Asset Allocation and Consumption Smoothing under Business Cycle and State Uncertainty

by

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Abstract

The paper investigates the optimal asset allocation and consumption problems under the assumption of mean-reverting stock return and unobservable state variable. The explicit solutions are given for the investors with power utility and habit formation, and the numerical solution is given for the investor with Epstein-Zin preference. Computation and simulations are conducted to examine the properties of the optimal strategies. We are especially interested in the consumption smoothing and find that the Epstein-Zin preference and habit formation will smooth the consumption stream while the classic investor will experience large changes of consumption.
1 Introduction

Optimal asset allocation and consumption strategy has been one of the heated topics in financial economics. The issue is not only highly theoretical but also highly practical. First, it’s highly theoretical. We need specific assumptions about asset return process and representative agent’s behavior to get some optimal strategies. Different assumptions would possibly lead to different results. Besides, in order to do some analysis we need to apply complicated mathematics. The closed-form solution can only be obtained for some special cases. For the general cases, we would get the numerical solutions or approximate solutions. At the same time it is also practical. The optimal strategies could give us some economic intuitions and a better understanding of certain phenomena. It can also be used as a guideline for investment and consumption in real life.

In this paper I would first try to solve the optimal asset allocation and consumption problems under certain assumptions. Many literatures and evidences have shown that stock returns have the mean-reverting phenomenon. So in this paper I would assume that the equity return exhibits mean reversion. Besides, some literatures have shown that there exists state uncertainty in reality. In the spirit of these literatures and evidences, I would assume the state variables are unobservable in the model.

Based on the above assumptions, I would try to solve the optimal asset allocation and consumption problems for three different categories of investors. The first category is the classic investors whose utility is the popular iso-elastic CRRA form. The second category is the investors with the Epstein-Zin preference whose utility function can disentangle the elasticity of intertemporal substitution from the risk aversion. Enlightened by the concept of habit formation and keeping up with the Joneses, the third category is assumed to be the investors with internal habit formation.
The utility gained from the currently consumption depends both on the currently amount of consumption and the benchmark of the consumption. That is to say, the utility function would take the form of $f(c_t - c_b)$ where $c_b$ is a benchmark of the consumption and it can be updated from period to period.

After that I would like to make some comparisons based on the results gained for the three different categories of investors. I would focus on the consumption smoothing exhibited by the investors.

The remainder of the paper is organized as follow. Section 2 would give us a brief review of the literatures on asset allocation strategy, consumption strategy and related fields. The setup of the model and the optimization problem would be defined in section 3A. In section 3B the optimization problem would be solved for the power utility function by the application of the martingale method developed by Cox and Huang (1989), and in section 3C for Epstein-Zin utility using numerical optimization. In section 3D new criteria of consumption smoothing would be constructed and used to measure the smoothness of consumption. Some discussions would be given in section 4. Section 5 concludes.

2 Literature review

This section would give us a brief review on the literatures in the field of asset allocation strategy, consumption strategy and other related fields.

Markowitz is the pioneer in the field of modern asset allocation. Markowitz (1952) introduced the mean-variance analysis which is often viewed as the start of the modern finance theory. He showed how investors should choose their optimal portfolios when they just care about the mean and variance of the assets return. In this analysis the investors would just look one period ahead. So it’s a static model.

To investigate the optimal asset allocation and consumption strategies for the
long horizon investors over multiple periods, it’s necessary to introduce the dynamic models. Samuelson (1969) and Merton (1969, 1971) are viewed to be the pioneers in this field.

Samuelson (1969) analyzed a discrete-time investment and consumption model. The investor aimed at maximizing his/her overall expected utility from consumption. By employing the dynamic programming method, he showed us the optimal investment and consumption strategies for the investor under the specific conditions.

Merton (1969) investigated a continuous-time model for the optimal investment and consumption problem. The two-asset model with CRRA utility was investigated in details in this paper. He showed us that the optimal proportion of the total wealth invested in stock is independent of the amount of total wealth and horizon\(^1\). That is to say, the optimal proportion is a constant.

Samuelson (1969) and Merton (1969) show us the two assumptions under which the investor behaves myopically even though his/her investment horizon is long enough. The first is that the investor has log utility, and the second is that the asset returns are IID and the investor has the power utility.

Optimal asset allocation and consumption problems with more general conditions such as stochastic opportunity set are investigated by the later literatures such as Kim and Omberg (1996), Wachter (2002), etc. Under these assumptions, the long-horizon investors would behave differently from the short-horizon investors.

A simple assumption about the equity return is random walk. But as mentioned in section 1, stock return exhibits mean reversion and there are many literatures on this issue. DeBondt and Thaler (1985) indicated that stock prices in the US stock market contained a strong mean reverting component. Fama and French (1988) and Poterba and Summers (1988) showed evidences that there existed mean reversion in the equity return. Exley et al. (2004) gave a detailed discussion about mean reversion.

\(^1\) The result depends on the specific assumptions such as the form of the bequest function.
The assumption of mean reversion is employed by Wachter (2002), Dai et al. (2010), etc.

Besides, it would be easier to deal with the model with complete market. But the investor would not have complete information about the status of the economy or the business conditions in reality. That is to say, there would be some uncertainty. William (1977), Klein and Bawa (1977) and Gennotte (1986) examined the unobservable state variable model which is employed by Dai et al. (2010).

The utility function plays a pivotal role in the models describing the optimal asset allocation and consumption problems. The classic investors are usually assumed to have the power utility. A drawback of the power utility is that it can’t disentangle the elasticity of intertemporal substitution from the risk aversion. The Epstein-Zin utility function introduced by Epstein and Zin (1989, 1991) is able to permit risk aversion to be disentangled from the elasticity of intertemporal substitution. The continuous-time Epstein-Zin utility function is defined recursively, and can’t be written explicitly as an expectation of future consumption. The explicit solution for the Epstein-Zin preference can only be obtained for some specific cases. Campbell (1993) employed the Epstein-Zin utility function to solve the intertemporal asset pricing issue.

Another important non-additive utility function is related with the concept of habit formation. It is based on the fact that taste and preference can possibly be cultivated and influenced by the past consumption. In other words, the satisfaction drawn from consuming certain amount of consumption may depend not only on the current amount of the consumption but also on some benchmarks. The habit is divided into two categories: internal habit and external habit. The internal habit assumes that the past consumption affects the habit process and the current and future consumption choice. The external habit which is also named as catching up with Joneses assumes that the past consumption only enters into the habit process but does not affect the
current and future consumption choice. Sundaresan (1989) built a model based on the internal habit to solve a general equilibrium example and a partial equilibrium example, and makes some comparisons with the corresponding results gained by time separable utility. Constantinides (1990) also applied the concept of habit formation. The indicator used in the paper is quite similar to the one used in Sundaresan’s paper. He used the setup to solve the risk premium puzzle with rational expectation and an economy allowed for production.

3A Setup of the Model

This section would demonstrate the basic setups of the model and the optimization problem.

There are two assets available in a frictionless continuous time market. One is the risk-free asset (bond), and the other is the risky asset (stock). For the purpose of simplicity, there is no dividend paid by the risky asset or non-financial income during the investment horizon.

The price of the risk-free bond at time $t$ is denoted by $B_t$. Assume that the risk-free interest rate is a constant during the investment horizon, and it is described by $r$. Then the price of the risk-free bond would follow the process:

$$dB_t = rB_t dt \quad (1)$$

Let $S_t$ denote the stock price at time $t$. It is assumed to follow the following stochastic process:

$$dS_t = \mu_S dt + \sigma_S S_t dZ_s \quad (2)$$

where $Z_s$ is a standard, one dimensional Brownian motion with zero drift and unit variance. The volatility term $\sigma_S$ is assumed to be a positive constant. The current drift term $\mu_S$ is the expected return of the stock at time $t$. It is assumed to follow an Ornstein-Uhlenbeck process:
where $\theta$, $\bar{\mu}$ and $\sigma_\mu$ are assumed to be three positive constants, $Z_{\mu t}$ is a second standard, one dimensional Brownian motion which is jointly normally distributed with $Z_{st}$, the correlation between them is given by

$$E[dZ_{st}dZ_{\mu t}] = \rho dt \quad |\rho| \neq 1$$

These assumptions reflect the characterization of the financial market that the risk premia of the stock is time-varying. Kim and Omberg (1996) used this characterization in the purpose of studying the dynamic nonmyopic portfolio behavior.

In the spirit of Merton (1980), Gennotte (1986) and Dai et al. (2010), the investors can’t observe the expected return $\mu_t$. But they are assumed to have known the values of the parameters. That is to say, they have a clear idea about the evolution of the expected return $\mu_t$. As mentioned by Dai et al. (2010) this is a relatively strong assumption and for more general models we may use the Bayesian method and take the parameter uncertainty into consideration. However, the simplified assumptions still reflects the essence of incomplete information in the market.

