlo Simulation Framework

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Abstract

This thesis focuses on the valuation of guarantee option contracts of two life insurance products (LOGA and Levensloop Rendement) by using Monte Carlo simulation procedures. The option contracts are in-the-money for a specific client if this client is alive and the underlying asset portfolio has not reached the guarantee value at the specified end date.

In this research project option values are derived in a Black-Scholes model, where asset prices are simulated, and in a Hull-White Black-Scholes model, where the short term interest rate is simulated as well. Since simulation time becomes a constraint for large client portfolios, I check whether the overall simulation time can be reduced by aggregating client portfolios, by using antithetic variables, and by simulating under another measure. Furthermore I check to what input parameters the option values of both individuals dummy clients and the entire portfolio are most sensitive. Finally, I propose an alternative life-cycle investment mix for the Levensloop Rendement product.

Within the Black-Scholes setting it is derived that in case of a low volatility level and a large interest rate parameter the option value is lowest for the LOGA product. The analysis in this setting for the Levensloop Rendement product is only based on the interest rate parameter. It shows low option values for high interest rate levels. Aggregation of client portfolios is investigated in the same setting. It turns out to be effective for the LOGA portfolio, but highly inaccurate for the Levensloop Rendement product. Antithetic variates, however, work well for both products in this setting.

In the Hull-White Black-Scholes model, results for the option values are derived as well. For the entire client portfolios these values are especially sensitive to interest rate related parameters. In the context of variance reduction, antithetic variables still work in this setting, but less effectively. The T-forward measure (instead of the risk-neutral measure), however, turns out to be extremely effective for dummies having a contract with a long time to maturity. These dummies also indicate that the option value as well as its sensitivities are very client related. By implementing a more defensive life-cycle investment strategy in the Levensloop Rendement product very different results are derived in comparison with the values of the original mix. The option value is lowered a lot, while it is in general relatively more sensitive to changes in the underlying parameter values.
Preface

This master thesis brings my beloved student life to an end and initiates my working career. Insurance company Loyalis N.V. gave me the opportunity to do an internship in the field of option pricing. I was able to extend my knowledge of insurance products and financial modeling and combine these aspects in my research project.

Therefore I firstly want to thank my colleagues of Loyalis N.V. for their time and effort to help me understand the insurance products in question, to check my computer program files, and to provide the relevant data. Furthermore I would like to thank my colleagues of the actuarial department for creating a very pleasant working atmosphere. I would especially like to thank supervisor Ramon van Oppen, who managed to understand even my most extensive programming files and came up with complementary insights to evaluate the subject, and Roel Cuypers who assessed my progress critically.

A special thank goes to Mr. Schumacher, my university supervisor, as well. I thank him for his patience to answer all my questions and reviewing my concept versions regularly. His ideas about modeling investment portfolios and structuring my thesis were definitely of added value to my thesis.

Last but not least, I would like to thank my friends and family for supporting me during the last four years. I really appreciate their constant belief in me. I thank them for visiting me even when I moved to the very south of the country. Finally, I want to thank my roommates who prepared my dinners daily, in order to provide me with vitamines during my unhealthy student life.

At this point, I have thanked the persons who deserve it in particular. What is left, is wishing you a pleasant time reading my master thesis!

Roel van Buul
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1 Introduction

Due to the latest developments in financial regulation, it becomes more and more important for pension funds and insurance companies to valuate their products properly. In fact, according to the Solvency II legislation, financial products should be valuated in a stochastic setting evolving over multiple time periods. Especially option contracts, for which the option value depends on financial results, demand such a valuation.

Insurance company Loyalis N.V. offers two products (LOGA and Levensloop Rendement) for which the evaluation of the embedded option values is of particular interest. Both products offer a guarantee value in case the investment returns do not provide an amount larger than this guarantee at the contract’s end date if the policy holder is still alive. Subject to the investment strategy, the insurance company gives away an option value. This value has to be determined for each individual contract separately, since a client’s option value is subject to his/her individual characteristics (e.g. current age, end date of the contract, gender, amount of premium payments etc.).

The main goal of this research project is to derive a value for the option of both products total client portfolios. Acquiring these numbers provides valuable information for a company’s balance sheet, while it is of minor importance from a risk management point of view. Therefore the second goal of this thesis is to analyze to what specific risks the total option values as well as the individual option values are in particular vulnerable. For the Levensloop Rendement product, which invests in a life-cycle mix, an alternative mix is proposed.

The analysis starts with a risk-neutral Black-Scholes model in which both interest rate and volatility are assumed to be constant. Later, the assumption of a constant interest rate is relaxed and a one-factor Hull-White short term interest rate model is implemented. The result is a so-called Hull-White Black-Scholes model. Due to the complex product construction an analytical solution is not apparent. Therefore, I resort to the more flexible but time consuming Monte Carlo simulation methods. These methods, however, allow for variance reduction techniques, which are meant to provide smaller confidence intervals of the value of interest in case the same amount of time is invested. The antithetic variates technique is used in this paper.

In the Hull-White Black-Scholes setting sensitivities with respect to input parameters are calculated for both products. These sensitivities are computed for client dummies with different end dates and financial characteristics as well. Since the client specific input for these dummies is based on real data, the results of these individuals indicate where the total risk of the portfolio in particular comes from. For these dummy clients simulation under the T-forward measure is performed as well.

Section 2 introduces the Black-Scholes World. Along with Section 3, which presents the fundamental information of interest rates and introduces the Hull-White model, this forms the basis for the more advanced Hull-White Black-Scholes model. In Section 4 this model is analyzed under both the risk-neutral measure and the T-forward measure. Together, these three sections represent the theory underlying this simulation study.

The most important aspects of the LOGA and Levensloop Rendement products are de-
scribed in Section 5. It forms the basis for Section 6. In this section the methods of deriving guarantee values at the end date of each contract are presented. In combination with Section 7, where the main market input parameters are introduced, and Section 8, where the Hull-White parameters are calibrated to market data, this section provides all the relevant data needed for the simulation.

Section 9 gives a detailed description of every simulation design which is evaluated. Furthermore it introduces the individual client dummies and it presents an alternative life-cycle mix. The results of all suggested simulation schemes are presented in Section 10, after which Section 11 concludes.
2 Black-Scholes World

One of the easiest and best-known procedures to value options is the Black-Scholes option pricing formula. Under strong and ideal conditions a closed-form formula for call and put options is available. Unfortunately, these assumptions are highly unrealistic and therefore this type of modelling is often considered as a useful benchmark rather than a realistic option valuation procedure. This section provides an introduction to the Black-Scholes option pricing framework.

The assumptions on the market and the underlying stock underpinning the Black-Scholes pricing equation are stated as follows by Black and Scholes (1973):

- The short term interest rate is known and constant through time.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate on the stock is constant.
- The stock pays no dividends or other distributions.
- The option is "European," that is, it can only be exercised at maturity.
- There are no transaction costs in buying or selling the stock or the option.
- It is possible to borrow any fraction of the price of a security to buy it or hold it, at the short term interest rate.
- There are no penalties to short-selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

Some of the assumptions above are more realistic than others. In this research project the following aspects are of main interest: the short term interest rate and the volatility parameter. The assumptions on the interest rate being constant will be relaxed. Furthermore, it is of crucial importance for my thesis that the option is European in both the Black-Scholes model and the option contract.

The input for the Black-Scholes option pricing formula is: the current stock price, the exercise price, the time to maturity, the variance rate of the return, and the interest rate. A call option price $C(S,t)$ can be calculated using the following formula:

$$C(S,t) = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$
Where $S_0$ is the spot price of the underlying asset, $\Phi(.)$ the cumulative normal density function, $K$ the strike price, $r$ the risk-free interest rate, $T$ the maturity time, and $\sigma_s$ the volatility of the returns of the underlying. This formula is easily converted to the price of a put option $P(S,t)$:

$$P(S,t) = Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

(2.4)

The guarantee option in both the LOGA and the Levensloop Rendement contracts has the form of a European put option, since it pays off in bad economic scenarios at the maturity date of the contract. Due to the future premium payments and the guarantee option based on the total of premium payments (not the premium payments separately), however, a closed-form formula based on the Black-Scholes option pricing formula is not readily available.

This does not mean that the Black-Scholes framework is irrelevant for the scope of this paper. One can for example choose to simulate under Black-Scholes assumptions. This is done in both the LOGA and the Levensloop Rendement setting. In the continuous-time Black-Scholes world the stock and bond price ($M_t$) evolve under the following stochastic differential equations, respectively:

$$dS_t = \mu S_t dt + \sigma_s S_t dW_t$$

(2.5)

$$dM_t = r M_t dt$$

(2.6)

The symbol $\mu$ denotes the drift term of the stock prices under the real world measure $\mathbb{P}$, $\sigma_s$ is its volatility, and $W_t$ is a Brownian motion (for the definition, please take a look in Appendix A.1) under $\mathbb{P}$. Now in case the bond is chosen as a numéraire (definition in Appendix A.2), the following stochastic differential equation appears:

$$d\frac{S_t}{M_t} = -\frac{S_t}{M_t} dM_t + \frac{1}{M_t} dS_t$$

$$= (\mu - r) \frac{S_t}{M_t} dt + \sigma_s \frac{S_t}{M_t} dW_t$$

(2.7)

Under the risk neutral measure the relative price process $\frac{S_t}{M_t}$ (2.7) is a martingale (see Appendix A.3). In order to evaluate this equation under an equivalent risk neutral measure, I apply Girsanov’s theorem (Appendix A.4).

$$d\frac{S_t}{M_t} = (\mu - r) \frac{S_t}{M_t} dt + \sigma_s \frac{S_t}{M_t} (d\tilde{W}_t - \lambda dt)$$

$$= (\mu - r - \lambda \sigma_s) \frac{S_t}{M_t} dt + \sigma_s \frac{S_t}{M_t} d\tilde{W}_t$$

(2.8)

If $\lambda$ is chosen such that $\lambda = \frac{\mu - r}{\sigma_s}$, then the equation (2.8) results in
By taking the expectation under the risk neutral measure, one is able to conclude that this equation is indeed a martingale. This conclusion results from the absence of a drift term in equation (2.9) and the fact that $\tilde{W}_t$ is a Brownian motion under the risk neutral measure.

Summarizing the analysis of this section leads to the evolution of the stock price under the risk neutral measure as defined by equation (2.10).

\[
dS_t = rS_t dt + \sigma_s S_t dW_t \tag{2.10}
\]

This establishes the basis for the analysis of the option value under Black-Scholes assumptions. One should note the importance of a correct specification of the interest rate $r$, on which the next section elaborates.
3 Modeling Interest Rates

In the Hull-White one factor model the short term interest rate dynamics are modeled. It can be characterized as an extension of the pioneering Vasicek short rate model in which Hull and White implemented the currently observed Yield curve. This section provides an indication of the use of interest rate models as well as a description of the Hull-White one factor model.

Section 2 ended by indicating the importance of the interest rate in the world of risk neutral pricing. The assumption of a constant interest rate as in the Black-Scholes world is not observed in the market and is therefore restrictive. This, however, is not the only modeling aspect of this research project in which the specification of an adequate interest rate is crucial. Since clients holding life insurance policies tend to have an end date in the far future, proper discounting to the current level is also of main importance. Section 3.1 provides the fundamental background in interest rates, while Section 3.2 discusses simple interest rate models which form the basic idea of the Hull-White one factor model. Section 3.3 elaborates on the Hull-White one factor model as such.

3.1 Background on Interest Rates

Let the yield from the current moment in time \((t = 0)\) until future time point \(t\) be denoted as \(y_t\), then a zero-coupon bond price which matures at time \(t\) can be calculated with:

\[
P(0,t) = e^{-ty_t}
\] (3.1)

Furthermore, one is able to determine forward interest rates known at time 0 from time point \(t_1\) to \(t_2\) using the following formula:

\[
f_0(t_1,t_2) = \frac{y_{t_2}t_2 - y_{t_1}t_1}{t_2 - t_1} = \frac{\log(P(0,t_1)) - \log(P(0,t_2))}{t_2 - t_1}
\] (3.2)

Let \(t_1 \uparrow t_2\), the resulting equation denotes the theoretical forward rate at time \(t_2\) contracted at time 0.

\[
f_0(t_2) = -\frac{\partial}{\partial t} \log(P(0,t_2))
\] (3.3)

In short rate models the spot rate \((r_t = f_t(t))\) is modeled. In the remainder of this article the observed forward rates \(f^*\) are distinguished from the theoretical forward interest rates \(f\). If one is able to specify the spot rates under an arbitrage free measure \((Q)\) at every moment in time in the future, one can derive prices of zero-coupon bonds by the following formula:

\[
P(t_1,t_2) = E_{t_1}^{Q}(e^{-\int_{t_1}^{t_2} r_s ds})
\] (3.4)

This is the price at time \(t_1\) of a zero-coupon bond paying off 1 at time \(t_2\) in the future. Let \(r_s\) denote the spot rate at time \(s\). The possibility to calculate such prices is of added value to the analytical tractability of the model. Two simple models, which form the basic idea behind the Hull-White model used in this paper, are presented in the next section.
3.2 Background on Interest Rate Models

During recent decades a lot of interest rate models have been proposed to describe the evolution of the spot rate. These range from relatively simple models which provide a thorough analytical tractability to very explicit models adapted to for example yield curves in order to link theory and practice. This section discusses two interest rate models.

An example of a simple model with high analytical tractability is developed in Vasicek (1977). In the analysis of Vasicek an Ornstein-Uhlenbeck description of the instantaneous spot rate is proposed. Under the risk neutral measure the short rate evolves as in the following formula:

$$dr_t = a(\theta - r_t)dt + \sigma_r d\tilde{W}_t$$  \hspace{1cm} (3.5)

The constant parameter \(\theta\) can be viewed as the long term mean of the interest rate, while constant \(a\) is the mean reverting parameter which specifies at what pace the short term interest rate converges to its \(\theta\). The volatility parameter is represented by another constant term \(\sigma_r\) (do not confuse this \(\sigma\) with the one in equation (2.5)), and the randomness of the process comes from \(\tilde{W}_t\), which is a Brownian motion under the risk neutral measure \(Q\).

The main advantage of the model is that it provides explicit formulas for the calculation of bond and option prices. This is a fast and efficient way of performing an option analysis. Unfortunately, this simple model suffers from some serious drawbacks (as indicated by for example Brigo and Mercurio (2006)). One of them is the fact that it is not explicitly linked to the term structure, since it is a mean-reverting process to the constant long-term average parameter \(\theta\). Another drawback comes from the fact that the process allows for negative interest rates. Conditional on an information set known at time \(s\) it is derived that \(r(t)\) (for \(t > s\)) follows a normal distribution. This has the consequence of having a positive probability of negative rates.

Ho and Lee were the first to propose a model which incorporates an initially specified yield curve. Hull (2006) indicates, that the continuous-time limit of the model is as follows:

$$dr_t = \theta(t)dt + \sigma_r d\tilde{W}_t$$  \hspace{1cm} (3.6)

In this model the parameter \(\theta(t)\) is time-dependent, which means that it can be chosen to fit the initial term structure of interest rates. At every point in time \(t\) the spot-rate has a drift term equal to \(\theta(t)\). Hull (2006) furthermore indicates, that although the analysis of Ho and Lee included a parameter for the market price of risk, this parameter proves to be irrelevant when the model is used in the context of interest rate derivatives pricing. Note that this result is analogous to the irrelevance of risk preferences in the pricing of equity derivatives.

Nevertheless, even this promising model has one major drawback, due to the lack of a mean-reverting term. To understand why this might be inappropriate, imagine a very high (low) interest rate due to realizations of the Brownian motion. Instead of reverting to a level which match the yield curve better, it in expectation grows parallel to the yield curve. This can
lead to levels of the interest rate which are unrealistically high (low). The next section discusses an interest rate model, which combines the advantages of the Vasicek model and the Ho-Lee model.

### 3.3 Hull-White One Factor Model

To overcome the problem of the Ho and Lee model, Hull and White (1990) decided to add a mean-reverting parameter to the model, and thus they combined the mean-reverting benefits of the Vasicek model with the link to the Yield curve of the Ho and Lee model. The Hull-White one factor model is the subject of this section. The formulas and derivations are based on Boshuizen et al. (2006) and Brigo and Mercurio (2006).

In formula form the Hull-White One Factor model looks like this:

\[
\begin{align*}
dr_t &= (\theta(t) - ar_t)dt + \sigma_r d\tilde{W}_t \\
&= (\theta(t) - ar_t)dt + \sigma_r d\tilde{W}_t
\end{align*}
\]

(3.7)

The parameters \(a\) and \(\sigma_r\) are constant and will be calibrated to market data in section 8. The first one denotes the strength of mean-reversion. Note that taking \(a = 0\) results in the Ho and Lee model discussed before. The parameter \(\sigma_r\) indicates the impact of the randomness due to the Brownian motion in the model. This model can be extended even more, by making for example the \(a\) and/or \(\sigma_r\) parameter time-dependent. Hull and White (1996) argue that although it is appealing to make use of all degrees of freedom offered in a model, it might not be very applicable for its real purpose. This is because in case an extended model is constructed to price for example swaptions, its effectiveness decreases for pricing other instruments. Therefore I decide to hold on to the model given by equation (3.7).

In order to derive an explicit expression for interest rate parameter \(r_s\), I calculate \(d(e^{at}r_t)\) by applying Itô’s formula.

\[
\begin{align*}
d(e^{at}r_t) &= ae^{at}r_t dt + e^{at}dr_t \\
&= ae^{at}r_t dt + e^{at}((\theta(t) - ar_t) dt + \sigma_r d\tilde{W}_t) \\
&= e^{at}\theta(t) dt + \sigma_r d\tilde{W}_t
\end{align*}
\]

(3.8)

By integrating (3.8) a proper expression is derived:

\[
r_s = e^{-as}r_0 + e^{-as}\int_0^s \theta(u)e^{au}du + \sigma_r e^{-as}\int_0^s e^{au}dW_u
\]

(3.9)

Now, I integrate formula (3.9) from \(t_1\) to \(t_2\), where \(t_2 > t_1\). The resulting expression is shown in (3.10). The extensive derivation can be found in Appendix B.1.

\[
\int_{t_1}^{t_2} r_s ds = B(t_1, t_2)r_{t_1} + \int_{t_1}^{t_2} B(u, t_2)\theta(u)du + \sigma_r \int_{t_1}^{t_2} B(u, t_2)dW_u
\]

(3.10)
The function $B(t_1, t_2)$ is defined as follows:

$$B(t_1, t_2) = 1 - e^{-a(t_2 - t_1)}$$  \hspace{1cm} (3.11)

When evaluating the terms on the right hand side of equation (3.10), one can conclude that the first two terms are discrete given the information up to time $t_1$, while the last term is normally distributed. This proves the following theorem already.

**Theorem 3.1.** Within the Hull-White short rate model the integral $\int_{t_1}^{t_2} r_s ds$ given the information up to time $t_1$ follows a normal distribution with mean

$$B(t_1, t_2)r_{t_1} + \int_{t_1}^{t_2} B(u, t_2)\theta(u)du$$

and variance parameter

$$\sigma_r^2 \int_{t_1}^{t_2} B^2(u, t_2)du.$$  

At this point, I am close to an expression for a zero-coupon bond price within the Hull-White model. Lemma 3.1 provides the last information needed to calculate this expression. The proof of this lemma can be found in Appendix B.2.

**Lemma 3.1.** If variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$, then $E(e^X) = e^{\mu + \frac{1}{2}\sigma^2}$.

Finally, all information needed to derive an analytic formula for zero-coupon bond prices is gathered. The following theorem states the formula.

**Theorem 3.2.** In the Hull-White model the price of a zero-coupon bond is given by

$$P(t_1, t_2) = e^{A(t_1, t_2) - B(t_1, t_2)r_{t_1}}$$  \hspace{1cm} (3.12)

Where $A(t_1, t_2)$ is defined as follows:

$$A(t_1, t_2) = \int_{t_1}^{t_2} \left( \frac{1}{2}\sigma_r^2 B^2(u, t_2) - \theta(u)B(u, t_2) \right) du$$  \hspace{1cm} (3.13)

**Proof.** The theorem is proved by calculating the expression in equation (3.4). From Theorem 3.1 it is known that $\int_{t_1}^{t_2} r_s ds$ is normally distributed and Lemma 3.1 provides an expression for $E(e^X)$ for a normally distributed variable $X$. Combining these observations proves the theorem. \(\square\)

Formula (3.12) in Theorem 3.2 shines new light on the theoretical forward rate as in (3.3). This can be seen in equation (3.14).

$$f_0(t) = \frac{\partial B(0,t)}{\partial t} r_t - \frac{\partial A(0,t)}{\partial t}$$  \hspace{1cm} (3.14)

Note that this equation involves the derivative with respect to $t$ of $B(0, t)$ and $A(0, t)$.
The $\theta(t)$ function in equation (3.7) is now chosen to fit the term structure of interest rates. This means that theoretical forward rates are chosen such that they match the observed ones. It also leads to a formula for the calculation of bond prices at every time $t$. The next theorem summarizes these statements.

**Theorem 3.3.** Let the parameters $a$ and $\sigma_r$ of equation (3.7) be given. Then by choosing

\[
\theta(t) = af_0^*(t) + \frac{\partial f_0^*(t)}{\partial t} + \sigma_r^2 B(0,t)(e^{-at} + \frac{1}{2}aB(0,t)) \tag{3.15}
\]

the observed prices match the prices calculated within the Hull-White setting. For a zero-coupon bond, the price is now given by

\[
P(t_1,t_2) = P(0,t_2) P(0,t_1) \exp(B(t_1,t_2)f_0^*(t_1) - \sigma_r^2 \frac{2}{a^2} (1-e^{-at_1}) - B(t_1,t_2)r_{t_1}) \tag{3.16}
\]

**Proof.** The equation to be solved is

\[
f_0^*(t) = f_0(t)
\]

Inserting the functions $B$ and $A$, given by (3.11) and (3.13) respectively, in equation (3.14) leads after differentiating and integrating to the following expression:

\[
f_0^*(t) = e^{-at}r_0 + \int_0^t e^{-a(T-u)}\theta(u)du - \sigma_r^2 \frac{2}{a^2} (1-e^{-at})^2
\]

Function $g$ solves the differential equation $g' + ag = \theta$, $g(0) = r_0$ and $h(t) = \sigma_r^2 \frac{B^2(0,t)}{2}$. This proves that the equation for $\theta$ is solved by

\[
\theta(t) = g'(t) + ag(t) = \frac{\partial f_0^*(t)}{\partial t} + \frac{\partial h(t)}{\partial t} + a(f_0^*(t) + h(t)) \tag{3.17}
\]

This completes the first statement of the theorem.

The proof of the last part of the theorem results from the implementation of the formula of $\theta(t)$ as displayed in equation (3.15) in equation (3.12).

The next theorem states how paths of the short rate can be simulated. Note that this simulation procedure does not involve approximation errors due to discretization. Appendix B.3 proves this theorem.

**Theorem 3.4.** If $r_t$ satisfies equation (3.7) under the risk-neutral measure (with the bank account as numéraire) and function $\theta(t)$ is given by equation (3.15), then the following equations hold:

\[
\alpha(t) = f_0^*(t) + \frac{\sigma_r^2}{2}B^2(0,t) \tag{3.18}
\]

\[
y_{t+\Delta t} = e^{-a\Delta t}y_t + \sqrt{\frac{1}{2}\sigma_r^2 B(0,2\Delta t)}Z_t \tag{3.19}
\]

\[
r_t = \alpha(t) + y_t \tag{3.20}
\]
$y_0 = 0$, $\Delta > 0$, and the $Z_t$ variables are independent and standard normally distributed.
4 Hull-White Black-Scholes Model

The Hull-White Black-Scholes model combines the Hull-White model for interest rates with the Black-Scholes model for stock prices. Although these models have been discussed separately in Section 2 and Section 3, Section 4.1 summarizes these results shortly. The Hull-White Black-Scholes model, however, allows to choose other numéraires as well. Section 4.2 discusses the evolution of interest rates and stock prices if a zero-coupon bond is chosen as a numéraire.

4.1 Risk-Neutral Measure

Since the main results concerning the Hull-White Black-Scholes model have already been derived in the two previous sections, this section mainly serves to present a short overview.

Equation (2.10) shows the evolution of stock prices under the risk-neutral measure if the interest rate is constant over time. This assumption no longer holds in the Hull-White model for interest rates as equation (3.7) indicates. It is easily verified, however, how the asset prices evolve under the bank-account measure, with a stochastic short rate ($r_t$ instead of $r$). If $r$ in (2.6) is replaced by $r_t$ and the expression $dS_t$ is worked out by applying Itô’s formula and Girsanov’s Theorem, one derives the following equation.

$$dS_t = r_t S_t dt + \sigma_s S_t dW_t \quad (4.1)$$

In combination with equation (3.7) the most important dynamics in the risk-neutral world are known. Note that the short rate can be simulated exactly within a Monte Carlo setting by making use of Theorem 3.4.

The main advantage of the risk-neutral measure compared with the T-forward measure, which is discussed in the next section, is that it is more flexible. E.g. to make use of the T-forward measure one has to choose a payoff date for the zero-coupon bond that matches the end-date of the contract.

4.2 T-Forward Measure

The previous section described the dynamics of the interest rate under the Hull-White assumptions when the money market is taken as a numéraire. This, however, is not the only measure which provides exact formulas for the Hull-White interest rate. This section elaborates on the so-called T-forward measure, which offers the dynamics of the interest rate as well as the dynamics of the stock price, in case a zero-coupon bond is taken as a numéraire.

Using another measure in some cases leads to more accurate results, implying more tight confidence intervals. For this reason a change of measure is performed in this thesis. The starting point of this analysis will be the dynamics of the stock, the zero-coupon bond, and the money market account under the money market account measure as in Van Haastrecht et al.
In the equations above, \( S_t \) denotes the stock price at time \( t \), \( W^s_t \) its Brownian motion, \( P_t \) a zero-coupon bond which pays off at time \( T^1 \), \( W^r_t \) is the Brownian motion of the interest rate, and \( M_t \) the money market account. Taking \( P_t \) as a numéraire and applying Itô’s rule leads to the following expressions:

\[
dS_t = r_t S_t dt + \sigma_s S_t dW^s_t \\
dP_t = r_t P_t dt - \sigma_r B(t, T) P_t dW^r_t \\
dM_t = r_t M_t dt
\]

Parameter \( \rho_{rs} \) denotes the correlation coefficient between the Brownian motions of the interest rate equation (3.7) and the Black-Scholes equation (4.1). Now I apply Girsanov’s theorem and choose \( \lambda \) such that the Brownian motions are related as follows:

\[
dW^s_t = -\sigma_r \rho_{rs} B(t, T) dt + d\tilde{W}^s_t \\
dW^r_t = -\sigma_r B(t, T) dt + d\tilde{W}^r_t
\]

In equation (4.8) and (4.9) \( \tilde{W} \) symbolizes a Brownian motion under the new T-forward measure. Therefore the evolution of the stock prices under the T-forward measure can be expressed as follows:

\[
dS_t = S_t (r_t - \sigma_s \sigma_r \rho_{rs} B(t, T)) dt + \sigma_s S_t d\tilde{W}^s_t
\]

The evolution of the stock price are not the only dynamics, which changed due to the switch of measure. Brigo and Mercurio (2006) shows that the evolution of the interest rate under the T-forward measure is very similar to the evolution of the interest rates under the risk-neutral measure. The difference is characterized by an extra term in equation (3.19). The exact

\[\text{Note that when financial products are analyzed, one wants to choose the } T \text{ such that it matches the end date of the product. Therefore it is less straightforward to value a contract with multiple payoff dates. The reason for this is discussed in Section 9.3.}\]
simulation method of the short term interest rate is displayed in the following equations.

\[ \alpha(t) = f_0(t) + \frac{\sigma^2}{2} B^2(0, t) \]  \hspace{1cm} (4.11)

\[ y(t + \Delta t) = e^{-a\Delta t} y_t - M^T(t, t + \Delta t) + \sqrt{\frac{1}{2}\sigma^2} B(0, 2\Delta t) Z_t \]  \hspace{1cm} (4.12)

\[ M^T(t, t + \Delta t) = \frac{\sigma^2}{2a^2} (1 - e^{-a\Delta t}) - \frac{\sigma^2}{2a^2} \left( e^{-a(T-t-\Delta t)} - e^{-a(T-t+\Delta t)} \right) \]  \hspace{1cm} (4.13)

\[ r_t = \alpha(t) + y_t \]  \hspace{1cm} (4.14)

\( y_0 = 0, \Delta t > 0, \) and the \( Z_t \) variables are independent and standard normally distributed. At this point the fundamental ingredients for a Hull-White Black-Scholes analysis under the T-forward measure are gathered. The next section presents the product types I analyze.
5 Product Description

In this section I give a detailed description of the financial products I am analyzing. The following sections provide information about the two products "LOGA" and "Levensloop Rendement" respectively.

5.1 LOGA

The LOGA product offers civil servants working as a fireman or as an ambulance employee a capital insurance. It is meant to fill their financial gap when they stop working at age 59 (or 60) and receive their first state pension income at the age of 62.

Participants pay premiums via their employers. It is assumed (and observed) that participants pay fixed monthly premiums which increase yearly with 2.75%. These premiums are invested in an asset mix, which is determined by the insurance company. At the moment, the investment portfolio consists of 15% equity and the remaining 85% is invested in bonds.

The insurance product only pays out if the participant is alive at the age of 59 or 60, specified by his/her employer. The client can opt for a 90% refund of his/her financial account to his/her heirs in case of death before the end date. Whether or not participants opt for this extra insurance affects the so-called 'Leven Bonus'. This is a bonus on the return on the deposits which rises when people get older. It is based on the fact that as time evolves, more and more people pass away (before reaching their end dates), and their money can be divided over the surviving participants.

The costs involved in this product are a yearly percentage of 0.8% of the deposit account. Although this percentage is expressed as a yearly percentage, it is applied every month. Therefore, the yearly percentage has to be converted to a monthly percentage.

The feature which is most relevant to my research project is the deposit guarantee at the end date of the contract. Clients are guaranteed a yearly return of 3% on the premiums paid during the accumulation phase. This can be seen as a put option in which the upside potential is borne by the customer. If the returns of the asset portfolio have not reached the yearly guarantee return of 3% at the end date of the product, then the option is in the money. Risky investments (investment strategies subject to large volatility) are therefore only interesting for clients holding the product and not for the insurance company in this setting.

If clients decide to switch to another product before the end date, then they can only transfer their accumulated deposit account without the guarantee option. Note the difference between the deposit account, which is subject to market returns and costs, and the guarantee fund, which is just the 3% return on the premiums paid.

5.2 Levensloop Rendement

Although the guarantee option in the Levensloop Rendement product is rather similar to the option discussed in the previous section, it contains some important differences. This section provides a product description.
The product can be classified as a life-cycle insurance in which the participants are allowed to choose their own end date (with a maximum of the retirement age of 65). The equity exposure of the participant’s deposit account is displayed in Figure 1. It is clear that the closer a participant comes to his/her end date, the less equity exposure his/her deposit account is subject to. Furthermore the investment mix is more diverse in the sense that it contains a bond portfolio as well as a cash portfolio (a bond portfolio with short term bonds). At the end date of the policy holder the insurance company has invested his/her deposit account entirely in cash.

Figure 1: Equity Exposure Levensloop Rendement

The premiums are paid via the employer. Participants are free to choose their premium payment profile. They can opt for a yearly amount as well as for a monthly amount. Premium payments are assumed to be constant (non-increasing) over time.

The Levensloop Rendement product can be seen as a life insurance product since it only pays out if the person is alive at maturity. Clients are allowed to choose for the 90% refund to their heirs in case of early death, but by doing this some of the 'Leven Bonus' as discussed in the previous section will be lost.

The costs involved in this product are withheld monthly. The fundamental difference between the costs in the LOGA setting and the costs in this Levensloop Rendement setting is that in the latter setting they depend on the deposit account on the moment in time they are settled. If the client has a large deposit account (more than €100,000) at the settlement date, the client pays a yearly percentage of 0.85% of the deposit account. However, if the client has a deposit account of less than €17,500, he pays a yearly amount of 1.25%. For deposit accounts between €17,500 and €100,000 the cost percentage is set by a stepwise function decreasing with respect to the amount of the deposit account.