Since $|\rho| \neq 1$ and $\mu_t$ is unobservable, we are dealing with an incomplete market with incomplete information. Generally speaking, it would be easier to deal with a complete market with complete information. According to the literatures we are able to do some transformation.

Dai et al. (2010) points out that the investor’s decision making process can be decomposed into two stages: inference stage and investment stage. In the first stage the investor uses the available information to form his/her own estimation about the unobservable expected return. In the second stage the investor tries to solve the optimization problem by using the already estimated expected return.

Following Dai et al. (2010), let $m_t$ and $v_t$ denote the conditional mean and

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variance of the investor’s estimation about the expected return. And the initial values
of $m_i$ and $v_i$ are assumed to be drawn from a normal distribution with a mean of $m_0$ and a variance of $v_0$. In that case the evolutions of the mean and variance can be expressed by the following two stochastic differential equations:

$$dm_i = \theta(\mu - m_i)dt + \frac{\sigma_i, \sigma + v_i}{\sigma_i^2} \left( \frac{dS_t}{S_t} - m_i dt \right)$$  (5)

$$dv_i = (-2 \theta v_i + \sigma_i^2 - \frac{(\sigma_i, \rho + v_i)^2}{\sigma_i^2})dt$$  (6)

As discussed in Gennotte (1986), $m_i$ is a diffusion process and $v_i$ is a deterministic function of time. These two equations describe the way in which the investor updates his/her estimation about the expected return and estimates the opportunity set.

Then enlightened by Gennotte (1986), we can define an innovation process $Z_i$ as the normalized deviation of the return from its conditional mean:\(^3\)

$$dZ_i = \frac{\left( \frac{dS_t}{S_t} - m_i dt \right)}{\sigma_i}$$

and $Z_i^0 = 0$  (7)

The innovation process $Z_i$ is observable, and it’s a standard one dimensional Brownian motion. $Z_i$ and $m_i$ determine the path of $S_t$, and $Z_i$ contains exactly the same information that is contained in original price path.

Substituting $Z_i$ for $Z_i$, the return over some infinitesimal time period at time $t$ would be given by:

$$dS_t = m_i S_t dt + \sigma_s S_t dZ_i$$  (8)

$$dm_i = \theta(\mu - m_i)dt + \frac{\sigma_i, \rho + v_i}{\sigma_i^2} dZ_i$$  (9)

In general it would be impossible for us to replicate arbitrage payoff in the

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\(^3\) Gennotte (1986) pp.783
original incomplete market. That is to say, it is possible the optimal consumption stream can’t be replicated. With the transformation we can now make sure the optimal consumption stream can be replicated. It seems as if it’s a magic since we transform the incomplete market into a complete market. In reality, we are now dealing with the subset of the original issues. And the subset is what we really concern about.

With the existence of unobservable state variable and uncertainty, the limit of $\nu_t$ is not zero as $t$ tends to infinity. So the estimation error denoted by $\nu_t$ can’t be ignored even if $t$ tends to infinity. The existence of $\nu_t$ would make the investor’s evaluation of his/her investment opportunity set imprecise. Dai et al. (2010) points out that there exists a stable phase in which $\nu_t$ converges to a constant. The constant is given by:

$$\nu = \sigma_s^2 \left( \sqrt{\theta^2 + \frac{\sigma_m^2}{\sigma_s^2} + 2\theta\rho \frac{\sigma_m}{\sigma_s} - \theta - \rho \frac{\sigma_m}{\sigma_s}} \right).$$  (10)

In that phase the investor would keep updating his estimation of the expected return, but the variance of estimated return converges to a constant. The situation is like an equilibrium where the newly arrived information is deducted by the new uncertainty. They have shown us that the expected return estimated by the investor is less volatile than that of the latent expected return in this phase.\(^4\) In the rest of the paper, I would assume that the economy is already in the stable phase since the financial market has existed for a long time.

Now let us consider the investor’s optimization problem. The investor uses the estimated expected return to solve his/her asset allocation and consumption problem.

Assume the investor starts with the amount of initial wealth denoted by $W_0$. Instead of being interested in the final wealth, the investor cares about his/her consumption stream between the beginning and some horizon $T$. At time $t$ ($0 < t < T$), the investor would have to make decisions about asset allocation and the current amount of consumption. Denote the proportion of the wealth invested in stock at time

\(^4\) Dai et al. (2010) proposition 3.1 demonstrates the property. The proof is given in their paper.
t by $\alpha_t$, so $1 - \alpha_t$ is the proportion of wealth invested in bond. Denote the amount of consumption at time $t$ by $c_t$.

### 3B The investor with power utility

In this section I would try to solve the optimization problem for the classic investor with the CRRA utility function under the above assumptions.

From the above assumptions, the investor would have to solve the following optimization problem:

$$
\max_{c_t, \gamma} E \left[ \int_0^T e^{\gamma - \eta} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]
$$

subject to

$$
dW_t = (1 - \alpha_t)W_t r dt + \alpha_t m_t W_t dt + \alpha_t \sigma_t dZ_t - c_t dt
$$

where $\gamma$ is the investor’s relative risk aversion parameter, and $\eta$ is his/her subjective rate of discount. The first constraint shows the increment of the wealth at $t$ is the return on investment minus the consumption. The second constraint shows that the investor should end with nonnegative wealth.

Enlightened by Wachter (2002), it would be easier to solve the above problem by employing the martingale method developed by Cox and Huang (1989). I would first present a brief introduction of the method and then show the solutions.

### Martingale method

The basic idea of the martingale method is that any payoff can be gained as long as one starts with enough initial wealth in a complete market. In that complete market,
the investor can create a unique Arrow-Debreu security for each state of the world.\textsuperscript{5} Then solving the optimization problem (11) is same as constructing the optimal Arrow-Debreu security.

Since it’s a complete market without arbitrage, there exists a unique Arrow-Debreu price (or pricing kernel) denoted by $\phi_t$ which is the price of the unit probability of unit payoff in each state. And it has the following property:

$$E_s[\phi_tA_t] = \phi_tA_t \quad \text{for} \quad s > t. \quad (12)$$

The process of the pricing kernel can be obtained from the stock price process when the Novikov’s condition\textsuperscript{6} applies. The process of $\phi_t$ is given by the following equation:

$$d\phi_t = -r\phi_tdt - \frac{m_t - r}{\sigma_t}\phi_tdZ_t'. \quad (13)$$

And let $x_t = \frac{m_t - r}{\sigma_t}$, then by Ito’s lemma:

$$dx_t = \theta(x - \bar{x})dt + \frac{\sigma_x \rho + \nu_t}{\sigma_t^2}dZ_t'$$

where $\bar{x} = \frac{\mu - r}{\sigma_t}$. Define $k = \frac{\sigma_x \rho + \nu_t}{\sigma_t}$ and $\sigma_x = -\frac{k}{\sigma_t}$, then $dx_t$ can be expressed by:

$$dx_t = \theta(x - x_t)dt - \sigma_x dZ_t'.$$

So equation (13) can be written as:

$$d\phi_t = -r\phi_tdt - x_t\phi_tdZ_t'. \quad (14)$$

\textsuperscript{5} Berkelaar et al. (2004) pp.975

\textsuperscript{6} $E[\exp\left(\frac{1}{2}\int_x^t \sigma_s^2 dt\right)] < \infty$
Optimal solution

By applying the martingale method, the budget constraint in equation (11) can be converted into a static budget constraint described by:

\[ E\left[\int_0^T c_t \phi_t dt\right] = W_0. \quad (15) \]

That is to say, the amount of consumption multiplied by the price in the corresponding state equals the initial wealth.

Combining equation (11) with equation (15), what we need to do now is to solve the following maximization problem:

\[
\max_{c_t} E\left[\int_0^T e^{-\eta r} c_t^{1-\gamma} dt\right] - \lambda (E\left[\int_0^T c_t \phi_t dt\right] - W_0) \quad (16)
\]

where \( \lambda \) is the Lagrange multiplier.

We can get the optimal amount of consumption at time \( t \):

\[ c_t^* = (\lambda \phi_t)^\frac{1}{\gamma} e^{-\frac{\eta r}{\gamma}}. \quad (17) \]

The wealth at time \( t \) should follow:

\[ \phi_t W_t^* = E_t[\int_t^T \phi_s c_s^* ds]. \quad (18) \]

It can be explained by saying the discounted value of wealth equals the discounted value of consumption.