In this contract, the nominal premium payments (independent of the costs) are guaranteed at the end date. If clients would cash out money before the end date, the guarantee value at that moment in time will be adapted in such a way that the relative relation between the deposit account and the guarantee value remains the same. The option holds, just as in the
LOGA case, only at the end date of the contract. Once again this option can be considered a put option which pays off (for the client) if the returns turn out to be insufficient at the end date.

In order to determine whether the deposit accounts are in-the-money or out-of-the-money, one has to know the guarantee strike value at the end date. The way this guarantee value is derived, is subject of the next section.
6 Guarantee Value

In order to calculate option prices, one needs to know the strike price. Because every policy holder has his/her own characteristics, like premium payments, end date, guarantee built up in the past, etc., it is impossible to derive one strike price for the entire portfolio of insurance policies. Therefore the guarantee value at the end date needs to be calculated for every policy holder separately. The methods I use to find these strikes are explained in the following sections.

6.1 Guarantee Value LOGA

To specify the guarantee value at the end date for a LOGA policy holder, one needs the following ingredients: the guarantees built up until the current date, the premium payment profile, the end date of the premium payments, and the end date of the policy. Furthermore it is important to realize that within the LOGA setting a 3% yearly return on premium payments is granted and that it is assumed that policy holders increase their premium payments with 2.75% every year.

I start with the guarantee value at the initiation point. At this point the time until the end point is known, therefore the guarantee value (without new premium payments) can be determined. This is done by multiplying the current guarantee value with the factor $e^{0.03 \times (T_{end} - T_{current})}$, where $T_{end}$ denotes the time of the end date, while $T_{current}$ denotes the current time point in years.

The next step is to determine the guarantees which will be built up in the future. Since all LOGA payments are assumed to be on a constant (monthly) basis, the guarantees can be determined separately. For every future premium payment I can derive how much will be paid at that moment in time and how much time is left until the maturity date. The value of a future premium payment at the end date is therefore the amount paid at a certain point in time ($T_i$) times $e^{0.03 \times (T_{end} - T_i)}$.

Summing all guarantees of the future premium payments at the end date and the guarantee value at the end date of the current guarantees, leads to the total guarantee of a certain participant. In this way the strike price for the simulation input is derived.

6.2 Guarantee Value Levensloop Rendement

Compared to the derivation of the LOGA guarantee values in Section 6.1, the calculation of the 'strike prices' within the Levensloop Rendement setting is subject to other constraints. This section describes how the guarantee values of the Levensloop Rendement product are determined and which assumptions are incorporated.

Once again the following policy holder details are crucial: the guarantees built up until the current date, the premium payment profile, the end date of the premium payments, and the end date of the policy. Contrary to the LOGA product, a 0% return guarantee is specified. This means that at the end date the policy holder can claim at least the amount of all his premium payments of the past.
Another major difference with respect to the LOGA calculations is related to the premium payment profile. Policy holders are assumed not to increase their premium payments over time. This means that they will hold on to their payment profile, which is set by themselves in the past. This is a fairly strong assumption since policy holders are free to choose their own premium payment profile. However, it is often observed in historical payment data and the clients have communicated their future premium payment intentions.

A final assumption needed to calculate the guarantee value at the end date has to do with the regularity of the premium payments. A relatively small amount of the policy holders has a yearly premium payment profile rather than a monthly scheme. If this is the case, the date of payment has to be set. It is assumed (and often observed in the corresponding data) that the premium payments in case of a yearly scheme take place in the month after the policy was started originally. For example, if a policy holder started a Levensloop Rendement agreement on March 2006, then he will pay in April 2006, April 2007, April 2008, etc. Until the end date is reached.

At this point everything one needs to know in order to set the ‘strike prices’ for the corresponding guarantee option is known. One can simply calculate the sum of the current guarantee value (the sum of all premiums paid in the past) and all the future premium payments. At the end date, this is the guarantee value.

At this point the most relevant input from the product’s point of view is mentioned. The next section introduces the most important information regarding the market input parameters.
7 Market Input Data

This section introduces the values of the market parameters I use for the simulation analysis. Firstly I will discuss the parameters needed in the Black-Scholes World (Section 7.1), and finally I discuss the relevant input for the Hull-White Black-Scholes model (Section 7.2). The data are based on information provided by external companies.

7.1 Black-Scholes

The analysis under Black-Scholes assumptions serves to indicate the importance and sensitivity of the option value with respect to the short term interest rate parameter and the volatility parameter. For the LOGA product I decide not to pick specific values for these parameters and simulate under several combinations of the parameter value.

This strategy is less appealing for the Levensloop Rendement product setting, because the stock volatility cannot be characterized as one value, since this changes along with the life-cycle investment mix. I therefore choose to set the relevant parameter values which determine a client’s investment volatility by the following values:

- Stock Volatility: 23%
- Bond Volatility: 5%
- Correlation Coefficient: 0.2

Section 9.2.1 states how I calculate individual volatilities from this input at every point in time. The interest rate parameter will be set at several values to indicate its impact. In comparison to the Black-Scholes model, the Hull-White Black-Scholes model needs more input. This is described in the next section.

7.2 Hull-White Black-Scholes

In the Hull-White Black-Scholes model the choice of parameter values is less straightforward. The input needed is a yield curve, Hull-White parameters $a$ and $\sigma$ (of equation (3.7)), the correlation coefficient between both stochastic differential equations $\rho$ and a volatility parameter for the Black-Scholes equation (4.1).

I use the yield curve of 12/31/2009 published by De Nederlandsche Bank (DNB) as input in the Hull-White Black-Scholes setting. This yield curve states rates at yearly time points only. The simulation strategy I use, however, has a monthly character. Therefore I decide to linearly interpolate this curve in order to derive rates for each month separately. For the first eleven months this method does not work optimally, since there is no yield stated for the duration of 0 years. To overcome this problem and to get at least some indication of the yield curve in this period, I resort to monthly Euribor rates stated at 12/31/2009. Note that these rates are average market rates settled between European banks, instead of the rates calculated by the DNB. With the knowledge of the yield rate for every month I am able to determine forward rates with equation (3.2).
The yield curve and its forward rates are input for the calibration of the Hull-White model. Section 8 describes how the parameters \( a \) and \( \sigma \) are calibrated on 16 swaptions with varying maturities of the option period and swap period.

For the parameter \( \rho \) I choose a value which is approximately in line with the value chosen in Section 7.1. Trial and error indicates that a choice of \(-0.2\) between the stochastic differential equations (3.7) and (4.1) is in line with the value 0.2 chosen between bond and asset prices.

And finally, for the equity volatility parameter I choose 0.23, which is identical to the value chosen in the previous section. At this point, all information needed to calibrate the model is gathered. This is the topic of the next section.
8 Calibration

For a proper estimation of the unknown parameters in the Hull-White one factor model ($a$ and $\sigma$ in equation (3.7)), they are calibrated to market data. The Levenberg-Marquardt Algorithm (for a description of how it is implemented in general, please take a look at Appendix C) is used to calibrate Hull-White’s mean-reverting parameter and volatility parameter to market data on at-the-money interest rate swaptions. The swaptions I use vary in both option and swap length. Every possible combination between option length of 1, 5, 10, and 20 and swap length of 1, 5, 10, 20 is taken into account.

I choose as objective function to be minimized the following least-squares function:

$$S(a, \sigma) = \sum_{i=1}^{n} (U_i - V_i(a, \sigma))^2 \quad (8.1)$$

Formula (8.1) sums the squared differences of the market prices ($U_i$) and the estimated prices based on the Hull-White model ($V_i(a, \sigma)$). Sections 8.1 and 8.2 analyze how, respectively, market prices and simulated prices are derived. Section 8.3 gives the output of the calibration algorithm.

8.1 Market Prices

This section deals with the determination of the $U_i$ variables in equation (8.1). Since the ‘prices’ of swaptions are given as implied volatilities, I convert them to market prices. To do this, a yield curve is needed as well. Because the client data I use is from ultimo 2009, I use the yield curve given at that time as stated in Section 7 and the implied volatilities of swaptions given at that particular moment in time in Bloomberg.

A swaption is defined as an option on a swap. In an interest rate swap contract two parties exchange a certain (constant) strike rate against a floating interest rate. The holder of the swaption has the right to enter the underlying swap contract at a predetermined moment in time. Note the difference between a receiver swaption, in which the holder has the right to receive the fixed payments and pay the floating, and the payer swaption, for which it works vice versa. I will focus on a receiver swaption. For the swap contract, both its length and its strike rate are set beforehand. Since at-the-money swaptions are used, the strike rate is determined in such a way that at the moment of settlement of the swaption the floating leg is equal to the fixed leg. Without loss of generality I assume that the underlying notional amount is equal to 1. The value at time 0 of the fixed leg of a swaption on an $n$-years swap starting at time $t$ is given by formula (8.2):

$$s_k(t, t+n) \sum_{i=1}^{n} P(0, t+i) \quad (8.2)$$

The term $s_k(t, t+n)$ is the strike rate of the swaption. The value of the floating leg is determined by:
\[ P(0, t) - P(0, t + n) \quad (8.3) \]

The moneyness of the swaption (the equality of the fixed and the floating leg) gives rise to the following formula of the forward swap rate at time 0:

\[ s_0(t, t + n) = \frac{P(0, t) - P(0, t + n)}{\sum_{i=1}^{n} P(0, t + i)} \quad (8.4) \]

This derivation of the forward swap rate is in line with Brigo and Mercurio (2006). At this point, all information is available that is needed to implement Black’s formula (from Black (1976)) to derive the market value \( U_i \) of a swaption with option length \( t \) and swap length \( n \). Equation (8.5) shows Black’s formula.

\[ U_i = \sum_{j=1}^{n} P(0, t + j)(s_k(t, t + n)N(-d_2) - s_0(t, t + n)N(-d_1)) \quad (8.5) \]

The variables \( d_1 \) and \( d_2 \) are calculated as follows:

\[ d_1 = \frac{\log \left( \frac{s_0(t, t + n)}{s_k(t, t + n)} \right) + \sigma^2 t}{\sigma \sqrt{t}} \quad (8.6) \]

\[ d_2 = d_1 - \sigma \sqrt{t} \quad (8.7) \]

In the last two equations the parameter \( \sigma \) denotes the implied volatility, which is input in the model. In short, by deriving forward swap rates one possesses all information needed to calculate market prices with Black’s formula if one knows the implied volatility, the option length, and the swap length. The next section discusses how market values can be simulated within the Hull-White framework.

### 8.2 Simulated Values

This section deals with the simulation method to derive swaption prices based on the Hull-White one factor model. It serves to determine the \( V_i(a, \sigma) \) terms in formula (8.1) by making explicit use of the derivation in Section 3.3 and Section 8.1. Furthermore the analysis in this section is based on Boshuizen et al. (2006).

For a better understanding of the pay-off of a swaption, one should take a closer look at the principle of a swap. In an interest rate swap contract the return on a one euro investment at time point \( t_{i-1} \) in \( t_i \)-bonds is swapped for the constant return of the strike rate \( s_k \). Boshuizen et al. (2006) actually expresses this statement in terms of bonds. Investing in a swap boils down to selling a floating interest rate bond and buying a coupon bond which pays off at a rate of \( s_k \) at each transaction moment. Concluding, the swap price at time 0 is given by the following formula:
A similar analysis can be applied for a swaption contract. The main differences, of course, come from the fact that it is an option on a swap and the start date lies in the future. Therefore the option will only be exercised when the pay-off at the maturity date of the option is positive. The pay-off of a swaption at maturity time $t$ with an underlying $n$-years swap is therefore given by:

$$\max \left( P(t, t + n) + s_k(t, t + n) \sum_{i=1}^{n} P(t, t + i) - 1, 0 \right)$$  \hspace{1cm} (8.9)$$

If one evaluates the swaption price at time 0, proper discounting is needed. Furthermore, one has to take the expectation under the risk neutral measure ($Q$), since it involves the pricing of a contract with future payoffs.

$$V_i(a, \sigma) = E^Q \left( e^{-\int_0^t r_s ds} \max \left( P(t, t + n) + s_k(t, t + n) \sum_{i=1}^{n} P(t, t + i) - 1, 0 \right) \right)$$  \hspace{1cm} (8.10)$$

Under the risk-neutral measure, interest rates can be simulated exactly in the Hull-White one factor model as in Theorem 3.4. By making use of the simulated values of the short term interest rate, one is able to derive zero-coupon bond prices. Implementing these simulated values in equation (3.16) leads to the bond prices needed in equation (8.10). In this way all swaption prices are set per simulated path of interest rates.

Repeating this 10,000 times for certain values of $a$ and $\sigma$, and approximating a Jordan matrix by slightly varying the parameters, leads to newly proposed parameter values of $a$ and $\sigma$. This is the idea of the Levenberg-Marquardt algorithm. At the end of this algorithm proper parameter values for $a$ and $\sigma$ are obtained. The results of the calibration are discussed in the next section.

### 8.3 Calibration Results

This section describes the results obtained from the analysis discussed in Section 8.1 and Section 8.2.

The optimal value for the parameter values $a$ and $\sigma$ are 3.41% and 0.98% respectively. I have checked several starting values in the Levenberg-Marquardt algorithm in order to avoid convergence to local minima, but all attempts resulted in the same parameter values.

Table 1 presents an overview of the calibrated values versus the values calculated by Black’s formula. The first and second column are expressed in years. The third and fourth column show the prices in basispoints of the notional amount. The last column represents the relative difference of the calibrated prices with respect to the Black prices.
Table 1: Black Prices vs Calibrated Prices

<table>
<thead>
<tr>
<th>Option Maturity</th>
<th>Swap Length</th>
<th>Black Price</th>
<th>Calibrated Price</th>
<th>Difference (%)</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.41</td>
<td>0.34</td>
<td>−17.41</td>
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<td>0.66</td>
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<td>6.03</td>
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</table>

Figure 2 plots the market value as calculated by Black’s formula relative to the value derived by the Hull-White procedure. In this figure the optimal values of $a$ and $\sigma$ are inserted. If the calibrated results would fit the market data perfectly, all plotted points in the figure would lie on the 45 degree line.

With $a$ and $\sigma$ known, all relevant market data is available, and the simulation procedure can be started. Next section describes how the option values are simulated. Furthermore it states what is calculated under which assumptions.
Figure 2: Calibration Results
9 Simulation

The main background on simulation techniques in general (Section 9.1), the simulation procedure under the Black-Scholes World assumptions (Section 9.2) as well as the fundamentals of the Hull-White Black-Scholes simulation procedure (Section 9.3) are discussed in this section. Section 9.4 presents a step by step description of the simulation method I use for the Hull-White Black-Scholes model. It is applicable for both products I analyze. An alternative life-cycle investment mix is presented in Section 9.5. Section 9.6 introduces representative client dummies, which will be used to analyze client specific sensitivities in the next section. This section ends with a subsection describing the calculation of sensitivity estimates.

All simulation runs I perform use common random numbers. This provides identical circumstances for every run, resulting in an ideal situation for comparing output results if the input parameters are slightly modified.

9.1 Simulation Techniques

This section focuses on the simulation of the asset return paths in a simple setting. For example under the Black-Scholes world assumptions randomness only comes from the asset returns, because all other relevant parameters (interest rate and volatility) are constant. This section indicates how asset return paths can be simulated using either the Euler method or the exact simulation procedure in case of a geometric Brownian motion.

The analysis in this section starts from equation (2.10). In this equation the evolution of the stock price under the risk neutral measure is displayed. The Euler method is applicable in various settings and is easily implemented if the stochastic differential equation is of the form of equation (9.1). The description of the Euler method is analogous to the description in Schumacher (2009).

\[
\frac{dS_t}{S_t} = f(S_t)dt + g(S_t)dW_t
\]  

(9.1)

This can be expressed in an exact formula, which shows the connection between \(S_t\) and \(S_{t+\Delta t}\), where \(\Delta t\) denotes the step size:

\[
S_{t+\Delta t} = S_t + \int_t^{t+\Delta t} f(S_u)du + \int_t^{t+\Delta t} g(S_u)dW_u
\]  

(9.2)

The idea of the Euler simulation procedure is approximating the first and the second integrand by \(f(S_t)\) and \(g(S_t)\) respectively. The resulting equation is:

\[
S_{t+\Delta t} = S_t + f(S_t)\Delta t + g(S_t)\Delta W_t
\]  

(9.3)

In this equation, \(\Delta W_t\) follows a normal distribution with mean 0 and variance \(t\). The Euler method serves as approximation. In some cases, however, one is able to derive an exact simulation procedure. In for example equation (2.10) the stochastic differential equation is actually a geometric Brownian motion, which can be solved analytically. Let \(Y_t\) be the logarithm
of \( S_t \). By making use of Itô’s formula the following derivation is obtained.

\[
dY_t = \frac{1}{S_t}dS_t - \frac{1}{S_t^2}[S_t, S_t] = \mu dt + \sigma_s dW_t - \frac{1}{2}\sigma_s^2 dt = (\mu - \frac{1}{2}\sigma_s^2) dt + \sigma_s dW_t \tag{9.4}
\]

Integrating (9.4) leads to:

\[
Y_t = Y_0 + (\mu - \frac{1}{2}\sigma_s^2)t + \sigma_s W_t \tag{9.5}
\]

Note that \( S_t = e^{Y_t} \). Implementing this observation, leads to an analytical expression for \( S_t \).

\[
S_t = S_0 e^{(\mu - \frac{1}{2}\sigma_s^2)t + \sigma_s W_t} \tag{9.6}
\]

This expression can be discretized to:

\[
S_{t+\Delta} = S_t e^{(\mu - \frac{1}{2}\sigma_s^2)\Delta t + \sigma_s \sqrt{\Delta} Z_t} \tag{9.7}
\]

In the equation above, \( Z_t \) denotes a standard normal variable.

All simulation runs I perform are in a Monte Carlo format. The basics of this type of simulation are described in Appendix D. Monte Carlo simulations are highly flexible and therefore allow for specific modifications. Variance reduction methods can be applied, for instance, to obtain more efficient results than in a completely ordinary Monte Carlo simulation. The so-called antithetic variables technique will be used in this research project. Its description can be found in Appendix E. The following sections discuss what techniques will be used in each specific setting.

9.2 Black-Scholes World

Last section described the simulation technique that is used for the Black-Scholes world analysis. This section states what technique is used and how it is adapted to the Levensloop Rendement product, which invests in a life-cycle mix. Furthermore, in Section 9.2.2 it is described how I aggregate client portfolios, which I use to check whether simulation time can be reduced by doing this.

9.2.1 Simulation

In the last section it was already mentioned that the randomness in the Black-Scholes world only comes from equation (2.10) and concluded that this allows an exact simulation procedure. Equation (9.7) is therefore implemented in the Black-Scholes world setting for both the LOGA product and Levensloop Rendement product. For the LOGA product simulations are run at 9 different combinations of fixed interest rate levels (i.e. 2%, 3%, and 4%) and fixed volatility.
levels (i.e. 5%, 10%, and 15%). The analysis in this simple setting serves to indicate to what extent the volatility parameter and interest rate parameter are crucial for the option value.

A similar analysis cannot be readily implemented for the Levensloop Rendement product, since it invests in a life-cycle mix. I therefore choose to analyze this product only at three different interest rate levels (i.e. 2%, 3%, and 4%). The volatility aspect cannot be seen as constant over time. To tackle this problem I make two simplifications.

The first simplification deals with the cash investment. Cash investments are modeled as bond investments. The result is that there are only two assets left: assets and bonds. The second simplification is that both assets are modeled as being one asset whose variance changes along with the position in the life-cycle fund. As input to determine the overall volatility $\sigma_T$ I take 23% as asset volatility $\sigma_s$ and 5% for the bond volatility $\sigma_b$ as stated in Section 7. The correlation coefficient $\rho$ of 0.2 completes the input for the following formula for the calculation of the overall volatility.

$$\sigma_T(t) = \sqrt{\omega_s^2(t)\sigma_s^2 + \omega_b^2(t)\sigma_b^2 + 2\rho\omega_s\omega_b\sigma_s\sigma_b} \quad (9.8)$$

At every moment in time the equity exposure $\omega_s(t)$ is known for every client $i$. Therefore for every client the total portfolio volatility $\sigma_T$ can be determined at any point in time using equation (9.8). Now all information needed to run simulations under Black-Scholes assumptions for both the LOGA product and the Levensloop Rendement product is gathered.

### 9.2.2 Aggregation

Another point of interest which is investigated in this setting is whether aggregated client portfolios can be used to derive reliable option values for the entire client portfolio. If so, simulation time can be reduced considerably.

The strategy I use for the LOGA product is replacing policy holders with the same age, end date, gender, and Leven Bonus choice by one new fictitious client for which the financial parameters are the sum of the original policy holders and compare with the original portfolio option value. These parameters are the financial accounts, guarantees at the end date, and the premium payments.

The client portfolio of the Levensloop Rendement product is larger and more diversified than the LOGA portfolio. Therefore fruitful aggregation would be of great interest in this product setting specifically. Aggregation on the characteristics mentioned before, however, might not lead to reliable results because of the differences in dates of birth and the different moneyness positions of the aggregated clients. To overcome the problem of current age, I choose to aggregate on this parameter by simply taking the average age of the individuals. The possible impact on moneyness is easily indicated by an example.

Let two clients ($a$ and $b$) with identical characteristics except for their deposit account be one month before their end date. Let the guarantee value of both clients be €2,000 at the end date, while they are not paying any premiums anymore and have deposit accounts of €1,000 and €10,000 respectively. Intuitively it is clear that for client $a$ the option value will
be approximately €1,000, while for client b the option value is approximately €0. Summing the options values individually leads to a total option value of approximately €1,000. By aggregating the guarantee value of both clients as well as their deposit accounts, a new portfolio arises with a deposit account of €11,000 and a guarantee value of €4,000. This aggregated portfolio will have a guarantee option value of approximately €0. Based on this observation I also choose to aggregate clients who are in-the-money separately from clients who are out-of-the-money.

The next section discusses the main insight needed to simulate in the Hull-White Black-Scholes world.

9.3 Hull-White Black-Scholes Model

The simulation procedure derived Section 9.2.1 does not apply to the Hull-White Black-Scholes model, since the short rate is time-dependent and needs to be simulated as well. The analysis of this section is based on the information on short rate simulation under Hull-White assumptions obtained in Section 3.3 and on the Euler simulation scheme from Section 9.1. Since simulation can be performed under the risk-neutral measure as well as the T-forward measure, this section is divided accordingly to accentuate the differences.

9.3.1 Risk-Neutral Measure

The main principles of a simulation study in the risk-neutral Hull-White Black-Scholes setting are presented here. The main idea of drawing interest rates and asset prices over time as well as the discounting at the end date is discussed here.

At every timestep I firstly simulate a short rate. I assume that this short rate is constant during a simulation period of one month. In order to simulate the short rate under the bank-account measure I use Theorem 3.4.

The next step is to implement this short rate in the Black-Scholes model. Since the short rate changes over time, the geometric Brownian motion describing the evolution of asset prices under the risk neutral measure (2.10) does not hold any longer. Therefore an analytical solution of the asset price process is not readily available and I resort to the Euler method.

The related Euler approximation, which I use in the risk-neutral Hull-White Black-Scholes model is then:

\[ S_{t+\Delta t} = S_t + r_t S_t \Delta t + \sigma_s S_t \sqrt{\Delta t} Z_t \]  

The stepsize is denoted by \( \Delta t \) and \( Z_t \) symbolizes a standard normal variate. At the end date of a contract, a value \( C(T) \) is derived. Since I use the bank account as a numéraire, this value should be ‘discounted’ as in equation (9.11).

\[ \frac{C(0)}{M(0)} = E^Q \left( \frac{C(T)}{M(T)} \right) \]  

\[ C(0) = E^Q \left( C(T) e^{-\int_0^T r_s ds} \right) \]
Because I assume interest rates being constant each month, equation (9.11) can be written as a sum. Section 9.4 makes this explicit.

### 9.3.2 T-Forward Measure

In comparison with Section 9.3.1 this section discusses the same simulation and discounting aspects. The only difference is that this section deals with the T-forward measure instead of the risk-neutral measure.

In line with the risk-neutral measure, I assume for the zero-coupon bond measure also the assumption of constant interest rates each month. The simulation of interest rates, however, is different. For the exact simulation of interest rates \( r_t \) under the T-forward measure I use equation (4.14) in combination with equations (4.11) and (4.12).

With the knowledge of the interest rate, I simulate asset prices by making use of the Euler approximation of formula (4.10).

\[
S_{t+\Delta t} = S_t + (r_t - \sigma_s \sigma_r \rho_r) S_t \Delta t + \sigma_s S_t \sqrt{\Delta t} Z_t
\] (9.12)

The evolutions of the stock prices as well as the short rate interest rate can now be simulated. When the price of the contract (in this case option guarantee value) is known at end date \( T \), it has to be 'discounted' back to start time 0. Since the zero-coupon bond (which pays off at the end date) is chosen as numéraire, the following equations hold.

\[
\frac{C(0)}{P(0,T)} = E^{QT} \left( \frac{C(T)}{P(T,T)} \right) \quad (9.13)
\]

\[
C(0) = P(0,T) E^{QT}(C(T)) \quad (9.14)
\]

\( E^{QT} \) stands for the expectation under the T-forward measure. Note that the simulated interest rates are no longer needed for discounting. Formula (9.14) indicates that if one wants to profit optimally from this measure, one has to choose a zero-coupon bond paying off at the end date of the contract. This, in combination with equation (9.12) where the duration of the zero is relevant as well, shows the relative inflexibility of this measure. A product related description of the simulation method I use, is described in the next section.

### 9.4 Simulation Procedure

This section describes how the guarantee option values are simulated in a specific product setting if in a Hull-White Black-Scholes model the asset returns as well as the short term interest rates are simulated. The investment mix is assumed to be a bucket of assets and bonds, in which the asset returns can be aggregated into one stochastic differential equation.

First of all one needs to gather the proper client data. For the products I analyze, the following information is of particular interest: Moments in time of premium payment, amount of premium to be paid at these payment dates, the current deposit account, gender, current age,
end date of the contract, and choice for partner restitution. With this information one is able to
determine the guarantee value at the end of the contract as discussed in Section 6. The current
age and gender are used to determine the survival probabilities of the client. Combining this
with the choice for extra Leven Bonus (or no partner restitution) one can set the extra amount
of return based on the Leven Bonus aspect of the products. Furthermore, the end date indicates
the investment strategy for Levensloop Rendement policy holders.

With this in mind, one is able to take the next step. Paths of the short rate can be simulated
now. Note that I assume, that the short rate is constant for every simulated period $\Delta t$. In
order to determine returns on a bond portfolio, I determine the returns on a rolling zero-coupon
bond with the same duration ($D$). I calculate these returns for scenario $i$ by determining the
price of a zero-coupon bond with length $D + \frac{1}{2} \Delta t$ at time $t$ ($P_i(t, t + D + \frac{1}{2} \Delta t)$), and calculating
(by making use of the new interest rate and equation (3.16)) $P_i(t + \Delta t, t + D - \frac{1}{2} \Delta t)$. The next
return is then calculated with the price of a zero with length $D + \frac{1}{2} \Delta t$ at $t + \Delta t$ and the price
of a zero with length $D - \frac{1}{2} \Delta t$ at $t + 2 \Delta t$, etcetera. I make use of relative returns, calculated as follows:

$$r^b_i(t, t + \Delta t) = \frac{P_i(t + \Delta t, t + D - \frac{1}{2} \Delta t) - P_i(t, t + D + \frac{1}{2} \Delta t)}{P_i(t, t + D + \frac{1}{2} \Delta t)}$$

(9.15)

Every time a new short rate is calculated, new asset prices are determined via Euler approxi-
mations. The Brownian motions involved in equation (4.1) and Theorem 3.4 are correlated with
correlation coefficient $\rho = -0.2$ (as stated in Section 7). This can be realized in a simulated
setting by drawing standard random numbers as follows:

$$Z_s = \tilde{Z}_1$$
$$Z_r = \rho \tilde{Z}_1 + \sqrt{1 - \rho^2} \tilde{Z}_2$$

(9.16)

In which $\tilde{Z}_1$ and $\tilde{Z}_2$ are uncorrelated standard normally distributed variables. Knowing the
asset price at time $S_i$ and $t + \Delta t$ enables me to determine the arithmetic return of the asset
portfolio in this period. The following formula shows the corresponding calculation method for
simulation $i$.

$$r^s_i(t, t + \Delta t) = \frac{S_i(t + \Delta t) - S_i(t)}{S_i(t)}$$

(9.17)

At this point, I know both the returns on bond portfolios (note that I can derive returns
for bond portfolios of any duration) and the asset portfolio. By keeping track of the number
of years the client has to go until his/her end date, I can determine the equity exposure of this
client in case he/she has a life-cycle mix. By adding these returns, a total return for a client
over a time period in a particular simulation is derived. The asset mix can maximally consist
of three types of products (discussed in Section 5): short term bonds (cash), long-term bonds,
and assets. The total return for simulation $i$, client $j$ and time period $(t, t + \Delta t)$ is defined by
\[ r^T_{ij}(t, t + \Delta t) = \omega^c_j(t) r^c_i(t, t + \Delta t) + \omega^b_j(t) r^b_i(t, t + \Delta t) + \omega^s_j(t) r^s_i(t, t + \Delta t) \] (9.18)

Where \( \omega^c_j(t) \), \( \omega^b_j(t) \), and \( \omega^s_j(t) \) denote the investment proportion in respectively cash, bonds, and assets for person \( j \) at time \( t \). Note that \( \omega^c_j(t) + \omega^b_j(t) + \omega^s_j(t) = 1 \). Since I know how to simulate new short rate values as well as asset returns, I am able to simulate from any time point \( t \) to \( t + \Delta t \). The next step is to determine what happens at every fixed timepoint \( t, t + \Delta t, t + 2\Delta t, \) etc.

By multiplying the deposit account at time \( t \) for person \( j \) in simulation \( i \) (\( DA_{ij}(t) \)) with the aggregated return over period \( (t, t + \Delta t) \), the deposit account at time \( t + \Delta t \) can be determined. However, this new deposit account needs to be corrected for several factors. First of all, the new premium payments are added (\( PREM_j(t + \Delta t) \)). Secondly, extra Leven Bonus is awarded \( LEV_j(t + \Delta t) \). Let the deposit account at this point be denoted by \( DA^*_ij(t + \Delta t) \). And finally the costs are withdrawn, leading to the actual deposit account on time point \( t + \Delta t \). Note that these costs are based on \( DA^*_ij(t + \Delta t) \) in case of the Levensloop Rendement product, while they are constant in the LOGA setting.

\[
DA^*_ij(t + \Delta t) = (DA_{ij}(t) r^T_{ij}(t, t + \Delta t) + PREM_j(t + \Delta t)) \times LEV_j(t + \Delta t)
\]

The last modification to be implemented at the node is the adjustment of the investment portfolio in case one analyzes a Levensloop Rendement product holder.

Continuing this process until the end date is reached, leads to a value of the deposit account at the end date. This can be compared with the previously determined guarantee value. The option value is only in the money if at the end date the deposit account is lower than the guarantee value \( GAR_j \). This option value needs to be corrected for survival probability from starting point \( (t = 0, \) with starting age \( x(j) \)) to end date \( T(j) (T(j)p_{x(j)}) \). The individual option value is now determined by discounting it properly (Section 9.3.1 for risk-neutral or Section 9.3.2 for T-forward). Furthermore, I take the sum across all clients within a simulation (let \( M \) be the number of persons holding a LOGA or Levensloop Rendement policy), and finally over all simulations \( N \). In equation (9.20) the estimated option value \( \hat{OV} \) is displayed. Note that the risk-neutral measure is used for the total client portfolio.

\[
\hat{OV} = \sum_{i=1}^{N} \sum_{j=1}^{M} e^{U_{ij}} \max (GAR_j - DA_{ij}(T), 0)_{T(j)p_{x(j)}}
\] (9.20)

Where

\[
U_{ij} = - \sum_{k=0}^{(T(j) - 1)} r_i(k) \Delta t
\] (9.21)

Note that \( r_i(k) \) in equation (9.21) denotes the interest at time \( k \) in scenario \( i \). When I do
not sum over all simulations as in equation (9.20), I derive $N$ different option values for which statistics like for example confidence intervals can be calculated.