Substituting equation (17) into the static budget constraint yields:

\[ \lambda = W_0^{-\gamma} \left[ E\left[\int_0^T \phi_t^{\frac{1}{\gamma}} e^{-\frac{\eta r}{\gamma}} dt\right]\right]. \quad (19) \]

Then define \( N_t = (\lambda \phi_t)^{-1} \). \quad (20)

By the definition of \( \phi_t \) and the Ito’s lemma, we can get:

\[ dN_t = (r + x_t^2) N_t dt + x_t N_t dZ_t'. \quad (21) \]

Substituting equation (17) and (20) into (19), we can get:

\[ W_t = N_t E_t\left[\int_t^T N_s^{\frac{1}{\gamma}} e^{-\frac{\eta r}{\gamma}} ds \mid N_t, x_t\right]. \quad (22) \]
In the spirit of Wachter (2002), define:

\[ G(N_t, x_t, t) = W_t, \]  

and

\[ F(N_t, x_t, t; s) = N_t E[N_{t+1}^{-1} | N_t, x_t]. \]  

From equation (23) and (24), we can have:

\[ G(N_t, x_t, t) = \int_t^T F(N_t, x_t, t; s) e^{-\gamma s} \, ds. \]  

Equation (18) shows us the wealth needed at time t to finance the optimal consumption. What we need to do now is to find the optimal portfolio strategy that replicate the wealth process. Since it’s a complete market and there’s no arbitrage, the instantaneous excess return should equal the price of risk multiplied by the risk.

Applying Ito’s lemma, the instantaneous expected return at time t follows:

\[
\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} (\theta(x - x)) + \frac{\partial G}{\partial N} N(r + x^2) + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2_x \\
+ \frac{1}{2} \frac{\partial^2 G}{\partial N^2} N^2 x^2 - \frac{\partial^2 G}{\partial x \partial N} N x \sigma_x + \frac{1}{2} \frac{\partial^2 G}{\partial N^2} e^{-\gamma t} 
\]

for the purpose of simplicity, the subscript t is omitted.

The instantaneous risk equals:

\[
\frac{\partial G}{\partial N} N x - \frac{\partial G}{\partial x} \sigma_x. 
\]  

Denote equation (26) by (*), we can have the equation:

\[
(*) - r G = \left( \frac{\partial G}{\partial N} N x - \frac{\partial G}{\partial x} \sigma_x \right) x. 
\]  

Enlightened by the solution in Wachter (2002), we can guess the form of the solution as:

\[ F(N_t, x_t, t; T) = \frac{1}{N_t} e^{\frac{2}{\gamma} (T-t)} H(x_t, T-t), \]  

and

\[ H(x_t, \tau) = \exp \left\{ -\frac{1}{\gamma} \left[ \frac{A_1(\tau) x_t^2}{2} + A_2(\tau) x_t + A_3(\tau) \right] \right\}. \]

From equation (25), (29) and (30), we can get:

\[ G(N_t, x_t, t) = \frac{1}{N_t} e^{-\frac{2}{\gamma} T} \int_0^{T-t} H(x_t, \tau) d\tau. \]
Substituting (31) into (28), and following the method used in Kim and Omberg (1996) leads to the equation for $A_1(\tau)$:

$$\frac{dA_1}{d\tau} = b_1 A_1^2 + b_2 A_1 + b_3$$ (32)

which is a Riccati equation

where

$$b_1 = \frac{1 - \gamma}{\gamma}$$ (33)

$$b_2 = 2(\frac{\gamma - 1}{\gamma} \sigma_x - \theta)$$ (34)

$$b_3 = \frac{1}{\gamma} \sigma_x^2$$ (35)

The condition for a formal solution is $b_2^2 - 4b_1b_3 > 0$. Define a new variable $\varphi = \sqrt{b_2^2 - 4b_1b_3}$.

Then the solutions for $A_1, A_2$ and $A_3$ would be given by:

$$A_1(\tau) = \frac{1 - \gamma}{\gamma} \frac{2(1 - e^{-\varphi})}{2\varphi - (b_2 + \varphi)(1 - e^{-\varphi})}$$ (36)

$$A_2(\tau) = \frac{1 - \gamma}{\gamma} \frac{4\theta \bar{x}(1 - e^{-\varphi})^2}{\varphi(2\varphi - (b_2 + \varphi)(1 - e^{-\varphi}))}$$ (37)

$$A_3(\tau) = \int_0^\tau \frac{1 - \gamma}{2} A_1^2 + \frac{1}{2} \sigma_x^2 A_1 + \vartheta \bar{x} A_2 + (1 - \gamma)r - \eta d\zeta$$ (38)

Then by equation (17), (23), (24) and (29) it’s easy to get the consumption to wealth ratio as:

$$\frac{c_t}{W_t} = \left[\int_0^{T-t} H(x_t, \tau)d\tau\right]^{-1}.$$ (39)

At the same time the optimal asset allocation can be gained by:

$$\alpha_i = \frac{\partial G}{\partial N} \frac{x_i}{\sigma_s} - \frac{\partial G}{\partial \sigma_s} \frac{\sigma_s}{\sigma_s}$$

$$= \frac{1}{\gamma} \frac{x_i}{\sigma_s} \int_0^{T-t} H(x_t, \tau)(A_1(\tau)x_i + A_2(\tau))d\tau$$ (40)
Numerical analysis

Since we have got the solutions for the optimal asset allocation and consumption strategies for the investor with power utility function under the assumption of mean-reverting stock return (business cycle) with state uncertainty, it is possible for us to make some computation to investigate the properties of the investment and consumption strategies.

First I would like to investigate the impact of horizon on the optimal stock weight. Figure 3.2.1 shows the impact of horizon on the optimal stock weight for investors with different rates of relative risk aversions. For the purpose of computation, we assume that the current expected return is equal to its long-run average. Regardless of the values of the relative risk-aversion parameters, the optimal stock weight is an increasing function of the horizon. It first increases quickly with the horizon and then converges to a constant when the horizon is long enough. Besides we should notice that the more risk-seeking (lower $\gamma$) investor would invest more in stock.

Figure 3.2.1: the impact of horizon on the optimal stock weight

The figure shows how the optimal stock weight changes with the horizon for different rates of relative risk aversion. We assume the expected return $m_t$ equals its long-run average for this computation.

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7 The parameter values used for the numerical analysis are drawn from Dai et al. (2010) which are calculated based on Barberis (2000) and Wachter (2002). The table is shown in the appendix A.
Then I would like to investigate the impact of the expected return $m_t$ on the optimal stock weight. Figure 3.2.2 shows how the optimal stock weight changes with the change of the expected return $m_t$ while the horizon is assumed to be 50 years. Regardless of the attitudes toward risk, the relationship between the optimal stock and the value of the expected return is linear. Since there is no longing and shorting constraints, the optimal stock weight can either be larger than 1 or negative when the value of the expected return is relatively extreme. It can be calculated that the 95% confidence interval for the expected return $m_t$ is $(-0.0102, 0.1254)$. The optimal stock weight seems to be reasonable in that range. The line for the more risk-seeking (lower $\gamma$) investor is steeper which means that they would time the market more aggressively and a small change of the expected return would induce a relatively big change of the optimal stock weight compared with that of the less risk-tolerant investor. The optimal stock weight would be around zero when the expected return is around the risk-free rate.

![Figure 3.2.2: the impact of the expected return $m_t$ on the optimal stock weight for investors with different attitudes toward risk](image)

After that let us have a look at the impact of the horizon on the optimal amount of consumption. Figure 3.2.3 shows how the optimal consumption changes with the horizon when the wealth and the expected return are given as 100 and its long-run
average respectively. Regardless of the attitudes toward risk, the optimal consumptions are almost the same. The consumption is relatively low when the horizon is long enough, since the investor needs wealth to finance the future consumption. It can be called as the long-horizon effect. The investor would consume what he/she has when it approaches the end which can be called as the short-horizon effect. It makes sense since anything left after the end is meaningless to the investor. But we should realize that the scale of figure 3.2.3 would be a bit misleading. It would be better for us to have a look at the long-horizon effect alone.

Figure 3.2.4 shows the long-horizon effect. As can be seen from the figure, the more risk-seeking (lower $\gamma$) investor would consume less compared with that of the less risk-tolerant investor who has the same wealth and faces the same investment opportunity set. It is especially obvious when the horizon is longer. The more risk-seeking investor has a relatively strong incentive to save. And they want to invest to gain more future consumption at the cost of decreasing the current amount of consumption when the horizon is long enough. It seems as if they gamble on the future investment return.

![Figure 3.2.4: the impact of horizon on the initial optimal consumption for investors with different attitudes toward risk](image)

The wealth is assumed to be 100 and the expected return is set to be its long-run average.
Figure 3.2.4: **the long horizon effect**  The impact of horizon on the optimal consumption when the horizon is in the range of 30 years to 100 years.

Figure 3.2.5 shows how the optimal consumption changes with the expected return for different horizons and attitudes toward risk. Panel A, B and C represent the optimal consumption for the horizons of 90, 50 and 10 years respectively. Regardless of the attitudes toward risk, the optimal consumption would be high when the expected return is either high or very negative. It shows a u shape. As can be seen from panel A and B, the more risk-seeking (lower $\gamma$) investor would consume a bit less when the horizon is long enough for some range of the expected return (the long horizon effect discussed above). But when the horizon is short (10 years), the more risk-seeking investor would consume a bit more which can be called as the short horizon effect. The more risk-seeking investor has the incentive to save to gain more future consumption and the incentive to consume since on average the more aggressive investment strategy would gain more. The incentive to save would dominate when the horizon is long enough, and the incentive to consume would dominate when the horizon is short.
Figure 3.2.5: the impact of the expected return on the optimal consumption for different attitudes toward risk and different horizons. The wealth is set to be 100.

Since we have got the explicit solutions for the investor with power utility function, it’s possible for us to conduct some simulations over a whole time period to figure out the possible paths of wealth, consumption, etc. Here I simulate different scenarios for the investor with the relative risk aversion parameter of 5 over a horizon of 100 years. The initial wealth is assumed to be 100, and the expected return starts with its long-run average. The results of the simulation are shown below.

Figure 3.2.6 shows the average optimal stock weight path, as well as the 5% and 95% quantiles over the whole horizon. The optimal stock weight decreases very slowly and remains almost a constant over a considerable long period. It starts to decrease quickly when it is less than 20 years to the end.
Figure 3.2.6: **the optimal stock weight over time**  The figure shows the average behavior of optimal stock weight over time, as well as 5% and 95% quantiles. Year 0 means a horizon of 100 years.

Figure 3.2.7 shows the average behavior of the optimal consumption to wealth ratio, as well as 5% and 95% quantiles. The ratio remains a constant over a considerable long time period and increases dramatically towards the end of the horizon. As can be seen from the figure, the three lines almost overlap each other. That is to say, there is little variation in the consumption to wealth ratio under different economic environments for a given horizon. We should notice the scale of the figure would be a bit misleading. It would be better for us to have a look at the optimal consumption to wealth ratio for the long horizon alone. The paths of the ratio for long horizon are shown by figure 3.2.8. The conclusions we have drawn above seem to be right.

Figure 3.2.7: **the optimal consumption to wealth ratio over the whole time period**  The figure shows the average path, as well as 5% and 95% quantiles.
Figure 3.2.8: **The optimal consumption to wealth ratio for long horizon (20 years to 100 years)**

The figure shows the average path, as well as 5% and 95% quantiles.

Figure 3.2.9 shows the paths of wealth and the optimal consumption over the whole time period. Panel A shows the average behavior of the wealth over time, as well as 5% and 95% quantiles. The amount of wealth decreases quickly as time goes by. It has dropped from 100 to about 10 at year 40. Panel B shows the average path of optimal consumption over the whole time period, as well as the 5% and 95% quantiles. It also decreases quickly as the time goes by. Though the optimal consumption to wealth ratio is a constant over a considerable long period (figure 3.2.7), the wealth and the amount of consumption are decreasing over time. During the first 80 years, the consumption to wealth ratio is higher than the expected return can be gained from the investment. The low consumption during the final two decades results from the negligible amount of wealth left. The consumption decreases from over 8 to almost 0 over the horizon. It seems as if the consumption is quite volatile over the whole horizon. We would wonder if the phenomenon is caused by the high subjective rate of discount used for the simulation. I redo the same simulation with the new subjective rate of discount which equals 0.02 (the risk-free rate is 0.0168). Again we can find that the average consumption decreases quickly over time. The paths of wealth and optimal consumption are shown by panel C and D figure 3.2.9. So this phenomenon should be caused by the property of the specific preference.
Figure 3.2.9: the paths of wealth and optimal consumption over the whole time period for different subjective rates of discount. The initial wealth is set to be 100. Panel A and B show the paths of wealth and consumption for the investor with the subjective rate of discount being 0.0678. Panel C and D show the paths of wealth and consumption for the investor with the subjective rate of discount being 0.02 (the risk-free rate is 0.0168).

3C The investor with Epstein-Zin preference

Now let us consider the investor with Epstein-Zin preference. This utility function is first introduced by Epstein and Zin (1989, 1991). The continuous time Epstein-Zin utility function is defined recursively, and can’t be written explicitly as an expectation of future consumption.

The most important feature of Epstein-Zin utility function is that it can disentangle the elasticity of intertemporal substitution from the risk aversion. That is to say, it can disentangle the investor’s attitude toward the smoothness of consumption over states from the investor’s attitude toward the smoothness of consumption over
The utility function takes the following form:

\[ V_t = E_t \int_t^T f(c_s, V_s) ds \]

where

\[ f(c, V) = \begin{cases} 
\frac{\eta}{1-\frac{1}{\psi}} ((1-\gamma)V)((c((1-\gamma)V)^{-\frac{1}{1-\gamma}}-1) & \psi \neq 1 \\
\eta((1-\gamma)V)(\log c - \frac{1}{1-\gamma}\log((1-\gamma)V)) & \psi = 1
\end{cases} \]

The explicit solutions for this asset allocation and consumption choice problem are only available when the elasticity of intertemporal substitution equals 1 or the elasticity of intertemporal substitution is the reciprocal of the risk aversion parameter \( (\gamma = 1/\psi , \text{ and } V_t \text{ is actually the power utility function}) \) and the market is complete.

So, let us try to get a numerical solution by carrying out a numerical optimization of the Epstein-Zin utility function. The numerical optimization is based on the trinomial tree method which is developed and employed by Hull and White (1990, 1993).

In order to carry out the numerical optimization we need to discretize in time variable and also in space variable. For a given time horizon \( T \) and the times \( 0 = t_0 < t_i < ... < t_N = T \) set \( \Delta t = t_{i+1} - t_i \) for each \( i \). Here we assume the time instants are equally spaced which means that \( \Delta t = T/N \).

According to Hull and White (1990, 1993) the value of the space variable \( m \) on the tree is equally spaced and set \( \Delta m = 2\sqrt{3}/3^k \). \( \nu \) is the standard deviation of \( m \) at time \( t_{i+1} \) conditional on the value of \( m \) at time \( t_i \). According to the definition, \( \nu \) is a constant in this paper.

Based on the above information we can get the corresponding probabilities of the value of expected return \( m \) at time \( t_{i+1} \) given by:

\[ m = \text{The grid of the space variable is a bit different from what is used by Hull and White (1993). The change is designed to make the simulation more accurate.} \]
\[
p_u = \frac{v^2}{2\Delta m^2} + \frac{\eta^2}{2\Delta m^2} + \frac{\eta}{2\Delta m}
\]
\[
p_m = 1 - \frac{v^2}{2\Delta m^2} - \frac{\eta^2}{\Delta m^2}
\]
\[
p_d = -\frac{v^2}{2\Delta m^2} + \frac{\eta^2}{2\Delta m^2} - \frac{\eta}{2\Delta m}
\]

where \( \eta = M_{i,j} - K\Delta m \), \( K = \text{round}\left(\frac{M_{i,j}}{\Delta m}\right) \) and \( M_{i,j} \) is the mean of \( m \) at time \( t_{i+1} \) conditional on the value of \( m \) at time \( t_i \). \( p_u \), \( p_m \) and \( p_d \) are the probabilities of the value of \( m \) being \( (K+1)\Delta m \), \( K\Delta m \) and \( (K-1)\Delta m \) at \( t_{i+1} \) respectively.

According to equation (9) and the definition of \( k \), the process of \( m \) can be written as:

\[
dm_i = \theta(\overline{m} - m_i)dt + k\sqrt{\theta}dZ'. \quad (41)
\]

Then we can get the solution of \( m_i \):

\[
m_i = \overline{m} + e^{-\theta(t-s)}(m_s - \overline{m}) + k\int_s^t e^{-\theta(t-u)}dZ'_\mu. \quad (42)
\]

Since \( Z'_\mu \) is an observable standard Brownian motion, we can get the conditional mean and variance of \( m \):

\[
E_s[m_i | m_s] = \overline{m} + e^{-\theta(t-s)}(m_s - \overline{m})
\]

\[
\text{var}[m_i | m_s] = \frac{k^2}{2\theta}[1 - e^{-2\theta(t-s)}]
\]

If we set \( t - s = \Delta t \), we can get the following approximations:

\[
E_s[m_i | m_s] \approx m_s + \theta\Delta t(\overline{m} - m_s)
\]

\[
\text{var}[m_i | m_s] \approx k^2\Delta t
\]

Here I would like to investigate two cases: fixed percentage investment strategy and the flexible investment strategy.

The stock price in the market may change so rapidly and there is some uncertainty around it, then it may not be a good idea to time the market rigorously. The investor may employ some strategies such as investing a fixed percentage of
his/her total wealth in stock. The fixed percentage strategy means that the investor would hold more stock when the price of stock is low and hold less stock when the price of stock is high. So it makes some sense to employ the fixed percentage investment strategy. The proportion of the wealth invested in stock would depend on the investor’s attitude towards risk.