The results in the Hull-White Black-Scholes World are presented in Section 10.2. The next section introduces an alternative life-cycle mix for the Levensloop Rendement product.

### 9.5 Alternative Life-Cycle Mix

This section provides an alternative Levensloop Rendement life-cycle mix. Under Hull-White Black-Scholes assumptions a simulation analysis for the entire client portfolio is performed in order to indicate the impact of the choice for a certain life-cycle mix. The alternative mix is presented in Figure 3.

#### Figure 3: Equity Exposure Levensloop Rendement: Alternative Life-Cycle Mix

Compared to the original life-cycle mix presented in Figure 1 in Section 5.2, this mix invests more in bonds and less in equity. Therefore it can be called less aggressive. The results of this alternative life-cycle mix are presented in Section 10.2.2.

### 9.6 Simulation Dummies

Except for the (aggregated) client portfolios for the LOGA and Levensloop Rendement products simulations are run for non-existing representative individual dummies as well. These simulation runs serve to indicate which clients carry in particular large option values and for what kind of risk they are most sensitive. Furthermore the effectiveness of the T-forward measure can be used and compared with the risk-neutral. This section describes which dummies are analyzed.

For the LOGA portfolio I analyze three at-the-money dummies (A, B, and C) for which the time until the end date is the main difference. Client data shows that on average policy holders with an end date in the near future tend to have a large deposit account and relatively large premium payments. While for young policy holders (with end dates in the far future) the current deposit accounts are relatively low accompanied by relatively low premium payments. These
observations determine the main financial characteristics of the dummies. Characteristics such as gender and the choice for partner restitution are also set beforehand. Their impact will be tested, however, by varying these characteristics. Furthermore I assume all dummy clients to pay premium until they have reached their end date being their 59th birthday. The characteristics for the dummies are displayed in Table 2.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years to End</td>
<td>1</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Deposit Account</td>
<td>€80,000</td>
<td>€20,000</td>
<td>€7,500</td>
</tr>
<tr>
<td>Yearly Premium</td>
<td>€25,000</td>
<td>€4,000</td>
<td>€1,000</td>
</tr>
<tr>
<td>Gender</td>
<td>Male</td>
<td>Male</td>
<td>Male</td>
</tr>
<tr>
<td>Partner</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Age</td>
<td>58</td>
<td>49</td>
<td>29</td>
</tr>
</tbody>
</table>

For the Levensloop Rendement product, I have constructed client dummies in a similar way. I focus again on three different dummies, which are at different stages in the life-cycle investment mix. I choose the same number of years until the clients have reached their end date (1, 10, and 30 years respectively). The financial parameters (deposit account and premium payments) tend to be lower than in the LOGA setting. The end date itself tends to be at the age of 61, while in the LOGA setting most end dates are at the age of 59.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years to End</td>
<td>1</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Deposit Account</td>
<td>€10,000</td>
<td>€5,000</td>
<td>€2,000</td>
</tr>
<tr>
<td>Yearly Premium</td>
<td>€2,000</td>
<td>€1,000</td>
<td>€500</td>
</tr>
<tr>
<td>Gender</td>
<td>Male</td>
<td>Male</td>
<td>Male</td>
</tr>
<tr>
<td>Partner</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Age</td>
<td>60</td>
<td>51</td>
<td>31</td>
</tr>
</tbody>
</table>

9.7 Sensitivity Analysis

Although the stand-alone guarantee option value can be of valuable information for any insurance company, information on the sensitivity of input parameters on the option value can give more information on the risks an insurance company is most vulnerable to. The determination of these sensitivities is subject of this section. I use simulations under risk-neutral assumptions within this section.

Sensitivities will be calculated with respect to a parallel shift of the yield curve, the stock volatility parameter $\sigma_s$, the deposit accounts at the start of the simulation, the correlation
coefficient in the Hull-White Black-Scholes setting, and the calibrated Hull-White parameters \( \sigma_r \) and \( a \). They are estimated by using the central finite-difference method and implementing percentual differences. The calculation underlying this approximation method is displayed in the following formula.

\[
\epsilon = \frac{\% \hat{O}V(\beta + h) - \% \hat{O}V(\beta - h)}{2h}
\]  

(9.22)

The variable \( \epsilon \) is the estimated sensitivity, \( \% \hat{O}V \) denotes the average percentual option value relative to the original option value over all independent replications, \( \beta \) is the input parameter value, and \( h > 0 \) indicates the deviation from the original parameter value. In order to minimize the bias occurring from the choice of \( h \), one in general wants to choose a very low value for \( h \). Glasserman (2003), however, states that although a small \( h \) is desirable, small values for \( h \) lead to very large variances of the sensitivity estimate. Therefore instead of taking very small values for \( h \), I choose \( h \) to be a percentage (2%) of \( \beta \). Of course this percentage remains an arbitrary choice.

This method is not applied for a shift of the yield curve and for the analysis of the correlation coefficient \( \rho \). Since the yield curve cannot be characterized as being one number I decide to shift the entire yield curve up and down by 5 basispoints instead of using the 2% shifts for \( h \). After doing this I apply formula (9.22). For \( \rho \) instead of making use of 2% shifts for \( h \), this sensitivity is determined by lowering and increasing this coefficient with 0.1.
10 Simulation Results

The simulation results obtained from the simulation settings described in Section 9 are presented in this section. Section 10.1 discusses the results obtained via the Black-Scholes method and Section 10.2 the results obtained using the Hull-White Black-Scholes model. The results of the dummy policy holders introduced in Section 9.6 are discussed in Section 10.3.

10.1 Black-Scholes World Results

This section deals with the simulation results obtained under the Black-Scholes World assumptions (i.e. constant interest rate and constant volatility). Since I analyze two different insurance products, I derive different results for any one of them. The results for the LOGA product are discussed in 10.1.1, while the results for the Levensloop Rendement product can be found in 10.1.2.

10.1.1 LOGA Results

This section provides an overview of the simulation results of the total LOGA client portfolio under Black-Scholes assumptions. In Section 9.2 the simulation strategy has been discussed. The results discussed here are per 12/31/2009 derived by 8,000 simulation runs.

I have chosen to simulate under 9 different input parameter combinations to indicate to what extent the option value is subject to the choice of these parameters. In the Black-Scholes world the only input parameters are the risk-free rate and the volatility. Three different interest rates are analyzed, e.g. 2%, 3%, and 4%. In combination with these interest rates, three different volatility levels are examined: 5%, 10%, and 15%.

In Figure 4 the option values of the total LOGA portfolio relative to the total portfolio deposit account value are displayed in a three-dimensional plot. The vertical axis symbolizes this percentage, while the other axes represent the input parameters (interest and volatility).

Drawing conclusions about the interest rate effect on the option value can be done by selecting a fixed value for the volatility. For example, let the volatility be fixed at the level of 10%. Out of the three scenarios simulated under this volatility level, the 4% interest rate level leads to the lowest relative option value (18.08%). The lower the interest rate (3% and 2%) the higher the option value (30.89% and 50.04%, respectively). Similar analyses can be done by evaluating the interest rate under fixed volatilities of 5% or 15%. The overall lowest percentage arises at the combination 4% interest rate and a 5% volatility level.

For the volatility parameter a similar analysis as with the interest rate parameter can be performed. Let the interest rate be fixed at a level of for instance 3%. Then a 5% volatility level leads to the lowest relative option value (19.04%). Larger volatility levels (10% and 15%) lead to larger relative option values (30.89% and 42.86%). This analysis can be implemented for interest rates of 2% and 4% as well.

Note that an interest level of 2% tends to lead to very high option values. This can be explained fairly easily by the fact that a yearly 3% growth of the guarantee values is granted...
by the insurance company, while under the risk-neutral measure the deposit accounts grow on expectation with only 2%.

Table 4 summarizes the option values (in millions) of the LOGA portfolio with respect to all calculated combinations of interest rate and volatility level. The values denoted between brackets are derived by aggregating policy holders as discussed in Section 9.2.2. Note that the results of the aggregated portfolio are close to the results of the original client portfolio, which indicates an effective aggregation procedure that reduces the processing time of a simulation run.

Two conclusions can be drawn out of the analysis in this section. First of all, one can conclude that high interest rates lead to low guarantee option values (ceteris paribus), and vice versa. Secondly, one can conclude that low volatility levels lead to low option values, and high
volatility levels to high option values. This is in line with the standard theory presented in academic literature, such as in the famous Black-Scholes (1973) paper.

A more comprehensive overview of the results, including confidence intervals of the value of interest, is presented in the tables in Appendix F.1. Table 10 and Table 11 show the results of the entire LOGA portfolio and the aggregated portfolio respectively. The resulting statistics of these portfolios derived with the use of antithetic variables are presented in Table 12 and Table 13. The confidence intervals are reduced very effectively for both the entire portfolio and the aggregated portfolio. Their sizes are reduced with more than 30% in all cases, and in some cases by even more than 80%.

The next section discusses results for the Levensloop Rendement portfolio under Black-Scholes assumptions.

10.1.2 Levensloop Rendement Results

The Black-Scholes world analysis for the Levensloop Rendement product based on Section 9.2 is presented in this section. Compared with the level of detail offered in Section 10.1.1, this section is less advanced in the sense that only the interest rate is varied. Furthermore, due to the size of the portfolio (and the resulting computation time) only 2,000 simulation runs are performed.

The option value is simulated at three different constant values for the short rate for the original portfolio and the two proposed aggregation methods of Section 9.2.2. The results of the original portfolio and this aggregated portfolio are displayed in columns 2 and 3 of Table 5.

Table 5: Option value of Levensloop Rendement portfolio in millions

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>Original Portfolio</th>
<th>Aggregated Portfolio</th>
<th>Aggregated Portfolio*</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>€24.17</td>
<td>€21.24</td>
<td>€21.36</td>
</tr>
<tr>
<td>3%</td>
<td>€13.23</td>
<td>€11.30</td>
<td>€11.38</td>
</tr>
<tr>
<td>4%</td>
<td>€6.83</td>
<td>€5.64</td>
<td>€5.69</td>
</tr>
</tbody>
</table>

In this table one can see that if the interest rate (denoted by the first column of Table 5) is high, the option value for the original portfolio as well as the aggregated portfolio is relatively low. For low interest rates substantially higher option values are derived. Although the aggregated portfolio shows a similar sensitivity to changes in interest rates, the overall option value is always more than one million lower.

The results of a portfolio, which is aggregated on moneyness as well (Aggregated Portfolio*) are displayed in the last column of Table 5. One can see that the results are very similar to the original aggregated portfolio and show therefore lower option values than the original portfolio does. In short, aggregating the client portfolio for the Levensloop Rendement product does not lead to reliable option values on the total portfolio level.
This is even better motivated by Table 14, Table 15, and Table 16 in Appendix F.2 where the simulation results of the three portfolios are displayed. One can see that at all three levels of the riskfree rate the confidence intervals of the aggregated portfolios do not match the original client portfolio. Therefore I decide to no longer work with aggregated portfolios.

For the original client portfolio I calculate the Monte Carlo simulation results in case antithetic variables are used as well. These results are displayed in Table 17. It indicates that all confidence intervals show overlap with the original ones and that the size of the confidence intervals can be reduced. In case of an interest rate of 2% the confidence interval is reduced maximally by 43.11%.

In the next section simulation results are presented if the assumption of constant interest rate is relaxed and the Hull-White model is implemented.

10.2 Hull-White Black-Scholes Results

This section describes the results for the simulation of entire client portfolios under Hull-White Black-Scholes assumptions. It is divided in a section describing the LOGA results (Section 10.2.1) and a section where Levensloop Rendement results are discussed (Section 10.2.2). The simulation strategy I use is discussed in Section 9.3.1 and Section 9.4.

10.2.1 LOGA Results

The results for the LOGA portfolio under Hull-White Black-Scholes assumptions are discussed in this section. Except for the option value itself, sensitivities with respect to the level of the yield curve, the stock volatility, the correlation coefficient, and the underlying deposit accounts are computed. Furthermore robustness of the calibration is investigated by estimating the sensitivity of the option value with respect to the calibrated parameters \( a \) and \( \sigma \).

The results of the 8,000 simulation runs of the entire LOGA client portfolio are summarized in Table 6. The upper part of this table shows the option value and the boundaries of a two-sided 95% confidence interval. All financial amounts stated in the table are in millions of euros. In order to give an indication of the size of this value, the relative option value with respect to the sum of the deposit accounts known at 12/31/2009 is included as well. The guarantee option value for the entire LOGA portfolio turns out to be €11.624 million.
Table 6: LOGA Results

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Original Monte Carlo</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>11.624</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td>% of Deposit Accounts</td>
<td>9.35</td>
<td>%</td>
<td></td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>11.251</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>11.998</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td><strong>Antithetic Variables</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>11.591</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>11.247</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>11.934</td>
<td>€(m)</td>
<td></td>
</tr>
<tr>
<td>Reduction of CI</td>
<td>8.03</td>
<td>%</td>
<td></td>
</tr>
<tr>
<td><strong>Sensitivities</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yield Curve</td>
<td>−83.809</td>
<td>Δ%pp</td>
<td></td>
</tr>
<tr>
<td>Deposit Accounts</td>
<td>−3.742</td>
<td>Δ%pp</td>
<td></td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>1.786</td>
<td>Δ%pp</td>
<td></td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>103.586</td>
<td>Δ%pp</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>−8.532</td>
<td>Δ%pp</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>6.108</td>
<td>Δ%pp</td>
<td></td>
</tr>
</tbody>
</table>

The middle part of Table 6 gives an overview of the results if antithetic variables are used. Again the option value is displayed along with its 95% two-sided confidence interval. The row 'Reduction of CI (%)' indicates the effectiveness of the variance reduction method. It shows the percentual reduction of the confidence interval by making use of antithetic variables. Note that the reduction of the confidence interval is less pronounced here than in the more simple Black-Scholes model.

The bottom segment of the table presents the results for the sensitivity calculations as explained in Section 9.7. The sensitivities need to be interpreted with care, since they do not have similar interpretations. The resulting meaning of the estimated sensitivities is summarized in the last column. I distinguish between three possible sensitivity units: $\frac{\Delta \%}{\Delta \%pp}$, $\frac{\Delta \%}{\Delta \%}$, and $\frac{\Delta \%}{\Delta \rho}$. They denote the impact of a change in percentage points of the underlying parameter on the percentual option value, a percentual change of the underlying parameter on the percentual option value, and an absolute change of the correlation coefficient on the percentual option value respectively. Note that the sensitivities are calculated in the neighborhood of the original parameter value, therefore the estimated sensitivities might not give proper results in case one makes calculations for large (relative) shifts in parameter values.

Interest rate risk is represented by three different sensitivity values (Yield Curve, $\sigma_r$, and $a$). An absolute rise of the entire yield curve of 10 basispoints approximately lowers the option value with 8.4%. The $a$ and $\sigma_r$ parameters both contribute to the volatility of the interest rate.
A stronger mean-reverting coefficient leads to lower interest rate volatility and therefore lowers the total LOGA portfolio option value, while a larger $\sigma_r$ produces opposite results.

If the equity volatility parameter $\sigma_s$ is increased with 1 percentage point the option value increases with approximately 1.8%. A ceteris paribus rise of all individual deposit accounts of 1% decreases the option value more extremely with approximately 3.7%. The influence of the correlation coefficient of the interest rate model and the Black-Scholes model is rather small. An absolute increase of the coefficient with 0.1 does not even change the option value with 1%.

The next section presents the results of a similar analysis for the Levensloop Rendement product.

10.2.2 Levensloop Rendement Results

In this section results of 2,000 simulation runs for the Levensloop Rendement product in the Hull-White Black-Scholes model are presented. The calculations made for the LOGA product (in Section 10.2.1) are made for this product as well. Of course, the fundamental differences within the two products (as discussed in Section 5) are maintained. Besides results of the original life-cycle mix (Figure 1), results of the alternative life-cycle mix (Figure 3) are presented as well.