For the numerical analysis of the fixed percentage strategy, the time interval is set to be 0.1, and the time horizon is set to be 100 years. The range of the expected return is chosen to avoid cutting the probabilities implied by the trinomial tree method. The relative risk aversion parameter is set to be 5 and the elasticity of intertemporal substitution is assumed to be 0.05. The corresponding results are shown below.

First I would like to investigate the impact of the expected return on the optimal consumption. Figure 3.3.1 shows how the optimal consumption changes with the expected return $m_t$ when the percentage of the wealth invested in stock is 0.2 and wealth is 1. The optimal consumption is an increasing function of the expected return. The relation between them can nearly be viewed as linear. The kinks in the figure are induced by the specific approximation method employed for the analysis.

![Figure 3.3.1: the impact of the expected return on the optimal consumption to wealth ratio](image)

Enlightened by the relation demonstrated by the figure, we can assume that the optimal consumption at time $t$ can be expressed as a function of the expected return...
and the wealth at time t. That is to say, we assume that there exists the following function:

\[ c_t = f(m_t)W_t \]

where \( f(m_t) \) is the optimal consumption to wealth ratio at time t which depends on the value of \( m_t \).

The function \( f \) does depend on the horizon. But I find that the values of the parameters change very slightly when the horizon is reduced from 100 years to 50 years. So we can get the following approximation for the function \( f \):

\[ f(m_t) = 0.0259 + 0.0113(m_t - \bar{m}) \]

\[ f(m_t) = 0.0348 + 0.0362(m_t - \bar{m}) \]

for the strategies of investing 20% and 50% in stock respectively.

With the above approximation of the function, it’s possible for us to simulate the optimal consumption path over some time period. Here I would like to show the simulation results of the optimal consumption over a horizon of 50 years. The results are shown by figure 3.3.2.

Panel A of figure 3.3.2 shows the average behavior of the optimal consumption path, as well as the 5% and 95% quantiles for the strategy of investing 20% in stock. Panel B shows basically the same thing for the more aggressive strategy which invests 50% in stock. Over a horizon of 50 year, the average consumption path for the conservative strategy decreases slowly from about 2.6 to a bit more than 2.4. The average consumption path for the aggressive strategy almost remains a constant of 3.5 (it decreases very slightly). With the same wealth and the expected return the consumption with the conservative strategy would be lower than that with the aggressive strategy. This is what we should have expected from the approximation form of the function \( f \). Besides we should notice that the aggressive investment strategy would lead to a larger spread, but the investor would be better off in many cases compared with the circumstance with the conservative investment strategy.
Generally speaking, the changes of the consumption over the horizon are insignificant. Panel C and D show the average wealth paths, as well as the 5% and 95% quantiles for the strategy of investing 20% and 50% in stock respectively. As can be seen from the panels, the average wealth would go down a bit for the conservative investment strategy, and the average wealth would almost remain a constant for the aggressive investment strategy.

In the above analysis, we assume different investment strategies for the same investor. In principle the different investors would employ different investment strategies, e.g. more risk-tolerant investor would employ more aggressive investment strategies. The specific assumptions and analysis are designed for the purpose of investigating the smoothness of consumption and making the results more comparable.

Now, let us have a look at the flexible investment strategy. For the flexible

Figure 3.3.2: the optimal consumption paths and the corresponding wealth paths for conservative and aggressive investment strategies The 5% and 95% quantiles are also given in the figure. The initial wealth is set to be 100 and the expected return starts with its long-run average.
investment strategy, the investment strategy is taken into consideration in the simulation process. We first set the feasible set for the percentage of the total wealth that can be invested in stock (-1 to 2 for this analysis) and the feasible range for consumption to wealth ratio. Then try to find the optimal investment strategy and the optimal consumption to wealth ratio for a given expected return. By repeating the above steps, we expect that we can get some form of relationship between the optimal asset allocation and the expected return and the relationship between the optimal consumption to wealth ratio and the expected return.

In order to find the relations I calculate backwards over a horizon of 100 year with the time step of 0.1. The results are shown below.

Figure 3.3.3 shows how the optimal stock weight for an investor with Epstein-Zin preference changes with the expected return. The relationship between them can be viewed as linear for the intermediate range of the expected return. The optimal investment strategy would be quite extreme when the absolute value of $m_i$ is large if there is no limit on the feasible range of the stock weight. In reality it will have some small impact on the future analysis. But the impact would be small enough to be ignored, since the 95% confidence interval for the expected return is (-0.01, 0.12). Besides I find that the investment strategy would remain the same as long as the horizon is long enough.

![Figure 3.3.3](image)

Figure 3.3.3: the impact of the expected return on the optimal stock weight

The horizontal part is caused by the constraint on the feasible range of the investment strategy.

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9 In reality it will have some small impact on the future analysis. But the impact would be small enough to be ignored, since the 95% confidence interval for the expected return is (-0.01, 0.12).

10 For this specific calculation, the investment strategy would remain the same as long as the horizon is longer than 14.5 years.
Now let us investigate the relation between the optimal consumption to wealth ratio and the value of the expected return. Figure 3.3.4 shows how the optimal consumption changes with the expected return with the wealth of 1 and a horizon of 100 years. As can be seen from the figure, the optimal consumption to wealth ratio would be an increasing function for the feasible range of the expected return when the horizon is long enough\textsuperscript{11}.

![Figure 3.3.4: the impact of the expected return on the optimal consumption to wealth ratio for the investor with flexible investment strategy](image)

Again we can try to guess there exist some functions to approximate the relations. The functions are given:

\[
\alpha_t = g(m_t) \\
c_t = f_2(m_t)W_t
\]

where \( \alpha_t \) is the optimal percentage of the wealth invested in stock.

In reality, it would be easier to use the results got from the former calculation to form the investment and consumption strategy directly for the simulation.\textsuperscript{12}

Based on these, we can simulate the optimal consumption paths over a certain time period for different scenarios. The result for the simulation over a horizon of 50 years and initial wealth of 100 are shown below by figure 3.3.5.

The panel A shows that the average consumption would go up for the investor

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\textsuperscript{11} The calculation results show that this is the true when the horizon is longer than 30 years.

\textsuperscript{12} Details are given in the appendix C.
with Epstein-Zin preference over the period of 50 years when the horizon is long enough. Compared with figure 3.3.2, the flexible investment strategy would do better than the conservative fixed percentage investment strategy. But it would lead to a pretty large spread. Generally speaking, the average consumption for the flexible investment strategy would increase over the 50 years. Panel B shows the average behavior of the wealth as well as the 5% and 95% quantiles. On average the wealth would go up over the 50 years.

![Figure 3.3.5: the optimal consumption paths and wealth paths for the investors with flexible investment strategy](image)

The average behaviors of consumption and wealth, as well as the 5% and 95% quantiles are shown. The initial wealth is 100.

### 3D The investor with habit formation

Now I want to investigate the asset allocation and consumption optimization with state uncertainty when the habit formation is taken into consideration. Then investigate the impact of habit formation with state uncertainty on investment strategy and consumption choice.

The concept of habit formation has been used for explaining the risk premium puzzle, the macroeconomic output persistence, etc. And some success has been achieved. Sundaresan (1989) and Constantinides (1990) are considered to be two of the major theoretical papers in this field.
Sundaresan (1989) used the weighted average of the consumer’s past consumption as the benchmark. It is given by:

$$c_{ht} = c_{h0} + \int_{0}^{t} e^{-\beta(t-s)}c_s ds.$$  

Constantinides (1990) used a similar notation for the benchmark of consumption which is given by:

$$c_{ht} = c_{h0}e^{-\beta t} + \alpha \int_{0}^{t} e^{-\beta(t-s)} c_s ds.$$  

Obviously it is a more general form. The original habit $c_{h0}$ is also discounted by $\beta$ which is used to determine to what extent the past habit is discounted, and the parameter $\alpha$ is used to determine to what extent the current habit is influenced by the current consumption. For this reason I would employ the habit defined by Constantinides (1990) in this paper.

From the definition of $c_{ht}$, we can derive the evolution of $c_{ht}$ which is given by:

$$dc_{ht} = (\alpha c_t - \beta c_{h0}) dt.$$  

The change of the habit benchmark consists of two parts: the contribution from the current consumption and the depreciation of the past habit. With $c_t = c_{ht}$, the change of $c_{ht}$ depends on the value of $\alpha$ and $\beta$. If $\alpha = \beta$, the habit benchmark will remain a constant. If $\alpha > \beta$ ($\alpha < \beta$), the habit benchmark will gradually increase (decrease).