The results for the option value for original investment strategy are displayed in the center column of Table 7, while the results for the alternative mix are presented in the right column. The first set of rows shows results of the ordinary Monte Carlo simulation, the second set of rows shows results when antithetic variates are used, and the last segment of rows shows how sensitive the option value is when input parameters are changed. The sensitivities are calculated in a model without variance reduction methods.

The guarantee option value of the original investment mix (approximately €7.3 million) is larger than the option value of the alternative investment mix (approximately €2.4 million). Also the 95% confidence intervals indicate a large difference between the two investment strategies. For both products the size of the confidence intervals can be reduced if antithetic variables are used.
In the table one can see that this lower option value for the alternative life-cycle mix comes with a price, since this option value is more sensitive to changes in the underlying parameters for five out of six parameters. Only the standard deviation of the equity portfolio has relatively more impact on the original life-cycle mix option value than on the original option value. The relatively larger impact of the interest rate parameters on the option value for the alternative mix, is readily explained by the fact that there is invested more in interest related products (bonds and cash) at every moment in time.

Note that the sensitivities concern relative changes of the option value. Since the option value is larger in case the original investment mix is used, a smaller relative impact for this option value might still be a larger absolute impact. The next section describes the results for dummy clients in a Hull-White Black-Scholes setting.

### 10.3 Dummy Analysis

Along with a broader study on the sensitivity of the input parameters, the results for the Monte Carlo simulation of the dummies introduced in Section 9.6 under Hull-White Black-Scholes assumptions are evaluated in this section. In line with the previous section, this section also divides its results in segments dealing with LOGA results (Section 10.3.1) and the Levensloop Rendement results (Section 10.3.2).

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Original Mix</th>
<th>Alternative Mix</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Monte Carlo</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>7.306</td>
<td>2.398</td>
<td>€(m)</td>
</tr>
<tr>
<td>% of Deposit Accounts</td>
<td>5.11</td>
<td>1.68</td>
<td>%</td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>6.699</td>
<td>2.128</td>
<td>€(m)</td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>7.913</td>
<td>2.667</td>
<td>€(m)</td>
</tr>
<tr>
<td>Antithetic Variables</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>6.824</td>
<td>2.124</td>
<td>€(m)</td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>6.374</td>
<td>1.901</td>
<td>€(m)</td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>7.273</td>
<td>2.346</td>
<td>€(m)</td>
</tr>
<tr>
<td>Reduction of CI</td>
<td>25.95</td>
<td>17.44</td>
<td>%</td>
</tr>
</tbody>
</table>

### Sensitivities

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Original Mix</th>
<th>Alternative Mix</th>
<th>Δ%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield Curve</td>
<td>-70.476</td>
<td>-102.210</td>
<td>Δ%</td>
</tr>
<tr>
<td>Deposit Accounts</td>
<td>-3.858</td>
<td>-7.201</td>
<td>Δ%</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>7.119</td>
<td>4.379</td>
<td>Δ%</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>62.990</td>
<td>147.338</td>
<td>Δ%</td>
</tr>
<tr>
<td>$a$</td>
<td>-4.746</td>
<td>-12.932</td>
<td>Δ%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>104.715</td>
<td>129.817</td>
<td>Δ%</td>
</tr>
</tbody>
</table>
10.3.1 LOGA Results

For dummies A, B, and C (discussed in Section 9.6) simulation results and sensitivities to input parameters are discussed. The analysis is broader in the sense that the impact on the option value of gender, the choice for end date, the choice for partner restitution, and the choice of numéraire is analyzed.

The results for 20,000 simulation runs are displayed in Table 8.

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Original Monte Carlo</th>
<th>Antithetic Variables</th>
<th>T-forward Measure</th>
<th>Sensitivities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>1,231.26</td>
<td>1,797.02</td>
<td>3,131.40</td>
<td>€</td>
</tr>
<tr>
<td>% of Deposit Account</td>
<td>1.54</td>
<td>8.98</td>
<td>41.75</td>
<td>%</td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>1,199.65</td>
<td>1,754.65</td>
<td>3,018.66</td>
<td>€</td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>1,262.87</td>
<td>1,839.39</td>
<td>3,244.14</td>
<td>€</td>
</tr>
<tr>
<td>Reduction of CI</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option Value</td>
<td>1,205.57</td>
<td>1,800.74</td>
<td>3,272.85</td>
<td>€</td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>1,178.73</td>
<td>1,765.84</td>
<td>3,163.21</td>
<td>€</td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>1,232.42</td>
<td>1,835.64</td>
<td>3,382.49</td>
<td>€</td>
</tr>
<tr>
<td>Reduction of CI</td>
<td>15.07</td>
<td>17.63</td>
<td>2.75</td>
<td>%</td>
</tr>
<tr>
<td>Option Value</td>
<td>1,230.29</td>
<td>1,800.52</td>
<td>3,250.04</td>
<td>€</td>
</tr>
<tr>
<td>Lowerbound CI</td>
<td>1,198.57</td>
<td>1,761.62</td>
<td>3,202.95</td>
<td>€</td>
</tr>
<tr>
<td>Upperbound CI</td>
<td>1,262.01</td>
<td>1,839.42</td>
<td>3,315.13</td>
<td>€</td>
</tr>
<tr>
<td>Reduction of CI</td>
<td>−0.35</td>
<td>8.19</td>
<td>50.25</td>
<td>%</td>
</tr>
<tr>
<td>Female</td>
<td>1,251.02</td>
<td>1,880.67</td>
<td>3,258.96</td>
<td>€</td>
</tr>
<tr>
<td>No Partner</td>
<td>1,018.16</td>
<td>1,247.06</td>
<td>2,704.54</td>
<td>€</td>
</tr>
<tr>
<td>Female &amp; No Partner</td>
<td>1,155.04</td>
<td>1,608.23</td>
<td>3,043.15</td>
<td>€</td>
</tr>
<tr>
<td>Yield Curve</td>
<td>−29.571</td>
<td>−90.845</td>
<td>−87.306</td>
<td>Δ%</td>
</tr>
<tr>
<td>Deposit Accounts</td>
<td>−22.146</td>
<td>−3.940</td>
<td>−0.633</td>
<td>Δ%</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>3.414</td>
<td>2.690</td>
<td>0.394</td>
<td>Δ%pp</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>84.968</td>
<td>65.244</td>
<td>100.985</td>
<td>Δ%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>−66.525</td>
<td>0.465</td>
<td>17.797</td>
<td>Δ%pp</td>
</tr>
</tbody>
</table>

The first rows of Table 8 indicate the results of Monte Carlo simulation without the use of a variance reduction method. It is clear that, although dummies B and C posses lower current deposit accounts and pay less premium than dummy A, their guaranteed option values are larger. The relative value with respect to the deposit accounts makes this even more explicit.

The second set of rows shows the results when making use of antithetic variates. It indicates that two-sided 95% confidence intervals can be reduced. For dummy C, with an end date in the far future, the confidence interval is only slightly reduced. Law (2007) argues that monotonicity
of random numbers on the response (in this setting the guarantee option value) is a fundamental requirement to make antithetic variates work. In the Hull-White Black-Scholes model I use, monotonicity might not be satisfied. This comes from the observation that a large random variate for the simulation of the interest rate rises the discounting rate and the drift term of the stock prices on one hand, but lowers bond prices on the other. Therefore it is unclear whether a draw of a large random number for the interest rate in all cases leads to lower option values.

Under the T-forward measure (in the third set of rows of Table 8) the size of the confidence intervals is not always reduced. In fact, for dummy A the size of the confidence interval is even larger than the original one. In case of dummy client C, however, a change of measure very effectively reduces confidence intervals. Because dummy C needs more calculations than the other dummy clients, since its end date is in the far future, variance reduction for this dummy is in particular desirable.

Under the sensitivity header, the absolute impacts of gender as well as the choice for partner restitution on the option value are displayed. Due to larger survival probabilities, the guarantee option value is larger for all dummies in case they were women. In case the dummies do not choose for partner restitution the guarantee option value is reduced considerably for all three dummies. This is due to the extra return granted for making this choice. Therefore the option value of a female client who opted for partner restitution is largest, while a male without partner restitution is lowest.

The bottom segment of the table presents sensitivities with respect to market parameters. The calculation of these parameters is discussed in Section 9.7 and an interpretation of the symbols of the last column is given in Section 10.2.1. Note that the signs of the sensitivities are almost always identical to the ones in Table 6, which displays the results for the entire LOGA client portfolio.

The results of the sensitivities indicate which client (younger or older) is in particular vulnerable to what specific risk. The sensitivity with respect to shifts in the yield curve and the interest rate related parameters $a$ and $\sigma_r$ indicate that the option value of client C suffers from great interest rate risk. The option value of dummy client A, however, seems most vulnerable to the level of the initial deposit account and the equity volatility parameter $\sigma_s$, since it lacks the time to recover from a bad position or equity shock. The correlation coefficient $\rho$ does not carry the same sign in the three evaluated cases. An absolute rise of $\rho$ lowers the option value for dummy A, while it increases the option value for dummies B and C.

In the next section the simulation results for Levensloop Rendement dummies D, E, and F are discussed.

### 10.3.2 Levensloop Rendement Results

This section presents similar analysis for the dummies D, E, and F (for the Levensloop Rendement product) as in the previous section for A, B, and C (for the LOGA product). Since D, E, and F are Levensloop Rendement policy holders, their deposit account is invested as a life-cycle mix (Figure 1). The position in this life-cycle mix is determined by the client’s number of years
left until he has reached his/her end date. Table 9 displays the simulation results when 20,000 runs are performed.

<table>
<thead>
<tr>
<th>Table 9: Levensloop Rendement Dummies Results</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimate</strong></td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td><strong>Original Monte Carlo</strong></td>
</tr>
<tr>
<td>Option Value</td>
</tr>
<tr>
<td>% of Deposit Account</td>
</tr>
<tr>
<td>Lowerbound CI</td>
</tr>
<tr>
<td>Upperbound CI</td>
</tr>
<tr>
<td><strong>Antithetic Variables</strong></td>
</tr>
<tr>
<td>Option Value</td>
</tr>
<tr>
<td>Lowerbound CI</td>
</tr>
<tr>
<td>Upperbound CI</td>
</tr>
<tr>
<td>Reduction of CI</td>
</tr>
<tr>
<td><strong>T-forward Measure</strong></td>
</tr>
<tr>
<td>Option Value</td>
</tr>
<tr>
<td>Lowerbound CI</td>
</tr>
<tr>
<td>Upperbound CI</td>
</tr>
<tr>
<td>Reduction of CI</td>
</tr>
<tr>
<td><strong>Sensitivities</strong></td>
</tr>
<tr>
<td>Female</td>
</tr>
<tr>
<td>No Partner</td>
</tr>
<tr>
<td>Female &amp; No Partner</td>
</tr>
<tr>
<td>Yield Curve</td>
</tr>
<tr>
<td>Deposit Accounts</td>
</tr>
<tr>
<td>$\sigma_s$</td>
</tr>
<tr>
<td>$\sigma_r$</td>
</tr>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
</tbody>
</table>

The first segment of the table shows that the option value is highly depending on the current age and time until the dummy’s end date. It is once more interesting to see, that although the current deposit account as well as the premium payments for dummy F are lower than dummies D and E, the option value is a lot larger. Making use of antithetic variables seems to work well for dummy D, since its antithetic confidence intervals is almost 50% smaller than its original one. For dummies E and F the results are less promising. While for dummy E the antithetic confidence interval is still only a fraction smaller, the confidence interval for dummy F is even
larger when antithetic variables are used. A potential reason is the monotonicity issue, discussed in the previous section.

In the next segment of Table 9, the results of the simulation analysis under the T-forward measure are displayed. The confidence interval is reduced most for the youngest client dummy, while it is not reduced at all for the oldest one.

The lower part of the table deals with sensitivities. It starts with presenting the option values of clients with different characteristics regarding gender and choice of partner restitution when holding all other client input constant. For all dummies the results turn out to be such that for all dummies it holds, that the female dummy with partner restitution yields the largest option value, followed by male with partner, female without partner, and male without partner respectively.

Sensitivities with respect to market parameters are displayed in the bottom segment of the table. The most striking results are derived for dummy D, whose option value tends to be extremely sensitive to changes in the level of the yield curve and the relative size of the deposit accounts. Since its option value is small compared to the option value of the other two dummies, this relative sensitivity would be less extreme in absolute sense. Furthermore, this dummy’s option value is insensitive to changes in the stock price volatility. This is readily explained by the fact that this dummy’s investment mix does not contain stocks anymore.

The results of the other two dummies are more logical in the sense that the sign and size of the sensitivity values can be explained by the later end date of dummy F in most cases. The only exceptions are the Yield Curve statistic, which has a more extreme relative impact on dummy E than on dummy F, and the correlation coefficient $\rho$ has more impact on E than on F.
11 Conclusions and Recommendations

11.1 Conclusions

In this research project guarantee option values of the insurance products LOGA and Levensloop Rendement have been analyzed in a Monte Carlo simulation framework. Firstly in a very standard setting under Black-Scholes assumptions. And finally under Hull-White Black-Scholes assumptions, where the constant interest rate assumption has been relaxed. In order to search for more accurate results without increasing the simulation time, I have used antithetic variables, aggregated client portfolios, and simulated under a different measure. Furthermore I have introduced dummy clients to indicate whether clients who are near their end date are more (less) sensitive to changes in input parameters and a new life-cycle investment mix to check whether the Levensloop Rendement option value can be reduced.

The results of the Black-Scholes simulation procedure indicated that the option value is lowest in case of large interest rate levels and low volatility parameters. This is in line with the perception of the sign of the sensitivities in the Black-Scholes world. I have tried to reduce simulation time by aggregating the client portfolio on common characteristics of clients. This works well for the smaller LOGA client database. But for the product for which it is most interesting, since there are more clients holding the product, it does not. The use of a variance reduction method called antithetic variables, however, does a better job. Especially for the LOGA product large reductions of confidence intervals are derived.

When the model is extended to a Hull-White Black-Scholes model, the guarantee option value of the entire LOGA client portfolio is approximately €11.6 million. In comparison with the sensitivity of this value with respect to the asset volatility, this value is extremely sensitive to changes in interest rate parameters. Dummy clients indicate that in particular option values of persons with end dates in the far future are sensitive to changes in the interest rate. Clients being close to their end dates, however, are most vulnerable to changes in their relative deposit account with respect to their guarantee value. Furthermore they show that being female and having a partner leads to larger option value.

In the Levensloop Rendement framework, the guarantee option value for the entire portfolio is approximately €7.3 million when the original investment life-cycle mix is used and about €2.4 million in case the alternative life-cycle mix is implemented. This indicates that in the field of lowering the option value for the product, changing the investment mix can be a very effective tool. Although the option value is lowered for the alternative mix, it is relatively more sensitive to the specification of most input parameters. Only the equity volatility parameter has a smaller impact on the option value when using the alternative mix. The main result of the analyses on dummy clients is that the option value for clients who are close to their end dates contain low option values which are extremely sensitive to changes in the level of the yield curve and the relative deposit account.

Reducing the variance in the Hull-White Black-Scholes setting can be done by using antithetic variates for both products. Although confidence intervals are more tight, the profit is...
only marginal. For the dummy client an analysis under the T-forward measure is performed as well. In the sense of variance reduction, the confidence intervals for younger dummies are reduced extremely. For clients being close to their end dates, this technique hardly works.

11.2 Recommendations

Although the modeling in this research project includes many relevant aspects regarding the guarantee option value, analyses can be improved in many ways. This section serves to indicate the main components on which progress can be made and the main components of which one should be aware of when using my model for future research.

This thesis indicates to what parameter shocks the guarantee option values are most sensitive to. In the context of risk management, this information is very important if an insurance company wants to eliminate these risks by making use of a hedge. Especially for the Levensloop Rendement which invests in a life-cycle mix, this can be a challenging process. Another possibility to lower financial risks involved in this product is by analyzing for which life-cycle mix the risks are minimized, without eliminating the upside profit potential completely.