Assume that the investor wants to maximize the utility with habit formation defined by:

$$\int_{0}^{T} e^{-\eta t} u(c_t, c_{ht}) dt$$

where $c_t$ is the consumption at time $t$, $c_{ht}$ is the habit benchmark at time $t$ and $\eta$ is the subjective rate of discount.

The exact form of $u(c_t, c_{ht})$ is given by:

$$u(c_t, c_{ht}) = \frac{(c_t - c_{ht})^{1-\gamma}}{1-\gamma}$$
where $\gamma$ is the risk aversion parameter.

With equation (8) and (9), the investor needs to solve the following maximization problem:

$$
\max_{c_t, \alpha_t} E\left[ \int_0^T e^{-\gamma \eta} \frac{\left( c_t - c_{ht} \right)^{1-\gamma}}{1-\gamma} dt \right]
$$

subject to:

$$
dW_t = (1-\alpha_t)W_t r dt + \alpha_t m_t W_t dt + \alpha_t W_t \sigma_t dZ_t - c_t dt
$$

$$
dc_{ht} = (\alpha_t c_t - \beta c_{ht}) dt
$$

$$
W_T \geq 0
$$

where $\alpha_t$ is the proportion of the wealth invested in stock.

Munk (2008) solved the portfolio choice problem with habit formation and stochastic investment opportunities. But the state uncertainty is not considered in that paper. The optimization with habit formation and state uncertainty can be solved by following the method employed by Munk.

According to the definition of $x_t$ defined in section (3B), the process of the stock return would follow the stochastic process:

$$
dS_t = (r + x_t \sigma_t) S_t dt + \sigma_t S_t dZ_t.
$$

In the spirit of Munk (2008), define:

$$
\zeta_t = \exp\left(-\int_0^t x_s dZ_s - \frac{1}{2} \int_0^t x_s^2 ds\right)
$$

and

$$
\xi_t = \zeta_t e^{-\eta t}.
$$

Then $\xi_t$ is the state price deflator (stochastic discount factor).

Again define:

$$
F_t = \frac{1-e^{-\left(r+\beta-\alpha\right)(T-t)}}{r+\beta-\alpha}
$$

and

$$
G_t = E_t \left[ \int_t^T e^{-r \frac{\eta}{7} (T-t)} \xi_t^{\frac{1}{7}} e^{-r \frac{T-t}{7}} (1+\alpha F_s)^{-\frac{1}{7}} dS_s \right].
$$

Following Munk (2008) theorem 2, define:

---

13 If $x_s$ is a constant, the $\zeta_t$ is just the $H_t$ defined in Kim and Lee (2009).

14 It is the definition of $\gamma(t)$ in Gupta (2009) and can be viewed as the cost of maintaining one unit of habit.
\[ b = \theta - \frac{\gamma^{-1}}{\gamma} \sigma_s \]

and

\[ d = \sqrt{b^2 + \frac{\gamma^{-1}}{\gamma^2} \sigma_s^2} \] \[15\].

Then \( G_t \) can be solved in the form of:

\[ G_t = \int_t^T (1 + \alpha F_s)^{\frac{1}{\gamma}} e^{\frac{1}{\gamma} g_1(s-t)x_t^2 + g_2(s-t)x_t + g_3(s-t)} ds \]

where \( g_0, \ g_1 \) and \( g_2 \) are given by

\[ g_0(\tau) = -\frac{\eta}{\gamma} \tau - \frac{\gamma^{-1}}{\gamma} \tau - \frac{1}{2} \ln \frac{2d - (d - b)(1 - e^{-2d\tau})}{2d} \]

\[ + \frac{\gamma^{-1}}{2\gamma^2} \left( \frac{\theta^2 x^2}{d} + \frac{\sigma_s^2}{b + d} \right) \tau + \frac{\theta^2 x^2 (d - 2b)e^{-2d\tau} + 4b e^{-d\tau} e - d - 2b}{2d - (d - b)(1 - e^{-2d\tau})} \]

\[ g_1(\tau) = \frac{\gamma^{-1}}{\gamma^2} \frac{1}{d} (1 - e^{-d\tau})^2 \]

\[ g_2(\tau) = \frac{\gamma^{-1}}{\gamma^2} \frac{1 - e^{-2d\tau}}{2d - (d - b)(1 - e^{-2d\tau})} \]

Then following eq. (13) and (15) in Munk (2008) the optimal investment and consumption strategies are given by:

\[ \alpha_t^* = \frac{x_t}{\gamma \sigma_s} \frac{W_t - c_{it} F_t}{W_t} \]

\[ \frac{(1 + \alpha F_s)^{\frac{1}{\gamma}}}{\sigma_s} \int_t^T (g_1(s-t) + g_2(s-t)x_t)(1 + \alpha F_s)^{\frac{1}{\gamma}} e^{\frac{1}{\gamma} g_1(s-t)x_t^2 + g_2(s-t)x_t + g_3(s-t)} ds \]

\[ c_t^* = c_{it} + (1 + \alpha F_s)^{\frac{1}{\gamma}} \frac{W_t - c_{it} F_t}{\int_t^T (1 + \alpha F_s)^{\frac{1}{\gamma}} e^{\frac{1}{\gamma} g_1(s-t)x_t^2 + g_2(s-t)x_t + g_3(s-t)} ds} \]

\[ ^{15} \text{According to theorem 2 in Munk (2008), we need to assume } b + d > 0 \text{. This assumption will hold if we use the data employed by Dai et al. (2010).} \]
**Numerical analysis**

In order to conduct some comparisons, I would do some numerical analysis based on the above results. For this computation, the values of the parameter $\alpha$ and $\beta$ are given as 0.3 and 0.4 respectively.

First I would like to investigate the impact of horizon on the optimal stock weight. Panel A of figure 3.4.1 shows how the optimal stock weight changes with the horizon for different attitudes toward risk with the benchmark of consumption and wealth set to be 3 and 100 respectively. Panel B shows the corresponding results when the benchmark of consumption is set to be 4. As can be seen from the figure, the more risk-seeking (low $\gamma$) investor would behave differently from what the more risk-averting investor would do. The optimal stock weight would first decrease and then increase slightly ($c_{ht} = 3$) or nearly remain a constant ($c_{ht} = 4$) for the more risk-seeking investor. The optimal stock weight would be higher for the more risk-seeking investors regardless of the benchmark of the consumption which is what we should have expected. Besides, we should notice that the optimal stock weight is different for the same investor with different benchmarks of consumption. With the benchmark of consumption being 4, the investor would invest a bit less in stock. Since the marginal utility is infinite at $c_t = c_{ht}$, the investor would never consume less than the benchmark of consumption. That is to say, the benchmark serves as the minimum consumption level for the investor. With higher benchmark, the investor lowers the stock weight to reduce the volatility of the future consumption/wealth and makes sure that he/she can always consume at the level $c_t \geq c_{ht}$. The investor would become conservative if the benchmark is higher. So we would be curious about the investor’s behavior if the investor faces even higher benchmark (such as a benchmark of 5).

Again, we can find that the investor would lower the stock weight a bit compared with

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16 The values of the parameters are drawn from Dai et al. (2010) which are calculated based on Barberis (2000) and Wachter (2002).
that under the circumstance with the benchmark of 4. That is to say, the investor would employ a safety first approach as long as the benchmark is affordable. So the investor would be more conservative with some higher affordable benchmark.

The wealth is set to be 100, and the benchmarks of consumption are set to be 3, 4 and 5 for panel A, panel B and panel C respectively.

The parameter value of the stock volatility is 0.151 for the previous analysis. With frequent crises in the financial market, we would like to investigate the impact of large volatility of stock.

Figure 3.4.2 shows how the optimal stock weight changes with the horizon when the volatility equals 0.25 and the benchmark of consumption is assumed to be 4. Compared with panel B in the previous figure, all the investors lower the corresponding stock weights remarkably and the optimal stock weights for the more risk-averting investors no longer increase with the horizon. These phenomena are induced by safety first approach employed by the investors. The higher volatility means higher uncertainty. The investors lower their stock weight to decrease the
uncertainty of future consumption and make sure that the consumption can be maintained above the corresponding benchmark.

Then I would investigate the impact of the expected return $m_t$ on the optimal stock weight. Figure 3.4.3 shows how the optimal stock weight changes with the expected return for different attitudes toward risk. As can be seen from the figure, the more risk-seeking investor would time the market more aggressively.

We assume that the benchmark of consumption equals 4.