The yield curve is modeled rather simply and inaccurate, by interpolating and combining two curves to obtain yields for all relevant moments in time. This leads to a stylized yield curve, which is not very smooth at all. Since this curve is input for the simulation of the Hull-White model as well as the calibration of its parameters, more accurate results can obtained by investing in this area.

Improvement can be made on the calibration segment of this thesis as well. Especially the number of swaptions on which the calibration is done can be extended. Furthermore the calibration can be performed on other financial products.

This research project is restricted to a number of assumptions, which might not be very realistic. For example the premium payment of Levensloop Rendement policy holders is assumed to be constant over time. This can be doubted already by the observation that in the data on average older policy holders pay larger premium amounts than younger policy holders. Furthermore, switching of clients is not modeled. Since the guarantee option cancels in case of early switching, involving this policy holder behaviour in the calculation will lead to lower option values.

The analysis in this thesis is mainly based on the one factor Hull-White model for interest rates in combination with the Black-Scholes stochastic differential equation for the modeling of asset prices. This framework can be extended in many ways. Although Hull and White (1996) argue that one should be careful when it comes to extending the model, better fit can be obtained by making the mean-reverting coefficient time-dependent. The Black-Scholes framework can be extended by making for example the volatility parameter stochastic, as is done in van Haastrecht et al. (2009). In order to model extreme scenarios, one could implement for example a poisson distribution to model accidental shocks in stock volatility.

Finally, I would like to state that one should be aware of the fact that although I modeled asset prices, interest rates, and bond prices, there are many economic variables which might be
relevant in the model and are not modeled at all. For example, the inflation rate can have its influence on the interest rate. Furthermore, one has to realize that I model the complete stock and bond portfolios individually by modeling just one asset.
Appendices

A Definitions and Theorems

A.1 Brownian Motion

The definition is as in Schumacher (2009). A continuous-time process $W_t \ (t \geq 0)$ is said to be a Brownian Motion if it satisfies the following properties.

- $W_0 = 0$.
- If $t_1 < t_2 < t_3 < t_4$, then the increments $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent.
- For any given $t_1$ and $t_2$ with $t_2 > t_1$, the distribution of the increment $W_{t_2} - W_{t_1}$ is the normal distribution with mean 0 and variance $t_2 - t_1$.

A.2 Numéraire

If an $m$–vector function satisfies the following properties:

- The (portfolio) price is in every state positive
- The investment strategy of the portfolio should be self-financing

Then it is called a numéraire.

A.3 Martingale

In continuous time a stochastic process is called a martingale relative to a given filtration $\mathcal{F}_t$ and a certain probability measure $\mathbb{P}$ if it satisfies the following two requirements:

- $E^\mathbb{P}(X_t | \mathcal{F}_s) = X_s$, for every $s < t$
- $E^\mathbb{P}(X_t | \mathcal{F}_s) < \infty$, for every $t \geq 0$

A.4 Girsanov’s Theorem

This description is identical to the Theorem described in Schumacher (2009). Let $W_t$ be a k-vector Brownian motion and let $\lambda_t$ be a k-vector process adapted to $W_t$. The scalar process $\theta_t$ defined by

$$d\theta_t = -\theta_t \lambda_t' dW_t, \quad \theta_0 = 1$$

is a $\mathbb{P}$-martingale and one may take it as a Radon-Nikodym process that defines a change of measure from the original measure $\mathbb{P}$ to a new measure $\mathbb{Q}$. Under this new measure, the process $\tilde{W}_t$ defined by

$$d\tilde{W}_t = \lambda_t dt + dW_t, \quad \tilde{W}_0 = 0$$

is a Brownian motion.
B Derivations

B.1 Equation (3.10)

Integrate \( r_x \) (which is defined by equation (3.9)) from \( t_1 \) to \( t_2 \):

\[
\int_{t_1}^{t_2} r_x ds = \int_{t_1}^{t_2} e^{-as} r_0 ds + \int_{t_1}^{t_2} \int_0^s e^{-as} \theta(u)e^{au} duds + \int_{t_1}^{t_2} \int_0^s \sigma r e^{-au} e^{au} dW_u ds \tag{B.1}
\]

The first term on the right hand side of (B.1) is most easily evaluated:

\[
\int_{t_1}^{t_2} e^{-as} r_0 ds = r_0 \left[ -\frac{1}{a} e^{-as} \right]_{t_1}^{t_2} = r_0 \left( \frac{1}{a} e^{-at_1} - \frac{1}{a} e^{-at_2} \right) = r_0 e^{-at_1} \frac{1-e^{-a(t_2-t_1)}}{a} \tag{B.2}
\]

The second term involves interchanging the order of integration. This is explicitly evaluated in the following derivation:

\[
\int_{t_1}^{t_2} \int_0^s e^{-as} \theta(u)e^{au} duds = \int_{t_1}^{t_2} \int_0^s e^{-as} \theta(u)e^{au} \, ds \, du + \int_{t_1}^{t_2} \int_0^s e^{-as} \theta(u)e^{au} \, duds \\
= \int_{t_1}^{t_2} \int_0^s e^{-as} \theta(u)e^{au} \, ds \, du + \int_{t_1}^{t_2} \int_0^s \theta(u)e^{au} \, e^{-as} \, duds \\
= \int_{t_1}^{t_2} \int_0^s e^{au} e^{-at_1} B(t_1, t_2) \, du + \int_{t_1}^{t_2} \int_0^s \theta(u)e^{au} e^{-au} B(u, t_2) \, du \\
= e^{-at_1} B(t_1, t_2) \int_0^s \theta(u)e^{au} \, du + \int_{t_1}^{t_2} \theta(u) B(u, t_2) \, du \tag{B.3}
\]

Note that the fourth equality sign in derivation (B.3) is based on the analysis performed in (B.2). At this point, there is only one term on the right hand side of equation (B.1) left to be evaluated. This is done in derivation (B.4).

\[
\int_{t_1}^{t_2} \int_0^s \sigma r e^{-au} e^{au} dW_u ds = \sigma_r \int_{t_1}^{t_2} \int_0^s e^{-au} e^{au} dW_u ds + \sigma_r \int_{t_1}^{t_2} \int_0^s e^{-as} e^{au} dW_u ds \\
= \sigma_r \int_{t_1}^{t_2} \int_0^s e^{au} \, dW_u + \sigma_r \int_{t_1}^{t_2} \int_0^s e^{au} \, e^{-as} \, dW_u ds \\
= \sigma_r \int_{t_1}^{t_2} e^{au} \left[ -\frac{1}{a} e^{-as} \right]_{t_1}^{t_2} \, dW_u + \sigma_r \int_{t_1}^{t_2} e^{au} \left[ -\frac{1}{a} e^{-as} \right]_{u}^{t_2} \, dW_u \\
= \sigma_r e^{-at_1} B(t_1, t_2) \int_0^s e^{au} \, dW_u + \sigma_r \int_{t_1}^{t_2} B(u, t_2) \, dW_u \tag{B.4}
\]

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Adding the last terms of the three derivations ((B.2), (B.3), and (B.4)) and executing one last manipulation, leads to the right hand side of equation (3.10).

\[
\int_{t_1}^{t_2} r_s ds = r_0 e^{-at_1} B(t_1, t_2) + e^{-at_1} B(t_1, t_2) \left( \int_{t_1}^{t_2} \theta(u) e^{au} du + \int_{t_1}^{t_2} \theta(u) B(u, t_2) du \right) + \sigma_r e^{-at_1} B(t_1, t_2) \left( \int_{t_1}^{t_2} e^{au} dW_u + \sigma_r \int_{t_1}^{t_2} B(u, t_2) dW_u \right) = B(t_1, t_2) r_{t_1} + \int_{t_1}^{t_2} \theta(u) B(u, t_2) du + \sigma_r \int_{t_1}^{t_2} B(u, t_2) dW_u
\]  

(B.5)

The last line of (B.5) equals exactly the right hand side of equation (3.10). Note that it is obtained by making use of equation (3.9) for \( s = t_1 \).

**B.2 Lemma 3.1**

*Proof.* The probability density function of a normally distributed variable \( X \) with mean \( \mu \) and variance \( \sigma^2 \) is

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

Now calculate \( E(e^X) \):

\[
E(e^x) = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx

= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2+2\mu x+2\sigma^2-\mu^2}{2\sigma^2}} dx

= e^{\mu + \frac{1}{2} \sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2-2\mu x+2\sigma^2-\mu^2-\sigma^2}{2\sigma^2}} dx

= e^{\mu + \frac{1}{2} \sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} dx

= e^{\mu + \frac{1}{2} \sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} dx
\]  

(B.6)

Note that the integrand in (B.6) is in fact the probability density function of a normal variable with mean \( (\mu - \sigma^2) \) and variance parameter \( \sigma^2 \). The total surface beneath a probability density function equals 1 by definition. Therefore it holds that:

\[
E(e^x) = e^{\mu + \frac{1}{2} \sigma^2}
\]  

(B.7)

\[\square\]
B.3 Theorem (3.4)

Proof. Let \( \alpha(t) \) and \( y_t \) be defined by (from equation (3.9)):

\[
\begin{align*}
\alpha(t) &= e^{-at} \left( r_0 + \int_0^t \theta(u) a^u du \right) \\
y_t &= \sigma_r e^{-at} \int_0^t e^{au} dW_u
\end{align*}
\] (B.8) (B.9)

If \( \theta(t) \) is now inserted in (B.8), the result in (3.18) is obtained.

\[
\begin{align*}
\alpha(t) &= e^{-at} \left( r_0 + \int_0^t \theta(u) a^u du \right) \\
&= e^{-at} r_0 + e^{-at} \int_0^t \theta(u) a^u du \\
&= e^{-at} r_0 + e^{-at} \int_0^t a f_0^*(u) e^{au} du + e^{-at} \int_0^t \frac{\partial f_0^*(u)}{\partial u} e^{au} du \\
&\quad + e^{-at} \alpha_r^2 \int_0^t B(0, u) e^{au} du + e^{-at} \frac{\sigma_r^2}{2} \int_0^t B^2(0, u) e^{au} du \\
&= e^{-at} r_0 + e^{-at} \left[ e^{au} f_0^*(u) \right]_0^t + e^{-at} \alpha_r^2 \int_0^t \frac{1 - e^{-au}}{a} du \\
&\quad + e^{-at} \frac{\sigma_r^2}{2} \int_0^t \left( \frac{1 - e^{-au}}{a} \right)^2 e^{au} du \\
&= e^{-at} r_0 + f_0^*(t) - e^{-at} f_0^*(0) + \alpha_r^2 e^{-at} \int_0^t \frac{1 - e^{-au}}{a} du \\
&\quad + e^{-at} \frac{\sigma_r^2}{2} \int_0^t \left( \frac{1 - e^{-au}}{a} \right)^2 e^{au} du \\
&= f_0^*(t) + \frac{\sigma_r^2}{2} B^2(0, t)
\end{align*}

In order to prove equation (3.19), take a look at the following formula:

\[ e^{\alpha(t+\Delta t)} y_{t+\Delta t} - e^{\alpha t} y_t = \sigma_r \int_t^{t+\Delta t} e^{au} dW_u \] (B.18)

Notice that the variables in (B.18) are independent and normally distributed with mean 0 and variance

\[ \sigma_r^2 \int_t^{t+\Delta t} e^{2au} du = \sigma_r^2 e^{2at} e^{2\Delta t} - \frac{1}{2a} \]

Rearranging the terms of equation (B.18) and dividing both sides by \( e^{\alpha(t+\Delta t)} \) completes the
proof.
C Levenberg-Marquardt Algorithm

The Levenberg-Marquardt Algorithm (LMA) is used to find a vector of parameters $p$ which satisfies a certain least squares curve fitting problem best. It combines the Gauss-Newton algorithm and the gradient descent method. Depending on the results of a certain parameter set a new parameter set might be adopted and more (or less) weight is given to one of the algorithms underlying the LMA (by the so-called ‘damping parameter’ $\lambda$). This section describes the Levenberg-Marquardt algorithm and is based on Vollrath and Wendland (2009).

The algorithm starts with selecting an objective function which has to be minimized. In the least squares setting this should take the following form:

\[
S(p) = \sum_{i=1}^{n} (U_i - V_i(p))^2 \tag{C.1}
\]

In this setting the $U_i$ symbolizes a certain data point, where $V_i(p)$ is the estimated value according to the model with input $p$.

The second step of the LMA is the choice of an initial vector of parameters $p_0$. Although, this might look like a rather trivial step it is crucial to choose parameters close to the overall minimum. In case of local minima, the solution might not converge to the overall one. If one is aware of reasonable values for the optimal vector $p$, one should choose values from this region.

Before a loop can be initiated to determine the best vector of parameters, one has to assign to the starting damping parameter $\lambda_0$ a suitable value. A low value (like 0.000001) tends to lead to the best results.

Now a loop can be started in order to derive the vector $p^*$ which minimizes the objective function (C.1). The easiest way to see how this loop works, is by taking a closer look at the formula which calculates the proposed new vector $p_{new}$:

\[
p_{new} = p + \left[ J^T J + \lambda \text{diag}(J^T J) \right]^{-1} J^T [U - V(p)], \lambda > 0
\]

The matrix $J$ symbolizes the Jacobian matrix of $V_i(p)$, and $J^T$ is simply its transpose. Note that the size of the Jacobian depends on the number of data points (the size of the vector $U$), which determines the number of rows, and the number of parameters ($p$) to be estimated, which determines the number of columns. The vector $V(p)$ denotes the predictions of the model under the parameter values $p$. When the new set of parameters is calculated it is tested whether it improves the solution or not. By calculating $S(p_{new})$ and comparing to $S(p)$, one should act according the following if statements:

- If $S(p_{new}) < S(p)$, then the proposed $p_{new}$ is accepted and $\lambda$ is decreased by a certain factor (usually 10).
- If $S(p_{new}) > S(p)$, the proposed $p_{new}$ is rejected and $\lambda$ is increased by a certain factor (usually 10).
A large $\lambda$ gives more weight to the gradient descent part, while a small $\lambda$ focusses on the Gauss-Newton method.

In the formulas of this thesis, the derivative of $V_i(p)$ is not analytically tractable. Therefore it is approximated by simulation.

After iterating this loop many times (depending on the pace of convergence), the vector $p_{new}$ has converged to the actual vector $p_{act}$.
D Monte Carlo Method

This Appendix follows the reasoning of Glasserman (2003) and Schumacher (2009). Monte
Carlo simulation is an easy implementable technique for estimating (conditional) expectations.
It is in particular applicable in finance when the pricing of financial derivatives involves such
expectations.

In the setting of this thesis the equivalent martingale measure is implemented, for example.
Monte Carlo can then be useful if it is possible to generate samples from the distribution of
the stochastic variable under this risk-neutral measure. Let the model be based on stochastic
differential equations, then exact samples can be generated if an analytic solution is known.
Otherwise, if no analytic solution is known, the stochastic differential equation needs to be
solved numerically. When the number of draws gets large, then the Law of Large Numbers
claims that the estimate converges to its correct value.

The expression for the Monte Carlo estimate of the expectation of $N$ independent random
draws $x_i$ of distribution $X$ is displayed in equation (D.1).

$$MCE = \frac{1}{N} \sum_{i=1}^{N} x_i$$ (D.1)

Therefore the Monte Carlo estimate (MCE) is a random variable itself and in expectation
equals the quantity of interest $X$. If it holds that $\text{Var}(X)$ is finite, the variance of the estimate
is also finite and is defined by equation (D.2).

$$\text{Var}(MCE) = \frac{1}{N^2} \text{Var} \left( \sum_{i=1}^{N} X \right) = \frac{1}{N^2} (N \ast \text{Var}(X)) = \frac{1}{N} \text{Var}(X)$$ (D.2)

For the computation of a confidence interval one needs the standard deviation of the MCE.
Since this is typically not known, one needs to estimate this itself by using equation (D.3).

$$\hat{s} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - MCE)^2}$$ (D.3)

With this all the knowledge is present to calculate a, for example two-sided 95%, confidence
interval. Formula (D.4) gives its definition.

$$[MCE + \Phi^{-1}(0.025) \frac{1}{\sqrt{N}} \hat{s}, MCE + \Phi^{-1}(0.975) \frac{1}{\sqrt{N}} \hat{s}]$$ (D.4)
E Antithetic Variables

This appendix covers the 'Antithetic Variables' method, which is a variance reduction method in the Monte Carlo simulation setting. Variance reduction methods fine-tune a simulation program in such a way that either the variance is reduced when using the same number of simulation runs, or similarly less simulation runs are needed to obtain a same variance level. This technique therefore leads to a faster computation of relatively tight confidence intervals.

While the method can be implemented in various forms, I focus primarily on the method I use. The description is based on the analysis of Glasserman (2003). The idea of antithetic variables is to reduce the variance by compensating large random variates drawn from the distribution of interest by small random variates. This negative correlation has to lead to an overall variance reduction.

If one takes a closer look at a Brownian motion (which is the driving process of both the formulas (2.10) and (3.7)) for example, one can see that increments are subject to a standard normal distribution. If on the interval $[0,1]$ a uniformly distributed variable $U_i$ is defined, then the corresponding random standard normal variate is defined by $Z_i = \Phi^{-1}(U_i)$. The simulation of a Brownian motion is based on a sequence $Z_1, Z_2, ...$ of independently and identically (i.i.d.) $N(0,\Delta t)$ distributed variables, given a time step of size $\Delta t$. Simulating a Brownian motion path by using variables $-Z_1, -Z_2, ...$ (where $-Z_i = -\Phi^{-1}(U_i) = \Phi^{-1}(1 - U_i)$) exactly mirrors the original path in the origin. Averaging both paths suggests variance reduction.

Suppose the objective is to derive an accurate value for $E[Y]$ and that the antithetic sampling procedure is used. This procedure leads to pairs of observations $(Y_1, \tilde{Y}_1), (Y_2, \tilde{Y}_2), ..., (Y_N, \tilde{Y}_N)$ having the following key features:

- The pairs $(Y_1, \tilde{Y}_1), (Y_2, \tilde{Y}_2), ..., (Y_N, \tilde{Y}_N)$ are i.i.d.
- For each $i$, $Y_i$ and $\tilde{Y}_i$ have the same distribution, though ordinarily they are not independent.

The variable $Y$ is used to indicate a random variable with the common distribution of the $Y_i$ and $\tilde{Y}_i$. The antithetic variates estimator is calculated straightforwardly:

$$\hat{Y}_{AV} = \frac{1}{2N} \left( \sum_{i=1}^{N} Y_i + \sum_{i=1}^{N} \tilde{Y}_i \right) = \frac{1}{N} \sum_{i=1}^{N} \left( Y_i + \tilde{Y}_i \right)$$

(E.1)

The expression on the right hand side of the second equality sign makes clear that $\hat{Y}_{AV}$ is the sample mean of the $N$ independent observations $\left(\frac{Y_i + \tilde{Y}_i}{2}\right)$, therefore the Central Limit Theorem can be applied:

$$\frac{\hat{Y}_{AV} - E[Y]}{\sigma_{AV}/\sqrt{N}} \to N(0,1)$$

(E.2)

Note that $\sigma^2_{AV} = \text{Var}\left[\frac{Y_i + \tilde{Y}_i}{2}\right]$. The importance of the variance parameter follows for example from the computation of a confidence interval. Note that the limit of formula (E.2) holds even if we replace $\sigma_{AV}$ by $s_{AV}$ (the sample standard deviation of the $N$ antithetic variables). An
Asymptotic 95% confidence interval can be calculated by:

\[ \hat{Y}_{AV} \pm \Phi^{-1}(0.025) \frac{s_{AV}}{\sqrt{N}} \]  

(E.3)

The question of interest is now: Under what conditions does the method of antithetic variables lead to variance reduction? In order to compare the effectiveness of antithetic variables with the Monte Carlo method without this variance reduction technique, I have to make two assumptions. First of all, I assume that the computational effort required to generate a pair \((Y_i, \tilde{Y}_i)\) is twice as much as the effort to determine one \(Y_i\) variable. The second assumption states that there are no computational savings for flipping a sign (from \(Z_i\) to \(-Z_i\)) rather than generating new standard random variables. As one can imagine the first assumption more or less implies the second. Summarizing the assumptions: It takes the same amount of effort to compute \(N\) variables \(Y_i\) (implementing the Monte Carlo method without variance reduction techniques) and computing \(\frac{1}{2}N\) variables of \(Y_i\) and their corresponding antithetics \(\tilde{Y}_i\) (applying the antithetic variables variance reduction method).

With this information one can state the definition of variance reduction as follows:

\[ Var[Y_i + \tilde{Y}_i] < 2Var[Y_i] \]  

(E.4)

Evaluating the left-hand side of inequality (E.4) boils down to:

\[ Var[Y_i + \tilde{Y}_i] = Var[Y_i] + Var[\tilde{Y}_i] + 2Cov[Y_i, \tilde{Y}_i] \]

\[ = 2Var[Y_i] + 2Cov[Y_i, \tilde{Y}_i] \]  

(E.5)

If equation (E.5) is subtracted from minus (E.4) the following inequality is derived:

\[ Cov[Y_i, \tilde{Y}_i] < 0 \]  

(E.6)

In other words, the criterium for variance reduction is satisfied in the context of antithetic variables when equation (E.6) holds. The antithetic variates simulating strategy tries to exploit the negatively correlated variables \(Z\) and \(-Z\) and hopes for negative covariance in the end.
F Output Results

This appendix section displays the additional output results of the Black-Scholes World. Section F.1 shows the results for the LOGA client portfolio, while Section F.2 presents the results for the Levensloop Rendement product.

The column headers need to be interpreted as follows: 'Interest' shows the level of the interest parameter, 'Volatility' shows the level of the volatility parameter, 'Option Value' denotes the corresponding value of the option (in millions), '% of Dep. Acc.' denotes the percentual option value with respect to the total amount of the deposit accounts, 'L.B. 95%' symbolizes the lowerbound of a two-sided 95% confidence interval (in millions), 'U.B. 95%' displays its upperbound (in millions), and 'C.I. Red.' denotes the percentual reduction of the size of the confidence interval when antithetic variables are used. Note that it is not possible to set one volatility parameter for the Levensloop Rendement product, therefore I display its bounds in the 'Volatility' column.

F.1 Black-Scholes LOGA

Table 10: Results of the Option Value of the LOGA Client Portfolio

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%</td>
<td>€49.47</td>
<td>39.80 %</td>
<td>€48.95</td>
<td>€50.00</td>
</tr>
<tr>
<td>2%</td>
<td>10%</td>
<td>€62.20</td>
<td>50.04 %</td>
<td>€61.38</td>
<td>€63.02</td>
</tr>
<tr>
<td>2%</td>
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<td>61.79 %</td>
<td>€75.78</td>
<td>€77.83</td>
</tr>
<tr>
<td>3%</td>
<td>5%</td>
<td>€23.67</td>
<td>19.04 %</td>
<td>€23.29</td>
<td>€24.05</td>
</tr>
<tr>
<td>3%</td>
<td>10%</td>
<td>€38.39</td>
<td>30.89 %</td>
<td>€37.77</td>
<td>€39.02</td>
</tr>
<tr>
<td>3%</td>
<td>15%</td>
<td>€53.28</td>
<td>42.86 %</td>
<td>€52.46</td>
<td>€54.10</td>
</tr>
<tr>
<td>4%</td>
<td>5%</td>
<td>€8.98</td>
<td>7.22 %</td>
<td>€8.76</td>
<td>€9.20</td>
</tr>
<tr>
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<td>10%</td>
<td>€22.47</td>
<td>18.08 %</td>
<td>€22.02</td>
<td>€22.92</td>
</tr>
<tr>
<td>4%</td>
<td>15%</td>
<td>€35.07</td>
<td>28.22 %</td>
<td>€34.43</td>
<td>€35.72</td>
</tr>
</tbody>
</table>
Table 11: Results of the Option Value of the aggregated LOGA Client Portfolio

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
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<td>€49.85</td>
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<td>10%</td>
<td>€62.48</td>
<td>50.27 %</td>
<td>€61.67</td>
<td>€63.29</td>
</tr>
<tr>
<td>2%</td>
<td>15%</td>
<td>€77.55</td>
<td>62.39 %</td>
<td>€76.53</td>
<td>€78.58</td>
</tr>
<tr>
<td>3%</td>
<td>5%</td>
<td>€23.66</td>
<td>19.03 %</td>
<td>€23.28</td>
<td>€24.04</td>
</tr>
<tr>
<td>3%</td>
<td>10%</td>
<td>€38.44</td>
<td>30.93 %</td>
<td>€37.82</td>
<td>€39.07</td>
</tr>
<tr>
<td>3%</td>
<td>15%</td>
<td>€53.43</td>
<td>42.99 %</td>
<td>€52.61</td>
<td>€54.26</td>
</tr>
<tr>
<td>4%</td>
<td>5%</td>
<td>€8.96</td>
<td>7.21 %</td>
<td>€8.74</td>
<td>€9.18</td>
</tr>
<tr>
<td>4%</td>
<td>10%</td>
<td>€22.52</td>
<td>18.11 %</td>
<td>€22.06</td>
<td>€22.97</td>
</tr>
<tr>
<td>4%</td>
<td>15%</td>
<td>€36.20</td>
<td>29.13 %</td>
<td>€35.55</td>
<td>€36.85</td>
</tr>
</tbody>
</table>

Table 12: Results of the Option Value of the LOGA Client Portfolio when using Antithetic Variables

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
<th>C.I. Red.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%</td>
<td>€49.60</td>
<td>39.90 %</td>
<td>€49.55</td>
<td>€49.65</td>
<td>90.46 %</td>
</tr>
<tr>
<td>2%</td>
<td>10%</td>
<td>€62.12</td>
<td>49.98 %</td>
<td>€61.98</td>
<td>€62.26</td>
<td>82.93 %</td>
</tr>
<tr>
<td>2%</td>
<td>15%</td>
<td>€77.14</td>
<td>62.06 %</td>
<td>€76.98</td>
<td>€77.31</td>
<td>83.90 %</td>
</tr>
<tr>
<td>3%</td>
<td>5%</td>
<td>€23.50</td>
<td>18.91 %</td>
<td>€23.38</td>
<td>€23.63</td>
<td>66.89 %</td>
</tr>
<tr>
<td>3%</td>
<td>10%</td>
<td>€38.14</td>
<td>30.69 %</td>
<td>€37.94</td>
<td>€38.36</td>
<td>66.53 %</td>
</tr>
<tr>
<td>3%</td>
<td>15%</td>
<td>€53.13</td>
<td>42.75 %</td>
<td>€52.89</td>
<td>€53.37</td>
<td>70.73 %</td>
</tr>
<tr>
<td>4%</td>
<td>5%</td>
<td>€8.88</td>
<td>7.14 %</td>
<td>€8.73</td>
<td>€9.02</td>
<td>33.33 %</td>
</tr>
<tr>
<td>4%</td>
<td>10%</td>
<td>€22.38</td>
<td>18.01 %</td>
<td>€22.14</td>
<td>€22.63</td>
<td>45.68 %</td>
</tr>
<tr>
<td>4%</td>
<td>15%</td>
<td>€35.98</td>
<td>28.95 %</td>
<td>€35.70</td>
<td>€36.26</td>
<td>56.56 %</td>
</tr>
</tbody>
</table>
Table 13: Results of the Option Value of the aggregated LOGA Client Portfolio when using Antithetic Variables

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
<th>C.I. Red.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%</td>
<td>€49.64</td>
<td>39.93 %</td>
<td>€49.58</td>
<td>€49.69</td>
<td>89.31 %</td>
</tr>
<tr>
<td>2%</td>
<td>10%</td>
<td>€61.97</td>
<td>49.86 %</td>
<td>€61.84</td>
<td>€62.10</td>
<td>83.94 %</td>
</tr>
<tr>
<td>2%</td>
<td>15%</td>
<td>€77.22</td>
<td>62.12 %</td>
<td>€77.05</td>
<td>€77.39</td>
<td>83.41 %</td>
</tr>
<tr>
<td>3%</td>
<td>5%</td>
<td>€23.63</td>
<td>19.01 %</td>
<td>€23.49</td>
<td>€23.76</td>
<td>64.24 %</td>
</tr>
<tr>
<td>3%</td>
<td>10%</td>
<td>€38.22</td>
<td>30.75 %</td>
<td>€38.00</td>
<td>€38.44</td>
<td>64.85 %</td>
</tr>
<tr>
<td>3%</td>
<td>15%</td>
<td>€53.26</td>
<td>42.85 %</td>
<td>€53.02</td>
<td>€53.50</td>
<td>70.94 %</td>
</tr>
<tr>
<td>4%</td>
<td>5%</td>
<td>€8.94</td>
<td>7.19 %</td>
<td>€8.80</td>
<td>€9.09</td>
<td>33.49 %</td>
</tr>
<tr>
<td>4%</td>
<td>10%</td>
<td>€22.48</td>
<td>18.08 %</td>
<td>€22.23</td>
<td>€22.73</td>
<td>45.05 %</td>
</tr>
<tr>
<td>4%</td>
<td>15%</td>
<td>€36.19</td>
<td>29.11 %</td>
<td>€35.90</td>
<td>€36.47</td>
<td>56.05 %</td>
</tr>
</tbody>
</table>

F.2 Black-Scholes Levensloop Rendement

Table 14: Results of the Option Value of the Levensloop Rendement Client Portfolio

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%/23%</td>
<td>€24.17</td>
<td>16.92 %</td>
<td>€23.05</td>
<td>€25.30</td>
</tr>
<tr>
<td>3%</td>
<td>5%/23%</td>
<td>€13.23</td>
<td>9.26 %</td>
<td>€12.46</td>
<td>€14.01</td>
</tr>
<tr>
<td>4%</td>
<td>5%/23%</td>
<td>€6.83</td>
<td>4.78 %</td>
<td>€6.32</td>
<td>€7.33</td>
</tr>
</tbody>
</table>

Table 15: Results of the Option Value of the aggregated Levensloop Rendement Client Portfolio

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%/23%</td>
<td>€21.24</td>
<td>14.86 %</td>
<td>€20.17</td>
<td>€22.30</td>
</tr>
<tr>
<td>3%</td>
<td>5%/23%</td>
<td>€11.30</td>
<td>7.91 %</td>
<td>€10.59</td>
<td>€12.02</td>
</tr>
<tr>
<td>4%</td>
<td>5%/23%</td>
<td>€5.64</td>
<td>3.95 %</td>
<td>€5.19</td>
<td>€6.10</td>
</tr>
</tbody>
</table>

Table 16: Results of the Option Value of the aggregated Levensloop Rendement Client Portfolio (including Moneyness)

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%/23%</td>
<td>€21.36</td>
<td>14.95 %</td>
<td>€20.30</td>
<td>€22.43</td>
</tr>
<tr>
<td>3%</td>
<td>5%/23%</td>
<td>€11.38</td>
<td>7.97 %</td>
<td>€10.67</td>
<td>€12.10</td>
</tr>
<tr>
<td>4%</td>
<td>5%/23%</td>
<td>€5.69</td>
<td>3.98 %</td>
<td>€5.23</td>
<td>€6.15</td>
</tr>
</tbody>
</table>
Table 17: Results of the Option Value of the Levensloop Rendement Client Portfolio when using Antithetic Variables

<table>
<thead>
<tr>
<th>Interest</th>
<th>Volatility</th>
<th>Option Value</th>
<th>% of Dep. Acc.</th>
<th>L.B. 95%</th>
<th>U.B. 95%</th>
<th>C.I. Red.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5%/23%</td>
<td>€24.69</td>
<td>17.28 %</td>
<td>€24.05</td>
<td>€25.33</td>
<td>43.11 %</td>
</tr>
<tr>
<td>3%</td>
<td>5%/23%</td>
<td>€13.48</td>
<td>9.43 %</td>
<td>€12.92</td>
<td>€14.04</td>
<td>27.74 %</td>
</tr>
<tr>
<td>4%</td>
<td>5%/23%</td>
<td>€6.93</td>
<td>4.9 %</td>
<td>€6.52</td>
<td>€7.35</td>
<td>17.82 %</td>
</tr>
</tbody>
</table>
References