Now I would like to investigate the impact of horizon and the expected return on the optimal consumption.
Figure 3.4.4 shows how the optimal consumption changes with horizon for different attitudes toward risk and different benchmarks of consumption. The wealth is set to be 100, and the expected return is assumed to be its long-run average. The benchmarks are 3 and 4 for panel A and B respectively. Figure shows that the investor would maintain a low level of consumption if the horizon is long enough (longer than 20 years). But the investor would consume a considerable part of his/her wealth if the horizon is short. This is the reasonable phenomenon. With habit formation higher consumption would increase the current utility as well as the habit. Higher habit means that the investor would have to consume more ever since. For an investor with long horizon this increase in habit would be costly, so he/she would maintain a relative low consumption level. As can be seen from the figure, it is almost a horizontal line when horizon is long enough. For the investors with short horizon, the increase in habit is not that costly, and since there is no bequest motivation the investor would try to consume all of his/her wealth. That’s why the consumption increases so rapidly towards the end. Besides we should notice the investor would consume more when the benchmark is higher. It makes sense since the benchmark functions as the minimum consumption level.

Figure 3.4.4: how the optimal consumption changes with the horizon for different attitudes toward risk  The benchmarks are 3 and 4 for panel A and B respectively. We assume the wealth equals 100.
The scale of figure 3.4.4 would be a bit misleading, and the three lines representing the different investors almost overlap each other. It would be better to have a look at the short horizon and long horizon respectively. Panel A figure 3.4.5 shows the optimal consumption for different investors when the horizon is short and panel B demonstrates the same content when the horizon is long. As can be seen from the figure, the more risk-seeking investor would consume more when the horizon is long enough. A reasonable explanation is that the more risk-seeking investor would employ more aggressive investment strategy and expect fruitful investment return (on average when the horizon is long enough) which makes it possible to meet the higher benchmark in the future. But the more risk-seeking investor would consume a bit less compared with that of the more risk-averting investor when the horizon is short. This is caused by the fact that the more aggressive investment strategy employed by the risk-seeking investor would induce more uncertainty. The risk-seeking investor lowers his/her consumption level to make sure he/she can maintain the consumption above the benchmark before the end of the horizon. The horizon effects are quite different from those for the classic investors. With the habit formation, investors would focus on different aspects.

![Figure 3.4.5: the short horizon and long horizon effect](image)

We assume that the wealth equals 100 and the benchmark equals 4 for this analysis.

Figure 3.4.6 show how the optimal consumption changes with the expected return for different attitudes toward risk and different values of benchmarks. The
horizon is 50 years and the wealth is 100. The benchmarks of consumption are 3 and 4 for panel A and B respectively. Regardless of the different rates of relative risk aversion, the relations between the optimal consumption and expected return all exhibit U shape. The consumption would be high when the expected return is very large or very negative. Besides, higher benchmark would increase the optimal consumption while keeping the other conditions constant.

Since we have got the explicit solution for the investor with habit formation, it’s possible for us to conduct some simulations over a whole time period to figure out the possible paths of wealth, consumption, etc. Here I simulate different scenarios for the investor with the relative risk aversion parameter of 5 over a horizon of 100 years. The initial benchmark of consumption is 2. The values of $\alpha$ and $\beta$ are assumed to be 0.3 and 0.4 again. The initial wealth is assumed to be 100, and the expected return starts with its long-run average. The results of the simulation are shown below.

First, let us have a look at the optimal stock weight. Figure 3.4.7 shows how the optimal stock weight changes with the time (year 0 means a horizon of 100 years). The figure shows the average path, as well as the 5% and 95% quantiles. For the investor with the relative risk aversion parameter of 5, the average optimal stock weight decreases with time. It reaches about 0.1 when the end approaches.
We are much more interested in the paths of consumption, wealth and benchmark. Next, let’s have a look at the corresponding results.

Panel A figure 3.4.8 shows the evolution of the benchmark over time. The average path, as well as the 5% and 95% quantiles is given. Panel B shows the average behavior of the optimal consumption path, as well as the 5% and 95% quantiles. Panel C demonstrates the average path of the wealth as well as the 5% and 95% quantiles. On average the consumption would first go up from 3 to about 4, and begin to decrease slowly at year 20 to about 1.5 as the end approaches. It always remains a bit higher than the corresponding benchmark over the whole time period. The evolution of the benchmark follows the consumption since the relation between them is given by the equation and the specific parameter value used for the simulation. On average the amount of wealth would gradually decrease and reaches zero at the end of the time period.

Figure 3.4.9 shows the paths of benchmark of consumption, optimal consumption and wealth when the initial benchmark of consumption equals 3. As panel A shows on average the benchmark would be maintained at the level of 3 for about 12 years and decrease after that. The average optimal consumption would remain at the level of 4 for about 12 years and decrease ever since. The average behaviors of the benchmark and the consumption differ from that when the initial
benchmark equals 2. With the higher initial benchmark, the investor would have to consume a bit more at the beginning. And he/she would try to lower the consumption so as to lower the benchmark and makes sure the future consumption level would always be above the future benchmark. The paths of the wealth are almost the same as they are when the initial benchmark is 2. They decrease linearly.

Figure 3.4.8: the evolution of benchmark, consumption and wealth over time  

The initial wealth and the initial benchmark are 100 and 2 respectively. The values of the parameters \( \alpha \) and \( \beta \) are set to be 0.3 and 0.4 respectively.

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\(^{17}\) I do simulation with the benchmark of 4, and find the results are basically the same (the consumption level is maintained above the corresponding benchmark, and the consumption would gradually decrease to lower the benchmark to make sure the future consumption can be financed). It should follow the same trend as long as the initial benchmark is not incredibly high which makes it impossible for the investor to finance the future consumption and maintain the consumption above the corresponding benchmark.
Figure 3.4.9: the evolution of benchmark, consumption and wealth over time  The initial wealth and the initial benchmark are 100 and 3 respectively. The values of the parameters $\alpha$ and $\beta$ are set to be 0.3 and 0.4 respectively.

4 Comparison of the results

Since we have done numerical analyses for the investors with power utility, Epstein-Zin preference and habit formation, it’s possible for us to make some comparisons between the results and check the smoothness of consumption.

In this paper we assume that the stock return exhibits mean-reversion. So the large increase of the stock return would imply some decrease of the expected return in the future. Taking this fact into consideration, the investors won’t consume extravagantly as if the expected return would remain at the high level. That’s one sort of the consumption smoothing. So, first we can check the slope of the reaction of the consumption with respect to the change of the expected return to investigate the
smoothness of consumption.

In reality, the change of the stock return would have two effects: the impact on the stock return (wealth) and the impact on the consumption. Generally speaking, the two impacts would be different. If the return is -20%, the wealth would go down by about 10% for an investor who invests half of his/her wealth in stock. But we expect that his/her consumption would be relatively stable since he/she forms the expectation that the future equity return would not be as bad as it is. That is to say, the response of consumption to the stock shock would not be as dramatic as the shock is. First, let us investigate the impact of the expected return when the wealth is assumed to be a constant.

As can be seen from panel A figure 3.2.5, the optimal consumption would increase from 8.5 to about 9.4 if the expected return is 0.1 rather than 0.05 for the investor with the relative risk aversion of 5 and the wealth of 100 when the horizon is 90 years. For the horizons of 50 and 10 years, it would increase from 8.3 to 9.2 and from 12.3 to 13.3 respectively.

As demonstrated by figure 3.3.1, the increment of the optimal consumption is insignificant when the expected return is 0.1 rather than 0.05 for the investor with Epstein-Zin preference and 20% stock investment strategy. With flexible investment strategy, the optimal consumption would increase from 5 to about 5.8.

For the investor with habit formation and the relative risk aversion of 5, the increment of the optimal consumption depends on the value of benchmark and horizon. Figure 3.4.6 shows that it would increase from 4.1 to 4.3 when the benchmark of consumption is given as 3 and the horizon is 50 years.

Since the above results are based on different conditions such as horizon, it would be imprecise to make direct comparisons. But the statistics can still shed some light on the smoothness of consumption for the change of the expected return. The investor with habit formation would change the consumption level slightly w.r.t to
change of the expected return. For the investor with Epstein-Zin preference, the change depends on the investment strategy. The changes of the optimal consumption for the classic investor are a bit stable (but large) for different horizons as long as the wealth remains the same.

Now let us have a look at the stock shock in detail. From the setup of the financial market and the parameter values used for the numerical analysis, we know that \((\sigma, \sigma, \rho + v_i) / \sigma_s\) is negative which means that the expected return is negatively correlated with the stock shock. That is to say, positive stock shock (high current realized stock return) would decrease the estimation of the future expected return and hence decrease the optimal consumption to wealth ratio.

In order to demonstrate it more clearly, let us do some numerical analysis. We assume that the expected return is originally equal to its long-run average. We introduce a negative stock shock which is set to be \(dZ_{st} = -1.96\). So the realized stock return would be -23.84%. The amount of the initial wealth is assumed to be 100. We assume the horizon is long enough (100 years for this analysis).

Column 1 table 4.1 shows us the impacts on the wealth, consumption to wealth ratio and consumption for the classic investor. The wealth would go down by 14.1% for the negative stock shock, but the optimal consumption to wealth ratio would increase by 12.2%. So the consumption would just go down by 11.8% rather than 14.1%. Column 2 demonstrates the impacts on the wealth, consumption to wealth ratio and consumption for the investor with Epstein-Zin preference and the 20% investment strategy. The wealth would go down by 3.5% and the optimal consumption to wealth ratio would increase by 3.3%. The consumption would just go down by 2.8%. The impacts on the wealth, consumption to wealth ratio and consumption for the investor with habit formation (the initial habit is assumed to be 3)

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18 We assume the stock shock equals -1.96, since (-1.96, 1.96) indicates the 95% confidence interval for x if x is a standard normal variable.
19 This is just the change of wealth caused by the investment return. As demonstrated in table 4.1, the wealth would be affected by the consumption in the previous period.
are shown by column 3. The wealth would go down by 13.5% and the optimal consumption to wealth ratio would increase by 6%. Due to the increase of the consumption to wealth ratio, the consumption would just go down by 11.9% instead of 13.5%. The results seem to be what we have expected.

<table>
<thead>
<tr>
<th>Items</th>
<th>Classic</th>
<th>Epstein-Zin preference</th>
<th>Habit formation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realized stock return</td>
<td>-23.84%</td>
<td>-23.84%</td>
<td>-23.84%</td>
</tr>
<tr>
<td>Change of wealth A&lt;sup&gt;20&lt;/sup&gt;</td>
<td>-8.5%</td>
<td>-3.5%</td>
<td>-4%</td>
</tr>
<tr>
<td>Change of wealth B&lt;sup&gt;21&lt;/sup&gt;</td>
<td>-14.1%</td>
<td>-3.5%</td>
<td>-13.5%</td>
</tr>
<tr>
<td>Change of the consumption to wealth ratio&lt;sup&gt;22&lt;/sup&gt;</td>
<td>12.2%</td>
<td>3.3%</td>
<td>6%</td>
</tr>
<tr>
<td>Change of the consumption&lt;sup&gt;23&lt;/sup&gt;</td>
<td>-11.8%</td>
<td>-2.8%</td>
<td>-11.9%</td>
</tr>
</tbody>
</table>

Table 4.1: the impacts of the stock shock for different investors  
We assume the stock shock equals -1.96.

We are more interested in the consumption smoothing in the long-run. So it would be better to rank the smoothness of consumption by investigating the evolutions of the consumption paths over certain time period<sup>24</sup> for the three different

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<sup>20</sup> This item shows the change of wealth caused by the consumption in the previous period.

<sup>21</sup> This item shows the change of wealth caused by the asset return in the current period.

<sup>22</sup> It is compared with the consumption to wealth ratio when the expected return equals its long-run average. The results are the percentage of change rather the absolute value of change.

<sup>23</sup> The change of consumption is caused by the two effects: the change of wealth and the change of consumption to wealth ratio. The change of wealth consists of two components: the consumption in the previous period and the investment return in this period. The row named change of wealth B just shows the change of wealth caused by the investment return.

<sup>24</sup> We would investigate the consumption smoothing when the horizon is long enough rather than the case when the end approaches. The factors such as bequest motivation would have a large impact on the consumption behavior when the horizon is short.
categories of investors.

Figure 3.2.9 shows the optimal consumption path for the investor with 100 initial wealth and a horizon of 100 years. The average consumption level has decreased quickly from 9 at the beginning to 1 at year 50.

Figure 3.3.2 shows the optimal consumption paths for investor with Epstein-Zin preference and different investment strategies. As the figure shows, the average consumption is quite stable over a horizon of 50 years, though the consumption level is a bit lower compared with the initial consumption level of the investor with power utility function. The result for flexible investment strategy is shown by figure 3.3.5. The average consumption demonstrates an upward trend.

Figure 3.4.8 shows the optimal consumption for investor with habit formation when the initial wealth is 100 and the initial benchmark of consumption is 2. The average consumption would first go up and then decrease slowly. Over the first 50 years, the average consumption changes slightly.

The optimal consumption for the classic investor changes a lot over time. It is stable for the investor with Epstein-Zin preference. But the stability depends on the investment strategy. The consumption is relatively stable for investors with habit formation. The stability would depend on the initial benchmark and the specific evolution of the benchmark.

Of course we can make a comparison by calculating the variances of the consumption paths over a given time period. Large variance would mean higher volatility of consumption. But the variance depends on the deviation from its own mean which make it less comparable to the other variance since the means are different.
5 Conclusion and future research

This paper demonstrates the optimal asset allocation and consumption issues for three different categories of investors, namely the investor with power utility, the investor with Epstein-Zin preference and the investor with habit formation under the assumption of mean-reverting stock return (business cycle) and state uncertainty.

According to Gennotte (1986) and Dai et al. (2010), the optimization problem with the state uncertainty can be solved in two stages, namely the inference and optimization. In the inference stage we can make a transformation to make sure that the optimal consumption stream can be replicated. It is easier to solve the optimization problems after the transformation.

This paper shows us the explicit solutions for the investors with power utility and habit formation and numerical solution for the investor with Epstein-Zin preference. After that we conduct some computation and simulations to investigate the properties of the optimal strategies for different investors.

We are especially interested in the consumption smoothing. The paper shows us that the Epstein-Zin preference and habit formation will smooth the consumption stream. However, the classic investor with power utility would experience large changes of consumption.

In the paper we assume the investor has a clear idea about the evolution of the state variable. For the more general models, we can take the parameter uncertainty and model uncertainty into consideration. Besides, it is possible that the risky asset return is partially predictable. We assume there is no non-financial income during the investment horizon. But it would be more general if we take the non-financial income (labor income for the individuals, donate for the charity funds, etc) into consideration.

It would be interesting and challenging to investigate the optimal asset allocation and consumption issues when the asset return is predictable, there exist parameter
uncertainty and model uncertainty and the non-financial income is taken into consideration.
References


Appendix

A

The table shows the parameter values used for the computation and simulation. They are drawn from Dai et al. (2010) which are calculated based on Barberis (2000) and Wachter (2002). The parameters are in real annual term.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free rate</td>
<td>0.0168</td>
</tr>
<tr>
<td>Volatility of stock price</td>
<td>0.1510</td>
</tr>
<tr>
<td>Unconditional mean of expected stock return</td>
<td>0.0576</td>
</tr>
<tr>
<td>Volatility of expected return</td>
<td>0.0343</td>
</tr>
<tr>
<td>Mean reversion parameter</td>
<td>0.2712</td>
</tr>
<tr>
<td>Subjective discount rate</td>
<td>0.0624</td>
</tr>
<tr>
<td>Correlation coefficient</td>
<td>-0.9351</td>
</tr>
</tbody>
</table>

Table 1: the parameter values used for the computation and simulation

B

The figure shows the distribution of the expected return $m_t$ when the trinomial tree method is employed. We assume the expected return starts with its long-run mean. Panel A and Panel B show the distribution of $m_t$ after 100 years when $\Delta m = \sqrt{3} / 3$ and $\Delta m = \sqrt{3} / 2$ respectively. Since the 95% confidence interval of $m_t$ is $(-0.0102, 0.1254)$, so it would be better to set $\Delta m = \sqrt{3} / 3$ when the time step is 0.1. Besides we should notice we need a wide range of $m_t$ to run the computation and simulation. It functions as the buffer for the numerical analysis.

Figure: the simulated distribution of the expected return $m_t$ when the trinomial tree method is employed
C

In order to simulate the optimal consumption path, we need to find the optimal asset allocation and consumption strategies. We can estimate the functions which can approximate the relation between the investment strategy and the expected return \((\alpha_t = g(m_t))\) and the relation between the consumption strategy and the expected return \((c_t = f_2(m_t)W_t)\).

But we can also do it by the following method. The backward calculation would give us the relation between the investment strategy and the expected return and the relation between the consumption strategy and the expected return. The relations are shown by figure 3.3.3 and 3.3.4. Instead of the smooth lines demonstrated by the figures, the results are stored by points in the matlab files. It is possible for us to extract three vectors representing the range of the expected return, the corresponding investment strategy and consumption strategy.

We can simulate the expected return under different scenarios. For each simulated expected return, we can locate it in the vector representing the range of the expected return. After that we can get the corresponding components in the other two vectors as the investment strategy and consumption strategy for this simulated expected return.

D

For the different investors with different strategies, we do tests to check if the budget constraint is satisfied. With the Girsanov's theorem, we do the simulation under “Q” (the risk-neutral measure). Then the model for the stock would have a constant drift \(rS_t\). The discounted future consumption should equal the initial wealth if the budget constraint is satisfied. The simulation results are around 100 (which is the initial wealth) such as 99.8310, 100.2190, etc. Taking the Monte Carlo error into consideration, we should be able to draw the conclusion that the budget constraint is satisfied.